LOCAL ACYCLICITY IN p-ADIC COHOMOLOGY

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ABSTRACT. We prove an analogue for p-adic coefficients of the Deligne–Laumon theorem on local acyclicity for curves. That is, for an overconvergent F-isocrystal E on a relative curve $f: U \to S$ admitting a good compactification, we show that the cohomology sheaves of $\mathbf{R}f_!E$ are overconvergent isocrystals if and only if E has constant Swan conductor at infinity.

CONTENTS

INTRODUCTION

Given a morphism $f: X \to S$ of algebraic varieties over a field k, it is natural to ask when the higher direct images $\mathbf{R}^i f_! E$ of some smooth coefficient object (such as a vector bundle with integrable connection, a lisse ℓ-adic sheaf, or an overconvergent F-isocrystal) are smooth coefficient objects on S. Of course, this will always happen 'generically', i.e. on a dense open subset of S, but one may hope to be able to say something about when this happens on the whole of S.

For example, the smooth and proper base change theorem in étale cohomology says that whenever f is smooth and proper, and E is a lisse ℓ-adic sheaf (with $\ell \neq \mathrm{char}(k)$), then the relative cohomology sheaves $\mathbf{R}^i f_! E = \mathbf{R}^i f_* E$ are also lisse. Similarly, Berthelot's conjecture (versions of whih have been proved by Shiho [Shi08] and Caro [Car15]) states that when k is perfect of characteristic $p > 0$, and E is an overconvergent F-isocrystal on X, then each $\mathbf{R}^i f_* E$ is an overconvergent F-isocrystal on S.

In characteristic 0 these smoothness results often persist for families of open varieties (at least, in the ℓ-adic case), provided that the morphism $f: X \to S$ admits a 'good' compactification, that is a compactification $\overline{X}$ smooth over S, such that the complement $\overline{X} \setminus X$ is a relative normal crossings divisor. However, the phenomenon of wild ramification means that the same is generally not true in positive characteristic, for example, one can use Artin–Schreier extensions to produce examples lisse $\mathbb{F}_\ell$-sheaves E on $\mathbb{A}^2_k$ (with $\ell \neq p$) such that the rank of the cohomology groups jumps in fibres of the projection $\mathbb{A}^2_k \to \mathbb{A}^1_k$.

This is explained by the fact that the Swan conductor of E at infinity, which is a numerical measure of the wild ramification of E, itself jumps along these fibres. It turns out, however, that for curves at least, the Swan conductor exactly controls the failure of the higher direct images to be lisse. Indeed, the main result of [Lau81] shows that for a relative smooth curve $f : U \to S$, and a lisse $\mathbb{F}_\ell$-sheaf E on U, "if the wild ramification of E at infinity is locally constant, then the higher direct images $\mathbf{R}^i f_! E$ are lisse". Concretely, the wild ramification of E being (locally) constant means that the Swan conductor of E at infinity is (locally) constant. One can use this to deduce a similar result with $\mathbb{Z}_\ell$ or $\mathbb{Q}_\ell$ coefficients.

The purpose of this article is to prove an analogue of this result for p-adic coefficients, that is for overconvergent F-isocrystals; in this case the correct analogue of the Swan conductor is the irregularity of a p-adic differential equation studied in [CM00]. We have two main results in this direction. The first of these, Theorem 3.7 is phrased in the language of relative Monsky–Washnitzer cohomology (in the spirit of [Ked06a]) and assumes that the base variety S is smooth, affine and connected. The second, Theorem 9.2, is phrased using the theory of arithmetic $\mathscr{D}^\dagger$-modules, as developed by Berthelot and Caro, and while it allows more general bases S, it assumes that k is perfect. In both cases, the result says that if E is an overconvergent F-isocrystal on a relative smooth curve $f : U \to S$, and E has constant irregularity at infinity, then appropriately defined higher direct images are overconvergent F-isocrystals. (In fact, we work everywhere with 'F-able isocrystals', that is extensions of subquotients of objects admitting some F^n-structure.) Results along these lines were previously obtained by Kedlaya [Ked06b, Proposition 3.4.3], the the proof of which provided part of the inspiration for the methods used in §7.

The majority of this article is concerned with the Monsky–Washnitzer case, that is Theorem 3.7; it is not too difficult to then use the general $\mathscr{D}^\dagger$-module machinery (nicely summarised in [AC13, §1]) to deduce Theorem 9.2 when k is perfect. The basic idea of the proof is rather simple, and the bulk of the work consists of facing down the technical difficulties involved in actually getting this idea to work. To explain the approach, suppose that we have some relative curve $f : U \to S$ over a smooth, affine, base, and an overconvergent F-isocrystal E on U. We know by results of Kedlaya (Theorem 3.5) that for some open subset $V \subset S$ the higher direct images $\mathbf{R}^i f_*(E|_{U_V})$ are overconvergent F-isocrystals on U, and by Noetherian induction we can assume that the complement $Z \subset S$ is also smooth over k, and that the higher direct images $\mathbf{R}^i f_*(E|_{U_Z})$ are themselves overconvergent F-isocrystals on Z of the same rank. The key Lemma 7.6 then tells us that we can use the overconvergence of these objects to 'glue' them together along a suitable punctured tube of Z inside some formal lift of S, to get an overconvergent F-isocrystal on the whole of S.

Interestingly enough, the deduction of the result for arithmetic $\mathscr{D}^\dagger$-modules only uses the fact that the higher direct images are convergent F-isocrystals on S. However, the above strategy would not work if we tried to work everywhere in the convergent category, since we would not be able to 'glue' along the stratification $V \hookrightarrow S \hookleftarrow Z$. Thus it is important to be working with overconvergent objects from the beginning.

Both of our main results are weaker than the Deligne–Laumon result in one crucial aspect. Namely, the curve $f : U \to S$ is assumed to have a good compactification, i.e. a smooth compactification $\bar{f} : C \to S$ such that the complement $C \setminus U$ is étale over S; in [Lau81] it is only assumed to be finite and flat. Our proof is completely different to that given in [Lau81], which uses the formalism of nearby and vanishing cycles in étale cohomology. While this article was being written, an analogue of the nearby and vanishing cycles formalism for p-adic coefficients appeared in [Abe18], which in fact allowed us to extend our main results to singular bases (at least when k is perfect). It would be interesting to see whether or not Abe's theory can be used to give another proof of local acyclicity, more similar in spirit to the ℓ-adic case, and that would be able to handle the more general case where the divisor at infinity is only finite flat.

Let us now give a summary of the various parts of the article. In §1 we introduce some basic notations and definitions concerning rigid cohomology and the theory of arithmetic $\mathscr{D}^\dagger$-modules, in particular the approach to the 6 operations formalism taken in [AC13]. In §2 we recall the definition of the irregularity of a p-adic differential equation, as well as some results of Kedlaya concerning extending break decompositions in families. In §3 we introduce the basic geometric setup that we will work with, in particular the notions of *good* and *simple* relative curves. We recall results of Kedlaya on generic higher direct images and state our first main result on relative Monsky–Washnitzer cohomology for good curves. In §4 we investigate the cohomology of ∇-modules over relative Robba rings, and prove a base change result under the assumption of constant irregularity, which in §5 is used to prove Theorem 3.7 for the 'lower' direct image $\mathbf{R}^0 f_*$. Section 6 is devoted to a detailed and grisly study of relative cohomology on tubes and punctured tubes, which forms the key input for the proof in §7 of finiteness and base change for $\mathbf{R}^1 f_*$ via the gluing argument outlined above. In §8 we then use this to deduce finiteness and base change for certain partially overconvergent cohomology groups, which in §9 then allows us to obtain our second main result, Theorem 9.2, by reduction to the smooth and affine case.

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1. Background on rigid cohomology and arithmetic $\mathscr{D}^\dagger$-modules

Throughout this article, we will denote by K (the ground field) a complete, normed field of characteristic 0, by $\mathscr{V}$ its ring of integers and by k (the residue field) its residue field, which will be assumed to be of characteristic $p > 0$. From §3 we will assume that K is discretely valued, and denote by ϖ a uniformiser for $\mathscr{V}$. To begin with, however, we will need to make certain constructions in the more general case, in which case ϖ will be a non-zero element of the maximal ideal $\mathfrak{m}$ of $\mathscr{V}$. In §9 we will want to assume that k is perfect, however, for the most part we will allow arbitrary k. We will assume that K admits a lifting σ of the absolute $q = p^a$-power Frobenius on k, and fix such a σ.

In this first section we will recall some definitions and constructions in rigid cohomology and the theory of arithmetic $\mathscr{D}^\dagger$-modules, and review the 6 opertaions formalism for varieties and couples introduced in [AC13]. We will generally assume the reader has some basic familiarity with the theory of rigid cohomology, as developed in [Ber96b], and will mostly use this section for fixing definitions and notations.

Definition 1.1. (1) A *variety* over k is a scheme X separated and of finite type over k.

(2) A *formal scheme* over $\mathscr{V}$ is a π-adic formal scheme $\mathfrak{X} \to \mathrm{Spf}(\mathscr{V})$ which is separated and topologically of finite type.
(3) A *rigid variety* over K is a rigid analytic space in the sense of Tate, which in addition is separated over $\mathrm{Sp}(K)$.
(4) A *couple* (X,Y) over k is an open immersion $j : X \hookrightarrow Y$ of k-varieties.
(5) A *frame* over $\mathscr{V}$ is a triple $(X,Y,\mathfrak{P})$ consisting of a couple (X,Y) and a closed immersion $Y \hookrightarrow \mathfrak{P}$ of formal $\mathscr{V}$-schemes.
(6) An *l.p. frame* over $\mathscr{V}$ is a quadruple $(X,Y,\mathfrak{P},\mathfrak{Q})$ such that $(X,Y,\mathfrak{P})$ is a frame, and $\mathfrak{P} \hookrightarrow \mathfrak{Q}$ is an open immersion of formal schemes, such that $\mathfrak{Q}$ is proper over $\mathscr{V}$.

A morphism of couples

$$(X',Y') \to (X,Y)$$

is said to be flat (resp. smooth, étale) if $X' \to X$ is, and proper (resp. finite) if $Y' \to Y$ is. It is said to be Cartesian if the diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & Y' \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

is Cartesian. A couple (X,Y) is flat (resp. smooth, étale, proper, finite) if the natural morphism

$$(X,Y) \to (\mathrm{Spec}\,(k), \mathrm{Spec}\,(k))$$

is. Similarly, a morphism of frames

$$(X',Y',\mathfrak{P}') \to (X,Y,\mathfrak{P})$$

is said to be flat (resp. smooth, étale) if $\mathfrak{P}' \to \mathfrak{P}$ is flat (resp. smooth, étale) in a neighbourhood of X', and proper (resp. finite) if $Y' \to Y$ is. It is said to be Cartesian if both squares in the diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & \mathfrak{P}' \arrow[d] \\ X \arrow[r] & Y \arrow[r] & \mathfrak{P} \end{tikzcd}$$

are Cartesian. A frame (X,Y) is flat (resp. smooth, étale, proper, finite) if the natural morphism

$$(X,Y,\mathfrak{P}) \to (\mathrm{Spec}\,(k), \mathrm{Spec}\,(k), \mathrm{Spf}\,(\mathscr{V}))$$

is. We will not need to define particular properties of morphisms of l.p. frames.

If (X,Y) is a pair, we will denote by $\mathrm{Isoc}((X,Y)/K)$ the category of isocrystals on X, overconvergent along Y. A *Frobenius structure* on an isocrystal E is an isomorphism

$$\varphi : F^{n*}E \xrightarrow{\sim} E$$

for some $n \geq 1$, and we will denote by

$$\mathrm{Isoc}_F((X,Y)/K) \subset \mathrm{Isoc}((X,Y)/K)$$

the full subcategory consisting of iterated extensions of subquotients of objects admitting Frobenius structures. Once checks that $E \in \mathrm{Isoc}_F((X,Y)/K)$ if and only if the irreducible constituents of E admit Frobenius structures, thus $\mathrm{Isoc}_F((X,Y)/K)$ is the thick abelian subcategory of $\mathrm{Isoc}((X,Y)/K)$ generated by objects admitting Frobenius structures. When $X = Y$ we will write $\mathrm{Isoc}_F(X/K)$ and $\mathrm{Isoc}(X/K)$, and when Y is proper over k these do not depend on Y and we will write $\mathrm{Isoc}_F^\dagger(X/K)$ and $\mathrm{Isoc}^\dagger(X/K)$. We will refer to objects in $\mathrm{Isoc}_F((X,Y)/K)$ as 'F-able isocrystals'.

1.1. **Monsky–Washnitzer cohomology.** For most of this article, we will use the Monsky–Washnitzer approach to rigid cohomology, the basics of which we very briefly review here. For the reader wishing for more details, a relatively thorough introduction is given in [Ked06a, §§2-3].

Definition 1.2. A K-dagger algebra is a K-algebra isomorphic to a quotient $K\langle x_1,\ldots,x_n\rangle^\dagger/I$ of an overconvergent power series algebra over K.

We will denote by $\|\cdot\|_{\sup}$ the supremum seminorm on a K-dagger algebra, if A is reduced then this is a norm, and equivalent to the norm induced by the Gauss norm on $K\langle x_1,\ldots,x_n\rangle^\dagger$ via any surjection $K\langle x_1,\ldots,x_n\rangle^\dagger \twoheadrightarrow A$. The image of $K\langle \lambda^{-1}x_1,\ldots,\lambda^{-1}x_n\rangle \subset K\langle x_1,\ldots,x_n\rangle^\dagger$ (for $\lambda > 1$) under any such surjection will be called a fringe algebra of A, this admits a norm $\|\cdot\|_\lambda$ coming from the supremum norm on $K\langle \lambda^{-1}x_1,\ldots,\lambda^{-1}x_n\rangle$. For A reduced, we denote by $A^{\mathrm{int}} \subset A$ the subring of integral elements, consisting of those elements of supremum norm ≤ 1, and by $\overline{A}$ the reduction of A, that is the quotient of A^{int} by the ideal of topologically nilpotent elements.

Definition 1.3. We say that A is of MW-type if it is integral, and its reduction $\overline{A}$ is smooth over k.

If A is a K-dagger algebra, we will let $\Omega^1_{A/K}$ denote the module of p-adically continuous differentials, a ∇-module over A is then by definition a finitely generated A-module M together with an integrable connection

$$\nabla : M \to M \otimes_A \Omega^1_{A/K}.$$

If A is of MW-type then the A-module underlying M is automatically projective. If A is a MW-type dagger algebra, with reduction $\overline{A}$, then there is a fully faithful functor

$$\mathrm{Isoc}^\dagger(\mathrm{Spec}\left(\overline{A}\right)/K) \to \underline{\mathrm{Mod}}^\nabla_A$$

from overconvergent isocrystals on $\overline{A}$ to ∇-modules on A, which we call ‘realisation on A’. A ∇-module is called overconvergent if it is in the essential image of this functor. This construction is compatible with pullback, and hence whenever we fix a lift σ of Frobenius on A, we obtain a functor

$$F\text{-}\mathrm{Isoc}^\dagger(\mathrm{Spec}\left(\overline{A}\right)/K) \to \underline{\mathrm{Mod}}^{(\varphi,\nabla)}_A$$

from overconvergent isocrystals on $\mathrm{Spec}\left(\overline{A}\right)$ to (φ,∇)-modules over A, that is ∇-modules equipped with a horizontal isomorphism $\varphi : \sigma^*M \overset{\sim}{\to} M$. This functor is an equivalence of categories. We will say that $M \in \underline{\mathrm{Mod}}^\nabla_A$ admits a Frobenius structure if there exists an isomorphism $\varphi : \sigma^{n*}M \overset{\sim}{\to} M$ for some $n \geq 1$, any such M is automatically overconvergent. We say that M is F-able if its irreducible constituents admit Frobenius structures, this is equivalent to being in the essential image of

$$\mathrm{Isoc}^\dagger_F(\mathrm{Spec}\left(\overline{A}\right)/K) \to \underline{\mathrm{Mod}}^\nabla_A.$$

Definition 1.4. Let u be a variable. The Robba ring $\mathscr{R}^u_A$ over A consists of those series

$$\sum_{i\in\mathbb{Z}} a_i u^i \in A[\![u,u^{-1}]\!]$$

satisfying the following convergence condition:

- there exists $\eta < 1$ such that for all $\eta < \rho < 1$ there there exists some fringe algebra $A_\lambda \subset A$ such that $a_i \in A_\lambda$ for all i and

$$\|m_i\|_\lambda \rho^i \to 0 \quad \text{as} \quad i \to \pm\infty.$$

The plus part $\mathscr{R}^{u+}_A$ of the Robba ring over A consists of those series for which $a_i = 0$ for all $i < 0$.

