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Parametrix for a hyperbolic initial value problem with dissipation in some region

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Abstract. We construct parametrices for initial value problems of the form

$$(\partial_z - \mathrm{i}A(z,x,D_x) + B(z,x,D_x))u = 0, \quad z > z_0, \quad u(z_0,\cdot) = u_0, \tag{*}$$

where $(z,x) \in \mathbb{R} \times \mathbb{R}^n$, $A(z,x,D_x)$ is a family of order 1 pseudodifferential operators with homogeneous real principal symbol $a(z,x,\xi)$, and $B(z,x,D_x)$ is a family of order $\gamma > 0$ pseudodifferential operators with non-negative homogeneous real principal symbol $b(z,x,\xi)$. The parametrix is a family of pseudodifferential operators when $A = 0$, and a Fourier integral operator with real phase function if $A \neq 0$. A priori this leads to symbols of type $(\rho,\delta) = (1-\frac{\gamma}{2}, \frac{\gamma}{2})$, which limits our construction to $\gamma < 1$, and leads to operators with a complicated symbol calculus in the case $\gamma = 1$. With an additional assumption on B we obtain symbols of type $(\rho,\delta) = (1-\frac{\gamma}{L}, \frac{\gamma}{L})$, for some $L \geqslant 2$. The assumption implies in particular that the first $L-1$ derivatives of b vanish where $b = 0$. Parametrices for (*) are constructed for the case when $2\gamma < L$.

Keywords: Fourier integral operators, pseudodifferential initial value problem

1. Introduction

In this paper we study pseudodifferential operators of the form

$$P = \partial_z - \mathrm{i}A(z,x,D_x) + B(z,x,D_x). \tag{1}$$

We assume that A and B satisfy

(i) $A = A(z,x,D_x)$ is a smooth family of pseudodifferential operators in $\operatorname{Op} S^1(\mathbb{R}^n \times \mathbb{R}^n)$, with homogeneous real principal symbol $a = a(z,x,\xi)$.
(ii) $B = B(z,x,D_x)$ is a smooth family of pseudodifferential operator in $\operatorname{Op} S^\gamma(\mathbb{R}^n \times \mathbb{R}^n)$, $\gamma > 0$, with non-negative real homogeneous principal symbol $b = b(z,x,\xi)$. To be precise we assume b homogeneous for $|\xi| \geqslant 1$, and smooth for $|\xi| < 1$.
(iii) The derivatives up to order $L-1$, $L \geqslant 2$ of b and of $B_{\mathrm{s}} = B - b$ satisfy

$$\begin{aligned} &\bigl|\partial_{(z,x)}^\alpha \partial_\xi^\beta b(z,x,\xi)\bigr| \leqslant C\bigl(1+|\xi|\bigr)^{-|\beta| + \frac{|\alpha|+|\beta|}{L}\gamma}\bigl(1+b(z,x,\xi)\bigr)^{1-\frac{|\alpha|+|\beta|}{L}}, \\ &|\alpha| + |\beta| < L, \end{aligned} \tag{2}$$

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$$\left|\partial^{\alpha}_{(z,x)}\partial^{\beta}_{\xi}B_{\mathrm{s}}(z,x,\xi)\right| \leqslant C\big(1+|\xi|\big)^{-|\beta|-1+\frac{|\alpha|+|\beta|+2}{L}\gamma}\big(1+b(z,x,\xi)\big)^{1-\frac{|\alpha|+|\beta|+2}{L}}, \qquad (3)$$
$$|\alpha|+|\beta|+2<L.$$

Without B the operator P would be a standard first order hyperbolic operator. The operator B introduces damping into the equation. Because B is of order $\gamma > 0$, this will lead to the suppression of singularities propagating in a region where $b \neq 0$. The third assumption means that B increases slowly near points where $b = 0$. It implies that the derivatives of b up to order $L-1$ vanish on the set of (z, x, ξ) where $b = 0$. It is automatically satisfied for $L = 2$, $\gamma = 1$ by the inequality $b'^2 \leqslant 2b\|b''\|_{L^\infty}$. By (ii) it is also satisfied for $|\alpha| + |\beta| \geqslant L$, with a constant $C_{\alpha,\beta}$ depending on α, β. We study the initial value problem for P with initial value at some $z_0 \in \mathbb{R}$, given by

$$Pu(z,\cdot)=0, \quad \text{for } z\in\,]z_0,Z[, \quad u(z_0,\cdot)=u_0. \qquad (4)$$

Operators of this type appear in one-way wave equations, see [7]. Here B is assumed to vanish on a large subset of $(\mathbb{R}\times\mathbb{R}^n)\times\mathbb{R}^n$, that has a non-empty interior. Outside this set the equation should not admit the propagation of singularities and B is assumed to be non-zero. As a consequence B vanishes to all orders at the boundary of the interior of the zero set. Operators of the type (1) also appear in drift diffusion equations. Estimates for exponential decay in hyperbolic first order systems, with $B = 0$, were given in [6].

The existence and uniqueness of solutions to (4) follows from the results in Section 23.1 of [3] for $\gamma \leqslant 1$. In Section 2 we extend this to the case where B satisfies (2), (3) with $2\gamma < L$. The solution operator will be denoted $E : H^s(\mathbb{R}^n) \to C([z_0, Z]; H^s(\mathbb{R}^n))$. By $E(z, z_0)$ we will denote the map $u_0 \to Eu_0(z, \cdot)$.

The solutions will be related to those of the purely hyperbolic operator P_0 defined by

$$P_0=\partial_z-\mathrm{i}A(z,x,D_x). \qquad (5)$$

By E_0 and $E_0(z, z_0)$ we denote the solution operator to the initial value problem for P_0. It is well known that E_0 is a Fourier integral operator, and can also be defined for $z \leqslant z_0$ if A and B are defined there, see [2] or a text on Fourier integral operators such as [1, Section 5.1] (note that $A(z, x, D_x)$ is not strictly a pseudodifferential operator in (z, x), but a z-family of pseudodifferential operators in x, but the argument remains valid, see, e.g., Theorem 18.1.35 of [3].) Let $p_0 = p_0(z, x, \zeta, \xi) = \mathrm{i}\zeta - \mathrm{i}a(z, x, \xi)$ denote the principal symbol of P_0. For $B = 0$ the singularities (elements of the wave front set of the solution) of the solution propagate on the set $p_0 = 0$ according to the Hamilton vector field of p_0/i, which reads

$$\frac{\partial}{\partial z}-\frac{\partial a}{\partial \xi}\cdot\frac{\partial}{\partial x}+\frac{\partial a}{\partial x}\cdot\frac{\partial}{\partial \xi}. \qquad (6)$$

The solution curves to this field are called bicharacteristics. They can be parameterized by z. We denote the (x, ξ) components of the solution curve with initial values (x_0, ξ_0) at z_0 by $(\Gamma_x(z, z_0, x_0, \xi_0), \Gamma_\xi(z, z_0, x_0, \xi_0))$.

Let $I = I(z, x, \xi)$ be the integral of B along a bicharacteristic with initial values x, ξ at z (not at z_0)

$$I(z,x,\xi)=\int_{z_0}^{z} b\big(z',\Gamma_x(z',z,x,\xi),\Gamma_\xi(z',z,x,\xi)\big)\,\mathrm{d}z'. \qquad (7)$$

Let Φ_{z,z_0} denote the bicharacteristic flow on $\mathbb{R}^n \times \mathbb{R}^n \setminus 0$, i.e.,

$$\Phi_{z,z_0}(x,\xi) = \big(\Gamma_x(z,z_0,x,\xi), \Gamma_\xi(z,z_0,x,\xi)\big). \tag{8}$$

We also define $\tilde{I} = \tilde{I}(z,x,\xi)$ by

$$\tilde{I}(z,x,\xi) = (I \circ \Phi_{z,z_0})(z,x,\xi) = \int_{z_0}^{z} b\big(z', \Gamma_x(z',z_0,x,\xi), \Gamma_\xi(z',z_0,x,\xi)\big)\,\mathrm{d}z'. \tag{9}$$

The factors $\exp(-I(z,x,\xi))$, $\exp(-\tilde{I}(z,x,\xi))$ will be used in the construction of solutions to (4).

We will construct a parametrix for (4), in the form of a family of pseudodifferential operators if $A = 0$, or a Fourier integral operator with real phase function if $A \neq 0$. The factor $\exp(-I)$ will be part of the amplitude. A complication is that $\exp(-I)$ is not a standard symbol in S^0. Instead we need the symbol classes $S^\mu_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ of type ρ, δ. By definition a function f in $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ is in the space $S^\mu_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ of symbols of order μ and type ρ, δ, if there are constants $C_{\alpha,\beta}$ such that

$$\big|\partial_x^\alpha \partial_\xi^\beta f(x,\xi)\big| \leqslant C_{\alpha,\beta}\big(1+|\xi|\big)^{\mu+|\alpha|\delta-|\beta|\rho}. \tag{10}$$

The case $\rho = 1, \delta = 0$ is the standard case. For $\gamma = 1$ and without additional assumptions on B the problem (4) leads to operators of type $(\rho,\delta) = (\frac{1}{2},\frac{1}{2})$. For example, when $A = 0$ and B is independent of z it is known that $E(z,z_0) = \exp(-(z-z_0)B(z,x,D_x))$ is in $\mathrm{Op}\, S^0_{1/2,1/2}$ (see the remarks in [9, p. 515]). This case has also been analyzed using Fourier integral operators with complex-valued phase function [4,5].

