

A HOMOTOPY EXACT SEQUENCE FOR OVERCONVERGENT ISOCRYSTALS

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ABSTRACT. In this article we prove exactness of the homotopy sequence of overconvergent p -adic fundamental groups for a smooth and projective morphism in characteristic p . We do so by first proving a corresponding result for rigid analytic varieties in characteristic 0, following dos Santos [dS15] in the algebraic case. In characteristic p , we then proceed by a series of reductions to the case of a liftable family of curves, where we can apply the rigid analytic result.

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INTRODUCTION

One of the basic principles in ‘algebraic’ approaches to homotopy theory is that a smooth and proper morphism $f : X \rightarrow S$ of schemes (in any characteristic) should behave like a Serre fibration of topological spaces. In particular, for any reasonable definition of homotopy groups, one expects a long exact sequence relating the homotopy groups of the base S , the total space X , and the fibre X_s over some point $s \in S$. For étale homotopy groups, for example, this was proved in [Fri73], at least after completing away from the residue characteristics of S .

While the étale fundamental group controls the category of ℓ -adic local systems on varieties in characteristics different from ℓ , the same is certainly not true for p -adic local systems in characteristic p . In this situation, the version of the fundamental group that is usually considered is the one defined using Tannakian duality; this is somewhat analogous to the full pro-algebraic completion of the topological fundamental group $\pi_1(X, x)$ of a complex algebraic variety. In this world of ‘pro-algebraic homotopy theory’

much less is known than in étale homotopy theory, even in the case of smooth varieties over the complex numbers.

For example, it is not completely clear what the correct analogues of the higher homotopy groups are (although see [Toë00] for some work in this direction), and hence even formulating the analogue of the homotopy long exact sequence is problematic. Even if one sticks to the well-understood terms, i.e. to the sequence

$$\pi_1(X_s) \rightarrow \pi_1(X) \rightarrow \pi_1(S) \rightarrow \pi_0(X_s) \rightarrow \pi_0(X) \rightarrow \pi_0(S) \rightarrow *$$

then showing exactness has in general proved to be rather difficult. Over \mathbb{C} , using de Rham fundamental groups, this follows from ‘right exactness of the pro-algebraic completion functor’, and more generally it was shown for fields of characteristic 0 using a mixture of algebraic and transcendental methods in [Zha14]. Other results along these sort of lines have been proved in [DPS16, EH06, Laz15].

A major new approach to these sorts of problems was introduced in [dS15], where the author showed how to construct push-forwards of certain kinds of ‘non-linear \mathcal{D} -modules’, i.e. stratified schemes over the total space X . He then used this construction to give a completely algebraic proof of exactness of the π_1 -part of the sequence, assuming geometric connectedness of X_s and S . One of the crucial insights of his article is that by replacing linear representations with projective representations, one can avoid completely one of the major difficulties in proving exactness of these sorts of ‘homotopy sequences’ (see §2).

Inspired by dos Santos’ methods, in this article we prove the following result.

Theorem (7.1). *Let $f : X \rightarrow S$ be a smooth, projective morphism of smooth varieties over a perfect field k of characteristic $p > 0$, with geometrically connected fibres and base. Then the sequence of fundamental groups*

$$\pi_1^\dagger(X_s) \rightarrow \pi_1^\dagger(X) \rightarrow \pi_1^\dagger(S) \rightarrow 1$$

classifying overconvergent isocrystals is exact.

If one tries to directly transport dos Santos’ construction to the overconvergent setting, one is very quickly confronted by a seemingly insurmountable list of problems and subtleties: even in the linear case the problem of constructing $\mathbf{R}^0 f_*$ of an overconvergent isocrystal (without F -structure!) along a smooth and proper morphism is unreasonably difficult (see §8). Instead we proceed in a much more roundabout fashion, advancing via a lengthy chain of reductions, which here we present in reverse order to that found in the body of the article.

First we cut our given morphism $f : X \rightarrow S$ by a sequence of hyperplane sections, which by some diagram chasing and a very weak form of the Lefschetz hyperplane theorem for fundamental groups allows us to reduce to the case of a family of curves. In this case, our morphism arises via pullback from the universal curve

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{C}_g \\ \downarrow & & \downarrow \\ S & \longrightarrow & \mathcal{M}_g \end{array}$$

and hence by lifting the morphism $S \rightarrow \mathcal{M}_g$ along some smooth lift of S (at least locally) we can assume that the whole family of smooth projective curves lifts to characteristic 0. (This is not quite what we do, but this is the basic idea anyway.)

In this case we can write the overconvergent fundamental group $\pi_1^\dagger(S)$ as a quotient

$$\varprojlim_{\lambda} \pi_1^{\mathrm{dR}}(V_{\lambda}) \twoheadrightarrow \pi_1^\dagger(S)$$

of the inverse limit of the de Rham fundamental groups $\pi_1^{\mathrm{dR}}(V_{\lambda})$ as V_{λ} ranges over strict neighbourhoods of $]S[$ inside the generic fibre of the given lift. Moreover, the same is true for $\pi_1^\dagger(X)$, using neighbourhoods W_{λ} , say (again, this is not strictly what we do but this is the essential idea). Now since we can choose these neighbourhoods so that each $W_{\lambda} \rightarrow V_{\lambda}$ is smooth and proper, some more diagram chasing allows us to reduce to the following result in rigid analytic geometry.

Theorem (1.5). *Let $f : W \rightarrow V$ be a smooth, projective morphism of smooth analytic varieties over a p -adic field K , with geometrically connected fibres and base. Let $v \in V(K)$. Then the sequence of fundamental groups*

$$\pi_1^{\mathrm{dR}}(W_v) \rightarrow \pi_1^{\mathrm{dR}}(W) \rightarrow \pi_1^{\mathrm{dR}}(V) \rightarrow 1$$

classifying coherent modules with integrable connection is exact.

The point is that now we have reduced to a statement solely concerning smooth projective morphisms of analytic K -varieties, with no reference to tubes or overconvergence. We are therefore in a situation where we really can directly apply dos Santos' ideas and arguments, as essentially all of the difficulties we originally faced have disappeared. This is now what we do: the proof of this 'de Rham' homotopy exact sequence consists entirely of translating dos Santos' proof from [dS15] into the analytic context.

In actual fact, we do much less than this. Rather than reprove all of dos Santos' results on 'push-forwards' of stratified schemes for rigid analytic varieties, we instead use various tricks to be able to reduce to cases where we can in fact *apply* his results. The basic idea is that it in fact suffices to show that for a stratified variety Z over W , the unit map $f^* f_{\mathrm{dR}*} Z \rightarrow Z$ for the *relative* push-forward is a closed immersion, and the image is stable under the stratification on Z . But now, by relative rigid analytic GAGA, these relative push-forwards simply arise as the analytification of those considered in [dS15]. Moreover, that the image is stable under the stratification can be checked after passing to the completed local ring at any rigid point, and hence to the various infinitesimal neighbourhoods of this point. The situation is now completely algebraic over the ground field K , and so once more we can use dos Santos' results. In fact a little care is needed, since these infinitesimal neighbourhoods will not be smooth over K , so *a priori* the results of [dS15] do not apply. However, it is straightforward to show that the *proofs* of these results apply in the situation we are interested in.

Finally, in §12, we discuss some applications of the homotopy exact sequence for overconvergent fundamental groups. First of all we prove a 'Lefschetz' type theorem, stating that if $Y \subset X$ is a smooth hyperplane section inside a smooth projective variety, then the induced map

$$\pi_1^\dagger(Y) \rightarrow \pi_1^\dagger(X)$$

on p -adic fundamental groups is surjective. We are then able to use this to show that when the ground field is algebraically closed, and X is smooth and projective, then there is a canonical isomorphism

$$\pi_0(\pi_1^\dagger(X)) \cong \pi_1^{\acute{\mathrm{e}}\mathrm{t}}(X)$$

between the component group of the p -adic fundamental group and the pro-finite étale fundamental group. This generalises a result of Crew [Cre92, Proposition 4.4], in which objects were assumed to have Frobenius structures.

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Notations and conventions.

- We will denote by k a perfect field of characteristic $p > 0$, \mathcal{V} a complete DVR with residue field k and fraction field K of characteristic 0. We will let ϖ denote a choice of uniformiser for \mathcal{V} .
- An algebraic variety over k (resp. K) will mean a separated scheme of finite type, the category of these will be denoted \mathbf{Var}_k (resp. \mathbf{Var}_K). If V is an algebraic variety over either k or K , we will denote by \mathbf{Var}_V the slice category of varieties over V .
- An analytic variety over K will mean an adic space, separated and locally of finite type over $\mathrm{Spa}(K, \mathcal{V})$. Since all rigid spaces will be locally of finite type over K , we may, without ambiguity, denote an affinoid adic space $\mathrm{Spa}(A, A^+)$ simply by $\mathrm{Spa}(A)$. We will let \mathbf{Rig}_K denote the category of analytic varieties over K , and for any such V the slice category will be denoted \mathbf{Rig}_V . The analytification of an algebraic variety over K will always be considered as an adic space. Throughout, we will implicitly use [FK13, Theorem II.A.5.2] to apply the results of [FK13] to objects of \mathbf{Rig}_K .
- If Y is a k -variety, we will denote by $\mathrm{Isoc}^\dagger(Y/K)$ the category of overconvergent isocrystals on Y/K .
- A closed subgroup of an affine group scheme will always mean a closed sub-scheme that is also a subgroup, a surjective homomorphism will be a group scheme homomorphism which is faithfully flat.
- Unadorned tensor or fibre products will be over k, K or \mathcal{V} , it will be clear which from the context. Sometimes, in order to avoid confusion, we will denote the fibre product of a diagram $X \xrightarrow{f} Z \xleftarrow{g} Y$ by one of $X \times_{f,Z,g} Y$, $X \times_{f,Z} Y$ or $X \times_{Z,g} Y$, depending on which structure morphism needs clarifying.

1. THE HOMOTOPY SEQUENCE FOR ANALYTIC K -VARIETIES

The first goal of this article will be the proof of a homotopy exact sequence for certain classes of families of smooth analytic K -varieties. To start with, we will need to define the de Rham fundamental group of such spaces. So let V be an analytic variety over K .

Proposition 1.1. *Assume that V is smooth, geometrically connected, and admits a rational point $v \in V(K)$. Then the category $\mathrm{MIC}(V/K)$ of coherent \mathcal{O}_V -modules with integrable connection is neutral Tannakian over K , with fibre functor v^* .*

Proof. We first claim that any coherent module with integrable connection is locally free. Indeed, this question is local, and we may assume $V = \mathrm{Spa}(A)$ to be affinoid. In particular, E comes from a coherent sheaf E^a on $\mathrm{Spec}(A)$ and it suffices to prove that E^a is locally free. But this may be checked after passing to the completed local ring $\widehat{A}_{\mathfrak{m}}$ at any closed point $\mathfrak{m} \in \mathrm{Spec}(A)$, which by enlarging K can be assumed to be K -valued. Now choosing étale co-ordinates $\mathrm{Spa}(A) \rightarrow \mathbb{D}_K^n$ in some neighbourhood of this given K -point

induces an isomorphism $\widehat{A}_m \cong K[[x_1, \dots, x_n]]$. Moreover the integrable connection on E induces a formal integrable connection on $E^a \otimes_A \widehat{A}_m$. Hence we may apply [Kat70, Proposition 8.9].

It therefore follows that

$$v^* : \text{MIC}(V/K) \rightarrow \text{Vec}_K$$

is a faithful, K -linear, exact tensor functor, and since V is connected, we can see that if $v^*(E)$ has dimension 1, then E is a line bundle. Hence applying [DMOS82, Ch. II, Proposition 1.20] it suffices to prove that the natural map

$$K \rightarrow H_{\text{dR}}^0(V/K) := H^0(V, \Omega_{V/K}^*)$$

is an isomorphism. Applying v^* we obtain a retraction

$$H_{\text{dR}}^0(V/K) \cong \text{End}_{\text{MIC}(V/K)}(\mathcal{O}_V) \rightarrow \text{End}_K(v^*\mathcal{O}_V) = K$$

of this map. In particular, if $H_{\text{dR}}^0(V/K)$ were strictly bigger than K , then $\Gamma(V, \mathcal{O}_V)$ would contain a non-trivial idempotent element, contradicting the connectedness of V . \square

Definition 1.2. Let (V, v) be as in Proposition 1.1. Then we define the de Rham fundamental group $\pi_1^{\text{dR}}(V, v)$ of V to be the Tannaka dual of $\text{MIC}(V/K)$ with respect to the fibre functor

$$v^* : \text{MIC}(V/K) \rightarrow \text{Vec}_K.$$

Now let $f : W \rightarrow V$ be a proper morphism of analytic K -varieties. Recall from [Con06] that a line bundle \mathcal{L} on W is said to be f -ample if it is so on each fibre W_v over a rigid point $v \in V$. In other words, for each rigid point $v \in V$, some tensor power $\mathcal{L}|_{W_v}^{\otimes n}$ defines a closed immersion $W_v \hookrightarrow \mathbb{P}_{K(v)}^{N, \text{an}}$.

Definition 1.3. We say that a proper morphism $f : W \rightarrow V$ of analytic K -varieties is projective if W admits an f -ample line bundle.

Remark 1.4. With this definition, a projective morphism admits a closed immersion $W \rightarrow \mathbb{P}_V^{N, \text{an}}$ locally on the base V , by [Con06, Theorem 3.2.7]. Such an embedding need not exist globally, although it will if the base is affinoid, or itself projective over $\text{Spa}(K)$. Note also that with this definition, a composition of projective morphisms is projective, but projectivity is not necessarily local on the base.

Suppose that $f : W \rightarrow V$ is a morphism of smooth, geometrically connected K -varieties, $w \in W(K)$ is a K -valued point, and set $v = f(w)$. If the fibre W_v is also smooth and geometrically connected then we call the sequence

$$\pi_1^{\text{dR}}(W_v, w) \rightarrow \pi_1^{\text{dR}}(W, w) \rightarrow \pi_1^{\text{dR}}(V, v) \rightarrow 1$$

of affine group schemes the *homotopy sequence* associated to the pair (f, w) . Then the main result of the first part of this article is the following.