The module of continuous derivations of $\mathscr{R}^u_A$ is isomorphic to

$$\Omega^1_{A/K} \otimes_A \mathscr{R}^u_A \oplus \mathscr{R}^u_A \cdot du,$$

and a ∇-module over $\mathscr{R}^u_A$ is defined to be a finitely presented module together with an integrable connection relative to K. The notion of a ∇-module over $\mathscr{R}^{u+}_A$ is defined similarly.

Remark 1.5. For any of the rings $A, \mathscr{R}_A^u$, $\mathscr{R}_A^{u+}$ (and some others that we will introduce later) a ∇-module will always (unless explicitly stated otherwise) mean a ∇-module relative to K, whose underlying module is finitely presented. Extra adjectives, such as projective, stably free, free &c. are understood to apply to the underling module.

Definition 1.6. A frame $(X, \overline{X}, \mathfrak{X})$ is called a Monsky–Washnitzer frame (or an MW frame for short) if:

(1) X is smooth, affine and connected;
(2) $X \hookrightarrow \overline{X}$ has dense image;
(3) $\mathfrak{X}$ is projective over $\mathscr{V}$ (and in particular algebraisable);
(4) the map $\overline{X} \to \mathfrak{X}_k$ is an isomorphism.

If $(X, \overline{X}, \mathfrak{X})$ is a MW frame, and $X = \operatorname{Spec}(\overline{A})$, then $A := \Gamma(\mathfrak{X}_K, j_X^\dagger \mathscr{O}_{\mathfrak{X}_K})$ is a K-dagger algebra of MW-type, whose reduction mod $\mathfrak{m}$ is exactly $\overline{A}$.

Definition 1.7. A modification of MW frames is a flat morphism

$$(X, \overline{X}, \mathfrak{X}) \to (Y, \overline{Y}, \mathfrak{Y})$$

such that $X \to Y$ is an isomorphism.

Note that a modification is proper, smooth and Cartesian, and induces an isomorphism

$$\Gamma(\mathfrak{Y}_K, j_Y^\dagger \mathscr{O}_{\mathfrak{Y}_K}) \xrightarrow{\sim} \Gamma(\mathfrak{X}_K, j_X^\dagger \mathscr{O}_{\mathfrak{X}_K})$$

on the level of K-dagger algebras. We will finish this section with a useful result about lifting étale morphisms from characteristic p to characteristic 0.

Lemma 1.8. *Assume that K is discretely valued, and that ϖ is a uniformiser for $\mathscr{V}$. Let $(X, \overline{X}, \mathfrak{X})$ and $(Y, \overline{Y}, \mathfrak{Y})$ be MW frames, and $X \to Y$ an étale morphism. Then there exists a modification*

$$(X, \overline{X}', \mathfrak{X}') \to (X, \overline{X}, \mathfrak{X})$$

and a proper, étale morphism of frames

$$(X, \overline{X}', \mathfrak{X}') \to (Y, \overline{Y}, \mathfrak{Y})$$

extending $X \to Y$.

Proof. Let $\overline{\mathscr{Y}}$ be a projective scheme over $\mathscr{V}$ such that $\mathfrak{Y} = \widehat{\overline{\mathscr{Y}}}$ and let $\mathscr{Y} \subset \overline{\mathscr{Y}}$ be an open affine subscheme with special fibre Y. Choose $\mathscr{X}$ and $\overline{\mathscr{X}}$ similarly. The morphism $X \to Y$ extends to a morphism $\widehat{\mathscr{X}} \to \widehat{\mathscr{Y}}$ on ϖ-adic completions, and hence by [Elk73, Théorème 2 bis.] we deduce that there exists a morphism $\mathscr{X}^h \to \mathscr{Y}$ from the ϖ-adic Henselisation of $\mathscr{X}$ to $\mathscr{Y}$ lifting $X \to Y$. Hence there exists an étale map $\mathscr{X}' \to \mathscr{X}$ inducing an isomorphism on the special fibres, and a lift $\mathscr{X}' \to \mathscr{Y}$ of $X \to Y$. Moreover, since the map $X \to Y$ is flat, after possibly replacing $\mathscr{X}'$ by an open subscheme we can also assume that $\mathscr{X}' \to \mathscr{Y}$ is flat. In particular, the induced map on ϖ-adic completions is étale.

Now choose a projective morphism $\overline{\mathscr{X}}' \to \overline{\mathscr{X}}$ compactifying $\mathscr{X}' \to \overline{\mathscr{X}}$, and a blowup $\overline{\mathscr{X}}'' \to \overline{\mathscr{X}}'$ resolving the indeterminacy locus of the rational map $\overline{\mathscr{X}}' \dashrightarrow \overline{\mathscr{Y}}$. Now taking $\mathfrak{X}' = \widehat{\overline{\mathscr{X}}''}$ and $\overline{X}'$ to be its special fibre, the modification

$$(X, \overline{X}', \mathfrak{X}') \to (X, \overline{X}, \mathfrak{X})$$

admits a proper, étale morphism to $(Y, \overline{Y}, \mathfrak{Y})$ extending $X \to Y$. $\square$

1.2. **Arithmetic $\mathscr{D}^\dagger$-modules on varieties and couples.** The purpose of this section is to recall how the 6 operations formalism works for arithmetic $\mathscr{D}^\dagger$-modules on k-varieties and couples, as described in [AC13]. We will assume that the ground field K is discretely valued, and that residue field k is perfect (these assumptions will be dropped again at the beginning of §2). Most unreferenced claims made in this section can be found in [AC13, §1].

Definition 1.9. (1) A variety X/k is realisable if there exists a frame $(X,Y,\mathfrak{P})$ such that $\mathfrak{P}$ is smooth and proper over $\mathscr{V}$.

(2) A couple $(X,Y)/k$ is realisable if there exists an l.p. frame $(X,Y,\mathfrak{P},\mathfrak{Q})$ such that $\mathfrak{Q}$ is smooth over $\mathscr{V}$.

Note that both these conditions are marginally stronger than might be expected. If $\mathfrak{P}$ is a smooth formal $\mathscr{V}$-scheme, we let

$$\mathrm{Hol}(\mathscr{D}^\dagger_{\mathfrak{P},\mathbb{Q}}) \quad \text{and} \quad D^b_{\mathrm{hol}}(\mathscr{D}^\dagger_{\mathfrak{P},\mathbb{Q}})$$

denote the categories of overholonomic (complexes of) $\mathscr{D}^\dagger_{\mathfrak{P},\mathbb{Q}}$-modules respectively. We denote by

$$\mathrm{Hol}_F(\mathscr{D}^\dagger_{\mathfrak{P},\mathbb{Q}}) \subset \mathrm{Hol}(\mathscr{D}^\dagger_{\mathfrak{P},\mathbb{Q}})$$

the thick abelian subcategory generated by objects which admit an F^n-Frobenius structure for some $n \geq 1$, and

$$D^b_{\mathrm{hol},F}(\mathscr{D}^\dagger_{\mathfrak{P},\mathbb{Q}}) \subset D^b_{\mathrm{hol}}(\mathscr{D}^\dagger_{\mathfrak{P},\mathbb{Q}})$$

the full subcategory of objects whose cohomology sheaves lie in $\mathrm{Hol}_F(\mathscr{D}^\dagger_{\mathfrak{P},\mathbb{Q}})$. If $\mathbb{X}=(X,Y)$ is a realisable couple, and $(X,Y,\mathfrak{P},\mathfrak{Q})$ is an l.p. frame with $\mathfrak{Q}$ smooth over $\mathscr{V}$, then Abe and Caro define the category

$$D^b_{\mathrm{hol},F}(\mathbb{X}/K) \subset D^b_{\mathrm{hol},F}(\mathscr{D}^\dagger_{\mathfrak{P},\mathbb{Q}})$$

of overholonomic complexes of $\mathscr{D}^\dagger$-modules on $\mathbb{X}$ to be the full subcategory of overholonomic complexes of $\mathscr{D}^\dagger_{\mathfrak{P},\mathbb{Q}}$-modules $\mathscr{M}$ which satisfy

$$\mathscr{M} \overset{\sim}{\to} \mathbf{R}\underline{\Gamma}^\dagger_Y \mathscr{M} \quad \text{and} \quad \mathscr{M} \overset{\sim}{\to} ({}^\dagger Y \setminus X)\mathscr{M}.$$

This does not depend on the choice of l.p. frame $(X,Y,\mathfrak{P},\mathfrak{Q})$ extending $\mathbb{X}$. There is a dual functor

$$\mathbf{D}_{\mathbb{X}} : D^b_{\mathrm{hol},F}(\mathbb{X}/K)^{\mathrm{op}} \to D^b_{\mathrm{hol},F}(\mathbb{X}/K)$$

and a tensor product functor

$$-\widetilde{\otimes}_{\mathbb{X}}- : D^b_{\mathrm{hol},F}(\mathbb{X}/K) \times D^b_{\mathrm{hol},F}(\mathbb{X}/K) \to D^b_{\mathrm{hol},F}(\mathbb{X}/K)$$

which are defined as follows. Let $(X,Y,\mathfrak{P},\mathfrak{Q})$ be an l.p. frame extending $\mathbb{X}$ with $\mathfrak{Q}$ smooth over $\mathscr{V}$. Then

$$\mathbf{D}_{\mathbb{X}} := \mathbf{R}\underline{\Gamma}^\dagger_Y \circ ({}^\dagger Y \setminus X) \circ \mathbf{D}_{\mathfrak{P}}.$$

Similarly, set

$$\mathscr{M}\widetilde{\otimes}_{\mathbb{X}}\mathscr{N} = \mathscr{M} \otimes^{\mathbf{L},\dagger}_{\mathscr{O}_{\mathfrak{P},\mathbb{Q}}} \mathscr{N}[-\dim \mathfrak{P}].$$

If $u : \mathbb{X}' \to \mathbb{X}$ is a morphism of couples then there are functors

$$u^!, u^+ : D^b_{\mathrm{hol},F}(\mathbb{X}/K) \to D^b_{\mathrm{hol},F}(\mathbb{X}'/K),$$

and if u is proper there are functors

$$u_!, u_+ : D^b_{\mathrm{hol},F}(\mathbb{X}'/K) \to D^b_{\mathrm{hol},F}(\mathbb{X}/K).$$

These are defined as follows: choose a morphism

$$\tilde{u} : (X', Y', \mathfrak{P}', \mathfrak{Q}') \to (X, Y, \mathfrak{P}, \mathfrak{Q})$$

of l.p. frames extending u, and set

$$u^! := \mathbf{R}\underline{\Gamma}^\dagger_{Y'} \circ ({}^\dagger Y' \setminus X') \circ \tilde{u}^!, \quad u^+ = \mathbf{D}_{\mathbb{X}'} \circ u^! \circ \mathbf{D}_{\mathbb{X}}$$
$$u_+ = \tilde{u}_+ \quad u_! = \mathbf{D}_{\mathbb{X}} \circ u_+ \circ \mathbf{D}_{\mathbb{X}'},$$

All these functors commute with Frobenius pullback. Both (u^+, u_+) and $(u_!, u^!)$ are adjoint pairs, and if

$$\begin{tikzcd} \mathbb{Y}' \arrow[r, "a'"] \arrow[d, "u'"'] & \mathbb{X}' \arrow[d, "u"] \\ \mathbb{Y} \arrow[r, "a"] & \mathbb{X} \end{tikzcd}$$

is a Cartesian morphism of couples, with u proper, then there is a natural isomorphism

$$a^! u_+ \cong u'_+ a'^!$$

of functors

$$D^b_{\mathrm{hol}}(\mathbb{Y}'/K)^F \to D^b_{\mathrm{hol}}(\mathbb{Y}/K)^F.$$

The triangulated category $D^b_{\mathrm{hol},F}(\mathbb{X}/K)$ admits a 'holonomic' t-structure, whose heart we will denote by $\mathrm{Hol}_F(\mathbb{X}/K)$. The duality functor $\mathbf{D}_{\mathbb{X}}$ is exact with respect to this t-structure, and hence induces an anti-equivalence

$$\mathbf{D}_{\mathbb{X}} : \mathrm{Hol}_F(\mathbb{X}/K)^{\mathrm{op}} \xrightarrow{\sim} \mathrm{Hol}_F(\mathbb{X}/K).$$

When $\mathbb{X}$ is *smooth* over k, then there exists a fully faithful functor

$$\mathrm{sp}_+ : \mathrm{Isoc}^\dagger_F(\mathbb{X}/K) \to \mathrm{Hol}_F(\mathbb{X}/K)$$

which is compatible with duality in the sense that there are isomorphisms

$$\mathrm{sp}_+ E^\vee \cong \mathbf{D}_{\mathbb{X}} \mathrm{sp}_+ E.$$

which are natural in E. We shall isolate two particularly important cases.

Definition 1.10. (1) If $\mathbb{X} = (X, \overline{X})$ is a couple proper over k, then $D^b_{\mathrm{hol},F}(\mathbb{X}/K)$ and $\mathrm{Hol}_F(\mathbb{X}/K)$ only depend on X up to canonical equivalence. In this case we write $D^{b,\dagger}_{\mathrm{hol},F}(X/K)$ and $\mathrm{Hol}^\dagger_F(X/K)$ respectively.

(2) For $\mathbb{X} = (X, X)$ then we write $D^b_{\mathrm{hol},F}(\mathbb{X}/K) = D^b_{\mathrm{hol},F}(X/K)$ and $\mathrm{Hol}_F(\mathbb{X}/K) = \mathrm{Hol}_F(X/K)$.

These categories correspond to overconvergent and convergent holonomic modules on X/K respectively, and there are obvious forgetful functors

$$D^{b,\dagger}_{\mathrm{hol},F}(X/K) \to D^b_{\mathrm{hol},F}(X/K) \quad \text{and} \quad \mathrm{Hol}^\dagger_F(X/K) \to \mathrm{Hol}_F(X/K)$$
$$\mathscr{M} \mapsto \hat{\mathscr{M}}.$$

The above defined duality and tensor product functors induce functors $\mathbf{D}_X$ and $\widetilde{\otimes}_X$ on $D^b_{\mathrm{hol},F}(X/K)$ and $D^{b,\dagger}_{\mathrm{hol},F}(X/K)$, such that $\mathbf{D}_X$ is exact for the holonomic t-structure. If $u : X' \to X$ is a morphism of varieties, then we have functors

$$u^!, u^+ : D^{b,(\dagger)}_{\mathrm{hol},F}(X/K) \to D^{b,(\dagger)}_{\mathrm{hol},F}(X'/K)$$

and

$$u_!, u_+ : D^{b,\dagger}_{\mathrm{hol},F}(X'/K) \to D^{b,\dagger}_{\mathrm{hol},F}(X/K),$$

as well as

$$u_!, u_+ : D^{b}_{\mathrm{hol},F}(X'/K) \to D^{b}_{\mathrm{hol},F}(X/K)$$

whenever u is proper. These have exactly the same properties as in the case of couples, and there is an analogous base change result. If X is smooth, then we have full subcategories

$$D^{b,(\dagger)}_{\mathrm{isoc},F}(X/K) \subset D^{b,(\dagger)}_{\mathrm{hol},F}(X/K)$$

consisting of objects whose cohomology sheaves are in the essential image of

$$\mathrm{sp}_+ : \mathrm{Isoc}^{(\dagger)}_F(X/K) \to \mathrm{Hol}^{(\dagger)}_F(X/K).$$

At least in the overconvergent case, it is explained how to extend all these definitions to the not-necessarily-realisable case in [Abe13]. That is, Abe shows that the category $\mathrm{Hol}^{\dagger}_F(X/K)$ is of a Zariski-local nature for realisable varieties, and thus for a general variety we may define $\mathrm{Hol}^{\dagger}_F(X/K)$ by taking an open affine cover and gluing (affine varieties being realisable). We can then define

$$D^{b,\dagger}_{\mathrm{hol},F}(X/K) := D^b(\mathrm{Hol}^{\dagger}_F(X/K)),$$

which is justified by the main result of [AC18], showing that this really does recover the previous definition in the realisable case. Abe explains in [Abe13, §2.3] how to define the 6 functors $u^+, u_+, u^!, u_!, \widetilde{\otimes}$ and $\mathbf{D}$ in the non-realisable case, and shows that all the same properties hold. Again, if X is smooth then we have

$$\mathrm{sp}_+ : \mathrm{Isoc}^{\dagger}_F(X/K) \to \mathrm{Hol}^{\dagger}_F(X/K)$$

and the corresponding full subcategory $D^{b,\dagger}_{\mathrm{isoc},F}(X/K) \subset D^{b,\dagger}_{\mathrm{hol},F}(X/K)$.

