# Curvilinear wave-equation angle transform: Caustics, turning rays, absence of kinematic artifacts

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## SUMMARY

Migration of seismic reflection data to common image-point gathers is an integral part of both migration velocity analysis (MVA) and amplitude (AVA) analysis. Its applicability in complex geology depends on whether these gathers will be artifact free, and is related to the formation of caustics and turning 'rays' due to the heterogeneity of the velocity model used. Here, we discuss an angle transform – which by methods of (survey-sinking or shot-geophone) wave-equation migration maps data into image gathers – in special curvilinear coordinates that remains artifact free in the presence of caustics while allowing turning 'rays' to illuminate steeply dipping reflectors.

## INTRODUCTION

We discuss an angle transform – which by methods of (survey-sinking or shot-geophone) wave-equation migration maps data into image gathers – in special curvilinear coordinates that remains artifact free in the presence of caustics while allowing turning 'rays'.

To define curvilinear coordinates, we begin with introducing a level set function that we will refer to as pseudodepth. The choice of level set function can adapt to the (local) geology and tectonics. The pseudolateral coordinates can be explicitly constructed in each constant pseudodepth surface (level set) from the coordinates at the earth's surface and the pseudodepth function.

We extend our wave-equation angle transform Stolk and De Hoop (2001, 2005, 2006) to (i) a shot-geophone formulation, thus admitting flexibility in source and receiver distributions, and (ii) special curvilinear coordinates (Riemannian manifolds) thus admitting not only the presence of caustics but also accounting for a large class of turning 'rays'. We show that the image gathers obtained by this angle transform, under a weak, so-called curvilinear DSR condition, are artifact free. This admits the introduction of annihilators in curved space-time for wave-equation migration velocity analysis exploiting the presence of caustics and turning 'rays' combined. The construction of the curvilinear wave-equation angle transform is based on one-way propagators on Riemannian manifolds derived from a Lagrangian action generating the wave equation Friedlander (1976). We consider the variable density wave equation, and simplify the derivation of the wave equation in curvilinear coordinates by using a variational formulation. The basic idea of one-way propagation in curvilinear coordinates has been applied before Sava and Fomel (2005); see also Brandsberg-Dahl and Etgen (2003). (Our introduction of curvilinear coordinates differs from the one by Sava and Fomel in as much as that they generate the new coordinate system using the rays associated with the wave propagation.) Here, we ensure true-amplitude behavior of the solutions to the curvilinear one-way wave equations; we derive the lower-order contribution to the curvilinear one-way wave equation. Moreover, the initial step towards forming image gathers can be interpreted as sinking a survey to a constant pseudodepth surface, a useful technique by itself.

While developing the curvilinear wave-equation angle transform, we address some of the misunderstandings related to the correct lower-order contributions published in the literature concerning the construction of 'true amplitude' one-way wave equations, in Cartesian as well as in curvilinear coordinates.

## NEW COORDINATES

In our notation we distinguish between horizontal coordinates $x = (x_1, x_2)$ and the vertical coordinate $z$. Similarly the new coordinates are denoted by $(\tilde{x}, \tilde{z}) = (\tilde{x}_1, \tilde{x}_2, \tilde{z})$. This notation is used because the pseudodepth $\tilde{z}$ will play a special role, different form $(\tilde{x}_1, \tilde{x}_2)$. By $q = q(x, z)$ we denote the pseudodepth function. In the following we construct new coordinates such that

1. $\tilde{z}(x, z) = q(x, z)$;
2. At each point $\frac{\partial(x,z)}{\partial \tilde{z}}$ is orthogonal to the surfaces $q(x, z) =$ constant.