Theorem 1.5. *Let $f : W \rightarrow V$ be a smooth projective morphism of smooth analytic K -varieties, with geometrically connected fibres and base, and let $w \in W(K)$. Then the homotopy sequence of the pair (f, w) is exact.*

2. EXACTNESS CRITERIA AND POLARISABLE G -VARIETIES

The strategy to prove Theorem 1.5 is essentially to translate dos Santos' proof of exactness of the homotopy sequence in [dS15] from the algebraic to the analytic setting. The need to work analytically will present us with several difficulties, and consequently at many points we will prove weaker results, and with extra hypotheses, than those obtained in [dS15]. In order to be able to get away with this, we will need to combine the 'projective' criteria for exactness of a sequence of affine group schemes discussed in [dS15, §4] with more traditional 'linear' versions considered for example in [EHS08, Appendix A]. To begin with then, let us quickly recall how these criteria work.

Theorem 2.1 ([dS15], Lemma 4.3). *Let*

$$L \xrightarrow{q} G \xrightarrow{p} A \rightarrow 1$$

be a sequence of affine group schemes such that p is faithfully flat. Then the sequence is exact if and only if for all $V \in \text{Rep}(G)$ the inclusion

$$\mathbb{P}(V)^{\ker p}(K) \subset \mathbb{P}(V)^L(K)$$

of K -points on the fixed schemes is an equality.

Theorem 2.2 ([EHS08], Theorem A.1(iii)). *Let*

$$1 \rightarrow L \xrightarrow{q} G \xrightarrow{p} A \rightarrow 1$$

be a sequence of affine group schemes, such that q is a closed immersion and p is faithfully flat. Then the sequence is exact if and only if the following three conditions hold.

- (1) *If $V \in \text{Rep}(G)$, then $q^*(V)$ is trivial in $\text{Rep}(L)$ if and only if $V \cong p^*(W)$ for some $W \in \text{Rep}(A)$;*
- (2) *for any $V \in \text{Rep}(G)$, if $W_0 \subset q^*(V)$ is the maximal trivial sub-object in $\text{Rep}(L)$, then there exists $W \subset V \in \text{Rep}(G)$ such that $q^*(W) = W_0 \subset q^*(V)$;*
- (3) *any object of $\text{Rep}(L)$ is a sub-object of one in the essential image of q^* .*

In practise, the first two of the conditions in Theorem 2.2 are (conceptually at least) very easy to verify, the third extremely difficult. It will therefore be useful to see what happens when we drop it. Note that the intersection of any collection of closed normal subgroups of an affine group scheme G is also a closed normal subgroup, hence we may define the normal closure $H^{\text{norm}} \subset G$ of a closed subgroup $H \subset G$ as the intersection of all closed normal subgroups containing it.

Definition 2.3. We say that a sequence of affine group schemes

$$L \xrightarrow{q} G \xrightarrow{p} A \rightarrow 1$$

is weakly exact if $G \xrightarrow{p} A$ is surjective, the composition $L \xrightarrow{p \circ q} A$ is trivial, and if $\ker(p) = q(L)^{\text{norm}}$. In other words, the sequence

$$1 \rightarrow q(L)^{\text{norm}} \rightarrow G \rightarrow A \rightarrow 1$$

is exact.

Weak exactness turns out to be exactly what we can prove without the third condition in Theorem 2.2.

Theorem 2.4. *Let $L \xrightarrow{q} G \xrightarrow{p} A \rightarrow 1$ be a sequence of affine group schemes over K such that p is faithfully flat. Assume that:*

- (1) *if $V \in \text{Rep}(G)$, then $q^*(V)$ is trivial in $\text{Rep}(L)$ if and only if $V \cong p^*(W)$ for some $W \in \text{Rep}(A)$;*

(2) for any $V \in \text{Rep}(G)$, if $W_0 \subset q^*(V)$ is the maximal trivial sub-object in $\text{Rep}(L)$, then there exists $W \subset V \in \text{Rep}(G)$ such that $q^*(W) = W_0 \subset q^*(V)$.

Then $L \rightarrow G \rightarrow A \rightarrow 1$ is weakly exact.

Proof. First note that by [EHS08, Theorem A.1] we may describe $\text{Rep}(q(L))$ as the full subcategory of $\text{Rep}(L)$ consisting of objects which are sub-quotients of objects in the essential image of $q^* : \text{Rep}(G) \rightarrow \text{Rep}(L)$. In particular, it is straightforward to verify that both conditions continue to hold if we replace L by $q(L)$, in other words we may assume that q is a closed immersion and L is in fact a closed subgroup of G .

We next claim that moreover the conditions continue to hold if we replace L by the normal subgroup L^{norm} it generates, the non-trivial one is (2). In this case, we know from condition (2) applied to L that for any representation V of G , the subspace V^L is in fact stable by G . Since V^L is therefore a G -representation on which L acts trivially, it follows that L^{norm} acts trivially, in particular we have $V^L = V^{L^{\text{norm}}}$, which suffices to prove that (2) also holds for L^{norm} .

In other words we may in fact assume that $L = L^{\text{norm}}$, and in particular that L is a normal subgroup of G . But now we note that by [EHS08, Theorem A.1(ii)] any object of $\text{Rep}(L)$ is a sub-object of one in the essential image of q^* , and hence applying Theorem 2.2 we can see that the sequence

$$1 \rightarrow L \rightarrow G \rightarrow A \rightarrow 1$$

is exact. □

As mentioned before, the conditions of Theorem 2.4 are often easy to verify, and in the situation of Theorem 1.5 we may do so as follows. Let $f : W \rightarrow V$, $w \in W(K)$ be as in Theorem 1.5, and suppose we are given $E \in \text{MIC}(W/K)$. We define

$$\begin{aligned} f_{\text{dR}*}E &:= \mathbf{R}^0 f_* \left(E \otimes_{\mathcal{O}_W} \Omega_{W/V}^* \right) \\ &= \ker \left(f_* E \rightarrow f_* (E \otimes \Omega_{W/V}^1) \right) \\ &= f_* \ker \left(E \rightarrow E \otimes \Omega_{W/V}^1 \right) \end{aligned}$$

to be the sheaf of relative horizontal sections. Since f is proper, $f_{\text{dR}*}E$ is a coherent sheaf on V , and exactly as in [KO68, §2] we may endow it with an integrable connection. One easily verifies that

$$f^* : \text{MIC}(V/K) \rightleftarrows \text{MIC}(W/K) : f_{\text{dR}*}$$

are adjoint functors, and that for any K -valued point $v \in V(K)$ there is a natural isomorphism

$$v^* f_{\text{dR}*}E \cong H_{\text{dR}}^0(W_v/K, E|_{W_v}).$$

Lemma 2.5. *In the situation of Theorem 1.5 the sequence*

$$\pi_1^{\text{dR}}(W_v, w) \rightarrow \pi_1^{\text{dR}}(W, w) \rightarrow \pi_1^{\text{dR}}(V, v) \rightarrow 1$$

is weakly exact.

Proof. Using the fact that $v^* f_{\text{dR}*}E \cong H_{\text{dR}}^0(W_v/K, E|_{W_v})$ one easily checks that the adjunction map $F \rightarrow f_{\text{dR}*}f^*F$ is an isomorphism for any $F \in \text{MIC}(V/K)$, thus the functor f^* is fully faithful. If we are given a sub-object $E \subset f^*F$, then again applying $f^* f_{\text{dR}*}$ we obtain

$$f^* f_{\text{dR}*}E \subset E \subset f^*F$$

and we claim that in fact $f^* f_{dR*} E = E$. But since this can be checked on fibres, it follows from the fact that any sub-object of a trivial object in $\text{MIC}(W_v/K)$ is itself trivial.

Hence the map $\pi_1^{\text{dR}}(W, w) \rightarrow \pi_1^{\text{dR}}(V, v)$ is faithfully flat. To show that condition (1) in Theorem 2.4 holds, we note that for $E \in \text{MIC}(W/K)$ the adjunction map $f^* f_{dR*} E \rightarrow E$ is an isomorphism iff it is so on fibres, which happens iff $E|_{W_v}$ is trivial. Similarly, for (2) we can take $f^* f_{dR*} E \subset E$ as the required sub-object. \square

The reason that this is useful is that now we can formulate an alternative version of dos Santos' criterion from Theorem 2.1.

Proposition 2.6. *Let $L \xrightarrow{q} G \xrightarrow{p} A \rightarrow 1$ be a weakly exact sequence of affine group schemes. Then the sequence is exact if and only if for any $V \in \text{Rep}(G)$ the fixed scheme*

$$\mathbb{P}(V)^L \subset \mathbb{P}(V)$$

is invariant under G .

Proof. By Theorem 2.1 we must prove that the inclusion $\mathbb{P}(V)^{\ker p}(K) \subset \mathbb{P}(V)^L(K)$ of K -points on the fixed scheme is an equality. If $\mathbb{P}(V)^L$ is invariant under G , then we obtain a homomorphism

$$\rho : G \rightarrow \mathbf{Aut}_K(\mathbb{P}(V)^L)$$

of functors on K -schemes that by definition satisfies $q(L) \subset \ker \rho$. Since $\mathbb{P}(V)^L$ is a projective variety the functor $\mathbf{Aut}_K(\mathbb{P}(V)^L)$ is representable by a group scheme over K , hence $\ker \rho$ is a *closed* normal subgroup of G . Since it contains $q(L)$, it must also contain $q(L)^{\text{norm}}$, from which we deduce that $\ker \rho$ must act trivially on $\mathbb{P}(V)^L$. Hence the claimed equality does indeed hold. \square

This shows the importance of considering projective schemes together with actions of the fundamental group, and many results from [dS15] involve extending the classical Tannakian duality to include these sorts of objects. We expect many of these results to also hold in the analytic context, but in our impatience to prove Theorem 1.5 (and consequently 7.1 below) we have not investigated this fully. Instead, we will stick to the more restrictive category of varieties together with a *polarisable* action.

Definition 2.7. Let G be an affine group scheme over K , Y a proper K -variety, and $\rho : G \rightarrow \mathbf{Aut}_K(Y)$ an action of G on Y . We say that the action is polarisable if:

- (1) ρ factors through an algebraic quotient $G \twoheadrightarrow H$;
- (2) there exists an ample line bundle \mathcal{L} on Y admitting a H -linearisation.

Note that ‘polarisable’ simply means that such an H and \mathcal{L} exist, we do not specify them as part of the data.

Lemma 2.8. *A G -action on Y is polarisable if and only if there exists some $V \in \text{Rep}(G)$ and a G -equivariant closed embedding*

$$Y \hookrightarrow \mathbb{P}(V).$$

Proof. Since the action on any such V must factor through an algebraic quotient, the existence of such an embedding clearly implies polarisability. For the converse, we may assume that G is algebraic and that the line bundle \mathcal{L} in condition (2) is very ample. In this situation, $H^0(Y, \mathcal{L})$ is a finite dimensional representation of G , and the natural map

$$Y \rightarrow \mathbb{P}(H^0(Y, \mathcal{L}))$$

is G -equivariant. \square

Remark 2.9. In fact, the proof of this lemma shows that the condition in Definition 2.7 that the action of G on either Y or the ample line bundle \mathcal{L} factors through some algebraic quotient is redundant.

Corollary 2.10. *If $G \rightarrow H$ is a homomorphism of affine group schemes, and Y is a proper K -variety with a polarisable H -action, then the induced G -action is also polarisable.*

3. STRATIFIED ANALYTIC SPACES

The proof of Lemma 2.5 demonstrates that the problem of proving weak exactness of the homotopy sequence is more or less that of constructing well-behaved ‘push-forwards’ of coherent modules with integrable connections along the given map $f : W \rightarrow V$. Similarly, one of the key insights of [dS15] is that the problem of proving that the conditions of Theorem 2.1 hold is essentially one of constructing push-forwards of more general, non-linear fibre bundles over W , endowed with ‘non-linear connections’. The construction of such push-forwards is exactly what we will want to imitate in the analytic setting. First, however, we will need to discuss the concept of a stratification on an analytic variety over some given base, which is the correct way to generalise integrable connections to non-linear objects.

So let V/K be an analytic variety, which for now we will not necessarily assume to be smooth. Let P_V^n denote the n th order infinitesimal neighbourhood of V inside $V \times V$, and $p_i^n : P_V^n \rightarrow V$ for $i = 0, 1$ the projection maps.

Definition 3.1. Let $Z \rightarrow V$ be an analytic variety over V . Then a stratification on Z is a collection of compatible isomorphisms

$$\varepsilon_n : Z \times_{p_0^n} P_V^n \xrightarrow{\sim} P_V^n \times_{p_1^n} Z,$$

of P_V^n -varieties such that $\varepsilon_0 = \text{id}$, and which satisfy the cocycle condition (see for example [Ber74, Ch. II, §1]). A morphism of stratified varieties is simply a morphism compatible with the maps ε_n , and the category of such objects will be denoted $\mathbf{Str}(V/K)$. We will denote the full subcategory of $\mathbf{Str}(V/K)$ consisting of varieties $Z \rightarrow V$ which are *projective* over V by $\mathbf{StrP}(V/K)$.

Example 3.2. Assume that V is smooth.

- (1) If $Z \rightarrow V$ is a bundle of analytic affine spaces, in other words is locally isomorphic to the projection $\mathbb{A}_V^{n,\text{an}} \rightarrow V$, and the stratification maps

$$Z \times_{p_0^n} P_V^n \xrightarrow{\sim} P_V^n \times_{p_1^n} Z$$

are linear, then we recover the notion of a coherent module with integrable connection on V .

- (2) If E is a coherent \mathcal{O}_V -module with integrable connection, with associated affine bundle $\mathbf{E} \rightarrow V$, then the projectivisation $\mathbb{P}(E) \rightarrow V$ of \mathbf{E} inherits a stratification from that on E .

More generally, a natural source of stratified varieties will be varieties equipped with an action of the fundamental group. From now on we will assume that V is smooth, connected, and with a K -rational point $v \in V(K)$.

Definition 3.3. We will denote the category of proper K -varieties together with a polarisable $\pi_1^{\text{dR}}(V, v)$ -action by \mathcal{R}_V .

It is worth pointing out that the category we have denoted \mathcal{R}_V is not the direct analogue in the analytic context of dos Santos’ category of the same name considered in [dS15, §6.2]. Our conditions are rather more restrictive, however, \mathcal{R}_V will still have enough objects for our purposes.