If X is not assumed to be smooth, then we can still consider overconvergent isocrystals as holonomic complexes on X using the approach of [Abe18]. Indeed, in [Abe13, §1.3] Abe defines another t-structure on $D^{b,\dagger}_{\mathrm{hol},F}(X/K)$, called the constructible t-structure. The heart of this t-structure is denoted $\mathrm{Cons}(X/K)$, and cohomology objects by ${}^c\mathscr{H}^i$. The pullback functor u^+ is t-exact for the constructible t-structure [Abe13, Lemma 1.3.4]. Abe constructs in [Abe18, §3] a fully faithful functor

$$\rho : \mathrm{Isoc}^{\dagger}_F(X/K) \to \mathrm{Cons}(X/K)$$

such that:

- $u^+\rho(E) \cong \rho(u^*E)$ for any morphism $u : X' \to X$;
- $\rho(E) \cong \mathrm{sp}_+E[-\dim X]$ whenever X is smooth.

We can therefore define

$$D^{b,\dagger}_{\mathrm{isoc},F}(X/K) \subset D^{b,\dagger}_{\mathrm{hol},F}(X/K)$$

to be the full subcategory whose *constructible* cohomology objects are in the essential image of ρ. This coincides with the previous definition when X is smooth, in which case

$$D^{b,\dagger}_{\mathrm{isoc},F}(X/K) \subset D^{b,\dagger}_{\mathrm{hol},F}(X/K)$$

is stable under $\mathbf{D}_X$, however, this is not true in general. It will be helpful to isolate the following result, which is simply a restatement of various results of Caro and Caro–Tsuzuki.

Lemma 1.11. *Let X/k be a smooth, realisable variety, and $\mathscr{M} \in D^{b,\dagger}_{\mathrm{hol},F}(X/K)$. Then $\mathscr{M} \in D^{b,\dagger}_{\mathrm{isoc},F}(X/K)$ if and only if $\hat{\mathscr{M}} \in D^{b}_{\mathrm{isoc},F}(X/K)$.*

Proof. We may assume that X is affine, and that there exists an immersion $X \hookrightarrow \mathfrak{P}$, with $\mathfrak{P}$ smooth and proper over $\mathscr{V}$, and such that there exists a divisor T of $P := \mathfrak{P}_k$ with X is closed in $P \setminus T$. Let $\mathfrak{U} \subset \mathfrak{P}$ be an open formal subscheme such that X is closed inside $\mathfrak{U}$. In this case, the t-structure on

$$D^{b,\dagger}_{\mathrm{isoc},F}(X/K) \subset D^b_{\mathrm{coh}}(\mathscr{D}^\dagger_{\mathfrak{P},\mathbb{Q}})$$

is the restriction of the obvious t-structure on $D^b_{\mathrm{coh}}(\mathscr{D}^\dagger_{\mathfrak{P},\mathbb{Q}})$, hence after replacing $\mathscr{M}$ by $\mathscr{H}^i(\mathscr{M})$ we may assume that $\mathscr{M}$ is simply an overholonomic $\mathscr{D}^\dagger_{\mathfrak{P},\mathbb{Q}}$-module, which by [Car15, Lemme 2.1.13] arises by restriction of scalars from a $\mathscr{D}^\dagger_{\mathfrak{P},\mathbb{Q}}({}^\dagger T)$-module. Let $\mathbf{D}_{\mathfrak{P},T}$ denote the dual functor for $\mathscr{D}^\dagger_{\mathfrak{P},\mathbb{Q}}({}^\dagger T)$-modules.

By [Car11, Théorème 6.1.11] our $\mathscr{D}^\dagger_{\mathfrak{P},\mathbb{Q}}$-module $\mathscr{M}$ is in the essential image of sp_+ if and only if it is an 'overcoherent isocrystal' in the sense of [Car06, Définition 6.2.1], that is, if both $\mathscr{M}$ and $\mathbf{D}_{\mathfrak{P},T}(\mathscr{M}) \cong (T^\dagger)\mathbf{D}_{\mathfrak{P}}(\mathscr{M})$ are overcoherent as $\mathscr{D}^\dagger_{\mathfrak{P},\mathbb{Q}}({}^\dagger T)$-modules, and if the restriction of $\mathscr{M}$ to $\mathfrak{U}$ is in the essential image of sp_+. To prove the lemma, then, we need to explain why the first two conditions are automatically satisfied.

In other words, we want to show that if $\mathscr{M}$ is a coherent $\mathscr{D}^\dagger_{\mathfrak{P},\mathbb{Q}}({}^\dagger T)$-module, overholonomic as a $\mathscr{D}^\dagger_{\mathfrak{P},\mathbb{Q}}$-module, then both $\mathscr{M}$ and $(T^\dagger)\mathbf{D}_{\mathfrak{P}}(\mathscr{M})$ are overcoherent as $\mathscr{D}^\dagger_{\mathfrak{P},\mathbb{Q}}({}^\dagger T)$-modules. However, these are both overholonomic as $\mathscr{D}^\dagger_{\mathfrak{P},\mathbb{Q}}$-modules, and since we are dealing with objects admitting Frobenius structures (or rather, the thick abelian subcategory generated by objects admitting a Frobenius structure), we may therefore appeal to [CT12, Theorem 2.3.17] to conclude. □

In the definition of the functor

$$u_+ : D^b_{\mathrm{hol},F}(\mathbb{X}'/K) \to D^b_{\mathrm{hol},F}(\mathbb{X}/K)$$

coming from a morphism of pairs $u : \mathbb{X}' \to \mathbb{X}$, we had to choose a morphism of l.p. frames $(X',Y',\mathfrak{P}',\mathfrak{Q}') \to (X,Y,\mathfrak{P},\mathfrak{Q})$ extending u. Note that neither the formal schemes $\mathfrak{Q}$ and $\mathfrak{Q}'$, nor the morphism between them, play any role in the definition of either the categories or the functors involved, however, one still needs to know that they exist. It will be important for us to shows that in certain situations we can completely ignore this technicality, and work simply with immersions of couples into smooth formal $\mathscr{V}$-schemes.

Our setup will be the following. We will take a base couple $\mathbb{S} = (S,S)$, with S smooth and affine, and admitting a smooth, affine lift $\mathfrak{S}$ to a formal scheme over $\mathscr{V}$. We assume that we are given a smooth and projective morphism $\tilde{u} : \mathfrak{X} \to \mathfrak{S}$, and an open immersion

$$U \hookrightarrow X := \mathfrak{X}_k$$

of k-varieties. We let $\mathbb{U} = (U,X)$, and we assume that both $\mathbb{U}$ and $\mathbb{S}$ are realisable as couples. The proper morphism $\tilde{u}$ induces a functor

$$\tilde{u}_+ : D^b_{\mathrm{hol}}(\mathscr{D}^\dagger_{\mathfrak{X},\mathbb{Q}}) \to D^b_{\mathrm{hol}}(\mathscr{D}^\dagger_{\mathfrak{S},\mathbb{Q}})$$

between the categories of complexes of overholonomic $\mathscr{D}^\dagger$-modules on $\mathfrak{X}$ and $\mathfrak{S}$ respectively, as in [Car09]. We define

$$D^b_{\mathrm{isoc}}(\mathscr{D}^\dagger_{\mathfrak{S},\mathbb{Q}}) \subset D^b_{\mathrm{hol}}(\mathscr{D}^\dagger_{\mathfrak{S},\mathbb{Q}})$$

to be the subcategory whose cohomology sheaves are coherent as $\mathscr{O}_{\mathfrak{S},\mathbb{Q}}$-modules. Again, the following lemma is just a rephrasing of various results of Caro.

Lemma 1.12. *There are fully faithful embeddings*

$$D^b_{\mathrm{hol},F}(\mathbb{U}/K) \hookrightarrow D^b_{\mathrm{hol}}(\mathscr{D}^\dagger_{\mathfrak{X},\mathbb{Q}}) \quad \text{and} \quad D^b_{\mathrm{hol},F}(\mathbb{S}/K) \hookrightarrow D^b_{\mathrm{hol}}(\mathscr{D}^\dagger_{\mathfrak{S},\mathbb{Q}})$$

such that the diagram

$$\begin{tikzcd} D^b_{\mathrm{hol},F}(\mathbb{U}/K) \arrow[r] \arrow[d, "u_+"'] & D^b_{\mathrm{hol}}(\mathscr{D}^\dagger_{\mathfrak{X},\mathbb{Q}}) \arrow[d, "\tilde{u}_+"] \\ D^b_{\mathrm{hol},F}(\mathbb{S}/K) \arrow[r] & D^b_{\mathrm{hol}}(\mathscr{D}^\dagger_{\mathfrak{S},\mathbb{Q}}) \end{tikzcd}$$

is 2-commutative. Moreover, the square

$$\begin{tikzcd} D^b_{\mathrm{isoc},F}(\mathbb{S}/K) \arrow[r] \arrow[d] & D^b_{\mathrm{isoc}}(\mathscr{D}^\dagger_{\mathfrak{S},\mathbb{Q}}) \arrow[d] \\ D^b_{\mathrm{hol},F}(\mathbb{S}/K) \arrow[r] & D^b_{\mathrm{hol}}(\mathscr{D}^\dagger_{\mathfrak{S},\mathbb{Q}}) \end{tikzcd}$$

is 2-Cartesian.

Proof. Since $\mathfrak{S}$ is affine, there exists an immersion

$$\mathfrak{S} \overset{i}{\hookrightarrow} \widehat{\mathbb{A}}^N_{\mathscr{V}} \hookrightarrow \widehat{\mathbb{P}}^N_{\mathscr{V}}$$

for some N. Similarly, since $\mathfrak{X} \to \mathfrak{S}$ is projective, we can extend this to a commutative diagram

$$\begin{tikzcd} \mathfrak{X} \arrow[r, hook] \arrow[d, "\tilde{u}"'] & \widehat{\mathbb{P}}^N_{\mathscr{V}} \times_{\mathscr{V}} \widehat{\mathbb{P}}^M_{\mathscr{V}} \arrow[d, "v"] \\ \mathfrak{S} \arrow[r, hook] & \widehat{\mathbb{P}}^N_{\mathscr{V}} \end{tikzcd}$$

where v is the first projection. Then $(S,S,\widehat{\mathbb{A}}^N_{\mathscr{V}},\widehat{\mathbb{P}}^N_{\mathscr{V}})$ is an l.p. frame, and setting $\mathfrak{Y} := v^{-1}(\widehat{\mathbb{A}}^N_{\mathscr{V}})$ gives a closed immersion $i' : \mathfrak{X} \hookrightarrow \mathfrak{Y}$, an l.p. frame $(U,X,\mathfrak{Y},\widehat{\mathbb{P}}^N_{\mathscr{V}} \times_{\mathscr{V}} \widehat{\mathbb{P}}^M_{\mathscr{V}})$, and a morphism of l.p. frames

$$v : (U,X,\mathfrak{Y},\widehat{\mathbb{P}}^N_{\mathscr{V}} \times_{\mathscr{V}} \widehat{\mathbb{P}}^M_{\mathscr{V}}) \to (S,S,\widehat{\mathbb{A}}^N_{\mathscr{V}},\widehat{\mathbb{P}}^N_{\mathscr{V}})$$

extending $\mathbb{U} \to \mathbb{S}$. By definition, $D^b_{\mathrm{hol},F}(\mathbb{S}/K)$ is a full subcategory of $D^b_{\mathrm{hol}}(\mathscr{D}^\dagger_{\widehat{\mathbb{A}}^N_{\mathscr{V}},\mathbb{Q}})$, consisting of objects supported on S. Hence by [Car09, Théorème 2.11] it is contained in the essential image of the fully faithful functor

$$i_+ : D^b_{\mathrm{hol}}(\mathscr{D}^\dagger_{\mathfrak{S},\mathbb{Q}}) \to D^b_{\mathrm{hol}}(\mathscr{D}^\dagger_{\widehat{\mathbb{A}}^N_{\mathscr{V}},\mathbb{Q}}),$$

in other words we can view it as a full subcategory of $D^b_{\mathrm{hol}}(\mathscr{D}^\dagger_{\mathfrak{S},\mathbb{Q}})$. An entirely similar argument applies for $D^b_{\mathrm{hol},F}(\mathbb{U}/K)$. For the claim concerning the pushforward functor, it suffices to verify that the diagram

$$\begin{tikzcd} D^b_{\mathrm{hol}}(\mathscr{D}^\dagger_{\mathfrak{X},\mathbb{Q}}) \arrow[r, "i'_+"] \arrow[d, "\tilde{u}_+"'] & D^b_{\mathrm{hol}}(\mathscr{D}^\dagger_{\mathfrak{Y},\mathbb{Q}}) \arrow[d, "\mathbf{R}\underline{\Gamma}^\dagger_S \circ v_+"] \\ D^b_{\mathrm{hol}}(\mathscr{D}^\dagger_{\mathfrak{S},\mathbb{Q}}) \arrow[r, "i_+"] & D^b_{\mathrm{hol}}(\mathscr{D}^\dagger_{\widehat{\mathbb{A}}^N_{\mathscr{V}},\mathbb{Q}}) \end{tikzcd}$$

commutes up to natural isomorphism, which follows for example from [Car09, Théorème 3.8]. The final claim simply follows from the construction of

$$\mathrm{sp}_+ : \mathrm{Isoc}(S/K) \to D^b_{\mathrm{coh}}(\mathscr{D}^\dagger_{\widehat{\mathbb{A}}^N_{\mathscr{V}},\mathbb{Q}})$$

as the composite

$$\mathrm{Isoc}(S/K) \xrightarrow{\mathrm{sp}_*} D^b_{\mathrm{coh}}(\mathscr{D}^\dagger_{\mathfrak{S},\mathbb{Q}}) \xrightarrow{i_+} D^b_{\mathrm{coh}}(\mathscr{D}^\dagger_{\widehat{\mathbb{A}}^N_{\mathscr{V}},\mathbb{Q}}),$$

where the first functor is Berthelot's equivalence [Ber96a, Proposition 4.1.4] between convergent isocrysals on S/K and $\mathscr{O}_{\mathfrak{S},\mathbb{Q}}$-coherent $\mathscr{D}^\dagger_{\mathfrak{S},\mathbb{Q}}$-modules. $\square$

2. Irregularity of p-adic differential equations

We next recall the definition and basic properties of the *irregularity* of an overconvergent ∇-module on (the germ of) a punctured (relative) curve, following Christol–Mebkhout [CM00]. For this section we will continue to allow K to be a general complete, normed field of residue characteristic p, unless specified otherwise. Let $\mathscr{R}^u_K$ denote the Robba ring over K, with co-ordinate u, say, and let M be a projective ∇-module over $\mathscr{R}^u_K$. Then for all $\rho < 1$ sufficiently close to 1, we can base change M to obtain a (necessarily free) ∇-module M_ρ over the completion K_ρ of $K(u)$ for the ρ-Gauss norm. Define the radius of convergence

$$R(M_\rho) := \min\left\{\rho, \liminf_{k\to\infty} |G_k|_\rho^{-1/k}\right\},$$

where G_k is the matrix of the operator $\frac{1}{k!}\frac{d^k}{du^k}$ acting on M_ρ.