The construction is carried out using the following differential equation, the solution of which are the curves $\tilde{x} =$ constant,

$$\frac{\mathrm{d}(x,z)}{\mathrm{d}\tilde{z}} = \frac{\nabla q(x,z)}{\|\nabla q(x,z)\|^2}. \tag{1}$$

We assume that the new coordinates are prescribed on the plane $q = 0$, i.e. the map $(x(\tilde{x}, 0), z(\tilde{x}, 0))$ is given. Denoting by $(X(\tilde{z}; x_0, z_0), Z(\tilde{z}; x_0, z_0))$ the solution curve to (1) with initial condition $(x(0), z(0)) = (x_0, z_0)$, the coordinate transformation is given by $x(\tilde{x}, \tilde{z}) = X(\tilde{z}; x(\tilde{x}, 0), z(\tilde{x}, 0))$, $z(\tilde{x}, \tilde{z}) = Z(\tilde{z}; x(\tilde{x}, 0), z(\tilde{x}, 0))$. In other words, $(x(\tilde{x}, \tilde{z}), z(\tilde{x}, \tilde{z}))$ is given by the flow over $\tilde{z}$ of differential equation (1):

$$(\tilde{x}, \tilde{z}) = \Phi_{\tilde{z}}(x(\tilde{x}, 0), z(\tilde{x}, 0)).$$

It can be easily verified that indeed $\frac{\partial(x,z)}{\partial \tilde{z}}$ is normal to the surfaces $\tilde{z} =$ constant. Moreover, $q(x, z) = \tilde{z}$ as required, since $\frac{\mathrm{d}}{\mathrm{d}\tilde{z}} q(x(\tilde{x}, \tilde{z}), z(\tilde{x}, \tilde{z})) = \nabla q \cdot \frac{\mathrm{d}(x,z)}{\mathrm{d}\tilde{z}} = 1$.

We will need the metric associated with the new coordinates. With the original coordinates we associate the metric $g_{ij} = \delta_{jk}$ (we will use upper and lower indices as in Riemannian geometry). The metric is

$$\tilde{g}_{il} = \frac{\partial(x,z)^j}{\partial(\tilde{x},\tilde{z})^i} \delta_{jk} \frac{\partial(x,z)^k}{\partial(\tilde{x},\tilde{z})^l}$$

We employ the summation convention: summation over repeated indices is implicit (in other words, this equation is a shorthand for $\tilde{g}_{il} = \sum_{j,k} \frac{\partial(x,z)^j}{\partial(\tilde{x},\tilde{z})^i} \delta_{jk} \frac{\partial(x,z)^k}{\partial(\tilde{x},\tilde{z})^l}$). The inverse metric equals

$$\tilde{g}^{il} = \frac{\partial(\tilde{x},\tilde{z})^i}{\partial(x,z)^j} \delta^{jk} \frac{\partial(\tilde{x},\tilde{z})^l}{\partial(x,z)^k}.$$

Since $\frac{\partial(x,z)}{\partial \tilde{x}_1} \perp \frac{\partial(x,z)}{\partial \tilde{z}}$, and $\frac{\partial(x,z)}{\partial \tilde{x}_2} \perp \frac{\partial(x,z)}{\partial \tilde{z}}$, the metric $\tilde{g}_{ij}$ must be of the form

$$\tilde{g}_{ij} = \begin{pmatrix} \tilde{g}_{11} & \tilde{g}_{12} & 0 \\ \tilde{g}_{21} & \tilde{g}_{22} & 0 \\ 0 & 0 & \tilde{g}_{33} \end{pmatrix}_{ij};$$

the inverse metric $\tilde{g}^{ij}$ is of the same form. By $\sigma, \sigma' = 1, 2$ we will denote indices for the 'horizontal' coordinates, and we let $\tilde{g}_{\sigma\sigma'} = \begin{pmatrix} \tilde{g}_{11} & \tilde{g}_{12} \\ \tilde{g}_{21} & \tilde{g}_{22} \end{pmatrix}_{\sigma\sigma'}$ be the horizontal part of the metric.