The construction of objects in $\mathbf{StrP}(V/K)$ from those in \mathcal{R}_V is relatively straightforward. Indeed, if $(Y, \rho) \in \mathcal{R}_V$ then we may choose an equivariant embedding $Y \hookrightarrow \mathbb{P}_K^N$ for some linear action of $\pi_1^{\mathrm{dR}}(V, \nu)$ on \mathbb{A}_K^{N+1} , and via this we may view the projective co-ordinate ring

$$S_Y := \bigoplus_n \Gamma(Y, \mathcal{O}_{\mathbb{P}_K^N}(n))$$

as a $\pi_1^{\mathrm{dR}}(V, \nu)$ -representation. By construction, we know that S_Y is the colimit of its finite dimensional sub-representations, and hence via the usual Tannakian correspondence we can construct an associated ind-coherent sheaf \mathcal{S}_Y of graded rings on V , equipped with an integrable connection. We now define

$$\mathcal{U}_V(Y, \rho) := \mathbf{Proj}_{\mathcal{O}_V}(\mathcal{S}_Y)$$

via the relative Proj construction of [Con06]. The integrable connection on \mathcal{S}_Y induces a stratification on $\mathcal{U}_V(Y, \rho)$, making it into an object of $\mathbf{StrP}(V/K)$. This generalises Example 3.2(2) in that if $Y = \mathbb{P}(E_V)$ for some $E \in \mathrm{MIC}(V/K)$, then $\mathcal{U}_V(Y) \cong \mathbb{P}(E)$.

Proposition 3.4. *This construction induces a functor $\mathcal{U}_V : \mathcal{R}_V \rightarrow \mathbf{StrP}(V/K)$ from polarisable $\pi_1^{\mathrm{dR}}(V, \nu)$ -varieties to projective stratified V -varieties. It is compatible with pull-back via morphisms $f : W \rightarrow V$ in the sense that the diagram*

$$\begin{array}{ccc} \mathcal{R}_V & \longrightarrow & \mathbf{StrP}(V/K) \\ \downarrow & & \downarrow \\ \mathcal{R}_W & \longrightarrow & \mathbf{StrP}(W/K) \end{array}$$

is 2-commutative.

Proof. There are two things to check: firstly that $\mathcal{U}_V(Y, \rho)$ does not depend on the choice of $\pi_1^{\mathrm{dR}}(V, \nu)$ -linearised ample line bundle \mathcal{L} , and secondly that we can make the association functorial in (Y, ρ) . For the first claim, we note that given two equivariant embeddings $Y \rightarrow \mathbb{P}(E_V)$ and $Y \rightarrow \mathbb{P}(F_V)$ we can simply consider their product

$$Y \hookrightarrow \mathbb{P}(E_V) \times_K \mathbb{P}(F_V)$$

and show that the two projection maps induce isomorphisms between appropriate relative Proj constructions. Similarly, to obtain functoriality, we can use the graph construction to reduce to considering closed immersions and projections. These can both be very easily handled.

Finally, functoriality in $f : W \rightarrow V$ follows from the facts that the homomorphism $\pi_1^{\mathrm{dR}}(W, w) \rightarrow \pi_1^{\mathrm{dR}}(V, \nu)$ corresponds to f^* on the level of modules with integrable connection, and that \mathbf{Proj} commutes with pull-back of ind-coherent modules by [Con06, Theorem 2.3.6]. \square

With additional polarisability assumptions, ‘Tannakian reconstruction’ theorems are very easy to prove using the classical ‘linear’ versions.

Definition 3.5. We say that $Z \in \mathbf{StrP}(V/K)$ is polarisable if there exists $E \in \mathrm{MIC}(V/K)$ and a closed embedding $Z \hookrightarrow \mathbb{P}(E)$ of stratified V -varieties.

Clearly, the functor \mathcal{R}_V lands inside the full subcategory $\mathbf{StrPol}(V/K) \subset \mathbf{StrP}(V/K)$ consisting of polarisable stratified V -varieties.

Theorem 3.6. *The functor*

$$\mathcal{U}_V : \mathcal{R}_V \rightarrow \mathbf{StrPol}(V/K)$$

is an equivalence of categories.

Proof. This follows very easily from ordinary Tannakian duality and the relative Proj construction introduced in [Con06]. We first claim that given $(Y, \rho) \in \mathcal{R}_V$, the functor \mathcal{U}_V induces a bijection between $\pi_1^{\mathrm{dR}}(V, v)$ -invariant sub-schemes of Y and closed stratified sub-varieties of $\mathcal{U}_V(Y, \rho)$. Indeed, injectivity is clear, since for $T \subset Y$ closed and $\pi_1^{\mathrm{dR}}(V, v)$ -invariant we may recover T as the fibre of $\mathcal{U}_V(T, \rho)$ over v .

For surjectivity, we note that by construction $\mathcal{U}_V(Y, \rho) = \mathbf{Proj}_{\mathcal{O}_V}(\mathcal{S}_Y)$ for some ind-coherent graded \mathcal{O}_V -algebra \mathcal{S}_Y , equipped with an integrable connection. The closed sub-variety $\mathcal{T} \hookrightarrow \mathcal{U}_V(Y, \rho)$ is therefore given by some quotient

$$\mathcal{S}_Y \rightarrow \mathcal{S}_{\mathcal{T}}$$

which, since T is a stratified sub-variety, must be horizontal. There is therefore an induced integrable connection on $\mathcal{S}_{\mathcal{T}}$. Moreover, since \mathcal{S}_Y is the colimit of its coherent, horizontal sub-bundles, the same is true of $\mathcal{S}_{\mathcal{T}}$. Hence by the usual Tannakian correspondence this has to come from some $\pi_1^{\mathrm{dR}}(V, v)$ -invariant quotient $S_Y = \mathcal{S}_{Y, v} \rightarrow S_{\mathcal{T}} := \mathcal{S}_{\mathcal{T}, v}$. Then $T = \mathrm{Proj}(S_{\mathcal{T}})$ is the required invariant closed sub-scheme of Y .

This immediately implies essential surjectivity of \mathcal{U}_V , and in fact also implies full faithfulness. Indeed, as in Proposition 3.4, to prove full faithfulness it suffices via the graph construction to treat closed immersions and projections from products. The latter is obvious, and we have just proved the former. \square

4. RELATIVE STRATIFICATIONS AND PUSH-FORWARDS

To construct appropriate ‘push-forwards’ of smooth, projective stratified varieties along a smooth and projective morphism in [dS15], dos Santos proceeds in two stages. First of all he considers the push-forward of a ‘relatively stratified variety’ Z , and then shows that when this arises from a variety with an ‘absolute’ stratification, there is a canonical induced stratification on this push-forward. The analogy to bear in mind from the ‘linear’ case is that the push-forward $f_{\mathrm{dR}*}E$ of some module with integrable connection $E \in \mathrm{MIC}(W/K)$ is constructed by first viewing it as an object in $\mathrm{MIC}(W/V)$, one then puts a connection on $f_{\mathrm{dR}*}E$ by using the fact that E came from $\mathrm{MIC}(W/K)$.

In this section we will achieve the first step by appealing to GAGA, which will tell us that we can actually apply dos Santos’ results to provide the required push-forwards. In Section 6 below, we will then find another way to complete the proof of Theorem 1.5 *without* having to develop the analytic analogue of the ‘infinitesimal equivalence relations’ used in [dS15], instead by reducing to the situation over the formal polydisc $\mathrm{Spf}(K[[x_1, \dots, x_n]])$ over K . We start by introducing certain ‘formal adic spaces’, which will allow a slightly better way of talking about stratifications.

Definition 4.1. Let $T \hookrightarrow V$ be a closed immersion of analytic K -varieties, it is therefore by [FK13, Proposition II.7.3.5] defined by a coherent ideal sheaf $\mathcal{I}_T \subset \mathcal{O}_Z$. Let $Z_T^{(n)}$ denote the closed sub-variety of V defined by the ideal sheaf \mathcal{I}_T^{n+1} (i.e. $Z_T^{(n)}$ is the n th infinitesimal neighbourhood of T in V). We define the ‘formal completion of V along T ’ to be the ind-object $V_{/T} := \{V_T^{(n)}\}_n \in \mathrm{Ind}(\mathbf{Rig}_K)$ in the category of analytic K -varieties.

Let $\mathrm{Sh}(\mathbf{Rig}_K)$ denote the category of sheaves on \mathbf{Rig}_K for the analytic topology. Since objects of \mathbf{Rig}_K are locally quasi-compact, we have a fully faithful embedding

$$\mathrm{Ind}(\mathbf{Rig}_K) \hookrightarrow \mathrm{Sh}(\mathbf{Rig}_K)$$

and we will use this to view V/T as such a sheaf.

Example 4.2. If $V = \text{Spa}(K\langle x \rangle, \mathcal{V}\langle x \rangle)$ and $T \hookrightarrow V$ is the zero section, then

$$V_T^{(n)} = \text{Spa}\left(\frac{K[x]}{(x^{n+1})}, \mathcal{V} + x\frac{K[x]}{(x^{n+1})}\right).$$

So we should think of V/T as being given by something like

$$\text{“Spa}(K[[x]], \mathcal{V} + xK[[x])\text{”}$$

where the topology on $K[[x]]$ has a basis of open subgroups of the form $\mathfrak{a}^m W[[x]] + x^n K[[x]]$. Note that with this topology, $K[[x]]$ is not an f -adic ring, and hence the pair $(K[[x]], \mathcal{V} + xK[[x]])$ is *not* an affinoid ring in the sense of [Hub96, §1.1]. It would be interesting to see if there is a more general category of adic spaces in which things like $\text{Spa}(K[[x]], \mathcal{V} + xK[[x]])$ make sense.

By considering the diagonal $\Delta : V \rightarrow V^2$ of a smooth, separated analytic K -variety, we obtain the ind-variety that we will denote by \widehat{P}_V , which comes equipped with two ‘projection’ maps $p_i : \widehat{P}_V \rightarrow V$. With this language, we can rephrase the data of a stratification on some variety $Z \rightarrow V$ as an isomorphism

$$Z \times_{p_0} \widehat{P}_V \xrightarrow{\sim} \widehat{P}_V \times_{p_1} Z$$

in the slice category $\text{Sh}(\mathbf{Rig}_K)_{/\widehat{P}_Z}$ of sheaves over \widehat{P}_V , subject to certain obvious conditions. If we let $s : \widehat{P}_V \rightarrow \widehat{P}_V$ denote the map switching the factors and $c : \widehat{P}_V \times_{p_1, V, p_0} \widehat{P}_V \rightarrow \widehat{P}_V$ the map induced by $((v_0, v_1), (v_1, v_2)) \mapsto (v_0, v_2)$, then exactly as in [Ber74, Ch. II, §1] we can show that the data

$$p_0, p_1 : \widehat{P}_V \rightrightarrows V, \quad c : \widehat{P}_V \times_{p_1, V, p_0} \widehat{P}_V \rightarrow \widehat{P}_V, \quad \Delta : V \rightarrow \widehat{P}_V, \quad s : \widehat{P}_V \rightarrow \widehat{P}_V$$

forms a ‘formal groupoid’ over $\text{Spa}(K)$, and that a stratification on a V -variety Z is equivalent to an action of this groupoid.

Similarly, if we are given some morphism $f : W \rightarrow V$, then we may consider the formal completion $\widehat{P}_{W/V}$ along the diagonal $\Delta : W \rightarrow W \times_V W$. We have

$$p_0, p_1 : \widehat{P}_{W/V} \rightrightarrows W, \quad c : \widehat{P}_{W/V} \times_{p_1, W, p_0} \widehat{P}_{W/V} \rightarrow \widehat{P}_{W/V}, \quad \Delta : W \rightarrow \widehat{P}_{W/V}, \quad s : \widehat{P}_{W/V} \rightarrow \widehat{P}_{W/V}$$

exactly as before, giving rise to a groupoid over V .

Definition 4.3. A V -linear stratification on a W -variety Z is an action of the groupoid $\widehat{P}_{W/V} \rightrightarrows W$. We denote the category of W -varieties with a V -linear stratification by $\mathbf{Str}(W/V)$, and the full subcategory of objects which are projective over W by $\mathbf{StrP}(W/V)$.

These notions satisfy all the usual functorialities, which can be summarised by say that for any commutative square

$$\begin{array}{ccc} W' & \longrightarrow & W \\ \downarrow & & \downarrow \\ V' & \longrightarrow & V \end{array}$$

there is a pull-back functor $\mathbf{Str}(W/V) \rightarrow \mathbf{Str}(W'/V')$, which is transitive in the obvious manner. For example taking $(W/V) \rightarrow (W/K)$ we obtain the forgetful functor $\mathbf{Str}(W/K) \rightarrow \mathbf{Str}(W/V)$.

Now let us suppose that we have a smooth, projective morphism $f : W \rightarrow V$ of analytic K -varieties, with geometrically connected fibres. Note that we *do not* assume at this point that the base V is smooth.

If $\mathbf{P}(V) = \mathbf{StrP}(V/V)$ denotes the category of projective V -varieties, then as we have just seen there is a pull-back functor

$$f^* : \mathbf{P}(V) \rightarrow \mathbf{StrP}(W/V).$$

We wish to construct an ‘adjoint’ to f^* . To do so, suppose therefore that we are given some $Z \in \mathbf{StrP}(W/V)$. Define a functor

$$\begin{aligned} f_*Z : \mathbf{Rig}_V &\rightarrow \mathbf{Sets} \\ T/V &\mapsto \{\text{sections of } Z \times_V T \rightarrow W \times_V T\} \end{aligned}$$

where sections are considered as certain closed sub-varieties of $Z \times_V T$. (Thus f_*Z is a sub-functor of the Hilbert functor, the usual flatness condition is redundant for f_*Z , since W is flat over V .)