Definition 2.1. We say that M is overconvergent if $\lim_{\rho\to 1} R(M_\rho) = 1$.

Remark 2.2. The more standard terminology for such a ∇-module is 'solvable', however, we will also want to work with ∇-modules over relative Robba rings $\mathscr{R}^u_A$ arising from overconvergent isocrystals. Thus we have chosen to use a more uniform terminology.

We say that M has uniform break b if for all ρ sufficiently closed to 1, and all sub-quotients N of M_ρ, we have $R(N) = \rho^{b+1}$.

Theorem 2.3. *[CM01, Corollaire 2.4-1] For any projective overconvergent ∇-module over $\mathscr{R}^u_K$, there exists a unique decomposition*

$$M = \bigoplus_{b\geq 0} M_b$$

of ∇-modules, called the break decomposition, such that each M_b has uniform break b.

Remark 2.4. In [CM01] the ground field K is assumed to be spherically complete, in which case M. It is explained how to extend this to the general case in [Ked07b, Lemma 2.7.3].

Definition 2.5. [CM00, Définition 8.3-8] The irregularity of M is defined to be $\mathrm{Irr}(M) := \sum_b b \cdot \mathrm{rank}_{\mathscr{R}^u_K} M_b$.

Remark 2.6. (1) We will often want to consider cases when K itself is equipped with a natural derivation ∂_t, for example when K is the completion of a rational function field $K_0(t)$ for the Gauss norm induced by a norm on K_0. In this case Kedlaya [Ked07b] has developed a more refined notion of irregularity, that takes this horizontal derivation ∂_t into account. We will only consider the 'naïve' irregularity coming from the vertical derivation ∂_u.

(2) If $K \to K'$ is an isometric extension of complete fields, then a projective ∇-module M over $\mathscr{R}^u_K$ is overconvergent if and only if $M \otimes \mathscr{R}^u_{K'}$ is, in which case they have the same irregularity [Meb02, Proposition 1.2-4].

Lemma 2.7. *Let L/K be a finite extension, and M an overconvergent ∇-module over $\mathscr{R}^u_L$. Let $\mathrm{Res}^L_K M$ denote M considered as an overconvergent ∇-module over $\mathscr{R}^u_K$ via the map $\mathscr{R}^u_K \to \mathscr{R}^u_L$. Then*

$$\mathrm{Irr}(\mathrm{Res}_{L/K} M) = [L:K]\mathrm{Irr}(M).$$

Proof. First assume that L/K is Galois. In this case, we have

$$\left(\mathrm{Res}^L_K M\right) \otimes_K L \cong \bigoplus_{\sigma \in \mathrm{Gal}(L/K)} M$$

as ∇-modules over $\mathscr{R}^u_L$, and so we can apply [Meb02, Proposition 1.2-4]. In the general case we take a Galois closure $F/L/K$ (recall that K is of characteristic 0) and deduce that

$$\mathrm{Irr}(\mathrm{Res}^L_K \mathrm{Res}^F_L (M \otimes_L F)) = \mathrm{Irr}(\mathrm{Res}^F_K (M \otimes_L F)) = [F:K]\mathrm{Irr}(M)$$

again using [Meb02, Proposition 1.2-4]. Finally, we use the fact that $\mathrm{Res}^F_L (M \otimes_L F) \cong M^{[F:L]}$ to conclude. □

2.1. **Irregularity in families.** We will be interested in studying how the irregularity varies in families, and so we will want to replace the field K in the above discussion by a K-dagger algebra A of MW-type. (It seems entirely likely that the results here will extend to more general K-dagger algebras, but we will only need this restricted case.) For such an A, suppose that we have a projective ∇-module M over the relative Robba ring $\mathscr{R}^u_A$. Let L be the completion of the fraction of A for the supremum norm.

Definition 2.8. We say that M is overconvergent if the generic fibre $M_L := M \otimes \mathscr{R}^u_L$ of M is overconvergent.

The generic fibre M_L therefore admits a break decomposition

$$M \otimes_{\mathscr{R}^u_A} \mathscr{R}^u_L = \bigoplus_{b \geq 0} M_{L,b}$$

by Theorem 2.3.

Theorem 2.9 ([Ked11], Theorem 1.3.2)**.** *There exists a dagger localisation $A \to B$ and a decomposition of $M \otimes_{\mathscr{R}^u_A} \mathscr{R}^u_B$ which restricts to the break decomposition over $\mathscr{R}^u_L$.*

Proof. The proof of [Ked11, Theorem 1.3.2] assumes that the derivation ∂_u is 'eventually dominant' relative to the derivations of L/K which also act on $M \otimes \mathscr{R}_L$. However, this assumption is only used to *interpret* the break decomposition obtained as a genuine break decomposition for the full collection of derivations, and is not used in showing that such a decomposition exists. □

It will be important to have conditions for extending this break decomposition over the whole of A. Let $\mathscr{M}(A)$ denote the Berkovich spectrum of A; for any $v \in \mathscr{M}(A)$ we let K_v denote the completed residue field at v. Thus base changing M via $\mathscr{R}^u_A \to \mathscr{R}^u_{K_v}$ we obtain a projective ∇-module M_v over $\mathscr{R}^u_{K_v}$. Let us also denote by ξ the unique point in the Shilov boundary of $\mathscr{M}(A)$, thus $K_\xi = L$, the completed fraction field of A.

Proposition 2.10. *For any point $v \in \mathscr{M}(A)$ the ∇-module M_v is overconvergent, and we have* $\mathrm{Irr}(M_v) \leq \mathrm{Irr}(M_L)$.

Before we can give the proof of this proposition, we need to introduce another function governing the variation of the irregularity along a 2-dimensional Berkovich space. Let M be a ∇-module over the ring of functions

$$K\langle \tau u^{-1}, \rho^{-1} u, x \rangle$$

converging for $\rho \leq |u| \leq \tau$ and $|x| \leq 1$. Then for any $(-\log\alpha, -\log\beta) \in [-\log\tau, -\log\rho] \times [0,\infty]$ we obtain by base change a ∇-module $M_{\alpha,\beta}$ over the field $K_{\alpha,\beta}$ obtained by completing $K(u,x)$ with respect to the norm for which $|u| = \alpha$ and $|x| = \beta$. (When $\beta = 0$ this should be interpreted as the completion of $K(u)$ for the norm $|u| = \alpha$.) For every irreducible constituent N of $M_{\alpha,\beta}$ we can therefore consider the radius of convergence $R(N)$ with respect to the u-derivation exactly as before, that is

$$R(N) := \min\left\{\rho, \liminf_{k\to\infty} |G_k|_\rho^{-1/k}\right\},$$

where G_k is the matrix of the operator $\frac{1}{k!}\frac{\partial^k}{\partial u^k}$ acting on N. We then define

$$F_M(-\log\alpha, -\log\beta) = -\sum_N \dim_{K_{\alpha,\beta}} N \cdot \log R(N),$$

the sum being over all such irreducible constituents N. This gives a function

$$F_M(-,-) : [-\log\tau, -\log\rho] \times [0,\infty] \to \mathbb{R}_{\geq 0}.$$

Proof of Proposition 2.10. It is harmless to replace A by its completion for the supremum norm, so we may instead prove the corresponding claim for a smooth, affinoid K-algebra A with good reduction. We let A_{K_v} denote the base change of A to K_v and consider the Cartesian diagram

$$\begin{tikzcd} \mathscr{M}(A_{K_v}) \arrow[r] \arrow[d] & \mathscr{M}(A) \arrow[d] \\ \mathscr{M}(K_v) \arrow[r] & \mathscr{M}(K). \end{tikzcd}$$

By construction, there exists a rigid point of $\mathscr{M}(A_{K_v})$ lying above $v \in \mathscr{M}(A)$, and the unique point in the Shilov boundary of $\mathscr{M}(A_{K_v})$ lies above ξ. By invariance under isometric extensions, we may replace K by K_v and thus assume that v is a rigid point of $\mathscr{M}(A)$. Taking a completed localisation of A around v, applying [Ked05, Theorem 1] and lifting we can assume we have a finite étale map $K\langle \boldsymbol{x}\rangle \to A$, for $\boldsymbol{x} = (x_1,\ldots,x_d)$. By Lemma 2.7 it suffices to prove the claim for the pushforward of M along $K\langle \boldsymbol{x}\rangle \to A$, hence we may assume that $A = K\langle \boldsymbol{x}\rangle$. By translating, and possibly increasing K, we may assume that $v = 0$.

We let v_i denote the image in $\mathscr{M}(K\langle \boldsymbol{x}\rangle)$ of the unique point in the Shilov boundary of

$$\mathscr{M}(K\langle x_1,\ldots,x_d\rangle/(x_1,\ldots,x_i))$$

under the canonical closed immersion. Thus $v_0 = \xi$ and $v_d = v$, and it therefore suffices to show that M_{v_i} overconvergent $\Rightarrow M_{v_{i+1}}$ overconvergent, and that $\mathrm{Irr}(M_{v_i}) \geq \mathrm{Irr}(M_{v_{i+1}})$. But now looking at the commutative (although not in general Cartesian) diagram

$$\begin{tikzcd} \mathscr{M}(K_{v_i}\langle x_i\rangle) \arrow[r] \arrow[d] & \mathscr{M}(K\langle \boldsymbol{x}\rangle) \arrow[d] \\ \mathscr{M}(K_{v_i}) \arrow[r] & \mathscr{M}(K) \end{tikzcd}$$

we can see that the zero point of $\mathscr{M}(K_{v_i}\langle x_i\rangle)$ lies above v_{i+1}, and the unique point in the Shilov boundary of $\mathscr{M}(K_{v_i}\langle x_i\rangle)$ lies above v_i. Again by invariance under isometric extensions we can therefore reduce to the case $d = 1$, i.e. $A = K\langle x\rangle$.

Now let ρ be close enough to 1 such that M comes from a ∇-module defined over the ring

$$\cap_{\rho\leq\tau<1} K\langle \rho u^{-1}, \tau^{-1}u, x\rangle$$

of functions converging for $\rho \leq |u| < 1$ and $|x| \leq 1$, and let

$$F_M(-,-) : (0, -\log \rho] \times [0, \infty] \to \mathbb{R}$$

be the function defined above. After possibly increasing ρ, we may assume by Theorem 2.3 quoted above that the function $F_M(r, 0)$ is given by

$$F_M(r,0) = (\mathrm{rank}_{\mathscr{R}^u_L} M_L + \mathrm{Irr}(M_L))r,$$

where M_L is the base change to $\mathscr{R}^u_L$. Now fix some $r_0 \in (0, -\log \rho]$ and consider the ∇-module $M \otimes K_{e^{-r_0}}\langle x \rangle$. The field $K_{e^{-r_0}}$ is of rational type in the sense of [KX10, Definition 1.4.1], hence we may apply [KX10, Theorem 2.2.6] to deduce that the function

$$F_M(r_0, -) : [0, \infty] \to \mathbb{R}_{\geq 0}$$

is decreasing and continuous. Thus we find that

$$\lim_{r \to 0} F_M(r, \infty) \leq \lim_{r \to 0} F_M(r, 0) = 0$$

from which we deduce that the base change M_0 of M to $\mathscr{R}^u_K$ via $x \mapsto 0$ is also overconvergent. Thus after possibly increasing ρ we may assume again by Theorem 2.3 that $F_M(r, \infty)$ is given by

$$F_M(r, \infty) = (\mathrm{rank}_{\mathscr{R}^u_K} M_0 + \mathrm{Irr}(M_0))r.$$

Now again using the fact that $F_M(r_0, -)$ is a decreasing function we can deduce that $\mathrm{Irr}(M_0) \leq \mathrm{Irr}(M_L)$ as required. □

We therefore obtain a function

$$\mathrm{Irr}_M : \mathscr{M}(A) \to \mathbb{Z}_{\geq 0}$$

bounded above by $\mathrm{Irr}(M_L)$. The main result we will need is the following.

Proposition 2.11. *Assume that K is discretely valued. Let M be an overconvergent, projective ∇-module over $\mathscr{R}^u_A$, and assume that the function Irr_M is constant. Then the break decomposition extends across A, that is, there exists a decomposition*

$$M = \bigoplus_b M_b$$

of ∇-modules over $\mathscr{R}^u_A$ which restricts to the break decomposition of $M \otimes_{\mathscr{R}^u_A} \mathscr{R}^u_L$. Moreover, for any closed point $s : A \to K'$ the induced decomposition

$$M_s = \bigoplus_b (M_b)_s$$

of $M_s := M \otimes_{\mathscr{R}^u_A, s} \mathscr{R}^u_K$ coincides with the break decomposition of M_s.

We start with a simple special case.

Lemma 2.12. *Let M be an overconvergent, projective ∇-module over $\mathscr{R}^u_{K\langle x \rangle^\dagger}$. Assume that the break decomposition extends across $K\langle x, x^{-1} \rangle^\dagger$ and that the function Irr_M is constant on $\mathscr{M}(K\langle x \rangle^\dagger)$. Then the break decomposition extends across $K\langle x \rangle^\dagger$.*

Proof. As usual, let L be the completed fraction field of $K\langle x\rangle^\dagger$. Take $\rho<1$ both close enough to 1 such that M comes from a ∇-module over

$$\cap_{\rho\leq\tau<1}\cup_{\lambda>1}K\langle\tau u^{-1},\rho^{-1}u,\lambda^{-1}x\rangle.$$

By the proof of [Ked11, Lemma 1.3.4] it suffices to show that for any such ρ the induced decomposition of $M\otimes K_\rho\langle x,x^{-1}\rangle^\dagger$ extends to a decomposition of $M\otimes K_\rho\langle x\rangle^\dagger$. having fixed ρ, we can choose $\lambda>1$ such that M comes from a ∇-module over $K_\rho\langle\lambda^{-1}x\rangle$. But we can now apply [KX10, Theorem 2.3.10] to see that there exists a unique decomposition of M over $K_\rho[\![\lambda^{-1}x]\!]_0$, the ring of convergent series on $|x|<\lambda$ which are bounded as $|x|\to\lambda$, which restricts to the break decomposition on $M\otimes K_{\rho,\eta}$ for each $\eta\in(0,\lambda)$. In particular, this therefore has to restrict to the break decomposition on $M\otimes K_\rho\langle x,x^{-1}\rangle^\dagger$. Thus for $\lambda'<\lambda$ we can find the required decomposition of $M\otimes K_\rho\langle\lambda'^{-1}x\rangle$, and hence of $M\otimes K_\rho\langle x\rangle^\dagger$. □

We can then reduce the general case to this as follows.

Proof of Proposition 2.11. Once we know that the break decomposition extends across A, the final claim that it induces the break decomposition at every closed point follows from Proposition 2.10 above.

To see that the break decomposition extends across A, we begin by using the dagger form of Tate's acyclicity theorem to show that if $\{A\to A_i\}_{i\in I}$ is a finite dagger open cover of A, and we set $A_{ij}=A_i\otimes_A^\dagger A_j$, then for any finite projective $\mathscr{R}_A^u$-module N the sequence

$$0\to N\to\prod_i N\otimes_{\mathscr{R}_A^u}\mathscr{R}_{A_i}^u\to\prod_{i,j}N\otimes_{\mathscr{R}_A^u}\mathscr{R}_{A_{ij}}^u$$

is exact. Applying this to $N=\underline{\mathrm{End}}(M)$ we can see that if the break decomposition extends across all A_i, then it extends across A. Thus the question is 'dagger local' on A.