# Curvilinear wave-equation angle transform

## TRANSFORMING TWO- AND ONE-WAY WAVE EQUATIONS

The transformation of the acoustic wave equation is most naturally done using a variational formulation. We define an action functional by

$$S = \tfrac{1}{2}\int_a^b \iint \left(\kappa \left|\frac{\partial u}{\partial t}\right|^2 - \rho^{-1}\|\nabla u\|^2 + uf\right) \mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t.$$

The wave equation follows from the Euler-Lagrange equations derived from this action. The variation of this action under $v$ (the derivative if $u \to u+v$) can be written as

$$\begin{aligned}\delta_v S &= \int_a^b \iint \left(\kappa \frac{\partial v}{\partial t}\frac{\partial u}{\partial t} - \rho^{-1}\nabla v\cdot\nabla u + vf\right) \mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t\\ &= \int_a^b \iint v\left(-\kappa\frac{\partial^2 u}{\partial t^2} + \nabla\cdot(\rho^{-1}\nabla u) + f\right) \mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t.\end{aligned}$$

where the second step was obtained by integration by parts, using that $v=0$ for $t=a$ and $t=b$. Since this must be true for all $v$, the wave equation follows.

We define the transformed wave field as $\tilde{u}(\tilde{x},\tilde{z}) = u(x(\tilde{x},\tilde{z}),z(\tilde{x},\tilde{z}))$. To obtain the wave equation in the new coordinates, we transform the action. In the new coordinates it becomes

$$\begin{aligned}S = \tfrac{1}{2}\int_a^b \iint \Bigg(\kappa\left|\frac{\partial \tilde{u}}{\partial t}\right|^2 - \rho^{-1}\left(\frac{\partial(\tilde{x},\tilde{z})}{\partial(x,z)}\frac{\partial\tilde{u}}{\partial(\tilde{x},\tilde{z})}\right)\cdot\left(\frac{\partial(\tilde{x},\tilde{z})}{\partial(x,z)}\frac{\partial\tilde{u}}{\partial(\tilde{x},\tilde{z})}\right)\\ + \tilde{u}f\Bigg)\left|\frac{\partial(x,z)}{\partial(\tilde{x},\tilde{z})}\right|\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{z}\,\mathrm{d}t.\end{aligned}$$

By a similar argument as above, we find that the wave equation has new coefficients (which are now anisotropic), $\kappa\left|\frac{\partial(x,z)}{\partial(\tilde{x},\tilde{z})}\right|$ and $\rho^{-1}\left|\frac{\partial(x,z)}{\partial(\tilde{x},\tilde{z})}\right|\tilde{g}^{ij}$, and reads

$$\kappa\left|\frac{\partial(x,z)}{\partial(\tilde{x},\tilde{z})}\right|\frac{\partial^2\tilde{u}}{\partial t^2} - \frac{\partial}{\partial(\tilde{x},\tilde{z})^i}\left(\rho^{-1}\left|\frac{\partial(x,z)}{\partial(\tilde{x},\tilde{z})}\right|\tilde{g}^{ij}\frac{\partial\tilde{u}}{\partial(\tilde{x},\tilde{z})^j}\right) = f\left|\frac{\partial(x,z)}{\partial(\tilde{x},\tilde{z})}\right|$$

or

$$\kappa\left|\frac{\partial(x,z)}{\partial(\tilde{x},\tilde{z})}\right|\frac{\partial^2\tilde{u}}{\partial t^2} - \frac{\partial}{\partial\tilde{z}}\left(\alpha\frac{\partial\tilde{u}}{\partial\tilde{z}}\right) - \frac{\partial}{\partial\tilde{x}^\sigma}\left(\rho^{-1}\left|\frac{\partial(x,z)}{\partial(\tilde{x},\tilde{z})}\right|\tilde{g}^{\sigma\sigma'}\frac{\partial\tilde{u}}{\partial\tilde{x}^{\sigma'}}\right) = f\left|\frac{\partial(x,z)}{\partial(\tilde{x},\tilde{z})}\right|,$$

with $\alpha = \rho^{-1}\tilde{g}^{33}\left|\frac{\partial(x,z)}{\partial(\tilde{x},\tilde{z})}\right|$.