Proposition 4.4. *The functor f_*Z is representable by an analytic variety over V which has the following property: for each open affinoid $\text{Spa}(A) \subset V$, the restriction of $f_*Z \times_V \text{Spa}(A) \rightarrow \text{Spa}(A)$ to each of its connected components arises as the analytification of a quasi-projective A -scheme.*

Proof. This is similar in spirit to [Con06, Theorem 4.1.3]. Since f_*Z is clearly a sheaf for the analytic topology on \mathbf{Rig}_k we may in fact assume that $V = \text{Spa}(A)$ is affinoid. Hence, by relative rigid analytic GAGA [Con06, Example 3.2.6], W is the analytification of a smooth projective A -scheme W^a , and $Z \rightarrow W$ is the analytification of a projective morphism $Z^a \rightarrow W^a$. We consider the corresponding functor

$$\begin{aligned} f_*^a Z^a : \mathbf{Sch}_A &\rightarrow \mathbf{Sets} \\ T/A &\mapsto \{\text{sections of } Z^a \times_A T \rightarrow W^a \times_A T\} \end{aligned}$$

of locally Noetherian A -schemes, which by [Gro61, §4, Variant c.] is representable by a disjoint union of quasi-projective A -schemes. It therefore suffices to show that the analytification of $f_*^a Z^a$ represents the functor f_*Z . Since both are sheaves for the analytic topology, it suffices to check this on affinoids $\text{Spa}(B) \rightarrow \text{Spa}(A)$. In this case, we can again appeal to rigid analytic GAGA, which says that any closed sub-variety of $Z \times_{\text{Spa}(A)} \text{Spa}(B)$ is algebraic, i.e. comes from a unique closed sub-scheme of Z_B^a . \square

Let $b : T \rightarrow f_*Z$ be a point of f_*Z , corresponding to a section $\tau_b : W \times_V T \rightarrow Z \times_V T$. Pulling back by the two projections $p_i : \widehat{P}_{W/V} \rightarrow W$, i.e. applying $\widehat{P}_{W/V} \times_{p_i, W}$, we obtain sections

$$p_i^*(\tau_b) : \widehat{P}_{W/V} \times_V T \rightarrow \widehat{P}_{W/V} \times_{p_i, W} Z \times_V T.$$

of $\text{id} \times_{p_i, W} g$. We say that b is *horizontal* if $\varepsilon(p_0^*(\tau_b)) = p_1^*(\tau_b)$, where ε is the stratification on Z .

Definition 4.5. We define $f_{\text{dR}*}Z \subset f_*Z$ to be the sub-functor of horizontal sections.

Proposition 4.6. *The sub-functor $f_{\text{dR}*}Z$ is representable by a closed analytic sub-variety of f_*Z . If Z is smooth over W , then for any open affinoid $\text{Spa}(A) \subset V$ the restriction of $f_{\text{dR}*}Z \times_V \text{Spa}(A) \rightarrow \text{Spa}(A)$ to each of its connected components is projective.*

Proof. Being a closed sub-variety is local, and hence we may in fact assume that $V = \text{Spa}(A)$ is affinoid. Now again the whole situation algebraises: we have some smooth projective $f^a : W^a \rightarrow \text{Spec}(A)$ and some projective $Z^a \rightarrow W^a$ giving rise to Z upon analytification. Moreover, since the algebraic infinitesimal neighbourhoods give rise to the analytic ones upon analytification, it follows that the analytic stratification on Z comes from a unique A -linear algebraic stratification on Z^a . Now we simply note that the results of [dS15, §10] apply over any separated, Noetherian base scheme, for example, $\text{Spec}(A)$. Translated into

algebraic terms, what we have termed ‘horizontal’ corresponds exactly to what dos Santos calls ‘tangential’, hence we may again use rigid analytic GAGA to show that our functor $f_{\mathrm{dR}*}Z$ is simply the analytification of dos Santos’ scheme $H_{f^a}(Z^a)$. \square

The defining property of $f_{\mathrm{dR}*}Z$ gives a section $W \times_V f_{\mathrm{dR}*}Z \rightarrow Z \times_V f_{\mathrm{dR}*}Z$, and by composing with the first projection we therefore obtain a morphism $\varepsilon_Z : f^* f_{\mathrm{dR}*}Z \rightarrow Z$ of W -varieties. Essentially all of the main properties of $f_{\mathrm{dR}*}Z$ can then be deduced from those proved in [dS15].

Proposition 4.7. *Let $Z \in \mathbf{StrP}(W/V)$.*

- (1) *The map $\varepsilon_Z : f^* f_{\mathrm{dR}*}Z \rightarrow Z$ is horizontal with respect to the pull-back (V -linear) stratification on $f_{\mathrm{dR}*}Z$ and the given (V -linear) stratification on Z .*
- (2) *Formation of $f_{\mathrm{dR}*}Z$ is compatible with base change: if $V' \rightarrow V$ is a morphism of smooth K -varieties, then $(f_{\mathrm{dR}*}Z) \times_V V' \cong f_{\mathrm{dR}*}(Z \times_V V')$, and, via this isomorphism, $\varepsilon_{Z \times_V V'} = \varepsilon_Z \times_V V'$.*
- (3) *If the base $V = \mathrm{Spa}(K)$ is a point, $w \in W(K)$ is a rational point, and $Z = \mathcal{U}_W(Y, \rho)$ for some $(Y, \rho) \in \mathcal{R}_W$, then $\varepsilon_Z : f^* f_{\mathrm{dR}*}Z \rightarrow Z$ is obtained by applying \mathcal{U}_V to the closed immersion*

$$Y^{\pi_1^{\mathrm{dR}}(W, w)} \rightarrow Y,$$

considered as a morphism of $\pi_1^{\mathrm{dR}}(W, w)$ -varieties.

Proof. Note that the first two are local on V , and therefore follow from their algebraic versions [dS15, Proposition 12.1, Corollary 12.4]. For the third, we wish to algebraise and apply [dS15, Proposition 14.5], the point is to check that the induced (algebraic) stratification on Z is simple. This follows from the argument in the paragraph preceding the proof of [dS15, Proposition 14.5]. \square

5. FORMAL STRATIFICATIONS, INTEGRABILITY AND BASE CHANGE

It will also be necessary for us to have a formal analogue of the above constructions, working for now over an arbitrary field F . Since the basic ideas are essentially identical to those in the previous section, we will not give too many details. Let $t = (t_1, \dots, t_d)$ be a collection of variables and set $S = \mathrm{Spf}(F[[t]])$ to be the d -dimensional formal polydisc over F (using the t -adic topology). We will let $f : X \rightarrow S$ be a smooth morphism of finite type. In this situation we may define the formal groupoids

$$\widehat{P}_X \rightrightarrows X, \quad \widehat{P}_{X/S} \rightrightarrows X, \quad \widehat{P}_S \rightrightarrows S$$

exactly as before, and consequently we have the notion of a formal stratification on some formal X -scheme Z .

If both Z/X and X/S are projective then we may define the push-forward f_*Z as a disjoint union of formal schemes over S , as well as the closed sub-scheme $f_{\mathrm{dR}*}Z \subset f_*Z$ exactly as in the previous section. If we let S_n, X_n, Z_n denote the mod t^{n+1} -reductions, then we could equally well construct $f_{\mathrm{dR}*}Z$ and f_*Z as the limits

$$f_{\mathrm{dR}*}Z = \mathrm{colim}_n(f_{n, \mathrm{dR}*}Z_n), \quad f_*Z = \mathrm{colim}_n(f_{n, *}Z_n)$$

of the algebraic push-forwards along $f_n : X_n \rightarrow S_n$, as considered by dos Santos in [dS15, §10] (and that he terms $H_0(Z_n)$ and $H_{f_n}(Z_n)$ respectively). The following represents a simple extension to formal schemes of the results of [dS15].

Theorem 5.1. *Assume that X is projective over S , and that Z is smooth and projective over X . Then there exists an F -linear stratification on $f_{dR*}Z$ as a formal S -scheme such that the map*

$$f^* f_{dR*}Z \rightarrow Z$$

is compatible with the F -linear stratifications on both sides.

Proof. The point is that this essentially follows from [dS15] upon ‘taking the limit in n ’, although a little care is needed to achieve this. Consider for all n the ‘mod t^{n+1} reduction’ $Z_n \xrightarrow{g_n} X_n \xrightarrow{f_n} S_n$ of everything in sight, we wish to construct a stratification on $f_{n,dR*}Z_n$ as an S_n -scheme such that $f_n^* f_{n,dR*}Z_n \rightarrow Z_n$ is compatible with the stratifications. We cannot directly apply the results of [dS15] since S_n is not smooth over F , but we can get around this as follows.

Firstly, let us recall that dos Santos views stratifications as particular kinds of ‘infinitesimal equivalence relations’, and *without* using any smoothness assumptions on the ‘base’ of the fibration (in our case S_n) he constructs an infinitesimal equivalence relation on $f_{n,dR*}Z_n$ such that $f_n^* f_{n,dR*}Z_n \rightarrow Z_n$ intertwines these two equivalence relations. The point is then to try to prove that this equivalence relation on $f_{n,dR*}Z_n$ actually comes from a stratification. This is the content of [dS15, Proposition 13.4], and this proposition is the *only* place where smoothness assumptions are used.

But here we can exploit the fact that our situation arises as the mod t^{n+1} reduction of $Z \xrightarrow{g} X \xrightarrow{f} S$ with S formally smooth over K . In particular, if we choose local étale co-ordinates $x = (x_1, \dots, x_k)$ for X/S and $z = (z_1, \dots, z_m)$ for Z/X , then the stratification on Z corresponds to some section $\mathcal{O}_Z[[dt, dx, dz]] \rightarrow \mathcal{O}_Z[[dt, dx]]$ of the natural inclusion $\mathcal{O}_Z[[dt, dx]] \rightarrow \mathcal{O}_Z[[dt, dx, dz]]$. In particular we may therefore choose m elements F_1, \dots, F_m generating the kernel, locally on X and Z . Now reducing mod t^{n+1} we can follow the proof of [dS15, Proposition 13.4] word for word to conclude. \square

For this to be useful to us, we will need to compare this with the set-up considered previously, let us therefore return to that situation. So we have some smooth projective map $f : W \rightarrow V$ of analytic K -varieties, with geometrically connected fibres, and $g : Z \rightarrow W$ some smooth and projective stratified variety over W . If $v \in V$ is a *smooth* rigid point of V , then we may consider the various infinitesimal neighbourhoods $v^{(n)} := V_v^{(n)}$ as before. The point is that now the base change $Z_{v^{(n)}} \rightarrow W_{v^{(n)}} \rightarrow v^{(n)}$ algebrises relative to the ground field K , so we may consider $Z_{v^{(n)}} \rightarrow W_{v^{(n)}}$ as a projective morphism of projective K -schemes, equipped with a K -linear stratification. Thus taking the limit in n we obtain a smooth projective stratified formal scheme $\widehat{Z}_v \xrightarrow{g_v} \widehat{W}_v \xrightarrow{f_v} \mathrm{Spf}(\widehat{\mathcal{O}}_{V,v})$, where $\widehat{\mathcal{O}}_{V,v}$ is simply considered with the maximal-adic topology (not any kind of p -adic topology). Since v was chosen to be a smooth point, we therefore find ourselves in the situation of Theorem 5.1.

Hence we have some projective stratified formal scheme $f_{v,dR*}\widehat{Z}_v$ over $\mathrm{Spf}(\widehat{\mathcal{O}}_{V,v})$. Alternatively, since the base change of $f_{dR*}Z$ to $v^{(n)}$ is a disjoint union of projective $v^{(n)}$ -schemes, we may take the limit in n to obtain $f_{dR*}Z \times_V \mathrm{Spf}(\widehat{\mathcal{O}}_{V,v})$ which is a disjoint union of projective formal $\widehat{\mathcal{O}}_{V,v}$ -schemes. Note that this is simply notation, since there is no actual map $\mathrm{Spf}(\widehat{\mathcal{O}}_{V,v}) \rightarrow V$ of locally ringed spaces.

Proposition 5.2. *There is a natural isomorphism*

$$f_{dR*}Z \times_V \mathrm{Spf}(\widehat{\mathcal{O}}_{V,v}) \xrightarrow{\sim} f_{v,dR*}\widehat{Z}_v$$

of disjoint unions of projective formal schemes over $\widehat{\mathcal{O}}_{V,v}$, such that the diagram

$$\begin{array}{ccc} f^* f_{\mathrm{dR}*} Z \times_V \mathrm{Spf}(\widehat{\mathcal{O}}_{V,v}) & \longrightarrow & f_v^* f_{v,\mathrm{dR}*} \widehat{Z}_v \\ \varepsilon_Z \times_V \mathrm{Spf}(\widehat{\mathcal{O}}_{V,v}) \downarrow & & \downarrow \varepsilon_{\widehat{Z}_v} \\ Z \times_V \mathrm{Spf}(\widehat{\mathcal{O}}_{V,v}) & \xlongequal{\quad} & \widehat{Z}_v \end{array}$$

of formal schemes with \widehat{V}_v -linear stratifications commutes.

Proof. This simply follows from applying Proposition 4.7(2) to the various infinitesimal neighbourhoods $v^{(n)} \rightarrow V$, and then taking the limit in n . \square

Corollary 5.3. *If $Z \in \mathbf{StrP}(W/K)$ is smooth over W , then for each connected open affinoid $\mathrm{Spa}(A) \subset V$ the fibre product*

$$f_{\mathrm{dR}*} Z \times_V \mathrm{Spa}(A)$$

is flat and projective over $\mathrm{Spa}(A)$.

Proof. We may assume that $V = \mathrm{Spa}(A)$, and that f, g come from algebraic maps

$$f^a : W^a \rightarrow \mathrm{Spec}(A), \quad g^a : Z^a \rightarrow W^a$$

of projective A -schemes. We have already observed in Proposition 4.6 that in this situation $f_{\mathrm{dR}*} Z$ is the analytification of a disjoint union $f_{\mathrm{dR}*}^a Z^a$ of projective A -schemes. Moreover, for any closed point $\mathfrak{m} \in \mathrm{Spec}(A)$, we know by Theorem 5.1 that the base change $f_{\mathrm{dR}*}^a Z^a \otimes_A \widehat{A}_{\mathfrak{m}}$ admits a formal stratification. In particular, by applying [dS15, Lemma 6.2], we can see that $f_{\mathrm{dR}*}^a Z^a \otimes_A \widehat{A}_{\mathfrak{m}}$ must be flat over $\widehat{A}_{\mathfrak{m}}$.