Now let $\mathscr{C}=\{A_i\}_{i\in I}$ denote the collection of all possible localisations of A such that the break decomposition extends across A_i. We will show by contradiction that $A\in\mathscr{C}$. Indeed, if $A\notin\mathscr{C}$ then by what we have just seen we know that after passing to the reductions modulo the maximal ideal of $\mathscr{V}$ the open immersion $\bigcup_{i\in I}\mathrm{Spec}\,(A_{i,0})\subsetneq\mathrm{Spec}\,(A_0)$ is strict. Thus after possibly making a finite extension of K (which is harmless) we may assume that there exists a smooth k-rational point z on the reduced complement $(\mathrm{Spec}\,(A_0)\setminus\bigcup_{i\in I}\mathrm{Spec}\,(A_{i,0}))_{\mathrm{red}}$. To contradict the maximality of $\mathscr{C}$, then, it suffices to produce a dagger localisation $A\to A'$ such that $z\in\mathrm{Spec}\,(A'_0)$ and such that the break decomposition extends across A'.

As we have already seen, the question is dagger local on A, hence we may localise around z, and use Lemma 1.8 together with [Ked05, Theorem 1] to obtain a finite étale map $K\langle x_1,\ldots,x_d\rangle^\dagger\to A$ such that z maps to the origin, and such that the break decomposition extends across $A\langle x_d^{-1}\rangle^\dagger$. Restricting along this finite étale map we may assume that $A=K\langle x_1,\ldots,x_d\rangle^\dagger$ and that the break decomposition extends across $K\langle x_1,\ldots,x_d,x_d^{-1}\rangle^\dagger$. Now let F be the completed fraction field of $K\langle x_1,\ldots,x_{d-1}\rangle^\dagger$, so we have

$$\mathscr{R}^u_{K\langle x_1,\ldots,x_d\rangle^\dagger}=\mathscr{R}^u_{K\langle x_1,\ldots,x_d,x_d^{-1}\rangle^\dagger}\cap\mathscr{R}^u_{F\langle x_d\rangle^\dagger}\subset\mathscr{R}^u_L.$$

Hence applying [Ked11, Lemma 1.2.7] to $\underline{\mathrm{End}}(M)$ we can see that it suffices to prove that the break decomposition extends across $\mathscr{R}^u_{F\langle x_d\rangle^\dagger}$. Now replacing K by F we can appeal to Lemma 2.12 above. □

3. Relative curves and generic pushforwards

We shall assume for the rest of the article that the ground field K is discretely valued. The residue field k will continue to be arbitrary of characteristic p.

3.1. **The basic geometric setup.** Here we will describe the basic geometric setup for our p-adic acyclicity theorems, and set some definitions and notations that will be used throughout the rest of the article.

Definition 3.1.
(1) An affine curve is a smooth, affine morphism $f: U \to S$ of k-varieties, of relative dimension 1.
(2) An affine curve is *good* if it admits a smooth compactification $\bar{f}: C \to S$ such that the complement $C \setminus U$ is étale over S. Such a compactification is called a good compactification of f.
(3) An affine curve is *simple* if it admits a good compactification $\bar{f}: C \to S$ such that $C \setminus U$ is a disjoint union of sections σ_i of $\bar{f}$, and there exists a neighbourhood of each σ_i on which it is defined by the vanishing of a single function $u_i \in \mathscr{O}_C$. Such a compactification is called a simple compactification of f.

Remark 3.2. It might be more usual to require any of these 'curves' to have geometrically connected fibres. However, it will be important for us *not* to assume this.

Lemma 3.3. *Let $f: U \to S$ be a good curve with good compactification $\bar{f}: C \to S$. Then there exists a finite étale cover $S' \to S$ and a Zariski open cover $\{S'_i\}$ of S' such that for all i the pullback $U_{S'_i} \to S'_i$ is simple, with simple compactification $C_{S'_i} \to S'_i$.*

Proof. Let S' be a common Galois closure of all the connected components of $C \setminus U$, which by assumption are finite étale over S. Then after base changing to S' the complement $C \setminus U$ is a disjoint union of sections, which by smoothness of $C \to S$ must all be regular closed immersions. Hence $C \to S$ becomes a simple compactification of $U \to S$ over a Zariski open cover of S'. $\square$

Fix an affine curve $f: U \to S$ admitting a smooth compactification $\bar{f}: C \to S$. Assume that the base $S = \mathrm{Spec}\left(\overline{A}\right)$ is smooth, affine and connected, and choose a MW-type frame $(S, \overline{S}, \mathfrak{S})$. Even without the assumption of geometrically connected fibres, we can still prove the following simple, yet crucial, result.

Lemma 3.4. *After possible replacing $(S, \overline{S}, \mathfrak{S})$ by a modification, there exists a smooth, proper, Cartesian morphism of frames*

$$\bar{f}: (C, \overline{C}, \mathfrak{C}) \to (S, \overline{S}, \mathfrak{S})$$

extending $\bar{f}$, such that the subframe $(U, \overline{C}, \mathfrak{C})$ is also of MW-type.

Proof. Since $\mathfrak{S}$ is projective it is algebraisable, say $\mathfrak{S} = \widehat{\overline{\mathscr{S}}}$ for some projective $\mathscr{V}$-scheme $\overline{\mathscr{S}}$. Choose an open affine subscheme $\mathscr{S} \subset \overline{\mathscr{S}}$ whose special fibre is S (since these are schemes, not formal schemes, there are many choices for such an $\mathscr{S}$). The smooth and proper curve $C \to S$ lifts to a smooth and proper curve $\mathscr{C}^h \to \mathscr{S}^h$ over the π-adic Henselisation of $\mathscr{S}$, and hence there exists an étale morphism $\mathscr{S}' \to \mathscr{S}$ of affine $\mathscr{V}$-schemes inducing an isomorphism on special fibres, and a smooth and proper curve $\mathscr{C}' \to \mathscr{S}'$ lifting $C \to S$. Now choose a compactification $\overline{\mathscr{S}}'$ of $\mathscr{S}'$ over $\mathscr{V}$, and a compactification $\overline{\mathscr{C}}' \to \overline{\mathscr{S}}'$ of $\mathscr{C}' \to \mathscr{S}'$.

There is a rational map $\overline{\mathscr{S}}' \dashrightarrow \overline{\mathscr{S}}$ induced by $\mathscr{S}' \to \mathscr{S}$, and since $\mathscr{S}$ is quasi-projective we may blowup the indeterminacy locus $\overline{\mathscr{S}}'' \to \overline{\mathscr{S}}'$ to obtain a morphism $\overline{\mathscr{S}}'' \to \overline{\mathscr{S}}$. Now set $\mathfrak{S}'' = \widehat{\overline{\mathscr{S}}''}$ and $\overline{S}''$ to be the special fibre of $\mathfrak{S}''$, thus the map

$$(S, S'', \mathfrak{S}'') \to (S, \overline{S}, \mathfrak{S})$$

is a modification of MW-type frames. Finally, taking $\mathfrak{C}$ to be the π-adic completion of $\overline{\mathscr{C}}' \times_{\overline{\mathscr{S}}'} \overline{\mathscr{S}}''$ and $\overline{C}$ to be the special fibre of $\mathfrak{C}$ we get a smooth, proper, Cartesian morphism of frames

$$(C, \overline{C}, \mathfrak{C}) \to (S, \overline{S}'', \mathfrak{S}'')$$

such that $(U,\overline{C},\mathfrak{C})$ is of MW-type. $\square$

For a morphism $\bar{f}:(C,\overline{C},\mathfrak{C})\to(S,\overline{S},\mathfrak{S})$ as in Lemma 3.4, we will set

$$A := \Gamma(\mathfrak{S}_K, j_S^\dagger \mathscr{O}_{\mathfrak{S}_K})$$
$$B := \Gamma(\mathfrak{C}_K, j_U^\dagger \mathscr{O}_{\mathfrak{C}_K}).$$

These are therefore MW-type K-dagger algebras, pullback induces a homomorphism $A\to B$ and the module of continuous differentials $\Omega^1_{B/A}$ is a finite projective B-module of rank 1.

'Normal' higher direct images will be defined in terms of the morphism of K-dagger algebras $A\to B$, but we will also need to make use of other higher direct images defined using relative Robba rings, as in [Ked06a]. We will only use these in the case when $f:U\to S$ is a simple affine curve and $\bar{f}:C\to S$ is a simple compactification (again, with base S smooth, affine and connected as above).

In this situation, let $\bar{\sigma}_i$ denote the closure of the image of σ_i inside $\overline{C}$, and for $\bar{f}:(C,\overline{C},\mathfrak{C})\to(S,\overline{S},\mathfrak{S})$ as in Lemma 3.4 set

$$\mathscr{R}^+_{A,\sigma_i} = \Gamma(]\bar{\sigma}_i[_{\mathfrak{C}_K}, j^\dagger_{\sigma_i}\mathscr{O}_{\mathfrak{C}_K})$$
$$\mathscr{R}_{A,\sigma_i} = \mathrm{colim}_V \Gamma(]\bar{\sigma}_i[_{\mathfrak{C}_K}\cap V, j^\dagger_{\sigma_i}\mathscr{O}_{\mathfrak{C}_K}),$$

the colimit in the second definition being over strict neighbourhoods V of $]\overline{C}\setminus\overline{\sigma}_i[_{\mathfrak{C}}$ inside $\mathfrak{C}_K$. Since each σ_i has a neighbourhood on which it is locally cut out by a single function u_i, by lifting these u_i to some dagger localisation of B and using the strong fibration theorem, we can identify

$$\mathscr{R}^+_{A,\sigma_i} \cong \mathscr{R}^{u_i+}_A$$
$$\mathscr{R}_{A,\sigma_i} \cong \mathscr{R}^{u_i}_A$$

with copies of the relative Robba ring over A. For all i there is a natural embedding

$$B\to\mathscr{R}_{A,\sigma_i}$$

of A-algebras, we define $\mathscr{Q}^{\{\sigma_i\}}_A$ to be the quotient

$$0\to B\to\bigoplus_i \mathscr{R}_{A,\sigma_i}\to\mathscr{Q}^{\{\sigma_i\}}_A\to 0.$$

3.2. **Generic pushforwards à la Kedlaya.** The setup and notations will be as in §3.1 above, thus we have an affine curve $f:U\to S$ over a smooth, affine, connected base S, admitting a smooth compactification $\bar{f}:C\to S$, and a morphism of frames

$$(C,\overline{C},\mathfrak{C})\to(S,\overline{S},\mathfrak{S})$$

extending $\bar{f}$ as in Lemma 3.4. Having set things up relatively geometrically, we will for a while revert to a more algebraic viewpoint on relative rigid cohomology, at least until §6. For any overconvergent isocrystal E on U/K, we can realise E on the frame $(U,\overline{C},\mathfrak{C})$ and take global sections to obtain an overconvergent ∇-module M over B, and thus define the cohomology groups

$$\mathbf{R}^0 f_* M := \ker\left(M\overset{\nabla}{\to}M\otimes_B\Omega^1_{B/A}\right) \quad\text{and}\quad \mathbf{R}^1 f_* M := \mathrm{coker}\left(M\overset{\nabla}{\to}M\otimes_B\Omega^1_{B/A}\right).$$

When $f : U \to S$ is simple, and $\bar{f} : C \to S$ is a simple compactification, we can define further higher direct images

$$\mathbf{R}^0_{\mathrm{loc}} f_* M := \bigoplus_i \ker \left(M \otimes_B \mathscr{R}_{A,\sigma_i} \xrightarrow{\nabla} M \otimes_B \mathscr{R}_{A,\sigma_i} \otimes_B \Omega^1_{B/A} \right)$$

$$\mathbf{R}^1 f_! M := \ker \left(M \otimes_B \mathscr{Q}_A^{\{\sigma_i\}} \xrightarrow{\nabla} M \otimes_B \mathscr{Q}_A^{\{\sigma_i\}} \otimes_B \Omega^1_{B/A} \right)$$

$$\mathbf{R}^1_{\mathrm{loc}} f_* M := \bigoplus_i \mathrm{coker} \left(M \otimes_B \mathscr{R}_{A,\sigma_i} \xrightarrow{\nabla} M \otimes_B \mathscr{R}_{A,\sigma_i} \otimes_B \Omega^1_{B/A} \right)$$

$$\mathbf{R}^2 f_! M := \mathrm{coker} \left(M \otimes_B \mathscr{Q}_A^{\{\sigma_i\}} \xrightarrow{\nabla} M \otimes_B \mathscr{Q}_A^{\{\sigma_i\}} \otimes_B \Omega^1_{B/A} \right)$$

using the notation from §3.1 above. These are (not necessarily finitely generated) A-modules which sit in an exact sequence

$$0 \to \mathbf{R}^0 f_* M \to \mathbf{R}^0_{\mathrm{loc}} f_* M \to \mathbf{R}^1 f_! M \to \mathbf{R}^1 f_* M \to \mathbf{R}^1_{\mathrm{loc}} f_* M \to \mathbf{R}^2 f_! M \to 0.$$

When $A = K$ (or a finite extension thereof) we will usually write

$$H^0(M),\ H^0_{\mathrm{loc}}(M),\ H^1_c(M),\ H^1(M),\ H^1_{\mathrm{loc}}(M),\ H^2_c(M)$$

instead. When $A \to A'$ is a morphism of MW-type K-dagger algebras, we will write B' for $B \otimes_A^\dagger A'$ and M' for $M \otimes_B B'$, thus M' is an overconvergent ∇-module over B'. The main result on existence of generic push-forwards is then the following.

Theorem 3.5. *[Ked06a, Theorem 7.3.3, Remark 7.2.2, Proposition 8.6.1] Assume that $f : U \to S$ is a simple affine curve, and that M is F-able.*

(1) *There exists a dagger localisation $A \to A'$ such that $\mathbf{R}^i f_* M'$, $\mathbf{R}^i_{\mathrm{loc}} f_* M'$, $\mathbf{R}^i f_! M'$ are finitely generated over A', formation of which commutes with flat base change $A' \to A''$ of MW-type dagger algebras.*

(2) *For any A' such that the conclusions of (1) hold for M' and $M'^\vee$, there are canonical perfect pairings*

$$\mathbf{R}^i f_* M' \otimes_{A'} \mathbf{R}^{2-i} f_! M'^\vee \to A'(-1)$$

$$\mathbf{R}^i_{\mathrm{loc}} f_* M' \otimes_{A'} \mathbf{R}^{1-i}_{\mathrm{loc}} f_* M'^\vee \to A'(-1)$$

of ∇-modules over A'.

Remark 3.6. (1) The base change claim implies that any Frobenius structure on M induces one on all of the higher direct images $\mathbf{R}^i f_* M'$, $\mathbf{R}^i_{\mathrm{loc}} f_* M'$, $\mathbf{R}^i f_! M'$, in a way compatible with the Poincaré pairings in (2).

(2) It was also shown in [Ked06a] that formation of these higher direct images commutes with base change to the completed fraction field L of A'.

Proof. In [Ked06a] the case when $U = \mathbb{A}^1_S$, $B = A\langle x \rangle^\dagger$ and M admits a Frobenius structure was treated, we will explain here how to reduce to this case. First of all, passing to the irreducible constituents of M we may assume that M itself admits a Frobenius structure. Applying [Ked05, Theorem 1] at the generic point of S and spreading out we can find an open immersion $S' \to S$ and a finite étale map $U_{S'} \to \mathbb{A}^1_{S'}$ of S'-schemes. Lifting to characteristic 0 via Lemma 1.8 we can therefore find a dagger localisation $A \to A'$ such that there exists a finite étale morphism $A'\langle x \rangle^\dagger \to B \otimes_A^\dagger A'$ of A'-algebras. Taking the pushforward along this finite étale map doesn't change any of the higher direct images, so we can replace A by A' and $B \otimes_A^\dagger A'$ by $A'\langle x \rangle^\dagger$, and thus reduce to considering the case where $B = A\langle x \rangle^\dagger$. ∎

Our goal (more or less) will be to use the irregularity of a ∇-module to give conditions under which we can take $A = A'$ in the above Theorem. If $f : U \to S$ is simple, and $\bar{f}$ is a simple compactification, then for each σ_i we can base change M along

$$B \to \mathscr{R}_{\sigma_i} \cong \mathscr{R}_A^{u_i}$$

to obtain an overconvergent ∇-module M^i_{loc} over $\mathscr{R}_A^{u_i}$, with an associated irregularity function

$$\mathrm{Irr}_{M^i_{\mathrm{loc}}} : \mathscr{M}(A) \to \mathbb{Z}_{\geq 0}.$$

We define the total irregularity of M to be the function

$$\mathrm{Irr}^{\mathrm{tot}}_M := \sum_i \mathrm{Irr}_{M^i_{\mathrm{loc}}} : \mathscr{M}(A) \to \mathbb{Z}_{\geq 0}.$$

We can also make a similar definition for a general good curve $f : U \to S$ over a smooth affine base, by appealing to Lemma 3.3 and descending. We can now state our first partial p-adic analogue of [Lau81, Corollaire 2.1.2] as follows.