To derive the one-way wave equations we write the wave equation as a first-order system in $\tilde{z}$. We will work in the time-Fourier domain. Let $\tilde{v} = -i\omega^{-1}\alpha\frac{\partial\tilde{u}}{\partial\tilde{z}}$, then the wave equation can be written as a first-order system in $\tilde{z}$,

$$\left(\frac{\partial}{\partial\tilde{z}} - i\omega A\right)\begin{pmatrix}\tilde{u}\\ \tilde{v}\end{pmatrix} = \begin{pmatrix}0\\ f\left|\frac{\partial(x,z)}{\partial(\tilde{x},\tilde{z})}\right|\end{pmatrix}, \tag{2}$$

where

$$A = \begin{pmatrix}0 & \alpha^{-1}\\ \kappa\left|\frac{\partial(x,z)}{\partial(\tilde{x},\tilde{z})}\right| + \omega^{-2}\frac{\partial}{\partial\tilde{x}^\sigma}\rho^{-1}\left|\frac{\partial(x,z)}{\partial(\tilde{x},\tilde{z})}\right|\tilde{g}^{\sigma\sigma'}\frac{\partial}{\partial\tilde{x}^{\sigma'}} & 0\end{pmatrix}.$$

A vertical slowness operator can be defined by the following expression, involving an operator square root,

$$\Gamma = (A_{12}^{1/2}A_{21}A_{12}^{1/2})^{1/2}.$$

It follows that

$$\Gamma = \sqrt{\rho\kappa\tilde{g}_{33} + \omega^{-2}\frac{\partial}{\partial\tilde{x}^\sigma}\tilde{g}^{\sigma\sigma'}\tilde{g}_{33}\frac{\partial}{\partial\tilde{x}^{\sigma'}}}.$$

More precisely, $\Gamma$ is a pseudodifferential operator, with principal symbol

$$\gamma_0 = \sqrt{\rho\kappa\tilde{g}_{33} - \omega^{-2}\tilde{g}^{\sigma\sigma'}\tilde{g}_{33}\tilde{k}_{x,\sigma}\tilde{k}_{x,\sigma'}},$$

where $\tilde{k}_x$ denotes the Fourier vector associated with $\tilde{x}$. The subprincipal symbol,

$$\begin{aligned}\gamma_{-1} &= \frac{i}{2\gamma_0}\omega^{-2}\frac{\partial(\tilde{g}^{\sigma\sigma'}\tilde{g}_{33})}{\partial\tilde{x}^\sigma}\tilde{k}_{x,\sigma'}\\ &- \frac{i}{4\gamma_0^3}\omega^{-2}\tilde{g}^{\sigma\sigma'}\tilde{g}_{33}\tilde{k}_{x,\sigma'}\left(\frac{\partial(\rho\kappa\tilde{g}_{33})}{\partial\tilde{x}^\sigma} - \omega^{-2}\frac{\partial(\tilde{g}^{\sigma''\sigma'''}\tilde{g}_{33})}{\partial\tilde{x}^\sigma}\tilde{k}_{\sigma''}\tilde{k}_{\sigma'''}\right)\end{aligned}$$

is needed to ensure true-amplitude behavior of the one-way wave equation.

To derive the one-way wave equations, we consider a transformation from $(\tilde{u},\tilde{v})$ to down- and upgoing (with respect to pseudodepth) constituents of the wavefield $(u_+,u_-)$. Let

$$\begin{aligned}L^{-1} &= \frac{1}{\sqrt{2}}\begin{pmatrix}\alpha^{-1/2}\Gamma^{-1/2} & \alpha^{-1/2}\Gamma^{-1/2}\\ \alpha^{1/2}\Gamma^{1/2} & -\alpha^{1/2}\Gamma^{1/2}\end{pmatrix},\\ L &= \frac{1}{\sqrt{2}}\begin{pmatrix}\Gamma^{1/2}\alpha^{1/2} & \Gamma^{-1/2}\alpha^{-1/2}\\ \Gamma^{1/2}\alpha^{1/2} & -\Gamma^{-1/2}\alpha^{-1/2}\end{pmatrix}\end{aligned}$$