Since this is true for all \mathfrak{m} , it follows that $f_{\mathrm{dR}*}^a Z^a$ is flat over A , so the restriction of $f_{\mathrm{dR}*}^a Z^a \rightarrow \mathrm{Spec}(A)$ to each of its connected components is flat and projective. Since $\mathrm{Spec}(A)$ is connected, each of these components must be set theoretically surjective over $\mathrm{Spec}(A)$, so each has a non-empty fibre over \mathfrak{m} . Hence the map

$$\pi_0(f_{\mathrm{dR}*}^a Z^a \otimes_A A/\mathfrak{m}) \rightarrow \pi_0(f_{\mathrm{dR}*}^a Z^a)$$

on connected components is surjective. Now applying [dS15, Proposition 6.4] and Proposition 4.7(3) to the fibre $f_{\mathrm{dR}*}^a Z^a \otimes_A A/\mathfrak{m}$ we know that this it has only finitely many connected components. Therefore so does $f_{\mathrm{dR}*}^a Z^a$, it is thus flat and projective over A . The claim now follows by taking the analytification. \square

6. INVARIANCE UNDER THE STRATIFICATION AND EXACTNESS OF THE HOMOTOPY SEQUENCE

Having constructed the relative push-forwards in §4, what we should do next is emulate the construction of [dS15, §13] to endow $f_{\mathrm{dR}*} Z$ with a stratification, at least when Z/W is smooth and comes from an object of $\mathbf{StrP}(W/K)$. We should then show that $f_{\mathrm{dR}*}$ is a ‘weak adjoint’ to f^* . In fact, to obtain the proof of Theorem 1.5 we can get away with the ‘formal’ version of this result, namely Theorem 5.1. Let us put ourselves in the situation of Theorem 1.5, so that $f : W \rightarrow V$ is smooth with geometrically connected fibres, and V is smooth and geometrically connected. Fix $w \in W(K)$ and set $v = f(w)$. Let $(Y, \rho) \in \mathcal{R}_W$ and assume that Y is smooth over K . Let $Z = \mathcal{U}_W(Y, \rho) \in \mathbf{StrP}(W/K)$. By applying the forgetful functor $\mathbf{StrP}(W/K) \rightarrow \mathbf{StrP}(W/V)$ we may construct $\varepsilon_Z : f^* f_{\mathrm{dR}*} Z \rightarrow Z$ as in the previous section.

Proposition 6.1. *The map $\varepsilon_Z : f^* f_{\mathrm{dR}*} Z \rightarrow Z$ is a closed immersion.*

Proof. We first note that by Proposition 4.7 the given map $\varepsilon_Z : f^* f_{dR^*} Z \rightarrow Z$ becomes a closed immersion on the fibre W_v over the given K -valued point $v \in V(K)$. By letting v vary and possibly increasing the base field K , we deduce that the same is true over any rigid point of V . Moreover, we know from Corollary 5.3 that on any connected open affinoid $\mathrm{Spa}(A) \subset V$, the base change

$$f_{dR^*} Z \times_V \mathrm{Spa}(A) \rightarrow \mathrm{Spa}(A)$$

is projective over A . Since ε_Z being a closed immersion is local on V , we therefore find ourselves in the following general situation. We have a smooth projective morphism $f : W \rightarrow \mathrm{Spa}(A)$ over an affinoid base, and a morphism $T \rightarrow Z$ of projective W -varieties, which is a closed immersion after passing to any rigid point of $\mathrm{Spa}(A)$. We wish to show that $T \rightarrow Z$ is a closed immersion. This is now a situation which can be algebrised, since by rigid analytic GAGA all three of W, T, Z come from projective A -schemes, as do all the morphisms between them.

Since the role of W can now be ignored, we can reduce to the following. Let A be a Noetherian ring, and $i : T^a \rightarrow Z^a$ a morphism of projective A -schemes, which is an isomorphism on the fibres over all closed points of A . Then we wish to show that i is a closed immersion. To see this, note that the quasi-finite locus of i must contain every closed point of Z^a , it must therefore be equal to Z^a by [Gro66, Théorème 13.1.5]. Therefore i is quasi-finite and projective, hence finite, say locally of the form $\mathrm{Spec}(C) \rightarrow \mathrm{Spec}(B)$. Moreover, for any maximal ideal \mathfrak{m} of B , the induced map $\frac{B}{\mathfrak{m}} \rightarrow C \otimes_B \frac{B}{\mathfrak{m}}$ is either the zero map or an isomorphism, in particular it is surjective. Hence i is a closed immersion as claimed. \square

Definition 6.2. We say that a closed sub-variety $T \hookrightarrow Z$ is stable under the stratification if the composite map

$$T \times_{W, \rho_0} \widehat{F}_W \rightarrow Z \times_{W, \rho_0} \widehat{F}_W \xrightarrow{\mathrm{pr}_Z \circ \varepsilon} Z$$

factors through T .

Theorem 6.3. Let $Z = \mathcal{U}_W(Y, \rho)$ for some $(Y, \rho) \in \mathcal{B}_W$ smooth over K . Then the closed immersion

$$f^* f_{dR^*} Z \hookrightarrow Z$$

is stable under the stratification.

Proof. Let us write T for $f^* f_{dR^*} Z$ to save on notation. First of all, the claim is local on V , which we may therefore assume to be affinoid, $V = \mathrm{Spa}(A)$. Applying Theorem 5.1 and Proposition 5.2 we know that for any rigid point $v \in V$ the base change $T \times_V \mathrm{Spf}(\widehat{\mathcal{O}}_{V, v})$ (again, this is just notation) is stable under the pull-back stratification. Now, since we are in characteristic zero, stability under the stratification (of either T or \widehat{T}_v) amounts simply to stability under the induced action of $\mathcal{D}er_K(\mathcal{O}_W)$. Therefore what we need to show is that for any derivation $\partial \in \mathcal{D}er_K(\mathcal{O}_W)$, and any local section $f \in \mathcal{I}_T$ of the ideal of T in Z , the section $\partial(f)$ is also in \mathcal{I}_T , i.e. maps to zero in \mathcal{O}_T .

Applying stability of \widehat{T}_v under the stratification, we know that $\partial(f)$ has to map to zero in $\mathcal{O}_T \otimes_A \widehat{A}_{\mathfrak{m}}$ (suitably interpreted!) for all maximal ideals of A . We can therefore reduce to the following general problem: we are given a regular Noetherian ring A , and an element $m \in M$ of some A -module, such that $m \mapsto 0$ in $M \otimes_A \widehat{A}_{\mathfrak{m}}$ for all maximal ideals \mathfrak{m} . We must show that $m = 0$. But this now follows from faithful flatness of $A \rightarrow \prod_{\mathfrak{m}} \widehat{A}_{\mathfrak{m}}$. \square

We can now complete the proof of Theorem 1.5.

Proof of Theorem 1.5. Suppose that we have some $E \in \text{MIC}(W/K)$. Then applying Theorem 6.3 we obtain a closed sub-variety

$$f^* f_{\text{dR}*} \mathbb{P}(E) \hookrightarrow \mathbb{P}(E)$$

which is stable under the stratification, and by Proposition 4.7 recovers the inclusion

$$\mathbb{P}(E_w)^{\pi_1^{\text{dR}}(W,w)} \hookrightarrow \mathbb{P}(E_w).$$

on the fibre over w . Since $f^* f_{\text{dR}*} \mathbb{P}(E)$ is stable under the stratification on $\mathbb{P}(E)$, it therefore acquires an induced stratification such that

$$f^* f_{\text{dR}*} \mathbb{P}(E) \hookrightarrow \mathbb{P}(E)$$

is a closed immersion of stratified W -varieties. We now apply Theorem 3.6: we can deduce that the closed immersion $f^* f_{\text{dR}*} \mathbb{P}(E) \hookrightarrow \mathbb{P}(E)$ must come from a unique $\pi_1^{\text{dR}}(W,w)$ -invariant closed sub-scheme of $\mathbb{P}(E_w)$. Put differently, we can see that the closed sub-scheme $\mathbb{P}(E_w)^{\pi_1^{\text{dR}}(W,w)} \subset \mathbb{P}(E_w)$ is invariant under $\pi_1^{\text{dR}}(W,w)$. Hence we may conclude by applying Proposition 2.6. \square

7. THE HOMOTOPY SEQUENCE FOR ALGEBRAIC k -VARIETIES

In the second part of this article, we will use Theorem 1.5 to deduce a corresponding result for algebraic k -varieties. Recall from [Cre92] that for a geometrically connected k -variety Y , the category $\text{Isoc}^\dagger(Y/K)$ of overconvergent isocrystals on Y/K is Tannakian, and any point $y \in Y(k)$ provides a fibre functor. Let $\pi_1^\dagger(Y,y)$ denote the corresponding fundamental group. (For brevity we have not included K in the notation, we hope that this will not present a problem.) Suppose that we are given a morphism $f : X \rightarrow S$ of geometrically connected k -varieties, and $x \in X(k)$. Write $s = f(x)$. If the fibre X_s is also geometrically connected, then we call the sequence

$$\pi_1^\dagger(X_s,x) \rightarrow \pi_1^\dagger(X,x) \rightarrow \pi_1^\dagger(S,s) \rightarrow 1$$

the *homotopy sequence* associated to the pair (f,x) .

Theorem 7.1. *Let $f : X \rightarrow S$ be a smooth and projective morphism of smooth k -varieties, with geometrically connected fibres and base, and let $x \in X(k)$. Then the homotopy sequence of the pair (f,x) is exact.*

The basic idea of the proof will be to reduce to the situation when X is a family of smooth projective curves. In this case we may lift the whole family to characteristic 0, and then standard results in rigid cohomology will enable us to deduce the exactness we require from Theorem 1.5. Let us note now that formation of π_1^\dagger commutes with taking finite extension of K (and hence of k), and exactness of a sequence of affine group schemes can be checked after such a finite extension. We will make use of this to freely take finite extensions of k throughout the proof.

Our first task in the proof of Theorem 7.1 will be to show the overconvergent analogue of Lemma 2.5, i.e. that the homotopy sequence of a smooth projective morphism is always weakly exact. While conceptually simple, the proof will require the existence of push-forwards for overconvergent isocrystals, and showing this will require some rather daunting heavy machinery from the theory of arithmetic \mathcal{D}^\dagger -modules.

Proposition 7.2. *Let $f : X \rightarrow S$ be a smooth projective morphism of smooth k -varieties, of constant relative dimension d . Then there exists a functor*

$$f_* : \text{Isoc}^\dagger(X/K) \rightarrow \text{Isoc}^\dagger(S/K),$$

right adjoint to

$$f^* : \text{Isoc}^\dagger(S/K) \rightarrow \text{Isoc}^\dagger(X/K)$$

such that for any $s \in S(k)$ we have $s^* f_* E \cong H_{\text{rig}}^0(X_s/K, E|_{X_s})$.

The proof will be postponed until §8 below, and for now we will deduce several important consequences.

Corollary 7.3. *Let $f : X \rightarrow S$, $x \in X(k)$, $s = f(x)$ be as in Theorem 7.1. Then the sequence*

$$\pi_1^\dagger(X_s, x) \rightarrow \pi_1^\dagger(X, x) \rightarrow \pi_1^\dagger(S, s) \rightarrow 1$$

of affine group schemes is weakly exact.

Proof. We will apply Theorem 2.4. Given Proposition 7.2, the proof is identical to Lemma 2.5 above. \square

Corollary 7.4. *To prove Theorem 7.1 it suffices to show that the image of*

$$\pi_1^\dagger(X_s, x) \rightarrow \pi_1^\dagger(X, x)$$

is a normal subgroup.

This observation will enable us to make some rather major simplifying assumptions in the proof of Theorem 7.1. Another corollary of weak exactness that will play an important role is the following very weak version of the Lefschetz hyperplane theorem for overconvergent fundamental groups.

Theorem 7.5. *Let $X \subset \mathbb{P}_k^n$ be smooth, projective and geometrically connected. Assume that $\dim X \geq 2$ and let $Y = H \cap X$ be a smooth hyperplane section. Let $y \in Y(k)$. Then the normal closure of the image of*

$$\pi_1^\dagger(Y, y) \rightarrow \pi_1^\dagger(X, y)$$

is the whole of $\pi_1^\dagger(X, y)$.

Proof. After possibly extending k we may assume (by Bertini's hyperplane section theorem) that there exists a hyperplane $H' \subset \mathbb{P}_k^n$ such that $y \in Y \cap H'$ and $Z = Y \cap H'$ is smooth. Let $\pi : \tilde{X} \rightarrow X$ be the blow-up of X along Z , and $\tilde{Y} \rightarrow Y$ the proper transform of π along $Y \rightarrow X$.

Claim. For any $x' \in \pi^{-1}(x)$ the induced map $\pi_* : \pi_1^\dagger(\tilde{X}, x') \rightarrow \pi_1^\dagger(X, x)$ is an isomorphism.

Proof of claim. We want to show that the functor $\text{Isoc}^\dagger(X/K) \rightarrow \text{Isoc}^\dagger(\tilde{X}/K)$ is an equivalence of categories. According to [Ked07, Proposition 5.3.6] the functor is essentially surjective. It is automatically faithful, hence we must demonstrate that it is full. So let $\pi^* E \rightarrow \pi^* F$ be a morphism. Since $\pi : \tilde{X} \setminus \pi^{-1}(Z) \rightarrow X \setminus Z$ is an isomorphism, this induces a morphism $E|_{X \setminus Z} \rightarrow F|_{X \setminus Z}$ which by [Ked07, Theorem 5.2.1] must come from a morphism $E \rightarrow F$. \square

Clearly, the claim also holds for the map $\tilde{Y} \rightarrow Y$, and hence it suffices to show that the normal closure of the image of $\pi_1^\dagger(\tilde{Y}, \tilde{y}) \rightarrow \pi_1^\dagger(\tilde{X}, \tilde{y})$ is the whole of $\pi_1^\dagger(\tilde{X}, \tilde{y})$. The pencil of hyperplane sections spanned by $Y = X \cap H$ and $X \cap H'$ furnishes a projective map $a : \tilde{X} \rightarrow \mathbb{P}_k^1$ whose generic fibre is smooth, and the pre-image of y with respect to π gives a section σ of a . Let $b : V \rightarrow U$ be the smooth locus of a , note that $\tilde{Y} \subset V$ is a fibre of b . Since b has geometrically connected fibres, Corollary 7.3 implies that the induced sequence of group schemes

$$\pi_1^\dagger(\tilde{Y}, \tilde{y}) \rightarrow \pi_1^\dagger(V, \tilde{y}) \rightarrow \pi_1^\dagger(U, a(\tilde{y})) \rightarrow 1$$

is weakly exact. We also have a diagram of affine group schemes

$$\begin{array}{ccccccc}
 & & & \sigma_* & & & \\
 & & & \swarrow & & & \\
 \pi_1^\dagger(\tilde{Y}, \tilde{y}) & \longrightarrow & \pi_1^\dagger(V, \tilde{y}) & \longrightarrow & \pi_1^\dagger(U, a(\tilde{y})) & \longrightarrow & 1 \\
 \parallel & & \downarrow & & \downarrow & & \\
 \pi_1^\dagger(\tilde{Y}, \tilde{y}) & \longrightarrow & \pi_1^\dagger(\tilde{X}, \tilde{y}) & \longrightarrow & \pi_1^\dagger(\mathbb{P}_k^1, a(\tilde{y})) & \longrightarrow & 1 \\
 & & \swarrow & \sigma_* & \searrow & & \\
 & & & \swarrow & & &
 \end{array}$$

such that the middle vertical arrow $\pi_1^\dagger(V, \tilde{y}) \rightarrow \pi_1^\dagger(\tilde{X}, \tilde{y})$ is surjective. Since $\pi_1^\dagger(V, \tilde{y})$ is generated by $\sigma_*(\pi_1^\dagger(U, a(\tilde{y})))$ and the normal closure of the image of $\pi_1^\dagger(\tilde{Y}, \tilde{y})$, it follows that $\pi_1^\dagger(\tilde{X}, \tilde{y})$ is generated by $\sigma_*(\pi_1^\dagger(\mathbb{P}_k^1, a(\tilde{y})))$ and the normal closure of the image of $\pi_1^\dagger(\tilde{Y}, \tilde{y})$. Since $\pi_1^\dagger(\mathbb{P}_k^1, a(\tilde{y})) = \{1\}$ by Lemma 7.6 below the result follows. \square

Lemma 7.6. *Let $g : Y \rightarrow Z$ be a map which Zariski-locally on Z is a product of projective bundles. Then the induced map*

$$g^* : \text{Isoc}^\dagger(Z/K) \rightarrow \text{Isoc}^\dagger(Y/K)$$

is an equivalence of categories.