Symbols of type ρ, δ do not always lead to a good calculus of pseudodifferential and Fourier integral operators. For pseudodifferential operators many of the usual results hold when $\rho - \delta > 0$. If on the other hand $\rho - \delta = 0$ then the commutator of two pseudodifferential operators is no longer of lower order. In that case a more refined analysis is required, which we will not discuss here. For Fourier integral operators one has $\rho = 1 - \delta$, and the requirement becomes $\frac{1}{2} < \rho \leqslant 1$, see, e.g., Theorem 2.4.1 in [1].

Here we show that when (2), (3) are satisfied with $2\gamma < L$, then we have symbols in classes $S^\mu_{1-\gamma/L,\gamma/L}$, so that a good calculus exists. A first indication of such behavior is that, if f is a symbol $[0,\infty[\to \mathbb{R}$, i.e., satisfies $|f^{(k)}(y)| < C_k(1+y)^{\delta-k}$, then $f \circ b$ is a symbol in $S^{\max(\delta,0)}_{1-\gamma/L,\gamma/L}$. The proof (almost immediate) is given in Section 2, where we also show that a pseudodifferential square root $\sqrt{1+B}$ exists modulo a regularizing operator. Our first main result is the following theorem about the solution operator E to the initial value problem (4). When we write that W is a bounded family of pseudodifferential operators in $S^{\mu_0}_{\rho,\delta}$ with $\partial_z^j W$ in $S^{\mu_j}_{\rho,\delta}$, then it will be understood that $\partial_z^j W$ is also a bounded family in this class.

Theorem 1. *Let A and B satisfy* (i)–(iii)*, and assume that $2\gamma < L$. Then there are bounded families of pseudodifferential operators $W = W(z,x,D_x)$ and $\widetilde{W} = \widetilde{W}(z,x,D_x)$ in* $\mathrm{Op}\, S^0_{1-\gamma/L,\gamma/L}(\mathbb{R}^n \times \mathbb{R}^n)$*, with $\partial_z^j W$ and $\partial_z^j \widetilde{W}$ in* $\mathrm{Op}\, S^{j\gamma}_{1-\gamma/L,\gamma/L}(\mathbb{R}^n \times \mathbb{R}^n)$*, such that*

$$W(z,x,D_x)E_0(z,z_0) = E(z,z_0) = E_0(z,z_0)\widetilde{W}(z,x,D_x). \tag{11}$$

The symbol $W(z,x,\xi)$ can be written as an asymptotic sum

$$W = \left(1 + \sum_{j=1}^{\infty} K^{(j)}\right) \exp(-I) + R, \tag{12}$$

where R is a smooth family of symbols in $S^{-\infty}(\mathbb{R}^n \times \mathbb{R}^n)$, and the $K^{(j)}$ satisfy

$$K^{(j)} \text{ is a smooth family of symbols in } S^{j(2\gamma-1)}(\mathbb{R}^n \times \mathbb{R}^n),\ z \in [z_0, Z],\ \text{and} \tag{13}$$

$$K^{(j)} \exp(-I) \text{ is a bounded family of symbols in } S^{-j(1-\frac{2\gamma}{L})}(\mathbb{R}^n \times \mathbb{R}^n),\ z \in [z_0, Z],$$
$$\text{with } \partial_z^k (K^{(j)} \exp(-I)) \text{ in } S^{k\gamma - j(1-\frac{2\gamma}{L})}(\mathbb{R}^n \times \mathbb{R}^n). \tag{14}$$

Furthermore, $K^{(j)}(z,x,\xi) = 0$ for (z,x,ξ) such that $I(z,x,\xi) = 0$. The same is true for $\widetilde{W}(z,x,\xi)$ with $\tilde{I}$ instead of I.

When $I(z,x,\xi) > 0$ for some point (z,x,ξ), then $\exp(-I(z,x,\lambda\xi))$ and its derivatives decay exponentially for $\lambda \to \infty$, and $\exp(-I)$ is in $S^{-\infty}$. Hence for a bicharacteristic with some finite part in the region of cotangent space where $B > 0$, the factor W is in $S^{-\infty}$, and the solution operator becomes regularizing. The difficult part is therefore the behavior near the boundary of the region $B > 0$.

For each z derivative our estimate of $W(z,x,\xi)$ worsens by a factor $(1 + |\xi|)^{\gamma}$. It turns out that in fact there is a better estimate. Let C_0 denote the canonical relation of E

$$C_0 = \big\{ \big(z, \Gamma_x(z,z_0,x_0,\xi_0), a(z_0,x_0,\xi_0), \Gamma_\xi(z,z_0,x_0,\xi_0); x_0, \xi_0\big) \mid$$
$$(x_0,\xi_0) \in \mathbb{R}^n \times \mathbb{R}^n \setminus 0, z > z_0 \big\}. \tag{15}$$

Theorem 2. *The symbols W and $\widetilde{W}$ are in $S^0_{1-\gamma/L,\gamma/L}((]z_0, Z[\times\mathbb{R}^n) \times \mathbb{R}^n)$, and the map E is a Fourier integral operator in $I^{-1/4}_{1-\gamma/L}(]z_0, Z[\times\mathbb{R}^n, \mathbb{R}^n; C_0)$.*

Recall that a symbol in $S^{\mu}_{\rho,\delta}((]z_0, Z[\times\mathbb{R}^n) \times \mathbb{R}^n)$ is assumed to satisfy estimates like (10) only locally. A function $f = f(z,x,\xi)$ is in $S^{\mu}_{\rho,\delta}((]z_0, Z[\times\mathbb{R}^n) \times \mathbb{R}^n)$ if for each α, β and each conically compact subset K of $(]z_0, Z[\times\mathbb{R}^n) \times \mathbb{R}^n)$ there is a constant $C_{\alpha,\beta,K}$ such that

$$\big|\partial^{\alpha}_{z,x} \partial^{\beta}_{\xi} f(z,x,\xi)\big| \leqslant C_{\alpha,\beta,K} \big(1 + |\xi|\big)^{\mu + |\alpha|\delta - |\beta|\rho}, \quad (z,x,\xi) \in K. \tag{16}$$

There need not be constants such that (16) is valid globally. Indeed we find bounds for the z-derivatives of $W(z,x,\xi)$ that blow up when $z \to z_0$.

The organization of the paper is as follows. In the next section we discuss the well-posedness of the problem (4). We then study the case $A = 0$. In Section 3 we estimate various quantities in terms of powers of $(1 + |\xi|)$ and $(1 + I)$, and we discuss symbols estimates for functions with a factor $\exp(-I)$ that result from this. We then prove Theorem 1 for the case $A = 0$ in Section 4. In Section 5 we use this result and Egorov's theorem to prove the case $A \neq 0$. We then prove the Fourier integral operator property in Section 6. In the final section we discuss a choice of b that satisfies the assumption (2).

To denote a constant we will use the letter C. The value of C can change between equations.

2. Well-posedness

In this section we construct the square root $\sqrt{1+B}$, modulo a smoothing operator. Using this square root we show that the problem (4) has unique solutions satisfying energy estimates. The following lemma concerns the square root.

However, we first show that if $f : [0, \infty[\to \mathbb{R}$ is a symbol of order δ, that is for each k there is a C_k such that $|f^{(k)}(y)| < C_k(1+y)^{\delta-k}$, then $f \circ b$ is a symbol in $S^{\max(\delta,0)}_{1-\gamma/L,\gamma/L}$. A simple computations shows that $\partial^{\alpha}_{(z,x)}\partial^{\beta}_{\xi} f(b)$ is a sum of terms of the form

$$cf^{(k)} \prod_{j=1}^{k} \partial^{\alpha_j}_{(z,x)}\partial^{\beta_j}_{\xi} b, \tag{17}$$

where c is a constant, $\sum_j \alpha_j = \alpha$, $\sum_j \beta_j = \beta$ and $(\alpha_j, \beta_j) \neq 0$. By (2) this is less or equal than

$$C'_{K,\alpha,\beta}\bigl(1+|\xi|\bigr)^{-\beta+\frac{|\alpha|+|\beta|}{L}\gamma}(1+b)^{\delta-\frac{|\alpha|+|\beta|}{L}} \leqslant C_{K,\alpha,\beta}\bigl(1+|\xi|\bigr)^{\max(\delta,0)-\beta+\frac{|\alpha|+|\beta|}{L}\gamma}, \tag{18}$$

hence $f \circ b$ is in $S^{\max(\delta,0)}_{1-\gamma/L,\gamma/L}$.

Lemma 3. *Assume that B is selfadjoint and satisfies* (ii) *and* (iii) *and that $2\gamma < L$. Then there is a smooth family of pseudodifferential operators*

$$Q(z, x, D_x) \in \operatorname{Op} S^{\gamma/2}_{1-\gamma/L,\gamma/L}\bigl(\mathbb{R}^n \times \mathbb{R}^n\bigr)$$

with $\partial^j_z Q(z, x, D_x) \in \operatorname{Op} S^{\gamma/2+j\gamma/L}_{1-\gamma/L,\gamma/L}(\mathbb{R}^n \times \mathbb{R}^n)$, such that Q is selfadjoint and $Q^2 = 1 + B + R$, with $R \in S^{-\infty}$.