while

$$\begin{pmatrix}u_+\\ u_-\end{pmatrix} = L^{-1}\begin{pmatrix}\tilde{u}\\ \tilde{v}\end{pmatrix}, \qquad \begin{pmatrix}f_+\\ f_-\end{pmatrix} = L^{-1}\begin{pmatrix}0\\ f\left|\frac{\partial(x,z)}{\partial(\tilde{x},\tilde{z})}\right|\end{pmatrix}.$$

The one-way wave equation can be derived by applying this transformation of variables $(\tilde{u},\tilde{v}) \to (u_+,u_-)$ to the system of differential equations (2). In the definitions above the fields $u_+$ and $u_-$ are normalized so that vertical acoustic flux is conserved. With $\Gamma_\pm = \pm\Gamma$, the one-way equations appear to be

$$\left(\frac{\partial}{\partial\tilde{z}} - i\omega\Gamma_\pm\right)u_\pm = f_\pm.$$

By $G_+ = G_+(\tilde{x},\tilde{z},\omega;\tilde{x}_0,\tilde{z}_0)$ we will denote the one-way fundamental solution, that is

$$\left(\frac{\partial}{\partial\tilde{z}} - i\omega\Gamma_+\right)G_+ = \delta(\tilde{x}-\tilde{x}_0)\delta(\tilde{z}-\tilde{z}_0).$$

## PSEUDODEPTH CONTINUATION, ANGLE TRANSFORM

Let $d(\tilde{x}_r,t;\tilde{x}_s) = d(x_r(\tilde{x}_r,0),z_r(\tilde{x}_r,0),t;x_s(\tilde{x}_s,0),z_s(\tilde{x}_s,0))$ denote the reflection data, observed in the hypersurface $\tilde{z}=0$. We now denote curvilinear subsurface midpoint coordinates by $(\tilde{x},\tilde{z})$, while curvilinear subsurface offset coordinates, contained in a level set of $q$, are written as $\tilde{h}_x$ (we have $\tilde{h}_z = 0$). The continuation of the data in pseudodepth, $\tilde{z}$, is described by

$$\begin{aligned}D(\tilde{x},\tilde{z},\tilde{h}_x,T) = \int \mathrm{d}S(\tilde{x}_r)\int\mathrm{d}S(\tilde{x}_s)\int\mathrm{d}t\,(\partial_t^2 d)(\tilde{x}_r,t;\tilde{x}_s)\\ \int\mathrm{d}\bar{t}\,G_+(\tilde{x}+\tilde{h}_x,\tilde{z},t-T-\bar{t};\tilde{x}_r,0)G_+(\tilde{x}-\tilde{h}_x,\tilde{z},\bar{t};\tilde{x}_s,0).\end{aligned} \tag{3}$$

Common image-point gathers are then obtained with the wave-equation angle transform, which is given by

$$\mathscr{A}\, d(\tilde{x},\tilde{z},\tilde{p}) = \frac{1}{2\pi}\iiint \exp[\mathrm{i}\omega(\langle\tilde{p},\tilde{h}_x\rangle - T)] \\ D(\tilde{x},\tilde{z},\tilde{h}_x,T)\,\mathrm{d}S(\tilde{h}_x)\mathrm{d}T\,\mathrm{d}\omega, \tag{4}$$

where $\langle\tilde{p},\tilde{h}_x\rangle = \tilde{p}_\sigma\tilde{h}_x^\sigma$ ($\tilde{p}$ transforms as a covector and $\tilde{h}$ transforms as a vector) and

$$\mathrm{d}S(\tilde{h}_x) = \left|j(\tilde{h}_x)^T j(\tilde{h}_x)\right|^{1/2}\mathrm{d}\tilde{h}_x,$$

in which

$$j(\tilde{h}_x) = \frac{\partial(x(\tilde{x}-\tilde{h}_x,\tilde{z}), z(\tilde{x}-\tilde{h}_x,\tilde{z}), x(\tilde{x}+\tilde{h}_x,\tilde{z}), z(\tilde{x}+\tilde{h}_x,\tilde{z}))}{\partial(\tilde{h}_x)}$$