Theorem 3.7. *Assume that $f : U \to S$ is a good affine curve over a smooth, affine, connected base, and that M is F-able. Then the following are equivalent:*

(1) *the total irregularity $\mathrm{Irr}^{\mathrm{tot}}_M : \mathscr{M}(A) \to \mathbb{Z}_{\geq 0}$ is constant;*
(2) *the higher direct images $\mathbf{R}^0 f_* M$ and $\mathbf{R}^1 f_* M$ are finitely generated over A, and their formation commutes with arbitrary base change $A \to A'$ of MW-type K-dagger algebras.*

Remark 3.8. As in Remark 3.6, it follows from the base change claim that any Frobenius structure on M induces one on $\mathbf{R}^i f_* M$. Formation of $\mathbf{R}^i f_* M$ also commutes with base change to the completed fraction field L of A.

Note that the implication (2)$\Rightarrow$(1) simply follows from the Grothendieck–Ogg–Shafarevich formula [CM01, Corollaire 5.0-12], the proof that (1)$\Rightarrow$(2) will occupy us until the end of §7. To start with, we will record a consequence of Theorem 3.5 that is not explicitly spelled out in [Ked06a], but can nonetheless be easily deduced from results there.

Lemma 3.9. *Let $f : U \to S$ be simple. Assume that the conclusions of Theorem 3.5 hold for M and $M^\vee$ without further localisation of A, and that $\mathbf{R}^0 f_* M = \mathbf{R}^0 f_* M^\vee = 0$. Then the formation of $\mathbf{R}^i f_* M$, $\mathbf{R}^i_{\mathrm{loc}} f_* M$ $\mathbf{R}^i f_! M$ commutes with arbitrary base change $A \to A'$ of MW-type dagger algebras.*

Proof. By choosing a set of topological generators for A' over A we can treat separately the cases when $A \to A'$ is surjective and when $A' = A\langle x_1, \ldots, x_n \rangle^\dagger$. The latter case is covered by Theorem 3.5, we will therefore consider the former. By Poincaré duality we know that $\mathbf{R}^2 f_! M = \mathbf{R}^2 f_! M^\vee = 0$. Since the base change map

$$(\mathbf{R}^1_{\mathrm{loc}} f_* M) \otimes_A A' \to \mathbf{R}^1_{\mathrm{loc}} f_* M'$$

is trivially surjective, we deduce that the latter is finitely generated over A', which is enough to show that the conclusions of Theorem 3.5 hold for M' without further localisation (see for example the proof of [Ked06a, Theorem 7.3.3]). Thus Poincaré duality also holds for M'. The map

$$(\mathbf{R}^2 f_! M) \otimes_A A' \to \mathbf{R}^2 f_! M'$$

is also trivially surjective, thus we deduce that $\mathbf{R}^2 f_! M' = 0$; replacing M by $M^\vee$ and applying Poincaré duality we can also see that $\mathbf{R}^0 f_* M' = 0$. Hence base change holds for $\mathbf{R}^0 f_* M$ and $\mathbf{R}^2 f_! M$. We now consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & (\mathbf{R}^0_{\mathrm{loc}} f_* M)\otimes_A A' \arrow[r] \arrow[d, hook] & (\mathbf{R}^1 f_! M)\otimes_A A' \arrow[r] \arrow[d, hook] & (\mathbf{R}^1 f_* M)\otimes_A A' \arrow[r] \arrow[d, two heads] & (\mathbf{R}^1_{\mathrm{loc}} f_* M)\otimes_A A' \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & \mathbf{R}^0_{\mathrm{loc}} f_* M' \arrow[r] & \mathbf{R}^1 f_! M' \arrow[r] & \mathbf{R}^1 f_* M' \arrow[r] & \mathbf{R}^1_{\mathrm{loc}} f_* M' \arrow[r] & 0
\end{tikzcd}$$

where the rows are the canonical exact sequences, and the vertical arrows come from base change. Immediately from the definitions we find that the right two vertical arrows are surjective; by replacing M by $M^\vee$ and using Poincaré duality we can see that the left hand vertical arrows are injective. Moreover, we know from [Meb02, Corollaire 1.3-2] and base change to the generic fibre that

$$\begin{aligned}
\mathrm{rank}_A \mathbf{R}^0_{\mathrm{loc}} f_* M &= \mathrm{rank}_A \mathbf{R}^1_{\mathrm{loc}} f_* M \\
\mathrm{rank}_{A'} \mathbf{R}^0_{\mathrm{loc}} f_* M' &= \mathrm{rank}_{A'} \mathbf{R}^1_{\mathrm{loc}} f_* M',
\end{aligned}$$

and hence base change has to hold for $\mathbf{R}^i_{\mathrm{loc}} f_* M$. We can also deduce that $(\mathbf{R}^1 f_* M)\otimes_A A'$, $(\mathbf{R}^1 f_! M)\otimes_A A'$, $\mathbf{R}^1 f_* M'$ and $\mathbf{R}^1 f_! M'$ all have the same rank over A', which gives base change for $\mathbf{R}^1 f_* M$ and $\mathbf{R}^1 f_! M$, and completes the proof. ☐

We can use this to give the following minor strengthening of Theorem 3.5

Corollary 3.10. *In the situation of Theorem 3.5 there exists a dagger localisation $A \to A'$ such that the higher direct images $\mathbf{R}^i f_* M'$, $\mathbf{R}^i_{\mathrm{loc}} f_* M'$, $\mathbf{R}^i f_! M'$ are finitely generated over A', whose formation commutes with arbitrary base change $A' \to A''$ of MW-type dagger algebras.*

Proof. As explained above, we can reduce to the case where $B = A\langle x\rangle^\dagger$, and $\mathbf{R}^i f_* M$, $\mathbf{R}^i_{\mathrm{loc}} f_* M$, $\mathbf{R}^i f_! M$ are finitely generated over A with formation commuting with flat base change, as well as to the completed fraction field L of A. In particular, we can verify that the natural map

$$\mathbf{R}^0 f_* M \otimes_A A\langle x\rangle^\dagger \to M$$

is injective, since it is so over L. After replacing A by a further localisation, we can also assume that the conclusions of Theorem 3.5 hold for the quotient N of M by $\mathbf{R}^0 f_* M \otimes_A A\langle x\rangle^\dagger$, as well as its dual $N^\vee$. The base change claim holds for $\mathbf{R}^0 f_* M \otimes_A A\langle x\rangle^\dagger$ by using the projection formula, hence by the five lemma it suffices to prove it for N. But again by comparing with the situation over L we can see that $\mathbf{R}^0 f_* N = 0$; in other words we may assume that $\mathbf{R}^0 f_* N = 0$. We now replace N by $N^\vee$, in order to produce an exact sequence

$$0 \to P \to N \to \left(\mathbf{R}^0 f_* N^\vee\right)^\vee \otimes_A A\langle x\rangle^\dagger \to 0,$$

again we can assume that the conclusions of Theorem 3.5 hold for P and $P^\vee$. Again appealing to the five lemma, it suffices to prove the result for P, and we may check that $\mathbf{R}^0 f_* P = \mathbf{R}^0 f_* P^\vee = 0$. Thus we may apply Lemma 3.9 ☐

4. UNIPOTENCE AND BASE CHANGE FOR ∇-MODULES OVER RELATIVE ROBBA RINGS

We will begin the proof of Theorem 3.7 with a study of the behaviour of $\mathbf{R}^0_{\mathrm{loc}} f_* M$ in the case when the generic fibre of M is unipotent. In this section we will not need the geometric setup of §3.1, and will instead simply work with dagger algebras and ∇-modules. We will let A be a K-dagger algebra of MW-type, L its completed fraction field, and M an overconvergent ∇-module over $\mathscr{R}^u_A$.

As part of the proof of [Ked06a, Theorem 7.3.3] Kedlaya shows that if M is free, and the generic fibre $M \otimes \mathscr{R}^u_L$ is unipotent as a ∇-module relative to L, then there exists a dagger localisation $A \to A'$ such that the ∇-module $M \otimes \mathscr{R}^u_{A'}$ admits a strongly unipotent basis $\{\boldsymbol{e}_i\}$ relative to A', that is one for which

$$u\nabla_u(\boldsymbol{e}_i) \in (A\boldsymbol{e}_1 + \ldots + A\boldsymbol{e}_{i-1}) \otimes du,$$

where ∇_u is the 'u-component' of the connection on M. It will be important for us to work with ∇-modules that are not known *a priori* to be free, and to still have a version of this result. Luckily, we will only need it in the case where $A = K\langle \boldsymbol{x}\rangle^\dagger = K\langle x_1, \ldots, x_d\rangle^\dagger$ is a free K-dagger algebra.

Theorem 4.1. *Let M be a projective ∇-module over $\mathscr{R}^u_{K\langle \boldsymbol{x}\rangle^\dagger}$, whose generic fibre $M \otimes \mathscr{R}^u_L$ is unipotent relative to L. Then M is free, and admits a strongly unipotent basis relative to $K\langle \boldsymbol{x}\rangle^\dagger$.*

The proof of this result will occupy the rest of §4. As in [Ked06a, §5], we will find it easier to introduce an auxiliary ring in place of $\mathscr{R}^u_{K\langle \boldsymbol{x}\rangle^\dagger}$.

Definition 4.2. Fix some $\eta \in \sqrt{|K^*|}$ and define $R^u_{L,\eta}$ to be the ring of series $\sum_i a_i u^i$ with $a_i \in L$, such that there exist $\eta_- < \eta < \eta_+$ such that

$$\|a_i\|\,\eta_\pm^i \to 0$$

as $i \to \pm\infty$. Similarly, define $R^u_{K\langle \boldsymbol{x}\rangle^\dagger,\eta}$ to be the ring of series $\sum_i a_i u^i$ with $a_i \in K\langle \boldsymbol{x}\rangle^\dagger$ such that there exist $\eta_- < \eta < \eta_+$ and $\lambda > 1$ such that $a_i \in K\langle \lambda^{-1}\boldsymbol{x}\rangle$ for all i and

$$\|a_i\|_\lambda\,\eta_\pm^i \to 0$$

as $i \to \pm\infty$.

As usual, a ∇-module over $R^u_{K\langle \boldsymbol{x}\rangle^\dagger,\eta}$ will mean a ∇-module relative to K, whose underlying $R^u_{K\langle \boldsymbol{x}\rangle^\dagger,\eta}$-module is finitely presented. Any extra adjectives such as projective, stably free, free, &c. are understood to apply to the underlying $R^u_{K\langle \boldsymbol{x}\rangle^\dagger,\eta}$-module.

Lemma 4.3. (1) *The map $K\langle \boldsymbol{x}\rangle^\dagger \to L$ is flat.*
(2) *The natural map $R^u_{K\langle \boldsymbol{x}\rangle^\dagger,\eta} \otimes_{K\langle \boldsymbol{x}\rangle^\dagger} L \to R^u_{L,\eta}$ is injective.*

Proof. The first claim is clear, since $K\langle \boldsymbol{x}\rangle^\dagger \to L$ is a monomorphism into a field. For the second, we choose a bijection $\mathbb{Z} \to \mathbb{N}$ and consider $R^u_{K\langle \boldsymbol{x}\rangle^\dagger,\eta}$ and $R^u_{L,\eta}$ as subspaces of the infinite products $\prod_{\mathbb{N}} K\langle \boldsymbol{x}\rangle^\dagger$ and $\prod_{\mathbb{N}} L$ respectively; it suffices to show that the natural map

$$\left(\prod_{\mathbb{N}} K\langle \boldsymbol{x}\rangle^\dagger\right) \otimes_{K\langle \boldsymbol{x}\rangle^\dagger} L \to \prod_{\mathbb{N}} L$$

is injective. If we let L_0 denote the (uncompleted) fraction field of $K\langle \boldsymbol{x}\rangle^\dagger$, then the map

$$\left(\prod_{\mathbb{N}} K\langle \boldsymbol{x}\rangle^\dagger\right) \otimes_{K\langle \boldsymbol{x}\rangle^\dagger} L_0 \to \prod_{\mathbb{N}} L_0$$

is always injective, with image exactly those elements of $\prod_{\mathbb{N}} L_0$ whose denominators all lie in some (non-dagger) localisation $K\langle \boldsymbol{x}\rangle^\dagger[f^{-1}]$. Hence it suffices to show that the map

$$\left(\prod_{\mathbb{N}} L_0\right) \otimes_{L_0} L \to \prod_{\mathbb{N}} L$$

is injective. We claim more generally that this is true with $L_0 \to L$ replaced by any field extension $F \to F'$. Indeed, suppose that we have an element

$$\sum_{i=1}^n (\lambda_{ij})_{j=1}^\infty \otimes f_i \in \left(\prod_{\mathbb{N}} F\right) \otimes_F F'$$

which maps to zero in $\prod_{\mathbb{N}} F'$, i.e. such that $\sum_i \lambda_{ij} f_i = 0$ for all j.

Let $V = F^n$ be the standard n-dimensional vector space equipped with the standard bilinear form. Let $\boldsymbol{\lambda}_j = (\lambda_{1j}, \ldots, \lambda_{nj}) \in V$ and $\boldsymbol{f} = (f_1, \ldots, f_n) \in V \otimes_F F'$. Let U_j denote the annihilator of $\boldsymbol{\lambda}_j$, and $W_j = \bigcap_{j'=1}^j U_{j'}$. Thus W_j is a descending sequence of subspaces of V, which must therefore eventually stabilise. Hence we have

$$(\cap_j W_j) \otimes_F F' = \bigcap_j \left(W_j \otimes_F F'\right).$$

Pick a basis $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_k$ for $\cap_j W_j$, and write these as $\boldsymbol{e}_l = (e_{l1}, \ldots, e_{ln})$ with $e_{lm} \in F$. The fact that $\boldsymbol{e}_l \in \cap_j W_j$ means that

$$\sum_i \lambda_{ij} e_{li} = 0$$

for all j, l. Since $\boldsymbol{f} \in \bigcap_j (W_j \otimes_F F')$ we must be able to write $\boldsymbol{f} = \sum_{l=1}^n \alpha_l \boldsymbol{e}_l$ for some $\alpha_l \in F'$. Putting this all together with have

$$\sum_{i=1}^n (\lambda_{ij})_{j=1}^\infty \otimes f_i = \sum_{i,l=1}^n (\lambda_{ij})_{j=1}^\infty \otimes \alpha_l e_{li} = \sum_{l=1}^n \left(\sum_{i=1}^n (\lambda_{ij} e_{li})_{j=1}^\infty\right) \otimes \alpha_l = 0$$

and the proof is complete. □

We can now extend some of the results in [Ked06a, §5] from free to stably free ∇-modules over $R^u_{K\langle \boldsymbol{x}\rangle^\dagger, r}$. For any ∇-module over any of the rings $R^u_{K\langle \boldsymbol{x}\rangle^\dagger, r}$, $R^u_{L,r}$, $\mathscr{R}^u_{K\langle \boldsymbol{x}\rangle^\dagger}$ or $\mathscr{R}^u_L$ we will write $H^0_{\nabla_u}$ for the kernel of the derivation $\frac{\partial}{\partial u}$. For M a ∇-module over $R^u_{K\langle \boldsymbol{x}\rangle^\dagger, \eta}$, we will write M_L for $M \otimes_{R^u_{K\langle \boldsymbol{x}\rangle^\dagger, \eta}} R^u_{L,\eta}$

Lemma 4.4. *Let M be a stably free ∇-module over $R^u_{K\langle \boldsymbol{x}\rangle^\dagger, \eta}$, such that M_L is unipotent relative to L. Then $H^0_{\nabla_u}(M)$ is a finite free ∇-module over $K\langle \boldsymbol{x}\rangle^\dagger$.*

Proof. All finitely generated ∇-modules over $K\langle \boldsymbol{x}\rangle^\dagger$ are projective, and therefore free, via the analogue of the Quillen–Suslin theorem for $K\langle \boldsymbol{x}\rangle^\dagger$. Thus $H^0_{\nabla_u}(M)$ will be free as soon as it is finitely generated. To see that it is finitely generated we may choose some n such that

$$M' = M \oplus R^{u,\oplus n}_{K\langle \boldsymbol{x}\rangle^\dagger, \eta}$$

is a free ∇-module over $R^u_{K\langle \boldsymbol{x}\rangle^\dagger, \eta}$. The claim for M can therefore be deduced from the claim for M', we may therefore assume that M is in fact free. In this case, [Ked06a, Proposition 5.2.6] shows that $H^0_{\nabla_u}(M_L)$ is finite dimensional over L, and it follows from [Ked06a, Lemma 7.3.4] that $H^0_{\nabla_u}(M)$ is finitely generated over $K\langle \boldsymbol{x}\rangle^\dagger$. □

Lemma 4.5. *Suppose that M is a stably free ∇-module over $R^u_{K\langle \boldsymbol{x}\rangle^\dagger,\eta}$, such that M_L is unipotent relative to L. Then the base change map*

$$H^0_{\nabla_u}(M)\otimes_{K\langle \boldsymbol{x}\rangle^\dagger} L \to H^0_{\nabla_u}(M_L)$$

is as isomorphism.