Proof. The proof is a variant of the standard construction as an asymptotic sum (see, e.g., Lemma II.6.2 in [8])

$$Q = Q^{(0)} + Q^{(1)} + \cdots. \tag{19}$$

In this construction the first term satisfies $Q^{(0)} = (1+b)^{1/2} + \text{l.o.t.}$ A remainder is defined

$$R^{(k)} = \left(\sum_{j=0}^{k-1} Q^{(j)}\right)^2 - (B+1), \tag{20}$$

and this is used to define the symbol of the next term, such that

$$Q^{(k)} = -\frac{1}{2}(1+b)^{-1/2}R^{(k)} + \text{l.o.t.} \tag{21}$$

We must show that the $Q^{(k)}$ defined in this way are symbols of decreasing order.

As an induction hypothesis we assume that $R^{(k)}$ is selfadjoint and a sum of terms of the form

$$c(1+b)^{1-l} \prod_{j=1}^{k'} \partial^{\alpha_j}_{x}\partial^{\beta_j}_{\xi} B_{\mathrm{s}} \prod_{j=k'+1}^{l} \partial^{\alpha_j}_{x}\partial^{\beta_j}_{\xi} b \tag{22}$$

modulo $S^{-\infty}$ and with $\sum_j |\alpha_j| + k' = \sum_j |\beta_j| + k' \geqslant k$. By (2) and (3) each of these terms is a bounded family of symbols in $S^{\gamma-(\sum_j |\alpha_j|+k')(1-2\gamma/L)}(\mathbb{R}^n \times \mathbb{R}^n)$. Set $Q^{(k)}$ equal to the $-\frac{1}{2}(1+b)^{-1/2}$ times the sum of the terms with $\sum_j |\alpha_j| + k' = k$, and make it selfadjoint by averaging with the adjoint. Then $Q^{(k)}$ is a sum of terms of the form (cf. Lemma 18.1.7 on adjoints of pseudodifferential operators in [3])

$$c(1+b)^{1/2-l} \prod_{j=1}^{k'} \partial_x^{\alpha_j} \partial_\xi^{\beta_j} B_{\mathrm{s}}, \prod_{j=k'+1}^{l} \partial_x^{\alpha_j} \partial_\xi^{\beta_j} b \tag{23}$$

modulo $S^{-\infty}$. It follows that $R^{(k+1)}$ is again a sum of terms (22), now with $\sum_j |\alpha_j| + k' = \sum_j |\beta_j| + k' \geqslant k+1$. The $Q^{(k)}$ are bounded families of symbols in $S^{\gamma/2-k(1-2\gamma/L)}(\mathbb{R}^n \times \mathbb{R}^n)_{1-\gamma/L,\gamma/L}$, with $\partial_z^j Q^{(k)}$ in $S^{\gamma/2+j\gamma/L-k(1-2\gamma/L)}_{1-\gamma/L,\gamma/L}(\mathbb{R}^n \times \mathbb{R}^n)$. It follows that the asymptotic sum Q exists and satisfies $Q^2 = 1 + B + R$, with $R \in S^{-\infty}$. $\square$

Next we consider the well-posedness of the Cauchy problem

$$Pu = f, \quad 0 < z < Z; \qquad u = u_0, \quad \text{when } z = 0. \tag{24}$$

For convenience we have set $z_0 = 0$ here. To show the well-posedness of this Cauchy problem we use the previous lemma and follow the argument in [3], Lemma 23.1.1 and Theorem 23.1.2, with minor modifications. Let $\tilde{\gamma} = \max(1, \gamma)$. We have the following equivalents to Lemma 23.1.1 and Theorem 23.1.2.

Lemma 4. *Suppose A and B satisfy* (i), (ii) *and* (iii) *with $2\gamma < L$. If $s \in \mathbb{R}$ and if λ is larger than some number depending on s, we have for every $u \in C^1([0,Z]; H^s) \cap C^0([0,Z]; H^{s+\tilde{\gamma}})$ and $p \in [1,\infty]$*

$$\left(\frac{1}{2}\int_0^Z \left\|\mathrm{e}^{-\lambda z} u(z,\cdot)\right\|_{H^s}^p\right)^{1/p} \leqslant \left\|u(0,\cdot)\right\|_{H^s} + 2\int_0^Z \mathrm{e}^{-\lambda z} \|Pu\|_{H^s}\,\mathrm{d}z, \tag{25}$$

with the interpretation as a maximum when $p = \infty$.

Theorem 5. *Suppose A and B satisfy* (i), (ii) *and* (iii) *with $2\gamma < L$. For every $f \in L^1(]0,Z[; H^s)$ and $u_0 \in H^s$, there is then a unique solution $u \in C([0,T]; H^s)$ of the Cauchy problem* (24)*, and* (25) *remains valid for this solution.*

Proof of Lemma 4 and Theorem 5. With the operators Q and R from Lemma 3 we find that for some constant c

$$\begin{aligned}\operatorname{Re}\big((\mathrm{i}A(z,x,D_x) + B(z,x,D_x))v, v\big) &= \operatorname{Re}\big((\mathrm{i}A(z,x,D_x) + R - 1)v, v\big) + (Qv, Qv) \\ &\geqslant -c(v,v), \quad v \in H^{\tilde{\gamma}},\end{aligned} \tag{26}$$

where to estimate $\operatorname{Re}(\mathrm{i}A(z,x,D_x)v, v)$ it is used that $A - A^*$ is pseudodifferential of order 0 and hence L^2-continuous. The case $s = 0$ now follows from the arguments of the proof of Lemma 23.1.1 in [3]. For $s \neq 0$ we set $E_s(D_x) = (1 + |D_x|^2)^{s/2}$. It follows from the composition formula for symbols and the assumptions (2), (3) that $E_s(D_x)A(z,x,D_x)E_{-s}(D_x)$ and $E_s(D_x)B(z,x,D_x)E_{-s}(D_x)$ satisfy the same assumptions as A and B. For $s \neq 0$ the estimate (25) now follows from the same estimate with s replaced by 0, P replaced by $E_s(D_x)PE_{-s}(D_x)$, and u replaced by $E_s(D_x)u$.

For the theorem we follow the proof of Theorem 23.1.2 in [3]. In the proof of the uniqueness in Theorem 23.1.2 we replace $s-1$ by $s-\tilde{\gamma}$. For the existence we find that the constructed solution $u \in C^1([0,Z];H^{s-2\tilde{\gamma}})$. Thus (on page 388) s is replaced by $s+2\tilde{\gamma}$ instead of $s+2$, and the approximating solution u_ν should be in $C^1([0,Z];H^{s+\tilde{\gamma}})$. The theorem then follows. □

3. Symbol estimates for $\exp(-I)$

In this section we will establish symbol estimates for functions of the form $K\exp(-I) = K(z,x,\xi)\exp(-I(z,x,\xi))$, $z \in [z_0,Z]$. Differentiation of $\exp(-I)$ brings out a derivative of I that is of order $(1+|\xi|)^\gamma$. We will improve on this by using the property (iii). We will first estimate the derivatives of I by powers of $(1+|\xi|)$ and I. After that symbol estimates are obtained using the fact that $(1+y)^\delta\exp(-y)$ is bounded for $y \geqslant 0$. We study the case $A = 0$, where the bicharacteristics of P_0 are the straight lines $(x,\xi) =$ constant and the integral I reduces to

$$I(z,x,\xi) = \int_{z_0}^{z} b(z',x,\xi)\,\mathrm{d}z'. \tag{27}$$

Lemma 6. *There are constants $C_{\alpha,\beta}$ such that*

$$\left|\partial_x^\alpha\partial_\xi^\beta I\right| \leqslant C_{\alpha,\beta}\left(1+|\xi|\right)^{-|\beta|+\frac{|\alpha|+|\beta|}{L}\gamma}(1+I)^{1-\frac{|\alpha|+|\beta|}{L}}, \quad z \in [z_0,Z]. \tag{28}$$

Proof. This is true automatically for $|\alpha|+|\beta| \geqslant L$. So suppose $|\alpha|+|\beta| < L$. By definition

$$\partial_x^\alpha\partial_\xi^\beta I = \int_{z_0}^{z} \partial_x^\alpha\partial_\xi^\beta b(z',x,\xi)\,\mathrm{d}z', \quad |\xi| > 1. \tag{29}$$