## FOCUSSING AT ZERO SUBSURFACE OFFSET – CAUSTICS

In this section we study which values of $\tilde{p}$ are useful in $\mathscr{A}\, d(\tilde{x},\tilde{z},\tilde{p})$. Secondly we argue that these angle gathers are free of kinematic artifacts. Migration aims to map a reflected signal (event) in the data to a signal in the image that is localized around the reflector position. This is indeed the case, even in the presence of multipathing, where binning-based migration methods in general fail. We assume that the source and receiver rays become nowhere horizontal in the curvilinear coordinate system. We refer to this assumption as the curvilinear DSR condition.

Ray tracing in the curvilinear coordinates, in general, implies for the tangent, or velocity, vector

$$(\tilde{v}_x(t),\tilde{v}_z(t)) = \frac{\mathrm{d}}{\mathrm{d}t}(\tilde{x},\tilde{z}) = -\frac{\partial\mathscr{H}}{\partial(\tilde{p}_x,\tilde{p}_z)}, \tag{5}$$

where $(\tilde{p}_x,\tilde{p}_z)$ is a slowness vector in the curvilinear coordinates, and $\mathscr{H}$ is the Hamiltonian, in this case given by

$$\mathscr{H} = \tfrac{1}{2}(\tilde{p}_x,\tilde{p}_z)_i g^{ij} c^2 (\tilde{p}_x,\tilde{p}_z)_j. \tag{6}$$

The length of the slowness vector is such that $\mathscr{H} = \frac{1}{2}$; the $\tilde{x}$-velocity satisfies

$$\tilde{v}_x(t) = c^2\tilde{g}^{\sigma\sigma'}\tilde{p}_{x,\sigma'},$$

While ignoring the $\tilde{z}$-component of velocity, it is immediate that

$$\tilde{v}_x^\sigma c^{-2}\tilde{g}_{\sigma\sigma'}\tilde{v}_x^{\sigma'} \leq 1. \tag{7}$$

For an event at $(x_s,x_r,t_{sr},\omega p_s,\omega p_r,\omega)$ to contribute to $D$ restricted to time $T$, it must hold that $(\tilde{x}_s,\tilde{z})$ is a point on the (source) ray, at time $t_s'$ say, which leaves the source at time $t=0$ with ray parameter $p_s$. Then $(\tilde{x}_r,\tilde{z})$ must be a point on the (receiver) ray, say at time $t_s''$, which arrives at the receiver $x_r$ with ray parameter $p_r$, at time $t_{sr}$. The geometry is displayed in Fig. 1. Moreover, the sum of travel times from the source to $(\tilde{x}_s,\tilde{z})$ and from the receiver to $(\tilde{x}_r,\tilde{z})$ must be equal to $t_{sr}-T$. From this, it follows that $t_s''-t_s'=T$. We now consider an event from a reflection at a point $(\tilde{x}_{\mathrm{scat}},\tilde{z}_{\mathrm{scat}})$ that is reached by the source ray at time $t_s$ and connects to the receiver by the receiver ray taking as initial time $t_s$.

