2 Exercises Radon transform part 2

Exercise 2.1. In this exercise we discuss a certain relationship between Sobolev spaces and ill-posedness.

(a) Let $\Xi^s = \mathrm{Op}(\langle\xi\rangle^s)$ as in Grubb Chapter 6. Let $\alpha > 0$. Show that $\Xi^{-\alpha}$ is continuous on $L_2(\mathbb{R}^n)$.

(b) Let $H = L_2(\mathbb{R}^n)$ Show that the problem

$$\text{given } g \in H, \text{ determine } f \text{ in } H \text{ such that } \Xi^{-\alpha} f = g$$

is ill-posed.

(c) Let A be a continuous operator on H. Suppose that there are constants c and C such that for all $f \in H$ we have

$$c\|f\|_{H^{-\alpha}(\mathbb{R}^n)} \le \|Af\|_{L_2(\mathbb{R}^n)} \le C\|f\|_{H^{-\alpha}(\mathbb{R}^n)} \tag{1}$$

Show that the inversion problem for A is ill-posed.

(d) We define a Sobolev like norm for functions of (θ, s) by

$$\|g\|^2_{H^\alpha(Z)} = \int_{S^{n-1}} \int_{\mathbb{R}} (1+\sigma^2)^\alpha \, |\hat{g}(\theta,\sigma)|^2 \, d\sigma d\theta.$$

Consider now the Radon transform R in n dimensions. Let Ω_n be the unit ball in $\mathbb{R}^n$. We consider $f \in C_0^\infty(\Omega_n)$. Show the left equality of (1) with $\alpha = (n-1)/2$, i.e. show that there is a constant C such that

$$c\|f\|_{H^{-(n-1)/2}(\Omega_n)} \le \|Rf\|_{L_2(Z)}$$

Hint: Use the Fourier slice theorem and show first that

$$\|Rf\|^2_{L_2(Z)} = 2(2\pi)^{n-1} \int_{S^{n-1}} \int_0^\infty |\hat{f}(\sigma\theta)|^2 \, d\sigma \, d\theta.$$

then substitute $\xi = \sigma\theta$ and complete the estimate.