The Hölder inequality implies that

$$\begin{aligned} &\int_{z_0}^{z} \left(\left(1+|\xi|\right)^{-\gamma}(1+b)\right)^{1-\frac{|\alpha|+|\beta|}{L}}\,\mathrm{d}z' \\ &\quad \leqslant \left\|\left(\left(1+|\xi|\right)^{-\gamma}(1+b)\right)^{1-\frac{|\alpha|+|\beta|}{L}}\right\|_{L^{1/(1-(|\alpha|+|\beta|)/L)}}\left\|I_{[z_0,z]}\right\|_{L^{L/(|\alpha|+|\beta|)}} \\ &\quad = (z-z_0)^{\frac{|\alpha|+|\beta|}{L}}\left(\frac{z-z_0+I}{(1+|\xi|)^\gamma}\right)^{1-\frac{|\alpha|+|\beta|}{L}} \leqslant C(z-z_0)^{\frac{|\alpha|+|\beta|}{L}}\left(\frac{1+I}{(1+|\xi|)^\gamma}\right)^{1-\frac{|\alpha|+|\beta|}{L}}, \end{aligned} \tag{30}$$

where $I_{[z_0,z]}$ is the indicator function and the $L^{\frac{L}{|\alpha|+|\beta|}}$ norm is taken on a z-interval in $\mathbb{R}$, for fixed (x,ξ). This inequality and (2) imply the estimate

$$\left|\partial_x^\alpha\partial_\xi^\beta I\right| \leqslant C_{\alpha,\beta}\left(1+|\xi|\right)^{-|\beta|+\frac{|\alpha|+|\beta|}{L}\gamma}(z-z_0)^{\frac{|\alpha|+|\beta|}{L}}(1+I)^{1-\frac{|\alpha|+|\beta|}{L}}, \quad |\alpha|+|\beta| < L, \tag{31}$$

from which (28) follows. □

Lemma 7. *Suppose there is a constant μ such that for each α, β, j, there are constants $C_{\alpha,\beta,j}$ and $p_{\alpha,\beta,j}$ such that*

$$\left|\partial_x^\alpha \partial_\xi^\beta \partial_z^j K(z,x,\xi)\right| \leqslant C_{\alpha,\beta,j}\big(1+|\xi|\big)^{\mu-|\beta|+(\frac{|\alpha|+|\beta|}{L}+j)\gamma}\big(1+I(z,x,\xi)\big)^{p_{\alpha,\beta,j}}, \quad z \in [z_0, Z]. \tag{32}$$

Then $K(z,x,\xi)\exp(-I(z,x,\xi))$ is a bounded family of symbols in $S^\mu_{1-\gamma/L,\gamma/L}(\mathbb{R}^n \times \mathbb{R}^n)$, with $\partial_z^j(K\exp(-I))$ in $S^{\mu+j\gamma}_{1-\gamma/L,\gamma/L}(\mathbb{R}^n \times \mathbb{R}^n)$.

Proof. For the derivative of an exponential we have

$$\begin{aligned}
&\partial_x^\alpha \partial_\xi^\beta \partial_z^j\big(K\exp(-I)\big) \\
&\quad= \sum_{m=1}^{|\alpha|+|\beta|+j} \sum_{\alpha^{(1)}+\cdots+\alpha^{(m)}=\alpha} \sum_{\beta^{(1)}+\cdots+\beta^{(m)}=\beta} \sum_{j^{(1)}+\cdots+j^{(m)}=j} c_{m,(\alpha^{(1)},\dots,\alpha^{(m)}),(\beta^{(1)},\dots,\beta^{(m)}),(j^{(1)},\dots,j^{(m)})} \\
&\qquad \times \exp(-I)\partial_x^{\alpha^{(1)}}\partial_\xi^{\beta^{(1)}}\partial_z^{j^{(1)}}K(z,x,\xi)\prod_{l=2}^m \partial_x^{\alpha^{(l)}}\partial_\xi^{\beta^{(l)}}\partial_z^{j^{(l)}}(-I).
\end{aligned} \tag{33}$$

Here $\sum_{j^{(1)}+\cdots+j^{(m)}=j}$ is the sum over all m-vectors with non-negative integer components satisfying $j^{(1)}+\cdots+j^{(m)}=j$, and for $\sum_{\alpha^{(1)}+\cdots+\alpha^{(m)}=\alpha}$ the $\alpha^{(l)}$ are itself multi-indices, and the sum is such that $(\alpha^{(l)},\beta^{(l)},j^{(l)}) \neq 0$ for $l \geqslant 2$. From the inequality (28) and the fact that $\partial_x^\alpha\partial_\xi^\beta\partial_z^j I \leqslant C(1+|\xi|)^{\gamma-|\beta|}$ it follows that

$$\left|\partial_x^{\alpha^{(1)}}\partial_\xi^{\beta^{(1)}}\partial_z^{j^{(1)}}K(z,x,\xi)\prod_{l=2}^m \partial_x^{\alpha^{(l)}}\partial_\xi^{\beta^{(l)}}\partial_z^{j^{(l)}}(-I)\right| \leqslant C\big(1+|\xi|\big)^{\mu-|\beta|+(\frac{|\alpha|+|\beta|}{L}+j)\gamma}(1+I)^q \tag{34}$$

for some q. Since $(1+y)^\delta\exp(-y)$ is bounded for any δ, $y \geqslant 0$ it follows that

$$\left|\partial_x^\alpha\partial_\xi^\beta\partial_z^j\big(K\exp(-I)\big)\right| \leqslant C\big(1+|\xi|\big)^{\mu-|\beta|+(\frac{|\alpha|+|\beta|}{L}+j)\gamma}. \tag{35}$$

This completes the proof. □

The lemma shows in particular that the function $\exp(-I)$ is a bounded family of symbols in $S^0_{1-\gamma/L,\gamma/L}(\mathbb{R}^n \times \mathbb{R}^n)$, with $\partial_z^j \exp(-I)$ a bounded family in $S^{j\gamma}_{1-\gamma/L,\gamma/L}(\mathbb{R}^n \times \mathbb{R}^n)$.

The class of K satisfying the assumption of the lemma is closed under multiplication and taking derivatives. That is, if K' satisfies the assumptions of Lemma 7, with constant μ', then its derivative $\partial_x^\alpha\partial_\xi^\beta\partial_z^j K'(z,x,\xi)$ satisfies the assumption with $\mu = \mu' - |\beta| + ((|\alpha|+|\beta|)/L + j)\gamma$. If K'' satisfies the assumption of the lemma with constant μ'', then the product $K'K''$ satisfies the assumptions with constant $\mu' + \mu''$.

4. Parametrix for the case $a = 0$

In this section we prove Theorem 1 for the case $A = 0$, for which the operator E_0 is given by $E_0(z, z_0) = \mathrm{Id}$. An important part of the proof is an order by order construction. We first prove a lemma that is important for the induction step. As usual # denotes the composition of symbols. From now on we will write $B^{(0)} = b, B^{(1)} = B - b$.

Lemma 8. *Assume $2\gamma < L$. Suppose $K = K(z,x,\xi)$ is a smooth family of symbols in $S^m(\mathbb{R}^n \times \mathbb{R}^n)$ that satisfies the estimate* (32) *with constant μ. Then $B\#(K\exp(-I))$ can be written as an asymptotic sum*

$$B(z)\#\big(K(z)\exp\big(-I(z)\big)\big) = \sum_{j=0}^{\infty} M^{(j)}\exp\big(-I(z)\big) + R, \tag{36}$$

with R a smooth family in $S^{-\infty}$, $M^{(0)} = bK$ and $M^{(j)}$ a smooth family of symbols in $S^{m+\gamma+j(\gamma-1)}(\mathbb{R}^n \times \mathbb{R}^n)$ that satisfies (32) *with constant $\mu' = \mu + \gamma - j + \frac{j}{L}\gamma$. The integral $\int_{z_0}^{z} M^{(j)}(z')\,\mathrm{d}z'$ satisfies* (32) *with $\mu' = \mu - j(1 - \frac{2}{L}\gamma)$.*

Proof. By the composition formula the symbol $B(z)\#(K(z)\exp(-I(z)))$ is given by an asymptotic sum (with $j = 0$ or 1)

$$\sum_{\tilde\alpha, j} \frac{(-\mathrm{i})^{|\tilde\alpha|}}{\tilde\alpha!}\partial_\xi^{\tilde\alpha} B^{(j)}(z)\partial_x^{\tilde\alpha}\big(K(z)\exp\big(-I(z)\big)\big) + R, \tag{37}$$

with R a smooth family in $S^{-\infty}(\mathbb{R}^n \times \mathbb{R}^n)$. We let

$$M^{(k)}(z) = \sum_{\tilde\alpha, j,\ |\tilde\alpha|+j=k} \frac{(-\mathrm{i})^{|\tilde\alpha|}}{\tilde\alpha!}\exp\big(I(z)\big)\partial_\xi^{\tilde\alpha} B^{(j)}(z)\partial_x^{\tilde\alpha}\big(K(z)\exp\big(-I(z)\big)\big), \tag{38}$$

so that (36) is satisfied. Each term in the sum (38) is a product of a constant, a factor $\partial_\xi^{\tilde\alpha} B^{(j)}(z)$, a factor $\partial_x^{\alpha'} K(z)$ and a factor $\exp(I(z))\partial_x^{\alpha''}\exp(-I(z))$, with $\alpha' + \alpha'' = \tilde\alpha$. These are smooth families of symbols of order $\gamma - j - |\tilde\alpha|$, μ and $|\alpha''|\gamma$, which shows the first statement about the $M^{(j)}$. Also, they satisfy the assumption of Lemma 7 with constant μ equal to $\gamma - |\tilde\alpha| - j$, $\mu + \frac{|\alpha'|}{L}\gamma$ and $\frac{|\alpha''|}{L}\gamma$, respectively. The remarks following the proof of Lemma 7 about the multiplication of such functions show that each term satisfies the assumptions of this lemma with constant $\mu + \gamma - |\tilde\alpha| - j + \frac{|\tilde\alpha|}{L}\gamma$. Since for $M^{(k)}$ we have $k = |\tilde\alpha| + j$ it follows that the $M^{(k)}$ satisfies (32) with constant $\mu' = \mu + \gamma - j + \frac{j}{L}\gamma$.