Proof. As in the proof of Lemma 4.4, we may assume that M is in fact free. Surjectivity of the base change map then follows from the proof of [Ked06a, Proposition 5.3.3]. To see injectivity, since $H^0_{\nabla_u}(M)\subset M$ we can deduce from flatness of $K\langle \boldsymbol{x}\rangle^\dagger \to L$ that $H^0_{\nabla_u}(M)\otimes_{K\langle \boldsymbol{x}\rangle^\dagger} L \to M\otimes_{K\langle \boldsymbol{x}\rangle^\dagger} L$ is injective, and since M is projective we deduce from injectivity of $R^u_{K\langle \boldsymbol{x}\rangle^\dagger,\eta}\otimes_{K\langle \boldsymbol{x}\rangle^\dagger} L\to R^u_{L,\eta}$ that

$$M\otimes_{K\langle \boldsymbol{x}\rangle^\dagger} L \to M\otimes_{R^u_{K\langle \boldsymbol{x}\rangle^\dagger,\eta}} R^u_{L,\eta}$$

is injective. Combining these two we can conclude. □

In fact, the restriction to stably free modules is essentially unnecessary.

Lemma 4.6. *Any finitely presented ∇-module over $R^u_{K\langle \boldsymbol{x}\rangle^\dagger,\eta}$ is projective, and becomes stably free after making a finite base extension $K\to K'$.*

Proof. The projectivity statement is straightforward, as any such module comes from a ∇-module over some regular strictly affinoid K-algebra. To see the claim on stable freeness, choose $\eta_- < \eta < \eta_+$ and $\lambda > 1$ such that M comes from a ∇-module over $K\langle \lambda^{-1}\boldsymbol{x}, \eta_- u^{-1}, \eta_+^{-1}u\rangle$. After replacing K by a finite extension we can assume that $\lambda,\eta_-,\eta,\eta_+\in |K^*|$, and thus after possibly shrinking the interval $[\eta_-,\eta_+]$ we can assume that $\eta_-/\eta_+ = |\pi|$. In this case we have an isomorphism

$$K\langle \lambda^{-1}\boldsymbol{x}, \eta_- u^{-1}, \eta_+^{-1}u\rangle \cong \frac{K\langle \boldsymbol{x},u,v\rangle}{(uv-\pi)}.$$

Again, any ∇-module over $K\langle \lambda^{-1}\boldsymbol{x}, \eta_- u^{-1}, \eta_+^{-1}u\rangle$ is projective, and hence it suffices to show that any projective $K\langle \boldsymbol{x},u,v\rangle/(uv-\pi)$-module is stably free. We consider the diagram

$$\begin{tikzcd}
K_0\left(\frac{K\langle \boldsymbol{x},u,v\rangle}{(uv-\pi)}\right) & K_0\left(\frac{\mathscr{O}_K\langle \boldsymbol{x},u,v\rangle}{(uv-\pi)}\right) \arrow[l] \arrow[r] & K_0\left(\frac{k[\boldsymbol{x},u,v]}{(uv)}\right) \\
K_0\left(\frac{K\langle u,v\rangle}{(uv-\pi)}\right) \arrow[u] & K_0\left(\frac{\mathscr{O}_K\langle u,v\rangle}{(uv-\pi)}\right) \arrow[l] \arrow[r] \arrow[u] & K_0\left(\frac{k[u,v]}{(uv)}\right). \arrow[u]
\end{tikzcd}$$

Since $\mathscr{O}_K\langle \boldsymbol{x},u,v\rangle/(uv-\pi)$ is regular and π-adically complete, we have that

$$K_0\left(\frac{\mathscr{O}_K\langle \boldsymbol{x},u,v\rangle}{(uv-\pi)}\right)\xrightarrow{\sim} K_0\left(\frac{k[\boldsymbol{x},u,v]}{(uv)}\right),\quad K_0\left(\frac{\mathscr{O}_K\langle \boldsymbol{x},u,v\rangle}{(uv-\pi)}\right)\twoheadrightarrow K_0\left(\frac{K\langle \boldsymbol{x},u,v\rangle}{(uv-\pi)}\right),$$

and by [Lan02, Ch. XXI, Theorem 2.8] we have

$$K_0\left(\frac{k[u,v]}{(uv)}\right)\xrightarrow{\sim} K_0\left(\frac{k[\boldsymbol{x},u,v]}{(uv)}\right).$$

Hence the map

$$K_0\left(\frac{K\langle u,v\rangle}{(uv-\pi)}\right)\to K_0\left(\frac{K\langle \boldsymbol{x},u,v\rangle}{(uv-\pi)}\right)$$

is surjective, so we can reduce to showing that

$$K_0\left(\frac{K\langle u,v\rangle}{(uv-\pi)}\right)\xrightarrow{\sim}\mathbb{Z}.$$

As $K\langle u,v\rangle/(uv-\pi)$ is a principal ideal domain, we are done. $\square$

We can now prove an analogue of Theorem 4.1 for $R^u_{K\langle \boldsymbol{x}\rangle^\dagger,\eta}$ in place of $\mathscr{R}^u_{K\langle \boldsymbol{x}\rangle^\dagger}$.

Proposition 4.7. *Suppose that M is a finitely presented ∇-module over $R^u_{K\langle \boldsymbol{x}\rangle^\dagger,\eta}$, such that M_L is unipotent relative to L. Then M is free, and admits a strongly unipotent basis relative to $K\langle \boldsymbol{x}\rangle^\dagger$.*

Proof. As observed above, M is projective; we will induct on the rank of M. If the rank is zero then there is nothing to prove. If the rank is >0, then we claim that $N=H^0_{\nabla_u}(M)$ is a non-zero free ∇-module over $K\langle \boldsymbol{x}\rangle^\dagger$, and the base change map

$$N\otimes_{K\langle \boldsymbol{x}\rangle^\dagger} R^u_{K\langle \boldsymbol{x}\rangle^\dagger,\eta}\to M$$

is injective. Both these claims may be verified after making a finite base extension $K\to K'$: the first then follows from combining Lemmas 4.5 and Lemma 4.6, and the second because the same injectivity holds after base changing to L. The quotient Q is a finitely presented ∇-module over $R^u_{K\langle \boldsymbol{x}\rangle^\dagger,\eta}$, thus again projective of strictly smaller rank, and we may apply the induction hypothesis to see that Q admits a strongly unipotent basis relative to $K\langle \boldsymbol{x}\rangle^\dagger$. This implies that M is free and unipotent relative to $K\langle \boldsymbol{x}\rangle^\dagger$, hence we may argue as in [Ked06a, Proposition 5.2.6] to show that it has a strongly unipotent basis relative to $K\langle \boldsymbol{x}\rangle^\dagger$. $\square$

Proof of Theorem 4.1. This is identical to the proof of [Ked06a, Proposition 5.4.1]. Let M be an projective ∇-module over $\mathscr{R}^u_{K\langle \boldsymbol{x}\rangle^\dagger}$ with unipotent generic fibre. Then for η close enough to 1, M comes from a projective ∇-module M_η over $\mathscr{R}^u_{K\langle \boldsymbol{x}\rangle^\dagger}\cap R^u_{K\langle \boldsymbol{x}\rangle^\dagger,\eta}$ such that $M_\eta\otimes(\mathscr{R}^u_L\cap R^u_{L,\eta})$ admits a strongly unipotent basis $\{\boldsymbol{e}_i\}$ relative to L. Hence $M_\eta\otimes R^u_{K\langle \boldsymbol{x}\rangle^\dagger,\eta}$ admits a strongly unipotent basis $\{\boldsymbol{f}_i\}$ relative to $K\langle \boldsymbol{x}\rangle^\dagger$ by Proposition 4.7. Moreover, these two bases must have the same L-span inside $M\otimes\mathscr{R}^u_L$. Hence $\{\boldsymbol{f}_i\}$ forms a basis of $M_\eta\cap(M\otimes\mathscr{R}^u_L)=M$. $\square$

For us, the most important consequence of this result is the following.

Corollary 4.8. *Let M be a projective, overconvergent ∇-module over $\mathscr{R}^u_{K\langle \boldsymbol{x}\rangle^\dagger}$ with constant irregularity, such that the generic fibre $M\otimes\mathscr{R}^u_L$ has rational exponents with denominators coprime to p. Then $H^0_{\nabla_u}(M)$ is finitely generated over $K\langle \boldsymbol{x}\rangle^\dagger$, and for any closed point $s:K\langle \boldsymbol{x}\rangle^\dagger\to K'$ the base change map*

$$H^0_{\nabla_u}(M)\otimes_{K\langle \boldsymbol{x}\rangle^\dagger}K'\to H^0_{\nabla_u}(M\otimes_{\mathscr{R}^u_{K\langle \boldsymbol{x}\rangle^\dagger}}\mathscr{R}^u_{K'})$$

is an isomorphism.

Proof. After possibly enlarging K and translating we may assume that $s=0$. If K contains all nth roots of unity, and n is coprime to p, then we have an identification

$$H^0_{\nabla_u}(M)=H^0_{\nabla_{u^{1/n}}}\left(M\otimes_{\mathscr{R}^u_{K\langle \boldsymbol{x}\rangle^\dagger}}\mathscr{R}^{u^{1/n}}_{K\langle \boldsymbol{x}\rangle^\dagger}\right)^{\mathbb{Z}/n\mathbb{Z}},$$

and similarly after base change via s. We are therefore free to make such a tamely ramified base change at any point we wish.

Write $M_L=M\otimes_{\mathscr{R}^u_{K\langle \boldsymbol{x}\rangle^\dagger}}\mathscr{R}^u_L$, and let $M_s=M\otimes_{\mathscr{R}^u_{K\langle \boldsymbol{x}\rangle^\dagger}}\mathscr{R}_K$ be the fibre over s. Let M_0 denote the break 0 part of M provided by Proposition 2.11, thus the fibre $M_{0,s}$ over s is the break 0 part of M_s. Since $M_{0,L}$ has rational exponents with denominators coprime to p, we may therefore by [Meb02, Théorème 1.3-1] take a tamely ramified base change $\mathscr{R}^u_L\to\mathscr{R}^{u^{1/n}}_L$ such that $M_{0,L}\otimes_{\mathscr{R}^u_L}\mathscr{R}^{u^{1/n}}_L$ is unipotent relative to L as a ∇-module over $\mathscr{R}^{u^{1/n}}_L$. Since such a base extension preserves the break 0 part we may therefore assume that $M_{0,L}$ is unipotent. But now since $M^{\nabla_u=0} = M_0^{\nabla_u=0}$ and $M_s^{\nabla_u=0} = M_{0,s}^{\nabla_u=0}$ we may moreover replace M by M_0 and therefore assume that M_L itself is unipotent relative to L.

Hence by Theorem 4.1 above, M is in fact unipotent relative to $K\langle \boldsymbol{x}\rangle^\dagger$, that is it is an iterated extension of ∇-modules pulled back from $K\langle \boldsymbol{x}\rangle^\dagger$. For any such pullback module, the base change claim is easily deduced using the projection formula, and the relatively unipotent case then follows by induction on the rank and the five lemma. □

5. Base change for $\mathbf{R}^0 f_*$

We can now use the results of the previous section to prove the $\mathbf{R}^0 f_* M$ case of Theorem 3.7. Let the setup and notation be as in §3.1, thus we have a good affine curve $f : U \to S$ with good compactification $\bar{f} : C \to S$, and

$$\bar{f} : (C, \overline{C}, \mathfrak{C}) \to (S, \overline{S}, \mathfrak{S})$$

a morphism of frames extending $\bar{f}$ as in Lemma 3.4. This induces a morphism $A \to B$ of the corresponding K-dagger algebras, and we let M be a ∇-module on B obtained as the realisation of an overconvergent isocrystal E on U/K. The main result of this section is the following.

Theorem 5.1. *Assume that M is F-able and has constant total irregularity $\mathrm{Irr}_M^{\mathrm{tot}}$. Then $\mathbf{R}^0 f_* M$ is finitely generated over A, and for any closed point $s : A \to K'$ the base change map*

$$\mathbf{R}^0 f_* M \otimes_A K' \to H^0(M_s)$$

is an isomorphism.

Remark 5.2. It follows from the theorem that in fact formation of $\mathbf{R}^0 f_* M$ commutes with arbitrary base change $A \to A'$ of MW-type K-dagger algebras. As in Remark 3.6, it then follows that any Frobenius structure on M induces one on $\mathbf{R}^0 f_* M$.

In fact, showing that $\mathbf{R}^0 f_* M$ is finitely generated is relatively straightforward, and does not depends on M having constant irregularity. The base change claim is much harder, and is false without this assumption. Our first reduction in the proof of Theorem 5.1 is to show that the claim is local on A.

Lemma 5.3. *Hypothesis as in Theorem 5.1. Let $\{A \to A_i\}_{i\in I}$ be a finite dagger open cover of A, and set $A_{ij} = A_i \otimes_A^\dagger A_j$. Write $M_{A_i} = M \otimes_A A_i$ and $M_{A_{ij}} = M \otimes_A A_{ij}$. If the conclusions of Theorem 5.1 hold for each M_{A_i} and each $M_{A_{ij}}$, then they hold for M.*

Proof. By assumption, formation of $\mathbf{R}^0 f_* M_{A_i}$ and $\mathbf{R}^0 f_* M_{A_{ij}}$ commute with base change to closed points of A_i and A_{ij} respectively. It follows that formation of $\mathbf{R}^0 f_* M_{A_i}$ commutes with base change along $A_i \to A_{ij}$. The dagger version of Tate's acyclicity theorem [GK00, Proposition 2.6] now gives the existence of a unique finite projective A-module N whose base change to each A_i is exactly $\mathbf{R}^0 f_* M_{A_i}$, and the fact that

$$\mathbf{R}^0 f_* M = \bigcap_i \mathbf{R}^0 f_* M_{A_i}$$

(intersection inside $\mathbf{R}^0 f_* M_L$) implies that $\mathbf{R}^0 f_* M$ is canonically isomorphic to N. Hence formation of $\mathbf{R}^0 f_* M$ commutes with each base change $A \to A_i$, and therefore to all closed points of A. □

One also easily checks that if $A \to A'$ is a finite étale morphism of MW-type K-dagger algebras, and the conclusions of Theorem 5.1 hold for $M_{A'}$, then they also hold for M_A. Hence applying Lemma 3.3

together with Lemma 1.8 we can reduce to the case when $f: U \to S$ is simple, and $\bar{f}: C \to S$ is a simple compactification.

