1 Exercises Radon transform part 1

Exercise 1.1. (a) Use the Fourier slice theorem to prove the formula for the Radon transform of $\partial_x^\alpha f$

$$R_\theta \partial_x^\alpha f = \theta^\alpha \partial_s^{|\alpha|} R_\theta f$$

(b) prove the formula for the Radon transform of the convolution $f * g$

$$R_\theta (f * g) = R_\theta f * R_\theta g$$

Exercise 1.2. In this exercise we consider the Radon transform of radial functions in $n = 2$ dimensions.

(a) The α-Abel transform is defined as

$$A_\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_t^\infty \frac{g(s)}{(s-t)^{1-\alpha}}\, ds.$$

Use the formula

$$\int_x^s \frac{dt}{(t-x)^\alpha (s-t)^{1-\alpha}} = \Gamma(\alpha)\Gamma(1-\alpha)$$

to show that, for sufficiently smooth g, we have have

$$(-\partial_x A_{1-\alpha} \circ A_\alpha) g = g,$$

i.e. we have a left-inverse for A_α.

(b) Suppose $f(x) = F(|x|)$ and $\tilde{F}(r) = F(\sqrt{r})$. Show that for such f, Rf is independent of θ and can be written as

$$Rf(s) = \sqrt{\pi} A_{1/2} \tilde{F}(s^2).$$

(c) Derive an inversion formula for the Radon transform for radial functions.

Exercise 1.3. (a) Assume $f(x)$ is of the form

$$f(x) = F(|x|) \left(\frac{x_1 + i x_2}{|x|} \right)^m, \qquad x \neq 0. \tag{1}$$

Show that in polar coordinates (r, ϕ) with $x_1 = r\cos\phi$, $x_2 = r \sin\phi$ the right hand side of this equation becomes $F(r) e^{im\phi}$.

(b) Suppose that f is given by (1). Show that $g = Rf$ can be written in the form

$$g(\theta, s) = G(s) \big(\theta_1 + i\theta_2\big)^m. \tag{2}$$

(c) Suppose that $g(\theta, s)$ can be written in the form (2). Show that $h = R^\# g$ can be written in similar form as (1), i.e.

$$h(x) = H(|x|) \left(\frac{x_1 + i x_2}{|x|} \right)^m.$$

(In [1, pp. 25-30] the map $F \mapsto G$ is studied and an alternative inversion formula for the Radon transform is given.)

References

[1] F. Natterer. *The mathematics of computerized tomography*, volume 32 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2001. Reprint of the 1986 original.