Let K' be given by $K' = \int_{z_0}^{z} M^{(l)}(z', x, \xi)\,\mathrm{d}z'$, for some l. For K' we must estimate the multiple derivative $\partial_x^\alpha\partial_\xi^\beta\partial_z^j K'(z,x,\xi)$ If $j \neq 0$, then this is equal to a multiple derivative of $M^{(l)}$, given by $\partial_x^\alpha\partial_\xi^\beta\partial_z^{j-1} M^{(l)}$, and the result follows from the result already proven for $M^{(l)}$. Next suppose that $j = 0$. In this case $\partial_x^\alpha\partial_\xi^\beta\partial_z^j K'(z,x,\xi)$ is a sum of terms

$$c\int_{z_0}^{z} \partial_x^{\alpha'+\tilde\alpha}\partial_\xi^{\beta'} B^{(j)}\exp(I)\partial_x^{\alpha''}\partial_\xi^{\beta''+\tilde\alpha}\big(K\exp(-I)\big)\,\mathrm{d}z', \tag{39}$$

where c is a constant and $\alpha' + \alpha'' = \alpha, \beta' + \beta'' = \beta$. For such term we can put outside the integral a factor $C(1+|\xi|)^{\mu - |\beta''| - |\tilde\alpha| + \frac{|\alpha''|+|\beta''|+|\tilde\alpha|}{L}\gamma}(1+I)^p$ that is an upperbound for $\exp(I)\partial_x^{\alpha''}\partial_\xi^{\beta''+\tilde\alpha}(K\exp(-I))$. Thus we obtain

$$|\partial_x^\alpha \partial_\xi^\beta K'| \leqslant C(1+|\xi|)^{\mu-|\beta''|-|\tilde{\alpha}|+\frac{|\alpha''|+|\beta''|+|\tilde{\alpha}|}{L}\gamma}(1+I)^p \times (1+|\xi|)^{\gamma-|\beta'|-j} \int_{z_0}^{z} \bigl((1+|\xi|)^{-\gamma}(1+b)\bigr)^{1-\frac{|\alpha'+\tilde{\alpha}|+|\beta'|+2j}{L}} \, \mathrm{d}z', \tag{40}$$

if $|\alpha'| + |\tilde{\alpha}| + |\beta'| + 2j < L$. The integral on the right-hand side can be estimated as in (30). Since $l = j + |\tilde{\alpha}|$ this yields that K' satisfies (32) with μ given by $\mu' = \mu - l(1 - \frac{2}{L}\gamma)$. If $|\alpha' + \tilde{\alpha}| + |\beta'| + 2j \geqslant L$, then

$$|\partial_x^\alpha \partial_\xi^\beta K'| \leqslant C(1+|\xi|)^{\mu-|\beta''|-|\tilde{\alpha}|+\frac{|\alpha''|+|\beta''|+|\tilde{\alpha}|}{L}\gamma+\gamma-|\beta'|-j}(1+I)^p. \tag{41}$$

Then

$$(1+|\xi|)^\gamma \leqslant (1+|\xi|)^{\frac{|\alpha'|+|\tilde{\alpha}|+|\beta'|+2j}{L}\gamma},$$

so that again K' satisfies (32) with μ given by $\mu' = \mu - l(1 - \frac{2}{L}\gamma)$. □

Proof of Theorem 1 for the case $A = 0$. In this case $E_0(z, z_0) = \mathrm{Id}$, and we can write $E = W$, where we view the family of pseudodifferential operators W as mapping functions of $x \in \mathbb{R}^n$ to functions of $(z, x) \in [z_0, Z] \times \mathbb{R}^n$. We construct a bounded family of pseudodifferential operators $\widehat{W}(z)$, $z \in [z_0, Z]$ in $S^0_{1-\gamma/L,\gamma/L}$, with $\partial_z^j \widehat{W}(z)$ a bounded family in $S^{j\gamma}_{1-\gamma/L,\gamma/L}$ satisfying

$$\widehat{W}(z_0, x, D_x) = \mathrm{Id}, \tag{42}$$

and

$$P\widehat{W} \text{ is a bounded map } H^s(\mathbb{R}^n) \to C^k\bigl([z_0, z_0 + Z], H^{s+l}(\mathbb{R}^n)\bigr), \tag{43}$$

for any $k,\ s,\ l$. After that we show that $W - \widehat{W}$ is a smoothing operator. From the construction of $\widehat{W}$ it will also follow that W has the properties described in the theorem.

The operator $\widehat{W}$ will be constructed as an asymptotic sum of $W^{(j)}$ with decreasing order $\widehat{W} = \sum_{j=0}^\infty W^{(j)}$. The symbols $W^{(j)}(z, x, \xi)$ will be of the form

$$W^{(j)}(z, x, \xi) = K^{(j)}(z, x, \xi) \exp\bigl(-I(z, x, \xi)\bigr), \tag{44}$$

with $K^{(0)} = 1$ and the $K^{(j)}$, $j \geqslant 1$, to be determined. We will assume that the $K^{(j)}$ will satisfy the assumptions of Lemma 7 for $\mu = -j(1 - \frac{2}{L}\gamma)$, and are symbols of order $j(2\gamma - 1)$.

The compositions $\partial_z \widehat{W}$ and $B\widehat{W}$ are again a family of pseudodifferential operators. We have

$$\partial_z W^{(j)} = \mathrm{Op}\biggl(\frac{\partial K^{(j)}}{\partial z} \exp(-I) - bK^{(j)} \exp(-I)\biggr). \tag{45}$$

Hence

$$PW^{(j)} = \mathrm{Op}\Big(\frac{\partial K^{(j)}}{\partial z}\exp(-I)\Big) + \mathrm{Op}\big(B(z,x,\xi)\big)\,\mathrm{Op}\big(K^{(j)}\exp(-I)\big) - \mathrm{Op}\big(b(z,x,\xi)K^{(j)}\exp(-I)\big). \tag{46}$$

We define operators $R = P\widehat{W}$ and $R^{(k)}$, given by

$$R^{(k)} = P\sum_{j=0}^{k-1} W^{(j)}. \tag{47}$$

We denote by $M^{(j,k)}$ the operators $M^{(k)}$ from the previous lemma, applied to $K = K^{(j-1)}$. For $k = 1$ we can write, using (46)

$$R^{(k)} = \sum_{l=0}^{\infty} r^{(k,l)}\exp(-I), \tag{48}$$

with

$$r^{(1,l)} = M^{(1,l)}, \quad l \geqslant 1, \tag{49}$$

$$r^{(1,0)} = M^{(1,0)} + \frac{\partial K^{(0)}}{\partial z} - b = 0. \tag{50}$$

For $k > 1$ we have (48) if we set

$$r^{(k,k-1+l)} = r^{(k-1,k-1+l)} + M^{(k,l)}, \tag{51}$$

$$r^{(k,k-1)} = r^{(k-1,k-1)} + M^{(k,0)} + \frac{\partial K^{(k-1)}}{\partial z} - bK^{(k-1)} = r^{(k-1,k-1)} + \frac{\partial K^{(k-1)}}{\partial z}. \tag{52}$$

Assume that $r^{(k,j)} = 0$, $j = 0,\dots,k-1$. Then let

$$K^{(k)}(z) = -\int_{z_0}^{z} r^{(k,k)}(z')\,\mathrm{d}z'. \tag{53}$$

By the previous lemma this is a bounded family of symbols in $S^{k(2\gamma-1)}$. It also satisfies the assumptions of Lemma 7 with $\mu = -k(1 - \frac{2}{L}\gamma)$. By (52) it follows that then $r^{(k+1,k)} = 0$. Thus by induction we find a series of $K^{(k)}$ such that $R^{(k)}$ is a bounded family of symbols in $S^{\gamma-\frac{\gamma}{L}-k(1-\frac{2}{L}\gamma)}_{1-\gamma/L,\gamma/L}$, for $z \in [z_0, Z]$. It follows that, for fixed z, R is continuous $H^s(\mathbb{R}^n)$ to $H^{s+l}(\mathbb{R}^{n+1})$, uniformly in z. The operators $\partial_z^j R$ are bounded families of symbols in $S^{(j+1)\gamma-\frac{\gamma}{L}}(\mathbb{R}\times\mathbb{R})$. The terms in their asymptotic expansions also vanish. Hence is $\partial_z^j R$ is also continuous $H^s(\mathbb{R}^n)$ to $H^{s+l}(\mathbb{R}^n)$, for any s, l. This shows (43).