Following the propagation of singularities in $D$, the 'horizontal' sunken source coordinates satisfy

$$\tilde{x}_{\mathrm{scat}} - (\tilde{x}-\tilde{h}_x) = \int_{t_s'}^{t_s}\mathrm{d}t\,\tilde{v}_{s,x}(t); \tag{8}$$

the 'horizontal' sunken receiver coordinates satisfy

$$-\tilde{x}_{\mathrm{scat}} + (\tilde{x}+\tilde{h}_x) = \int_{t_s''}^{t_s}\mathrm{d}t\,\tilde{v}_{r,x}(t). \tag{9}$$

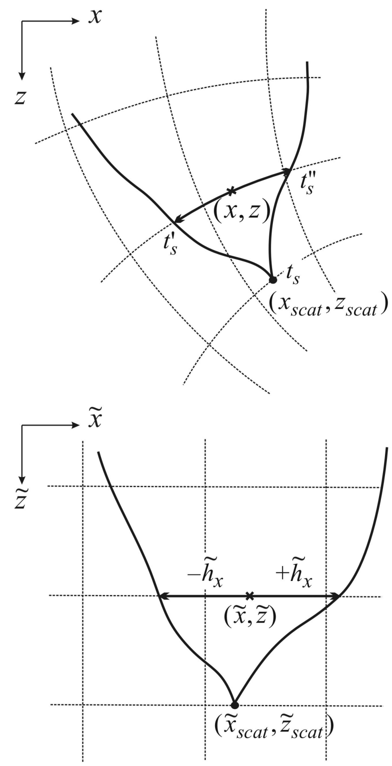


Figure 1: Scattering rays geometry associated with the angle transform, and the transformation of coordinates.

Adding up these equations results in

$$2\tilde{h}_x = \int_{t_s'}^{t_s''}\mathrm{d}t\,\tilde{v}_x(t), \tag{10}$$

where $\tilde{v}_x(t)$ is taken from the source ray for $t<t_s$ and from the receiver ray for $t<t_s$.

We introduce a tensor $B_{\sigma\sigma'}$ that is assumed to satisfy the 'bound' (cf. (7))

$$w^\sigma B_{\sigma\sigma'}w^{\sigma'} \leq w^\sigma c^{-2}\tilde{g}_{\sigma\sigma'}w^{\sigma'}. \tag{11}$$

Using the particular structure of the metric tensor, we obtain the estimate

$$2\,(\tilde{h}_x^\sigma B_{\sigma\sigma'}\tilde{h}_x^{\sigma'})^{1/2} \leq \int_{t_s'}^{t_s''}\left(\tilde{v}_x^\sigma c^{-2}\tilde{g}_{\sigma\sigma'}\tilde{v}_x^{\sigma'}\right)^{1/2}\mathrm{d}t \leq |t_s'-t_s''| = |T|. \tag{12}$$

We conclude that the energy in $D$ is located within the cone in $(\tilde{h}_x,T)$ space defined by this equation.

The angle transform in an integral of $D$ over a plane in $(\tilde{h}_x,T)$ space given by

$$T = \tilde{p}_\sigma\tilde{h}_x^\sigma.$$

Let $B^{\sigma\sigma'}$ denote the elements of the inverse of the matrix $B_{\sigma\sigma'}$. Suppose now that

$$p_\sigma B^{\sigma\sigma'}p_{\sigma'} < 2. \tag{13}$$

With

$$|\tilde{p}_\sigma\tilde{h}_x^\sigma| \leq (p_\sigma B^{\sigma\sigma'}p_{\sigma'})^{1/2}(\tilde{h}_x^\sigma B_{\sigma\sigma'}\tilde{h}_x^{\sigma'})^{1/2} \tag{14}$$

it then follows that

$$|T| = |\tilde{p}_\sigma\tilde{h}_x^\sigma| < 2(\tilde{h}_x^\sigma B_{\sigma\sigma'}\tilde{h}_x^{\sigma'})^{1/2}. \tag{15}$$

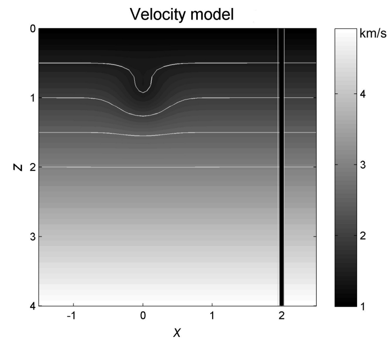


Figure 2: Model example: A low velocity lens embedded in a constant gradient background and a vertical reflector.