Proof. Applying Proposition 7.2 we get unit and counit maps

$$\begin{aligned}
 g^* g_* E &\rightarrow E, \quad E \in \text{Isoc}^\dagger(Y/K) \\
 F &\rightarrow g_* g^* F, \quad F \in \text{Isoc}^\dagger(Z/K),
 \end{aligned}$$

we must prove that these are isomorphisms. But this can be checked fibre by fibre, we are therefore reduced to the case of the structure map $\mathbb{P}_k^{m_1} \times_k \dots \times_k \mathbb{P}_k^{m_r} \rightarrow \text{Spec}(k)$. Hence using GAGA we may reduce to the statement that over a field K of characteristic 0, every vector bundle with integrable connection on $\mathbb{P}_K^{m_1} \times_K \dots \times_K \mathbb{P}_K^{m_r}$ is trivial. \square

8. PUSHFORWARD OF OVERCONVERGENT ISOCRYSTALS

The purpose of this section is to prove Proposition 7.2, stating that if $f : X \rightarrow S$ is a smooth and proper morphism of k -varieties, then $f^* : \text{Isoc}^\dagger(S/K) \rightarrow \text{Isoc}^\dagger(X/K)$ has a right adjoint $f_* : \text{Isoc}^\dagger(X/K) \rightarrow \text{Isoc}^\dagger(S/K)$, which commutes with base change and on fibres recovers the zeroth cohomology group H_{rig}^0 with coefficients. This will require some heavy machinery from the theory of arithmetic \mathcal{D} -modules, as developed by Berthelot and Caro, and while the result we require is essentially contained in work of Caro, it will take a little care to extract it in the form that we need. This section is extremely technical, and the casual reader will gain very little from going through it in detail; they are advised to simply take Proposition 7.2 on trust.

With these warnings out of the way, let us begin. To start with, by uniqueness of adjoints, the question is local on the base S , which we may therefore assume to be affine. Hence we may assume that there exists a smooth and proper formal scheme \mathcal{Q} over W , a divisor $H \subset \mathcal{Q} \times_{\psi} k$ and a locally closed immersion $S \hookrightarrow \mathcal{Q}$ such that $S = \bar{S} \setminus H$. In other words, $(\mathcal{Q}, H, \bar{S})$ is a ‘triplet lisse en dehors du diviseur’ in the sense of [Car11, Définition 3.1.6]. By choosing a projective embedding of X over S , we may construct another

smooth and proper formal scheme \mathfrak{P} over W together with a commutative diagram of embeddings

$$\begin{array}{ccc} X & \longrightarrow & \mathfrak{P} \\ f \downarrow & & \downarrow g \\ S & \longrightarrow & \Omega \end{array}$$

such that $\mathfrak{P} \rightarrow \Omega$ is smooth, and $X = \overline{X} \setminus g^{-1}(H)$. Set $T = g^{-1}(H)$, so that again $(\mathfrak{P}, T, \overline{X})$ is a ‘triplet lisse en dehors du diviseur’.

We will let $D_{\text{surcoh}}^b(\mathfrak{P}, T, \overline{X}/K)$ denote the category of bounded complexes of overcoherent \mathcal{D}^\dagger -modules on the triple $(\mathfrak{P}, T, \overline{X})$ in the sense of [Car15, Notations 1.2.3], in other words overcoherent complexes of $\mathcal{D}_{\mathfrak{P}, \mathbb{Q}}^\dagger(\dagger T)$ -modules \mathcal{E} such that $\mathbf{R}\Gamma_{\overline{X}}^\dagger(\mathcal{E}) \cong \mathcal{E}$. We will similarly denote by $D_{\text{surcoh}}^b(\Omega, H, \overline{S}/K)$ the category of bounded complexes of overcoherent \mathcal{D}^\dagger -modules on the triple $(\Omega, H, \overline{S})$. Following [Car15, Définition 1.2.5], we have full subcategories

$$\begin{aligned} \text{Isoc}^{\dagger\dagger}(\mathfrak{P}, T, \overline{X}/K) &\subset \text{Surcoh}(\mathfrak{P}, T, \overline{X}/K) \\ \text{Isoc}^{\dagger\dagger}(\Omega, H, \overline{S}/K) &\subset \text{Surcoh}(\Omega, H, \overline{S}/K) \end{aligned}$$

consisting of ‘overcoherent isocrystals’ and by [Car11, Corollaire 3.5.10, Théorème 4.2.2] canonical equivalences of categories

$$\begin{aligned} \text{sp}_{\overline{X} \rightarrow \mathfrak{P}, T, +} : \text{Isoc}^\dagger(X/K) &\xrightarrow{\sim} \text{Isoc}^{\dagger\dagger}(\mathfrak{P}, T, \overline{X}/K) \\ \text{sp}_{\overline{S} \rightarrow \Omega, H, +} : \text{Isoc}^\dagger(S/K) &\xrightarrow{\sim} \text{Isoc}^{\dagger\dagger}(\Omega, H, \overline{S}/K), \end{aligned}$$

we will denote inverse functors by sp_-^{-1} . There are full subcategories

$$\begin{aligned} D_{\text{isoc}}^b(\mathfrak{P}, T, \overline{X}/K) &\subset D_{\text{surcoh}}^b(\mathfrak{P}, T, \overline{X}/K) \\ D_{\text{isoc}}^b(\Omega, H, \overline{S}/K) &\subset D_{\text{surcoh}}^b(\Omega, H, \overline{S}/K) \end{aligned}$$

consisting of objects whose cohomology sheaves are overcoherent isocrystals.

Let d denote the relative dimension of f , \mathbf{D}_T (resp. \mathbf{D}_H) the $\mathcal{D}_{\mathfrak{P}, \mathbb{Q}}^\dagger(\dagger T)$ -linear (resp. $\mathcal{D}_{\Omega, \mathbb{Q}}^\dagger(\dagger H)$ -linear) dual functor, and $g_+, g^!$ the direct and inverse image functors between $\mathcal{D}_{\mathfrak{P}, \mathbb{Q}}^\dagger(\dagger T)$ and $\mathcal{D}_{\Omega, \mathbb{Q}}^\dagger(\dagger H)$ -modules. Applying [Car11, Proposition 3.1.7, Corollaire 3.5.10] and [Car15, Théorème 3.3.1] we have factorisations

$$\begin{aligned} f^! &:= \mathbf{R}\Gamma_{\overline{X}}^\dagger \circ g^! : D_{\text{isoc}}^b(\Omega, H, \overline{S}/K) \rightarrow D_{\text{isoc}}^b(\mathfrak{P}, T, \overline{X}/K) \\ f_+ &:= g_+ : D_{\text{isoc}}^b(\mathfrak{P}, T, \overline{X}/K) \rightarrow D_{\text{isoc}}^b(\Omega, H, \overline{S}/K) \\ \mathbf{D} &:= \mathbf{D}_T : D_{\text{isoc}}^b(\mathfrak{P}, T, \overline{X}/K) \rightarrow D_{\text{isoc}}^b(\mathfrak{P}, T, \overline{X}/K) \\ \mathbf{D} &:= \mathbf{D}_H : D_{\text{isoc}}^b(\Omega, H, \overline{S}/K) \rightarrow D_{\text{isoc}}^b(\Omega, H, \overline{S}/K). \end{aligned}$$

Set $f^+ := \mathbf{D} \circ f^! \circ \mathbf{D}$. By [Car11, Proposition 4.2.4] the diagram

$$\begin{array}{ccc} \text{Isoc}^\dagger(S/K) & \xrightarrow{\text{sp}_{\overline{S} \rightarrow \Omega, H, +}^{-1}} & D_{\text{isoc}}^b(\Omega, H, \overline{S}/K) \\ f^* \downarrow & & \downarrow f^+[d] \\ \text{Isoc}^\dagger(X/K) & \xrightarrow{\text{sp}_{\overline{X} \rightarrow \mathfrak{P}, T, +}^{-1}} & D_{\text{isoc}}^b(\mathfrak{P}, T, \overline{X}/K) \end{array}$$

is 2-commutative, and by combining [Car06, Théorèmes 1.2.7, 1.2.9] we can see that f_+ and f^+ are adjoint functors. Putting this all together, we obtain a natural isomorphism

$$\mathrm{Hom}_{\mathrm{Isoc}^\dagger(X/K)}(f^*F, E) \cong \mathrm{Hom}_{D_{\mathrm{isoc}}^b(\Omega, H, \bar{S}/K)}(\mathrm{sp}_{\bar{S} \rightarrow \Omega, H, +}^{-1} F, f_+ \mathrm{sp}_{\bar{X} \rightarrow \mathfrak{P}, T, +} E[-d])$$

for any $E \in \mathrm{Isoc}^\dagger(X/K)$ and $F \in \mathrm{Isoc}^\dagger(S/K)$. To complete the proof of Proposition 7.2, it suffices to show that we have

$$s^* \mathrm{sp}_{\bar{S} \rightarrow \Omega, H}^{-1} \mathcal{H}^{i-d}(f_+ \mathrm{sp}_{\bar{X} \rightarrow \mathfrak{P}, D, +} E) \cong H_{\mathrm{rig}}^i(X_s/K, E|_{X_s})$$

since this will imply that $f_+ \mathrm{sp}_{\bar{X} \rightarrow \mathfrak{P}, D, +} E[-d]$ is concentrated in degrees ≥ 0 , and hence that

$$\mathrm{Hom}_{D_{\mathrm{isoc}}^b(\Omega, H, \bar{S})}(\mathrm{sp}_{\bar{S} \rightarrow \Omega, H, +}^{-1} F, f_+ \mathrm{sp}_{\bar{X} \rightarrow \mathfrak{P}, D, +} E[-d]) = \mathrm{Hom}_{\mathrm{Isoc}^\dagger(S/K)}(F, \mathrm{sp}_{\bar{S} \rightarrow \Omega, H}^{-1} \mathcal{H}^{-d}(f_+ \mathrm{sp}_{\bar{X} \rightarrow \Omega, D, +} E)).$$

Therefore taking

$$f_* E = \mathrm{sp}_{\bar{S} \rightarrow \Omega, H}^{-1} \mathcal{H}^{-d}(f_+ \mathrm{sp}_{\bar{X} \rightarrow \mathfrak{P}, D, +} E)$$

will do the trick. To prove the fibre-wise comparison with cohomology, we note that by combining [Car11, Proposition 4.2.4, Théorème 5.2.5] and [Car15, Théorème 4.4.2], we have an isomorphism

$$s^* \mathrm{sp}_{\bar{S} \rightarrow \Omega, H}^{-1} \mathcal{H}^{i-d}(f_+ \mathrm{sp}_{\bar{X} \rightarrow \mathfrak{P}, D, +} E) \cong H^{i-d}(f_{s, +} \mathrm{sp}_{X_s \rightarrow \mathfrak{P}_{\tilde{s}}, +} E|_{X_s}),$$

where \tilde{s} denotes some lift of s to a \mathcal{V} -point of Ω , and $\mathfrak{P}_{\tilde{s}}$ the fibre of g over \tilde{s} . We may therefore reduce to Lemma 8.1 below.

Lemma 8.1. *Let $X \hookrightarrow \mathfrak{P}$ be a closed embedding of a smooth d -dimensional k -variety into a smooth formal \mathcal{V} -scheme, and let $f : \mathfrak{P} \rightarrow \mathrm{Spf}(\mathcal{V})$ denote the structure morphism. Let E be a convergent isocrystal on X . Then for all $i \in \mathbb{Z}$ we have an isomorphism*

$$H^{i-d}(f_+ \mathrm{sp}_{X \rightarrow \mathfrak{P}, +} E) \cong H_{\mathrm{conv}}^i(X/K, E)$$

of K -vector spaces.

Proof. Since both sides satisfy Zariski descent, we may reduce to the corresponding question for both \mathfrak{P} and X affine. Further localising, we may assume that $X \hookrightarrow \mathfrak{P}$ lifts to a closed embedding of smooth formal \mathcal{V} -schemes $i : \mathfrak{X} \hookrightarrow \mathfrak{P}$. In this case we have by construction (see [Car09, §2]) that $\mathrm{sp}_{X \rightarrow \mathfrak{P}, +} E \cong i_+ \mathrm{sp}_{\mathfrak{X} \rightarrow \mathfrak{X}, +} E$. Hence by the transitivity of push-forward we can reduce to the case $X = \mathfrak{P} \times_W k$, which follows for example from [Ber02, (4.3.6.3)]. \square

9. EXACTNESS FOR LIFTABLE MORPHISMS

In this and the following sections, we will slowly build up to the proof of Theorem 7.1 in stages, starting from very particular situations and then reducing the general case to these. The first situation in which we will prove Theorem 7.1 is under some very strong liftability assumptions on the morphism f .