By definition $W^{(0)}(z_0) = 1$ and $W^{(j)} = 0$, $j \geqslant 1$. We can assume the asymptotic sum of symbols is such that (42) is satisfied. (Because $\widehat{W}$ is a bounded family, with continuous symbol, it follows that $W(z)u_0$ is a continuous function of z with values in H^s, for any $u_0 \in H^s$.)

Eq. (43) and the energy estimates for this problem (see Section 2) imply that

$$W - \widehat{W} \in S^{-\infty}\big(([z_0, Z] \times \mathbb{R}^n) \times \mathbb{R}^n\big), \tag{54}$$

It follows that Eqs. (11) are satisfied.

From the construction of $\widehat{W}$ it follows that (13) and (14) are satisfied. It is clear that when $I(z, x, \xi) = 0$, then $b(z', x, \xi) = 0$, $z' \in [z_0, z]$, and then all the $K^{(j)}(z, x, \xi)$ are zero. □

5. The case $A \neq 0$

In this section we complete the proof Theorem 1, by proving the case $A \neq 0$. The general case will be derived from the case $A = 0$, by using Egorov's theorem (see, e.g., [8, p. 147]). Consider the transformed function $\widetilde{u}$ defined from u by

$$\widetilde{u}(z, \cdot) = E_0(z, z_0)^{-1} u(z). \tag{55}$$

Of course we have

$$\frac{\partial E_0}{\partial z}(z, z_0) = \mathrm{i}A(z, x, D_x)E_0(z, z_0). \tag{56}$$

We will use the notation

$$\widetilde{B}(z, x, D_x) = E_0(z, z_0)^{-1} B(z, x, D_x) E_0(z, z_0). \tag{57}$$

The pseudodifferential equation now becomes

$$\big(\partial_z + \widetilde{B}(z, x, D_x)\big)\widetilde{u} = 0. \tag{58}$$

By Egorov's theorem $\widetilde{B}$ is a pseudodifferential operator of order γ with homogeneous real non-negative principal symbol

$$\tilde{b} = b \circ \Phi_{z,z_0}. \tag{59}$$

We have

$$\tilde{I}(z, x, \xi) = \int_{z_0}^{z} \tilde{b}(z', x, \xi)\, \mathrm{d}z'. \tag{60}$$

The following lemma states that the properties (2), (3) and (12)–(14) are conserved under the mapping $B \mapsto \widetilde{B}$.

Lemma 9. *The symbol $\widetilde{B}$, defined by* (57) *is a smooth family of symbols in $S^{\gamma}(\mathbb{R}^n \times \mathbb{R}^n)$ satisfying* (2) *and* (3) *with $\tilde{b}$ instead of b, if and only if B has these properties. Let $W(z, x, D)$ be a bounded family of pseudodifferential operators. Then $\widetilde{W}(z) = \exp(z, z_0)^{-1} W(z) \exp(z, z_0)$ satisfies* (12)–(14) *if and only if W satisfies* (12)–(14).

Proof. We use Egorov's theorem in the form given by Taylor [8, p. 147]. Taylor assumes that the subprincipal symbol $A - a$ is polyhomogeneous, but it can be checked from the proof that this assumption may be omitted, and that the result applies to our case as well. The map $Q \mapsto \widetilde{Q} = E(z, z_0)^{-1} Q E(z, z_0)$ maps a bounded set of symbols $Q(x, \xi)$ to a bounded set of symbols, where the principal symbol is given by $\tilde{q} = q \circ \Phi_{z,z_0}$.

To apply this to a family of symbols $B = B(z, x, \xi)$, observe that the derivatives $\partial_z^j \widetilde{B}$ are given by

$$\partial_z^j \widetilde{B} = E(z, z_0)^{-1} \left(\partial_z - \mathrm{i}[A, \cdot]\right)^j B E(z, z_0). \tag{61}$$

It follows that if B is a smooth family of symbols in $S^\gamma(\mathbb{R}^n \times \mathbb{R}^n)$ then $\widetilde{B}$ is a smooth family of symbols in $S^\gamma(\mathbb{R}^n \times \mathbb{R}^n)$.

It follows from (59) that $\widetilde{B}$ has homogeneous, real, non-negative principal symbol $\tilde{b}$. To establish that $\widetilde{B}$, $\tilde{b}$ satisfy the properties (2), (3) we recall the construction of the asymptotic series for $\widetilde{B}$ in the proof of Egorov's theorem in [8, p. 147]. Consider the transformation of a symbol Q independent of z. From the transformation $\widetilde{Q} = E(z, z_0)^{-1} Q E(z, z_0)$ a z_0-family of operators $\widetilde{Q}$ is obtained, that satisfies the differential equation

$$\partial_{z_0} \widetilde{Q}(z_0) = \mathrm{i}\left[A(z_0, x, D), \widetilde{Q}(z_0)\right], \quad \widetilde{Q}(z) = Q. \tag{62}$$

The asymptotic series is obtained by solving a series of differential equations for the $\widetilde{Q}^{(j)}$, $j = 0, 1, \ldots,$

$$(\partial_{z_0} - H_a) \widetilde{Q}^{(j)}(z_0, x, \xi) = a_{j-1}, \tag{63}$$

with initial condition

$$\widetilde{Q}^{(j)}(z_0, x, \xi) = \begin{cases} Q(x, \xi), & \text{at } z_0 = z, \quad j = 0, \\ 0, & \text{at } z_0 = z, \quad j \geqslant 1, \end{cases} \tag{64}$$

where $a_{-1} = 0$ and

$$a_j = \{A - a, \widetilde{Q}^{(j)}\} + \sum_{|\alpha| \geqslant 2} \frac{\mathrm{i}^{-|\alpha|+1}}{\alpha!} \left(\partial_\xi^\alpha A \partial_x^\alpha \widetilde{Q}^{(j)} - \partial_\xi^\alpha \widetilde{Q}^{(j)} \partial_x^\alpha A\right). \tag{65}$$

Thus $\widetilde{Q}^{(0)}$ is constant along the integral curves of $\partial_z - H_a$, hence we have

$$\widetilde{Q}^{(0)}(z_0, x, \xi) = Q\left(\Phi_{z_0,z}^{-1}(x, \xi)\right). \tag{66}$$

For $j \geqslant 1$ we find

$$\widetilde{Q}^{(j)}(z_0, x, \xi) = -\int_{z_0}^{z} a_{j-1}\left(z', \Phi(z_0, z')^{-1}(x, \xi)\right). \tag{67}$$

It follows that we can write the asymptotic series for Q as

$$\left(\sum_{\alpha,\beta} S_{\alpha,\beta} \partial_x^\alpha \partial_\xi^\beta Q\right) \circ \Phi_{z_0,z}^{-1}, \tag{68}$$

where $S_{\alpha,\beta}$ are smooth families of symbols of order $-|\alpha| + \lfloor(|\alpha| + |\beta|)/2\rfloor$ and $S_{0,0} = 1$.

This shows that (2), (3) are satisfied at least when the z-component of α is 0. For terms with a non-zero number of z-derivatives it follows using (61).

This shows the if part. The map $B \mapsto \widetilde{B}$ has an inverse given by $B = E(z, z_0)\widetilde{B}E(z, z_0)^{-1}$, which satisfies a similar differential equation in z. The only if part is obtained by applying similar arguments.

The statement about W follows from (12) and the expression for the asymptotic series (68). □

Proof of Theorem 1 for the general case. By the previous lemma, and the proof of the theorem for $A = 0$ it follows that (58) has a solution operator $\widetilde{E}$ that is a family of pseudodifferential operators $\widetilde{W}(z, x, D)$ with the properties of the theorem. By the definition of $\widetilde{u}$ it follows that

$$E(z, z_0) = E_0(z, z_0)\widetilde{E}(z, z_0). \tag{69}$$

This yields the second equality in (11). Obviously we have

$$E = E_0\widetilde{E}E_0^{-1}E_0. \tag{70}$$

By Lemma 9, $E_0\widetilde{E}E_0^{-1}$ is a family of pseudodifferential operators $W(z, x, D)$ with the properties of given in the theorem. This yields the first equality in (11). □

6. The Fourier integral operator property

In this section well establish the Fourier integral operator property. So far we had that for each z derivative the bounds for symbols of the form $K\exp(-I)$ increased with a factor $(1 + |\xi|)^\gamma$, uniformly in z. To obtain symbols estimates also w.r.t. the z-derivatives we establish improved bounds in the following lemmas.

Lemma 10. *Let b be as in* (ii), (iii). *Suppose I is given by* (27). *Then there are constants C, C' such that*

$$b(z', x, \xi) \leqslant C(z - z_0)^{-\frac{L}{L+1}}\left(1 + |\xi|\right)^{\frac{1}{L+1}\gamma} I(z, x, \xi)^{\frac{L}{L+1}} \tag{71}$$

and

$$b(z', x, \xi) \leqslant C'\left(1 + |\xi|\right)^{\gamma}\left(1 + (z - z_0)(1 + \xi)^{\gamma}\right)^{-1+\frac{1}{L+1}}\left(1 + I(z, x, \xi)\right)^{1-\frac{1}{L+1}} \tag{72}$$

when $z \in\,]z_0, Z]$ and $z' \in [z_0, z]$. There are constants $C_{\alpha,\beta,j}$ such that

$$\left|\partial_x^\alpha \partial_\xi^\beta \partial_z^j I\right| \leqslant C_{\alpha,\beta,j}\left(1 + |\xi|\right)^{j\gamma - |\beta|}\left(1 + (z - z_0)(1 + |\xi|)^{\gamma}\right)^{-j+\frac{j+|\alpha|+|\beta|}{L}}(1 + I)^{1-\frac{j+|\alpha|+|\beta|}{L}}, \tag{73}$$

when $z \in\,]z_0, Z]$.