Thus the planes do not lie in the earlier mentioned cone. The only points where the planes of integration intersect the set where energy of $D$ is located, are points $T = 0$, $\tilde{h}_x = 0$. It follows that the energy in the angle transform is located only at the true scattering point independent of $\tilde{p}$.

## DISCUSSION AND EXAMPLE

In a 2D synthetic example, we compute the $(x_s, x_r, t_{sr}, \omega p_s, \omega p_r, \omega)$ of reflection data. The model is illustrated in Fig. 2: It contains a vertical reflector (representing a salt flank), a low velocity lens that causes the formation of caustics, embedded in a constant velocity gradient that causes rays to turn. Indeed, to illuminate and image the vertical reflector, one needs to make use of turning rays. Some incident and (specularly) reflected rays for a particular shot are shown in Fig. 3. In the same figure, we illustrate a proper curvilinear coordinate system with respect to which it becomes evident that the angle transform is free from artifacts (that is, the curvilinear DSR condition is satisfied). We then, numerically, computed the propagation of singularities by the angle transform, taking points $(x_s, x_r, t_{sr}, p_s, p_r, 1)$ of the modelled reflection data as input; the wavefront (also referred to as singular support) of the outcome, that is, the image gather, is illustrated in Fig. 4. We used a limited acquisition aperture so that the effect of illumination in the image plane is visible.

The transform introduced, here, also defines annihilators of the data, essentially by including a derivative with respect to $\tilde{p}$ ('differential semblance'). The transform is thus designed for wave-equation migration velocity analysis (MVA) in the presence of caustics including turning 'rays', extending our earlier approach to velocity analysis with steeply dipping (and vertical) reflectors. It involves the formulation of the wave-equation angle transform on manifolds with particular Riemannian metrics. We have shown the absence of artifacts, which is fundamental to the success of MVA in such complex environments. Moreover, the approach presented here can, in principle, be applied in global seismology. The curvilinear formulation admits a variety of possible fast propagation or continuation algorithms, while the shot-geophone representation (rather than our previous DSR representation) admits an irregular spaced source distribution.

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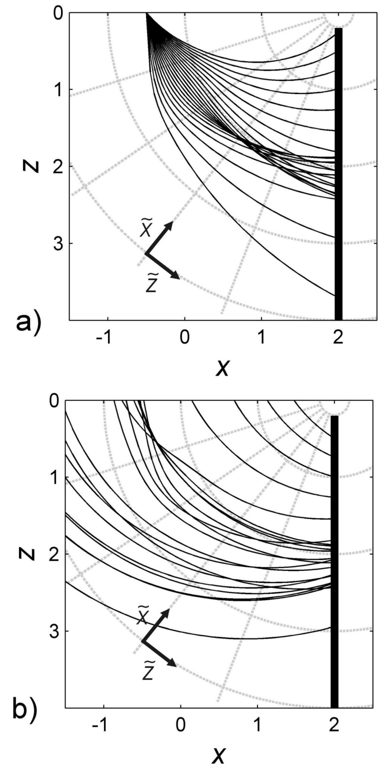


Figure 3: Incident (top) and reflected (bottom) and turning rays and an appropriate choice of curvilinear coordinates.

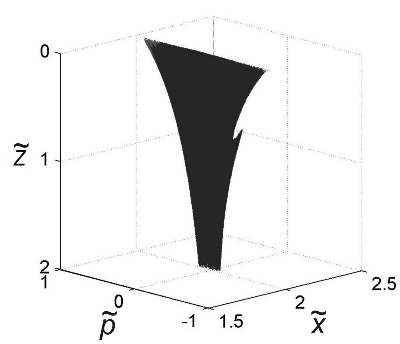


Figure 4: Wavefront of the image gather, numerically computed. Limited acquisition aperture induces the illumination revealed by the shading of the plane.