So suppose that we have some smooth affine variety $S = \mathrm{Spec}(A_0)$ over k . Then by [Eik73, Théorème 6] we know that S lifts to a smooth and affine \mathcal{V} -scheme $\mathcal{S} = \mathrm{Spec}(A)$. Choosing a presentation of A gives us embeddings

$$\mathcal{S} \hookrightarrow \mathbb{A}_{\mathcal{V}}^N \hookrightarrow \mathbb{P}_{\mathcal{V}}^N$$

and we let \mathfrak{S} denote the completion of the closure of \mathcal{S} inside $\mathbb{P}_{\mathcal{V}}^N$. Let \bar{S} denote the closure of S inside $\mathfrak{S} \times_{\mathcal{V}} k$, so we have a smooth and proper frame $(S, \bar{S}, \mathfrak{S})$ over \mathcal{V} , in the sense of [LS07, Definitions 3.3.5, 3.3.10].

Definition 9.1. We will call any frame of the form $(S, \bar{S}, \mathfrak{S})$, as just constructed, a ‘Monsky–Washnitzer’ frame.

The main result of this section is then the following.

Theorem 9.2. *Let $f : X \rightarrow S$, $x \in X(k)$ and $s = f(x)$ be as in Theorem 7.1. Assume that S is affine, and that there exists a morphism of frames*

$$\begin{array}{ccccc} X & \longrightarrow & \bar{X} & \longrightarrow & \mathfrak{X} \\ f \downarrow & & \downarrow & & \downarrow g \\ S & \longrightarrow & \bar{S} & \longrightarrow & \mathfrak{S} \end{array}$$

extending f such that:

- (1) $(S, \bar{S}, \mathfrak{S})$ is a Monsky–Washnitzer frame;
- (2) both squares in the above diagram are Cartesian;
- (3) $\mathfrak{X} \rightarrow \mathfrak{S}$ is projective, and smooth in a neighbourhood of X .

Then the homotopy sequence for (f, x) is exact.

Remark 9.3. Note that by GFGA together with the Monsky–Washnitzer assumption on $(S, \bar{S}, \mathfrak{S})$, the morphism $g : \mathfrak{X} \rightarrow \mathfrak{S}$ in the statement of the theorem arises as the formal completion of a morphism of projective \mathcal{Y} -schemes.

In the situation of Theorem 9.2 we may choose a cofinal system of neighbourhoods V_λ of $]S[_{\mathfrak{S}}$ inside \mathfrak{S}_K , for $\lambda \rightarrow 1^-$, such that each V_λ is smooth and geometrically connected over K . Then $W_\lambda := g^{-1}(V_\lambda)$ form a cofinal system of neighbourhoods of $]X[_{\mathfrak{X}}$ inside \mathfrak{X}_K . Let

$$\mathrm{MIC}(S, \mathfrak{S}/K) = 2\text{-colim}_\lambda \mathrm{MIC}(V_\lambda/K)$$

denote the category of coherent $j_S^\dagger \mathcal{O}_{\mathfrak{S}_K}$ -modules with integrable connection. Similarly let

$$\mathrm{MIC}(X, \mathfrak{X}/K) = 2\text{-colim}_\lambda \mathrm{MIC}(W_\lambda/K)$$

denote the category of coherent $j_X^\dagger \mathcal{O}_{\mathfrak{X}_K}$ -modules with integrable connection. (For the equivalence between these two interpretations see [LS07, Proposition 6.1.15].)

Proposition 9.4. *Choose a lift $\tilde{x} \in]X[_{\mathfrak{X}}(K)$ of $x \in X(k)$, and let $\tilde{s} = g(\tilde{x})$. Then the categories $\mathrm{MIC}(X, \mathfrak{X}/K)$ and $\mathrm{MIC}(S, \mathfrak{S}/K)$ are neutral Tannakian over K , with fibre functors provided by \tilde{x}^* and \tilde{s}^* respectively.*

Proof. It follows from Proposition 1.1 that objects in $\mathrm{MIC}(X, \mathfrak{X}/K)$ (resp. $\mathrm{MIC}(S, \mathfrak{S}/K)$) are locally free, and the rest of the proof is word for word the same as the proof of Proposition 1.1. \square

We will let $\pi_1^{\mathrm{colim}}(]X[_{\mathfrak{X}}, \tilde{x})$ and $\pi_1^{\mathrm{colim}}(]S[_{\mathfrak{S}}, \tilde{s})$ denote the corresponding Tannaka duals.

Proposition 9.5. *The sequence of affine group schemes*

$$\pi_1^{\mathrm{dR}}(\mathfrak{X}_{K, \tilde{s}}, \tilde{x}) \rightarrow \pi_1^{\mathrm{colim}}(]X[_{\mathfrak{X}}, \tilde{x}) \rightarrow \pi_1^{\mathrm{colim}}(]S[_{\mathfrak{S}}, \tilde{s}) \rightarrow 1.$$

is exact.

Proof. By combining the push-forward functors $g_{\mathrm{dR}*} : \mathrm{MIC}(W_\lambda/K) \rightarrow \mathrm{MIC}(V_\lambda/K)$ considered in the proof of Lemma 2.5 it is entirely straightforward to construct a push-forward functor

$$g_{\mathrm{dR}*} : \mathrm{MIC}(X, \mathfrak{X}/K) \rightarrow \mathrm{MIC}(S, \mathfrak{S}/K)$$

which is adjoint to g^* , and which on fibres recovers H_{dR}^0 . Now arguing exactly as in the proof of Lemma 2.5 we can see that the claimed sequence is weakly exact. By Theorem 2.1 it therefore suffices to show that for any $E \in \text{MIC}(X, \mathfrak{X}/K)$, with associated monodromy representation

$$\pi_1^{\text{colim}}(\text{]X}[\mathfrak{X}, \bar{x}) \rightarrow \text{GL}(E_{\bar{x}}),$$

the inclusion

$$\mathbb{P}(E_{\bar{x}})^{K_{\text{colim}}}(K) \subset \mathbb{P}(E_{\bar{x}})^{\pi_1^{\text{dR}}(\mathfrak{X}_{K, \bar{s}, \bar{x}})}(K)$$

is in fact an equality. Note that any such object is pulled back from some W_λ via the map $\text{]X}[\mathfrak{X} \rightarrow W_\lambda$. Let K_{colim} denote the kernel of $\pi_1^{\text{colim}}(\text{]X}[\mathfrak{X}, \bar{x}) \rightarrow \pi_1^{\text{colim}}(\text{]S}[\mathfrak{S}, \bar{s})$ and K_λ the kernel of $\pi_1^{\text{dR}}(W_\lambda, \bar{x}) \rightarrow \pi_1^{\text{dR}}(V_\lambda, \bar{s})$. We therefore have a natural map $K_{\text{colim}} \rightarrow K_\lambda$. Applying Theorem 1.5 the sequence

$$\pi_1^{\text{dR}}(\mathfrak{X}_{K, \bar{s}, \bar{x})} \rightarrow \pi_1^{\text{dR}}(W_\lambda, \bar{x}) \rightarrow \pi_1^{\text{dR}}(V_\lambda, \bar{s}) \rightarrow 1$$

is exact, and hence again applying Theorem 2.1 we can deduce that

$$\mathbb{P}(E_{\bar{x}})^{\pi_1^{\text{dR}}(\mathfrak{X}_{K, \bar{s}, \bar{x}})}(K) = \mathbb{P}(E_{\bar{x}})^{K_\lambda}(K).$$

But since we have $K_{\text{colim}} \rightarrow K_\lambda$ it follows that

$$\mathbb{P}(E_{\bar{x}})^{\pi_1^{\text{dR}}(\mathfrak{X}_{K, \bar{s}, \bar{x}})}(K) = \mathbb{P}(E_{\bar{x}})^{K_\lambda}(K) \subset \mathbb{P}(E_{\bar{x}})^{K_{\text{colim}}}(K)$$

and the proof is complete. \square

Proof of Theorem 9.2. It follows from [Ber96, Proposition 2.2.7] that the functors

$$\text{Isoc}^\dagger(X/K) \rightarrow \text{MIC}(X, \mathfrak{X}/K)$$

$$\text{Isoc}^\dagger(S/K) \rightarrow \text{MIC}(S, \mathfrak{S}/K)$$

$$\text{Isoc}^\dagger(X_s/K) \rightarrow \text{MIC}(\mathfrak{X}_s/K)$$

are all fully faithful with image stable by sub-quotients. In particular in the commutative diagram

$$\begin{array}{ccccccc} \pi_1^{\text{dR}}(\mathfrak{X}_{K, \bar{s}, \bar{x}}) & \longrightarrow & \pi_1^{\text{colim}}(\text{]X}[\mathfrak{X}, \bar{x}) & \longrightarrow & \pi_1^{\text{colim}}(\text{]S}[\mathfrak{S}, \bar{s}) & \longrightarrow & 1 \\ \downarrow & & \downarrow & & \downarrow & & \\ \pi_1^\dagger(X_s, x) & \longrightarrow & \pi_1^\dagger(X, x) & \longrightarrow & \pi_1^\dagger(S, s) & \longrightarrow & 1 \end{array}$$

all the vertical maps are surjective. Since the top sequence is exact, it follows that the image of $\pi_1^\dagger(X_s, x) \rightarrow \pi_1^\dagger(X, x)$ must be normal subgroup, and hence the bottom sequence is exact by Corollary 7.4. \square

10. THE CASE OF A SMOOTH FAMILY OF CURVES

While the liftability condition in Theorem 9.2 is extremely strong, it will always hold for a family of curves over a smooth affine base. That this is true is the key result of this section; we will then use this to deduce Theorem 7.1 in relative dimension 1. In order to do so, we will need to show that it suffices to treat the case when the base is affine, which is the content of the following lemma. Throughout, let $f : X \rightarrow S$, $x \in X(k)$ and $s = f(x)$ be as in the statement of Theorem 7.1

Lemma 10.1. *Let $U \subset S$ be an open subset containing s and $f_U : X_U \rightarrow U$ the base change. If the homotopy sequence for (f_U, x) is exact, then so is the homotopy sequence for (f, x) .*

Proof. We consider the diagram

$$\begin{array}{ccccccc} \pi_1^\dagger(X_s, x) & \longrightarrow & \pi_1^\dagger(X_U, x) & \longrightarrow & \pi_1^\dagger(U, s) & \longrightarrow & 1 \\ \parallel & & \downarrow & & \downarrow & & \\ \pi_1^\dagger(X_s, x) & \longrightarrow & \pi_1^\dagger(X, x) & \longrightarrow & \pi_1^\dagger(S, s) & \longrightarrow & 1. \end{array}$$

where the vertical arrows are surjective by [Ked07, Theorem 5.2.1, Proposition 5.3.1]. If the top sequence is exact, then the image of $\pi_1^\dagger(X_s, x) \rightarrow \pi_1^\dagger(X, x)$ is normal, and hence the bottom sequence is exact by Corollary 7.4. \square

This enables us to prove exactness of the homotopy sequence for curves.

Theorem 10.2. *Theorem 7.1 is true if $f : X \rightarrow S$ has relative dimension 1.*

Proof. Let g denote the genus of f . If $g = 0$ then we are done by the proof of Lemma 7.6 - to make the argument work, we only need a rational point on the fibre over s , which we have by assumption. We will treat the case $g \geq 2$ and then point out where the argument needs to be modified to work for $g = 1$.

By Proposition 10.1 we are free to replace S by any open sub-scheme containing s , in particular we may assume that $S = \text{Spec}(A_0)$ is affine, and $f : X \rightarrow S$ can be tri-canonically embedded in \mathbb{P}_S^{5g-6} . If we let H_g^0 denote the moduli scheme (over \mathbb{Z}) of such tri-canonically embedded curves, and $Z_g^0 \rightarrow H_g^0$ the universal curve, then we obtain a Cartesian diagram of schemes

$$\begin{array}{ccc} X & \longrightarrow & Z_g^0 \\ \downarrow & & \downarrow \\ S & \longrightarrow & H_g^0. \end{array}$$

After possibly shrinking S further we may assume that there exists an open affine sub-scheme $V \subset H_g^0$ through which $S \rightarrow H_g^0$ factors.

Now by [Elk73, Théorème 6] we may lift A_0 to a smooth \mathcal{V} -algebra A , let A^h denote the \mathfrak{w} -adic Henselisation of A . Since V is smooth and affine over \mathbb{Z} by [DM69, Corollary 1.7] we may apply [Ray72, Théorème 2] to deduce that the given morphism $S \rightarrow V$ lifts to a morphism

$$\text{Spec}(A^h) \rightarrow V.$$

Since V is of finite type over \mathbb{Z} , it follows that after possibly passing to some étale A -algebra with the same special fibre, we may assume that the family of curves $X \rightarrow S$ lifts to a family $\mathcal{X} \rightarrow \mathcal{S} = \text{Spec}(A)$ over A , which is moreover a closed sub-scheme of $\mathbb{P}_{\mathcal{S}}^{5g-6}$. Now choose a projective embedding

$$\mathcal{S} \hookrightarrow \mathbb{A}_{\mathcal{V}}^N \hookrightarrow \mathbb{P}_{\mathcal{V}}^N,$$

let $\overline{\mathcal{S}}$ be the closure of \mathcal{S} inside $\mathbb{P}_{\mathcal{V}}^N$, and let $\overline{\mathcal{X}}$ be the closure of \mathcal{X} inside $\mathbb{P}_{\overline{\mathcal{S}}}^{5g-6}$. Setting $\mathfrak{X} = \widehat{\overline{\mathcal{X}}}$ and $\mathfrak{S} = \widehat{\overline{\mathcal{S}}}$ we find ourselves in the situation of Theorem 9.2.