Proof. An assumption on b is that

$$\left(1 + |\xi|\right)^{-\gamma}\left|\frac{\partial b}{\partial z}\right| \leqslant C\left(\left(1 + |\xi|\right)^{-\gamma} b\right)^{1-\frac{1}{L}}. \tag{74}$$

It follows from this that for some constant C_1

$$\big((1+|\xi|)^{-\gamma}b(z')\big)^{\frac{1}{L}} - \big((1+|\xi|)^{-\gamma}b(z'')\big)^{\frac{1}{L}} \leqslant C_1|z'-z''|. \tag{75}$$

Assuming for the moment $z' > z''$, it follows that

$$\big((1+|\xi|)^{-\gamma}b(z'')\big) \geqslant \big(\big((1+|\xi|)^{-\gamma}b(z')\big)^{\frac{1}{L}} - C_1(z'-z'')\big)^{L}. \tag{76}$$

Let

$$\tilde{z}_{\pm} = z' \pm \frac{((1+|\xi|)^{-\gamma}b(z'))^{\frac{1}{L}}}{C_1}.$$

Denote $I(z, z_0, x, \xi) = \int_{z_0}^{z} b(z', x, \xi)\,\mathrm{d}z'$. By integrating the previous inequality over z'' it follows that

$$\big(1+|\xi|\big)^{-\gamma} I(z', \tilde{z}_-, x, \xi) \geqslant \frac{1}{L+1}\big((1+|\xi|)^{-\gamma}b(z', x, \xi)\big)^{\frac{L+1}{L}}. \tag{77}$$

With $z'' > z'$ we find the same inequality for $(1+|\xi|)^{-\gamma}I(\tilde{z}_+, z', x, \xi)$ by a similar argument. Hence if $z - z_0 \leqslant C((1+|\xi|)^{-\gamma}b(z', x, \xi))^{\frac{1}{L}}$, then

$$\frac{I}{z-z_0} \geqslant \frac{I(\tilde{z}_-, z')}{z' - \tilde{z}_-} \geqslant Cb, \tag{78}$$

while if $Z - z_0 \geqslant z - z_0 \geqslant C((1+|\xi|)^{-\gamma}b(z', x, \xi))^{\frac{1}{L}}$, then by (77)

$$\big(1+|\xi|\big)^{-\gamma} I \geqslant C\big((1+|\xi|)^{-\gamma}b\big)^{\frac{L+1}{L}}. \tag{79}$$

It follows that

$$\big(1+|\xi|\big)^{-\gamma} b \leqslant C\left(\frac{I}{(z-z_0)(1+|\xi|)^{\gamma}}\right)^{\frac{L}{L+1}}. \tag{80}$$

The inequality (71) follows from this.

Of course we also have that $b \leqslant C(1+|\xi|)^{\gamma}$, hence

$$b \leqslant C\big(1+|\xi|\big)^{\frac{\gamma}{L+1}}(1+I)^{\frac{L}{L+1}} \min\big((1+|\xi|)^{\gamma}, (z-z_0)^{-1}\big)^{\frac{L}{L+1}}. \tag{81}$$

The minimum on the right-hand side can be estimated by

$$\min\big((1+|\xi|)^{\gamma}, (z-z_0)^{-1}\big) \leqslant 2\left(\frac{1}{(1+|\xi|)^{\gamma}} + z - z_0\right)^{-1} = \frac{2(1+|\xi|)^{\gamma}}{1 + (1+|\xi|)^{\gamma}(z-z_0)}. \tag{82}$$

Thus we have (72).

Suppose first that $j+|\alpha|+|\beta|<L$. If $j=0$ the inequality (73) follows from (31). In case $j\geqslant 1$ we have for the derivatives w.r.t. (z,x,ξ) of I

$$\partial_x^\alpha\partial_\xi^\beta\partial_z^j I=\partial_x^\alpha\partial_\xi^\beta\partial_z^{j-1}b(z,x,\xi). \tag{83}$$

From (2), (72) and (83) it follows that

$$\left|\partial_x^\alpha\partial_\xi^\beta\partial_z^j I\right|\leqslant C\big(1+|\xi|\big)^{\gamma-|\beta|}\big(1+(z-z_0)\big(1+|\xi|\big)^\gamma\big)^{\frac{-L-1+j+|\alpha|+|\beta|}{L+1}}(1+I)^{\frac{L+1-j+|\alpha|+|\beta|}{L+1}}. \tag{84}$$

The inequality (73) follows, using that

$$\frac{1+I}{1+(z-z_0)(1+|\xi|)^\gamma}$$

is bounded. For any (α,β,j) there is a constant C such that

$$\left|\partial_x^\alpha\partial_\xi^\beta\partial_z^j I\right|\leqslant C(z-z_0)\big(1+|\xi|\big)^{\gamma-|\beta|}. \tag{85}$$

This implies that (73) is also true when $j+|\alpha|+|\beta|\geqslant L$. □

Analogously as in Lemma 7, symbols estimates for functions of the form $K\exp(-I)$ in

$$S^\mu_{1-\gamma/L,\gamma/L}\big((]z_0,Z[\times\mathbb{R}^n)\times\mathbb{R}^n\big)$$

follow if K satisfies certain estimates in terms of powers of $(1+I)$:

Lemma 11. *Suppose I is given by* (27)*. Suppose that there are μ and κ such that for each α,β,j there are constants $C_{\alpha,\beta,j}$ and $p_{\alpha,\beta,j}$ such that*

$$\begin{aligned}\left|\partial_x^\alpha\partial_\xi^\beta\partial_z^j K(z,x,\xi)\right|&\leqslant C_{\alpha,\beta,j}\big(1+|\xi|\big)^{\mu+j\gamma-|\beta|}\\&\quad\times\big(1+(z-z_0)\big(1+|\xi|\big)^\gamma\big)^{\kappa-j+\frac{j+|\alpha|+|\beta|}{L}}\big(1+I(z,x,\xi)\big)^{p_{\alpha,\beta,j}},\quad z\in\,]z_0,Z].\end{aligned} \tag{86}$$

Then there are constants $C'_{\alpha,\beta,j}$ such that

$$\begin{aligned}&\left|\partial_x^\alpha\partial_\xi^\beta\partial_z^j\big(K(z,x,\xi)\exp\big(-I(z,x,\xi)\big)\big)\right|\\&\quad\leqslant C'_{\alpha,\beta,j}\big(1+|\xi|\big)^{\mu+j\gamma-|\beta|}\big(1+(z-z_0)\big(1+|\xi|\big)^\gamma\big)^{\kappa-j+\frac{j+|\alpha|+|\beta|}{L}},\quad z\in\,]z_0,Z],\end{aligned} \tag{87}$$

and hence $K\exp(-I))\in S^{\mu+\gamma\kappa}_{1-\gamma/L,\gamma/L}((]z_0,Z[\times\mathbb{R}^n)\times\mathbb{R}^n)$.

The class of K satisfying the assumption of the lemma for some μ and κ is closed under multiplication and taking derivatives. That is, if K' satisfies the assumptions of Lemma 11, with constant μ' and κ', then its derivative $\partial_x^\alpha\partial_\xi^\beta\partial_z^j K'(z,x,\xi)$ satisfies the assumption with $\mu=\mu'-|\beta|$ and $\kappa=\kappa'-j+$

$(j+|\alpha|+|\beta)/L$. If K'' satisfies the assumption of the lemma with constant μ'' and κ'', then the product $K'K''$ satisfies the assumptions with constants $\mu'+\mu''$ and $\kappa'+\kappa''$. From Lemma 10 it follows that I satisfies the assumption of Lemma 11 with $\mu=\kappa=0$, and b with $\mu=\gamma$ and $\kappa=-1+\frac{1}{L}$.

Proof of Theorem 2. To show that the $W^{(j)}$ are in $S^{-j(1-\frac{2\gamma}{L})}_{1-\gamma/L,\gamma/L}((]z_0,Z[\times\mathbb{R}^n)\times\mathbb{R}^n)$, we first give an improvement of Lemma 8. We will show that in fact the $M^{(j)}$ satisfy (86) with constants $(\mu+\gamma-j,-1+(1+j)/L)$, and that the integral $\int_{z_0}^{z} M^{(j)}(z',z_0,x,\xi)\,\mathrm{d}z'$ satisfies (86) with constants $(\mu,\kappa+2j/L)$.

Recall that the $M^{(j)}$ were defined in (38). Each term in the sum (38) is a product of a constant, a factor $\partial_\xi^{\tilde\alpha}B^{(j)}(z)$, a factor $\partial_x^{\alpha'}K(z)$ and a factor $\exp(I(z,z_0))\partial_x^{\alpha''}\exp(-I(z,z_0))$. These satisfy the assumptions of Lemma 11 with constant (μ,κ) equal to $(\gamma-|\tilde\alpha|-j,-1+(1+2j+|\tilde\alpha|)/L)$, $(\mu,\kappa+|\alpha'|/L)$ and $(0,|\alpha''|/L)$, respectively. The remarks following the proof of Lemma 11 about the multiplication of such functions show that each term satisfies the assumptions of this lemma with constants $(\mu+\gamma-|\tilde\alpha|-j,-1+(1+2j+2|\tilde\alpha|)/L)$. Since for $M^{(k)}$ we have $k=|\tilde\alpha|+j$ it follows that the $M^{(k)}$ satisfies the assumptions of Lemma 11 with constants $(\mu+\gamma-k,-1+(1+k)/L)$.

Let K' be given by $K'=\int_{z_0}^{z} M^{(l)}(z',x,\xi)\,\mathrm{d}z'$, for some l. For K' we must estimate the multiple derivative $\partial_x^\alpha\partial_\xi^\beta\partial_z^j K'(z,x,\xi)$. If $j\neq 0$, then this is equal to a multiple derivative of $M^{(l)}$, given by $\partial_x^\alpha\partial_\xi^\beta\partial_z^{j-1}M^{(l)}$, and the result follows from the result already proven for $M^{(l)}$. Next suppose that $j=0$. In this case $\partial_x^\alpha\partial_\xi^\beta\partial_z^j K'(z,x,\xi)$ is a sum of terms

$$c\int_{z_0}^{z}\partial_x^{\alpha'+\tilde\alpha}\partial_\xi^{\beta'}B^{(j)}\exp(I)\partial_x^{\alpha''}\partial_\xi^{\beta''+\tilde\alpha}\big(K\exp(-I)\big)\,\mathrm{d}z',\tag{88}$$

where c is a constant and $\alpha'+\alpha''=\alpha$, $\beta'+\beta''=\beta$. For such term we can put outside the integral a factor

$$C\big(1+|\xi|\big)^{\mu}\big(1+(z-z_0)\big(1+|\xi|\big)^{\gamma}\big)^{\kappa+\frac{|\alpha''|+|\beta''|+|\tilde\alpha|}{L}}(1+I)^p$$

that is an upperbound for $\exp(I)\partial_x^{\alpha''}\partial_\xi^{\beta''+\tilde\alpha}(K\exp(-I))$. Thus we obtain

$$\begin{aligned}\big|\partial_x^\alpha\partial_\xi^\beta K'(z,x,\xi)\big| &\leqslant C\big(1+|\xi|\big)^{\mu}\big(1+(z-z_0)\big(1+|\xi|\big)^{\gamma}\big)^{\kappa+\frac{|\alpha''|+|\beta''|+|\tilde\alpha|}{L}}(1+I)^p\\ &\quad\times\big(1+|\xi|\big)^{\gamma}\int_{z_0}^{z}\big(\big(1+|\xi|\big)^{-\gamma}(1+b)\big)^{1-\frac{|\alpha'|+|\tilde\alpha|+|\beta'|+2j}{L}}\,\mathrm{d}z'.\end{aligned}\tag{89}$$

The integral on the right-hand side can be estimated as in (30). Since $l=j+|\tilde\alpha|$ this yields that K' satisfies the assumptions of Lemma 11 with μ and κ given by $\mu'=\mu$ and $\kappa'=\kappa+\frac{2l}{L}$.

Since the $W^{(j)}$ are in $S^{-j(1-\frac{2\gamma}{L})}_{1-\gamma/L,\gamma/L}$, we can assume that also the asymptotic sum $\widehat{W}=\sum_{j=0}^{\infty}W^{(j)}$ is in $S^0_{1-\gamma/L,\gamma/L}((]z_0,Z[\times\mathbb{R}^n)\times\mathbb{R}^n)$. Then by (54) the symbol of W is also in $S^0_{1-\gamma/L,\gamma/L}((]z_0,Z[\times\mathbb{R}^n)\times\mathbb{R}^n)$. By the Egorov theorem and (61) it follows that $\widetilde{W}$ is also in $S^0_{1-\gamma/L,\gamma/L}((]z_0,Z[\times\mathbb{R}^n)\times\mathbb{R}^n)$.

Let $\psi_1\in C_0^\infty(]z_0,Z[)$ be a function of z only. It is sufficient to show that for each such ψ_1, the operator $\psi_1 E$ is a Fourier integral operator. Let $\psi_2\in C_0^\infty(]z_0,Z[)$ be 1 on $\operatorname{supp}(\psi_1)$. Let $\chi=\chi(D_z,D_x)$ be in $\operatorname{Op}S^0(\mathbb{R}^{n+1}\times\mathbb{R}^{n+1})$, with symbol $\chi(\zeta,\xi)$ that is 1 on a small conic neighborhood of $\xi=0,\zeta\neq 0$, and 0 for (ζ,ξ), with $|(\zeta,\xi)| \geqslant 1$ and outside a larger conic neighborhood of $\xi = 0, \zeta \neq 0$. We assume in particular that $\chi = 0$ on a neighborhood of the set $\zeta = a(z,x,\xi)$. Since ψ_2 is only a function of z it commutes with W, and we have

$$\psi_1 E = \psi_1 W\big((1-\chi)+\chi\big)\psi_2 E_0. \tag{90}$$

The operator $\chi\psi_2 E_0$ is a smoothing operator. The proof of Theorem 18.1.35 of [3] shows that $\psi_1 W(1-\chi)$ is a pseudodifferential operator in $\mathrm{Op}\, S^0_{1-\gamma/L,\gamma/L}(\mathbb{R}^{n+1}\times\mathbb{R}^{n+1})$. Since E_0 is a Fourier integral operator in $I_1^{-1/4}(]z_0,Z[\times\mathbb{R}^n,\mathbb{R}^n;C_0)$, it follows that $\psi_1 W((1-\chi)\psi_2 E_0$ is a Fourier integral operator in $I^{-1/4}_{1-\gamma/L}(]z_0,Z[\times\mathbb{R}^n,\mathbb{R}^n;C_0)$. Thus $\psi_1 E$ is a Fourier integral operator. □

7. Symbols b satisfying the assumption

We discuss a class of examples such that (2) holds for b. Let h be a scalar function $h:\mathbb{R}\to [0,\infty[: y\mapsto h(y)$ that satisfies

$$h \text{ is } C^\infty; \quad h(y)=0 \quad \text{for } y\leqslant 0; \quad h(y)>0 \quad \text{for } y>0. \tag{91}$$

We assume that for $|\xi|>1$, b is given by

$$b(z,x,\xi) = |\xi|^\gamma W\left(z,x,\frac{\xi}{|\xi|}\right) h\left(\rho\left(z,x,\frac{\xi}{|\xi|}\right)\right), \tag{92}$$

$|\xi|>1$, where ρ and W are real valued C^∞ functions.

For h we assume that there are an interval $[-\beta,\beta]$ an integer L and a constant C such that on $[-\beta,\beta]$

$$\left|\frac{\mathrm{d}^j h}{\mathrm{d}y^j}(y)\right| < Ch(y)^{1-j/L}, \quad j=1,\dots,L-1. \tag{93}$$

We have in mind the well known example

$$h(y) = \begin{cases} 0, & y\leqslant 0, \\ \exp(-1/y), & y>0. \end{cases} \tag{94}$$

It can be seen that this choice of h satisfies the properties (91) and (93) by computing the successive derivatives and using that $y^{-j}\exp(-(1-\alpha)/y)$ is bounded for each j, $\alpha<1$ and $y>0$. We have the following proposition.

By computing the derivatives of b and the assumption (93) we have the following proposition.

Proposition 12. *Let h be given by* (94) *or otherwise let h satisfy* (91) *and suppose that there are $\beta>0$, an integer L and a constant C such that on $[-\beta,\beta]$ the inequalities* (93) *hold. Suppose $\rho=\rho(z,x,\xi/|\xi|)$ and $W=W(z,x,\xi/|\xi|)$ are real valued C^∞ function such that*

$$W\left(z,x,\frac{\xi}{|\xi|}\right) \geqslant \textit{constant} > 0. \tag{95}$$

Let b satisfy (92) *for $|\xi|>1$. Then b satisfies* (2).

Proof. We can use new local coordinates (y, λ) in a subset of $\mathbb{R}^{2n} \times \mathbb{R}_+$ instead of (z, x, ξ), such that y depends only on $(z, x, \xi/|\xi|)$, and $\lambda = |\xi|$. We simply write $b(\lambda, y), W(y), \rho(y)$ for the functions b, W, ρ considered as functions of the new coordinates. Then $b(\lambda, y) = \lambda^\gamma W(y)h(\rho(y))$. For the derivatives of b we have

$$\left|\partial_\lambda^j \partial_y^\alpha b\right| \leqslant C\lambda^{\gamma-j} h^{(|\alpha|)}\big(\rho(y)\big) \leqslant C(1+\lambda)^{\frac{|\alpha|+j}{L}\gamma - j}\big(\lambda^\gamma W(y) h\big(\rho(y)\big)\big)^{1-\frac{|\alpha|+j}{L}}. \tag{96}$$

Going back to the original coordinates leads to (2). □

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