When $g = 1$ we can argue as follows. First of all, we consider the base change $X \times_S X \rightarrow X$ of f by itself, equipped with the rational point (x, x) . Then we have a commutative diagram

$$\begin{array}{ccccccc} \pi_1^\dagger(X_s, x) & \longrightarrow & \pi_1^\dagger(X \times_S X, (x, x)) & \longrightarrow & \pi_1^\dagger(X, x) & \longrightarrow & 1 \\ \parallel & & \downarrow & & \downarrow & & \\ \pi_1^\dagger(X_s, x) & \longrightarrow & \pi_1^\dagger(X, x) & \longrightarrow & \pi_1^\dagger(S, s) & \longrightarrow & 1 \end{array}$$

where the surjectivity of the vertical arrows follows from Corollary 7.3. If the top sequence is exact, then it follows that the image of $\pi_1^\dagger(X_s, x) \rightarrow \pi_1^\dagger(X, x)$ is a normal subgroup, and hence the bottom sequence is exact by Corollary 7.4. In particular, after replacing $X \rightarrow S$ by $X \times_S X \rightarrow X$, we may assume that $f : X \rightarrow S$ admits a section, i.e. is an elliptic curve.

Hence after possibly localising on S and using Lemma 10.1, we may assume that we have a smooth Weierstrass model $X \hookrightarrow \mathbb{P}_S^2$ of X . We now replace the scheme H_g^0 in the previous argument with the smooth moduli scheme over \mathbb{Z} parametrising Weierstrass models of elliptic curves. \square

11. CUTTING A SMOOTH PROJECTIVE MORPHISM BY CURVES

Using Theorem 7.5, we can now finally complete the proof of Theorem 7.1 by reducing to the case of a family of smooth projective curves, and hence to Theorem 10.2. So suppose that we are in the situation of Theorem 7.1; by Lemma 10.1 we may assume that S is quasi-projective, and hence that there exists a global closed immersion $X \hookrightarrow \mathbb{P}_S^n$. Let \tilde{S} denote the product $\tilde{\mathbb{P}}_S^n \times_S \dots \times_S \tilde{\mathbb{P}}_S^n$ of $d - 1$ copies of the dual projective space, and define

$$\tilde{X} \subset X \times_S \tilde{S}$$

to be the sub-scheme of tuples (x, H_1, \dots, H_{d-1}) such that $x \in H_1 \cap \dots \cap H_{d-1}$. We therefore have a diagram

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & X \\ \tilde{f} \downarrow & & \downarrow f \\ \tilde{S} & \longrightarrow & S \end{array}$$

such that \tilde{f} is projective, with generic fibre a smooth, projective, geometrically connected curve of some genus g . Moreover, we may choose an open sub-scheme $U \subset \tilde{S}$, surjective over S , such that the pull-back $\tilde{f}_U : \tilde{X}_U \rightarrow U$ is smooth with geometrically connected fibres. In particular, after possibly making a finite extension of k , we may assume that there exists some k -rational point $u \in U$ lifting $s \in S(k)$, and some $\tilde{x} \in \tilde{X}_U(k)$ such that $\tilde{f}_U(\tilde{x}) = u$.

In particular, combining Lemma 7.6 with [Ked07, Theorem 5.2.1, Proposition 5.3.1] we have a commutative diagram

$$\begin{array}{ccccccc} \pi_1^\dagger(\tilde{X}_u, \tilde{x}) & \longrightarrow & \pi_1^\dagger(\tilde{X}_U, \tilde{x}) & \longrightarrow & \pi_1^\dagger(U, u) & \longrightarrow & 1 \\ \parallel & & \downarrow & & \downarrow & & \\ \pi_1^\dagger(\tilde{X}_u, \tilde{x}) & \longrightarrow & \pi_1^\dagger(\tilde{X}, \tilde{x}) & \longrightarrow & \pi_1^\dagger(\tilde{S}, u) & \longrightarrow & 1 \\ \downarrow & & \downarrow \cong & & \downarrow \cong & & \\ \pi_1^\dagger(X_s, x) & \longrightarrow & \pi_1^\dagger(X, x) & \longrightarrow & \pi_1^\dagger(S, s) & \longrightarrow & 1. \end{array}$$

Proposition 11.1. *If the homotopy sequence of (\tilde{f}_U, \tilde{x}) is exact, then so is that of (f, x) .*

Proof. We first claim that under the hypothesis of the proposition, the sequence

$$\pi_1^\dagger(\tilde{X}_u, \tilde{x}) \rightarrow \pi_1^\dagger(\tilde{X}, \tilde{x}) \rightarrow \pi_1^\dagger(\tilde{S}, u) \rightarrow 1$$

satisfies the conditions of Theorem 2.4 (note that this does *not* follow from Corollary 7.3). Indeed, surjectivity of $\pi_1^\dagger(\tilde{X}, \tilde{x}) \rightarrow \pi_1^\dagger(\tilde{S}, u)$ follows from that of $\pi_1^\dagger(X, x) \rightarrow \pi_1^\dagger(S, s)$, and one half of (1) is clear. For the other half of (1), suppose that $E \in \text{Isoc}^\dagger(\tilde{X})$ is such that $E|_{\tilde{X}_u}$ is trivial. We may assume by Lemma 7.6 that E comes from an object E' of $\text{Isoc}^\dagger(X/K)$. Applying Theorem 7.5 to the map $\tilde{X}_u \rightarrow X_s$ we can see that in fact $E'|_{X_s}$ is trivial, and so $E' \cong f^*(F')$ for some $F' \in \text{Isoc}^\dagger(S/K)$. Hence $E \cong \tilde{f}^*(F)$ for some $F \in \text{Isoc}^\dagger(\tilde{S}/K)$ as required. To prove (2) we note that if the top sequence is exact then the image of

$$\pi_1^\dagger(\tilde{X}_u, \tilde{x}) \rightarrow \pi_1^\dagger(\tilde{X}, \tilde{x})$$

is a normal subgroup, and hence for any representation V of $\pi_1^\dagger(\tilde{X}, x)$, we know that $V^{\pi_1^\dagger(\tilde{X}_u, \tilde{x})}$ is in fact stable under $\pi_1^\dagger(\tilde{X}, \tilde{x})$.

Since we have already noted that the image of $\pi_1^\dagger(\tilde{X}_u, \tilde{x}) \rightarrow \pi_1^\dagger(\tilde{X}, \tilde{x})$ is a normal subgroup it follows that the sequence $\pi_1^\dagger(\tilde{X}_u, \tilde{x}) \rightarrow \pi_1^\dagger(\tilde{X}, \tilde{x}) \rightarrow \pi_1^\dagger(\tilde{S}, u) \rightarrow 1$ is exact; the exactness of

$$\pi_1^\dagger(X_s, x) \rightarrow \pi_1^\dagger(X, x) \rightarrow \pi_1^\dagger(S, s) \rightarrow 1$$

now follows from a simple diagram chase. □

Proof of Theorem 7.1. By Proposition 11.1 we may assume that f has relative dimension 1, in which case we apply Theorem 10.2. □

12. APPLICATIONS

In this final part we deduce a couple of corollaries of Theorem 7.1.

Theorem 12.1. *Let $X \subset \mathbb{P}_k^n$ be smooth, projective and geometrically connected. Assume that $\dim X \geq 2$ and let $Y = H \cap X$ be a smooth hyperplane section. Let $y \in Y(k)$. Then the map*

$$\pi_1^\dagger(Y, y) \rightarrow \pi_1^\dagger(X, y)$$

is surjective.

Proof. We simply copy the proof of Theorem 7.5, replacing all instances of Corollary 7.3 with Theorem 7.1. □

Corollary 12.2. *Let X, Y be as in the statement of Theorem 12.1. Then any irreducible $E \in F\text{-Isoc}(X/K)$ remains irreducible upon restriction to Y .*

Proof. If $E' \subset E|_Y$ is a sub- F -isocrystal, then by Theorem 12.1 there exists some sub-isocrystal $E'' \subset E$ such that $E''|_Y = E'$. To check stability of E'' under the Frobenius of E boils down to showing the equality of E'' and $F^*E'' \subset F^*E \cong E$ as sub-objects of E , which can clearly be checked after restricting to Y . □

We can also use Theorem 7.1 to compare the p -adic fundamental group $\pi_1^\dagger(X)$ with the étale one $\pi_1^{\text{ét}}(X)$. So let us assume that k is algebraically closed, X is smooth, projective and connected over k and that $x \in X(k)$. Then there is an obvious functor

$$\text{Ét}(X) \rightarrow \text{Isoc}^\dagger(X/K)$$

which sends a finite étale cover $f : Y \rightarrow X$ to $f_* \mathcal{O}_{Y/K}^\dagger$. This gives rise to a homomorphism of pro-algebraic groups

$$\pi_1^\dagger(X, x) \rightarrow \pi_1^{\text{ét}}(X, x)$$

and hence to a homomorphism of pro-finite groups

$$\pi_0(\pi_1^\dagger(X, x)) \rightarrow \pi_1^{\text{ét}}(X, x)$$

from the component group of $\pi_1^\dagger(X, x)$ to the étale fundamental group. The following strengthens a result of Crew [Cre92, Proposition 4.4].

Theorem 12.3. *The map $\pi_0(\pi_1^\dagger(X, x)) \rightarrow \pi_1^{\text{ét}}(X, x)$ is an isomorphism.*

We can translate this into Tannakian terms as follows. Let $E \in \text{Isoc}^\dagger(X/K)$, and consider the associated monodromy representation

$$\pi_1^\dagger(X, x) \rightarrow \text{GL}(E_x).$$

By definition the image of this homomorphism is the monodromy group $\text{DGal}_x(E)$ of E . Then Theorem 12.3 amounts to the claim that if $E \in \text{Isoc}^\dagger(X/K)$ has finite monodromy group, then it is trivialised by some finite étale cover $f : Y \rightarrow X$. Note that by [Cre92, Proposition 4.3] it suffices to show that we can find some injection $E \hookrightarrow E'$ where E' has a Frobenius structure. The point is that we can now use Theorem 12.1 to reduce this claim to the case of curves.

Lemma 12.4. *To prove Theorem 12.3 it suffices to treat the case when X is a curve.*

Proof. First of all, since finite groups are reductive, we may assume that E is semi-simple (as an object of $\text{Isoc}^\dagger(X/K)$), and therefore moreover simple. By Theorem 12.1 E remains simple upon restriction to some iterated smooth hyperplane section $C \subset X$ of dimension 1, and the monodromy groups coincide. So applying the result for curves, we obtain some $E|_C \hookrightarrow E'$ with E' admitting a Frobenius structure φ .

For all $n \geq 0$ we can therefore view the Frobenius pullback $(F^n)^*E|_C$ as lying inside E' via φ^n . Let $E'' \subset E'$ denote the sum of all the $(F^n)^*E|_C$ for $n \geq 0$, this is therefore stable by the Frobenius φ of E' . By simplicity of E , and therefore of $(F^n)^*E|_C$ (since F^* is an equivalence of categories by [Ogu84, Corollary 4.10]) we can see that we can write $E'' \cong \bigoplus_i (F^{n_i})^*E|_C$. Hence E'' extends to X , and again applying Theorem 12.1 we can see that both the Frobenius structure on E'' and the inclusion $E|_C \hookrightarrow E''$ also extend to X , so we are done. \square

In fact, arguing similarly we can make another reduction.

Lemma 12.5. *To prove Theorem 12.3 it suffices to show that whenever C is a smooth, projective, connected curve, and $E \in \text{Isoc}^\dagger(C/K)$ is simple with finite monodromy, then there exists some smooth connected curve C' , and a non-constant morphism $C' \rightarrow C$ such that $E|_{C'}$ is trivial.*

Proof. First of all, by extending $C' \rightarrow C$ to the compactification $\overline{C'} \rightarrow C$ and applying the fact that $\pi_1^\dagger(C') \rightarrow \pi_1^\dagger(\overline{C'})$ is surjective we may assume that C' is finite over C . Now by throwing away the branch locus we may instead assume that $C' \xrightarrow{f} U \rightarrow C$ is finite étale over some non-empty open $U \subset C$. It follows that $E|_U \hookrightarrow f_* \mathcal{O}_{C'/K}^\dagger$ is a sub-object of something admitting a Frobenius structure. Now since $\pi_1^\dagger(C) \rightarrow \pi_1^\dagger(U)$ is surjective we may argue exactly as in Lemma 12.4 above. \square

Remark 12.6. We have used in the proof the fact that for a smooth, connected affine curve C , the Frobenius pullback $F^* : \text{Isoc}^\dagger(C/K) \rightarrow \text{Isoc}^\dagger(C/K)$ is an equivalence of categories. To see this, we note that the tube of a subscheme is a topological invariant (i.e. does not change under nilpotent thickenings) so we may simply apply the proof of [Ogu84, Corollary 4.10].

We can now complete the proof of Theorem 12.3.

Proof of Theorem 12.3. Let us suppose that we have C a smooth projective connected curve and $E \in \text{Isoc}^\dagger(C/K)$ with finite monodromy. Choose a lift \mathcal{C} of C to \mathcal{V} , with generic fibre \mathcal{C}_K a smooth, projective, geometrically connected curve over K . Then E corresponds to a module with integrable connection on \mathcal{C}_K .

Since $\pi_1^{\text{dR}}(\mathcal{C}_K) \rightarrow \pi_1^\dagger(C)$ is surjective, it follows that E has finite monodromy when considered as a representation of $\pi_1^{\text{dR}}(\mathcal{C}_K)$. Therefore by comparison with the complex situation we know that there exists a finite étale cover $\mathcal{C}'_K \rightarrow \mathcal{C}_K$ trivialising E as a module with integrable connection. Let \mathcal{C}' denote the normalisation of \mathcal{C} inside the function field extension $K(\mathcal{C}_K) \rightarrow K(\mathcal{C}'_K)$.

By de Jong's theorem on alterations [dJ96, Theorem 8.2] we can find, after possibly increasing K (this does not change the problem), some alteration $\mathcal{C}'' \rightarrow \mathcal{C}'$ with strictly semistable reduction, let C'' denote the special fibre. Then the map from the smooth locus of C'' to C is dominant, so we may choose some connected component C''' of $\text{sm}(C'')$ which is non-constant over C . Then the formal completion \widehat{C}''' is smooth over \mathcal{V} in a neighbourhood of C''' , and so the pull-back of isocrystals along $C''' \rightarrow C$ can be identified with the pull-back of modules with integrable connection along

$$]C'''[_{\widehat{C}'''} \subset \mathcal{C}_K^{\text{an}} \rightarrow \mathcal{C}'_K^{\text{an}} \rightarrow \mathcal{C}_K^{\text{an}}.$$

In particular, the pull-back of E to C''' as an overconvergent isocrystal is trivial, and so we may apply Lemma 12.5. \square

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