Further localising on S, we may assume that S admits a finite étale map to $\mathbb{A}^d_k$. Using Lemma 1.8 and the fact that modifications do not change anything on the level of dagger algebras, we may therefore assume that there exists an étale, proper morphism of frames

$$g : (S, \overline{S}, \mathfrak{S}) \to (\mathbb{A}^d_k, \mathbb{P}^d_k, \widehat{\mathbb{P}}^d_{\mathcal{V}}),$$

which induces a finite étale map $K\langle \boldsymbol{x} \rangle^\dagger \to A$. Corresponding to this we have a finite étale map $\mathscr{R}^{u_i}_{K\langle \boldsymbol{x} \rangle^\dagger} \to \mathscr{R}^{u_i}_A$ for each i.

Lemma 5.4. *If $\mathrm{Irr}^{\mathrm{tot}}_M$ is constant, then each $M \otimes_B \mathscr{R}^{u_i}_A$ has constant irregularity.*

Proof. If $M \otimes_B \mathscr{R}^{u_i}_A$ does not have constant irregularity, then by Proposition 2.10 there exists a point of $\mathscr{M}(A)$ at which the irregularity of $M \otimes_B \mathscr{R}^{u_i}_A$ is strictly smaller than the irregularity at the maximal point. Since all the other irregularities cannot increase under such a specialisation, it follows that $\mathrm{Irr}^{\mathrm{tot}}_M$ cannot be constant. □

Consider $M \otimes_B \mathscr{R}^{u_i}_A$, as a ∇-module over $\mathscr{R}^{u}_{K\langle \boldsymbol{x} \rangle^\dagger}$. By Lemma 2.7 this has constant irregularity, and since M is F-able we can apply Corollary 4.8 to deduce that $\mathbf{R}^0_{\mathrm{loc}} f_* M$ is finitely generated over $K\langle \boldsymbol{x} \rangle^\dagger$, and its formation commutes with base change to closed points of $K\langle \boldsymbol{x} \rangle^\dagger$. Hence $\mathbf{R}^0_{\mathrm{loc}} f_* M$ is finitely generated over A, and for every closed point $s : A \to K'$, the base change map

$$\mathbf{R}^0_{\mathrm{loc}} f_* M \otimes_A K' \to H^0_{\mathrm{loc}}(M_s)$$

is an isomorphism. The fact that $\mathbf{R}^0 f_* M$ is finitely generated (and thus projective) over A now follows from the injection

$$\mathbf{R}^0 f_* M \hookrightarrow \mathbf{R}^0_{\mathrm{loc}} f_* M.$$

(There is in fact an easier way of seeing this, using [Ked06a, Lemma 7.3.4] but we will still need the full force of Corollary 4.8 anyway). It remains to show the base change claim, and the key remaining input is the following.

Lemma 5.5. *Assume that A admits a finite étale morphism $K\langle \boldsymbol{x} \rangle^\dagger \to A$. Then the direct image with compact support $\mathbf{R}^1 f_! M$ is finitely generated projective over A, and for all closed points $s : A \to K'$ the base change map*

$$(\mathbf{R}^1 f_! M) \otimes_A K' \to H^1_c(M_s)$$

is injective.

Proof. To see that $\mathbf{R}^1 f_! M$ is finitely generated, we use projectivity of M to embed $M \otimes \mathscr{Q}^{\{\sigma_i\}}$ inside a number of copies of $\mathscr{Q}^{\{\sigma_i\}}$, and then apply [Ked06a, Lemma 7.3.4]. Since it has a natural ∇-module structure it is thus projective.

For the base change claim, we may assume that $K = K'$. By translating we may assume that s maps to the origin under the given finite étale morphism $K\langle \boldsymbol{x} \rangle^\dagger \to A$. For $1 \le i \le d$ let $A_i = A/(x_1, \ldots, x_i)$; since A_d is an étale K-algebra, it in fact suffices to show that the base change map

$$(\mathbf{R}^1 f_! M_A) \otimes_A A_d \to \mathbf{R}^1 f_! M_{A_d}$$

is injective. As we have already shown that each $\mathbf{R}^1 f_! M_{A_i}$ is finite projective (and hence flat), it suffices to show that for all i the base change map

$$(\mathbf{R}^1 f_! M_{A_i}) \otimes_{A_i} A_{i+1} \to \mathbf{R}^1 f_! M_{A_{i+1}}$$

is injective. By induction on d, then, what we must show is that

$$(\mathbf{R}^1 f_! M) \otimes_A A_1 \to \mathbf{R}^1 f_! M_{A_1}$$

is injective. One easily checks that the sequence

$$0 \to x_1 \mathscr{Q}_A^{\{\sigma_i\}} \to \mathscr{Q}_A^{\{\sigma_i\}} \to \mathscr{Q}_{A_1}^{\{\sigma_i\}} \to 0$$

is exact, for example, by using the corresponding fact for each $\mathscr{R}_A^{u_i}$ and applying the nine lemma. Since M is finite projective and $\ker \nabla$ is left exact, we deduce that

$$0 \to x_1 \mathbf{R}^1 f_! M \to \mathbf{R}^1 f_! M \to \mathbf{R}^1 f_! M_{A_1}$$

is exact. This finishes the proof. $\square$

We can now complete the proof of Theorem 5.1. Consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & (\mathbf{R}^0 f_* M) \otimes_{K\langle \boldsymbol{x} \rangle^\dagger} K' \arrow[r] \arrow[d] & (\mathbf{R}^0_{\mathrm{loc}} f_* M) \otimes_{K\langle \boldsymbol{x} \rangle^\dagger} K' \arrow[r] \arrow[d] & (\mathbf{R}^1 f_! M) \otimes_{K\langle \boldsymbol{x} \rangle^\dagger} K' \arrow[r] \arrow[d] & {} \\ 0 \arrow[r] & H^0(M_s) \arrow[r] & H^0_{\mathrm{loc}}(M_s) \arrow[r] & H^1_c(M_s) \arrow[r] & {} \end{tikzcd}$$

which has exact rows, the top row being exact because $\mathbf{R}^0 f_* M$, $\mathbf{R}^0_{\mathrm{loc}} f_* M$ and $\mathbf{R}^1 f_! M$ are all finite projective over A. We already observed (using 4.8) that the middle vertical map is an isomorphism, and the right hand vertical arrow is injective by Lemma 5.5. The left hand vertical map is therefore an isomorphism by the five lemma.

6. The strong fibration theorem and the cohomology of punctured tubes

To prove Theorem 3.7 for $\mathbf{R}^0 f_* M$ we more or less worked completely algebraically, that is, within the language of dagger algebras. In order to deal with the $\mathbf{R}^1 f_* M$ case we will need to do a little more geometry. In this section, we will only consider the case when the base frame

$$(S, \overline{S}, \mathfrak{S}) = (\mathbb{A}^d_k, \mathbb{P}^d_k, \widehat{\mathbb{P}}^d_{\mathscr{V}})$$

is the natural MW frame enclosing affine space over k. Fix a smooth, proper, Cartesian morphism of frames

$$\bar{f} : (C, \overline{C}, \mathfrak{C}) \to (S, \overline{S}, \mathfrak{S})$$

such that $C \to S$ is a good compactification of a good affine curve $f : U \to S$. Let $A \to B$ be the induced morphism of dagger algebras (thus $A = K\langle x_1, \ldots, x_d \rangle^\dagger$), and M a ∇-module over B arising from an overconvergent isocrystal on U/K. We will assume that M is F-able. Let

$$\mathfrak{S}_0 = \widehat{\mathbb{P}}^{d-1}_{\mathscr{V}} \subset \widehat{\mathbb{P}}^d_{\mathscr{V}}$$

be the hyperplane given in affine co-ordinates by $x_d = 0$, and denote fibre product with $\mathfrak{S}_0$ over $\mathfrak{S}$ by $(-)_0$. Thus we have varieties $S_0, \overline{S}_0, U_0, C_0, \overline{C}_0$ and a formal scheme $\mathfrak{C}_0$. Set

$$\mathscr{R}^+_{(S_0, \mathfrak{S})} = \Gamma(]\overline{S}_0[_{\mathfrak{S}}, j^\dagger_{S_0} \mathscr{O}_{]\overline{S}_0[_{\mathfrak{S}}}),$$

we thus have an identification

$$\mathscr{R}^+_{(S_0,\mathfrak{S})} = \mathscr{R}^{x_d+}_{K\langle x_1,\ldots,x_{d-1}\rangle^\dagger}.$$

We can also define

$$\mathscr{R}_{(S_0,\mathfrak{S})} := \mathrm{colim}_V \Gamma(V \cap]\overline{S}_0[_{\mathfrak{S}}, j^\dagger_{S_0}\mathscr{O}_{]\overline{S}_0[_{\mathfrak{S}}})$$

where the colimit is over all strict neighbourhoods V of $]\overline{S}\setminus\overline{S}_0[_{\mathfrak{S}}$ inside $\mathfrak{S}_K$ (equivalently: over all strict neighbourhoods of $]\overline{S}\setminus S_0[_{\mathfrak{S}}$, or of $]S\setminus S_0[_{\mathfrak{S}}$). Again, we have an identification

$$\mathscr{R}_{(S_0,\mathfrak{S})} = \mathscr{R}^{x_d}_{K\langle x_1,\ldots,x_{d-1}\rangle^\dagger}.$$

We similarly define

$$\mathscr{R}^+_{(U_0,\mathfrak{C})} = \Gamma(]\overline{C}_0[_{\mathfrak{C}}, j^\dagger_{U_0}\mathscr{O}_{]\overline{C}_0[_{\mathfrak{C}}})$$

and

$$\mathscr{R}_{(U_0,\mathfrak{C})} = \mathrm{colim}_V \Gamma(V \cap]\overline{C}_0[_{\mathfrak{C}}, j^\dagger_{U_0}\mathscr{O}_{]\overline{C}_0[_{\mathfrak{C}}}),$$

the colimit this time being over all strict neighbourhoods V of $]\overline{C}\setminus\overline{C}_0[_{\mathfrak{C}}$ inside $\mathfrak{C}$ (equivalently: all strict neighbourhoods of $]\overline{C}\setminus U_0[_{\mathfrak{S}}$ or of $]U\setminus U_0[_{\mathfrak{C}}$). Since the closed immersion $U_0 \to U$ may no longer admit a smooth retraction (even locally on U_0), we cannot necessarily identify these rings with ordinary relative Robba rings. The point of this section will be to use the strong fibration theorem to show that nevertheless we can do so cohomologically. We will first need the analogue of "Theorem B" for coherent $j^\dagger$-modules on $]\overline{C}_0[_{\mathfrak{C}}$.

Proposition 6.1. *Let E be a coherent $j^\dagger_{U_0}\mathscr{O}_{]\overline{C}_0[_{\mathfrak{C}}}$-module on $]\overline{C}_0[_{\mathfrak{C}}$. Then we have*

$$H^i(]\overline{C}_0[_{\mathfrak{C}}, E) = 0$$

for all $i > 0$. Similarly, we have

$$\mathrm{colim}_V H^i(V \cap]\overline{C}_0[_{\mathfrak{C}}, E) = 0$$

where the colimit is over all strict neighbourhoods V of $]\overline{C}\setminus\overline{C}_0[$ inside $\mathfrak{C}_K$.

Proof. To show that $H^i(]\overline{C}_0[_{\mathfrak{C}}, E) = 0$, we note that the embedding

$$]U_0[_{\mathfrak{C}} \hookrightarrow]\overline{C}_0[_{\mathfrak{C}}$$

is as an admissible open inside a partially proper analytic K-variety. Hence by [GK00, Theorem 2.27] we can put an overconvergent structure sheaf $\mathscr{O}^\dagger_{]U_0[_{\mathfrak{C}}}$ on the G-topological space underlying $]U_0[_{\mathfrak{C}}$ to form a dagger analytic space $]U_0[^\dagger_{\mathfrak{C}}$. There is a natural morphism of ringed spaces

$$\varphi :]U_0[^\dagger_{\mathfrak{C}} \to \left(]\overline{C}_0[_{\mathfrak{C}}, j^\dagger_{U_0}\mathscr{O}_{]\overline{C}_0[_{\mathfrak{C}}}\right)$$

and (φ^*, φ_*) are inverse equivalences on the categories of coherent sheaves. By choosing a projective embedding of $\mathfrak{C}$, we obtain a closed embedding

$$]U_0[^\dagger_{\mathfrak{C}} \hookrightarrow \mathbb{B}^{1,\circ}_K \times_K \mathbb{B}^{m,\dagger}_K$$

of the quasi-Stein dagger space $]U_0[^\dagger_{\mathfrak{C}}$ into a product of an open unit disc with a dagger closed polydisc (the open part coming from $|x_d| < 1$) and hence by [Bam16, Corollary 4.22] the higher cohomology of coherent sheaves on $]U_0[^\dagger_{\mathfrak{C}}$ vanishes. Moreover, this remains true on a basis of open subsets of $]U_0[^\dagger_{\mathfrak{C}}$, hence we can also see that $\mathbf{R}^i\varphi_*E = 0$ for any coherent sheaf E on $]U_0[^\dagger_{\mathfrak{C}}$ and any $i > 0$. This implies that

$$H^i(]\overline{C}_0[_{\mathfrak{C}}, E) = H^i(]\overline{C}_0[_{\mathfrak{C}}, \mathbf{R}\varphi_*\varphi^*E) = H^i(]U_0[^\dagger_{\mathfrak{C}}, \varphi^*E) = 0$$

for $i > 0$ as required. For the second claim, concerning the cohomology groups $H^i(]\overline{C}_0[_{\mathfrak{C}} \cap V, E)$, we can argue as follows. There exists a cofinal system of such strict neighbourhoods such that each $V \cap]\overline{C}_0[_{\mathfrak{C}}$ is partially proper (for example, we choose neighbourhoods defined by strict rather than weak inequalities). Again,

$$]U_0[_V :=]U_0[_{\mathfrak{C}} \cap V$$

then admits an overconvergent sheaf of rings $\mathscr{O}^\dagger_{]U_0[_V}$, with corresponding dagger analytic space $]U_0[^\dagger_V$. This dagger analytic space is quasi-Stein and admits a closed embedding into a product of open polydiscs and dagger closed polydiscs (the open part will now have dimension > 1 in general). There is a morphism

$$\varphi :]U_0[^\dagger_V \to \left( ]\overline{C}_0[_{\mathfrak{C}} \cap V, j^\dagger_{U_0}\mathscr{O}_{]\overline{C}_0[_{\mathfrak{C}}} \right)$$

of ringed spaces, and we can argue exactly as above to show that

$$H^i(]\overline{C}_0[_{\mathfrak{C}} \cap V, E) = H^i(]U_0[^\dagger_V, \varphi^*E) = 0$$

for any coherent $j^\dagger_{U_0}\mathscr{O}_{]\overline{C}_0[_{\mathfrak{C}}}$-module E. In other words, we have shown that $H^i(]\overline{C}_0[_{\mathfrak{C}} \cap V, E) = 0$ for a cofinal system of strict neighbourhoods V, and the result follows. □

From M, we obtain via base change ∇-modules $M \otimes_B \mathscr{R}^+_{(U_0,\mathfrak{C})}$ and $M \otimes_B \mathscr{R}_{(U_0,\mathfrak{C})}$, with cohomology groups

$$\mathbf{R}^i f_*(M \otimes \mathscr{R}^\#_{(U_0,\mathfrak{C})}) := H^i\left(M \otimes_B \mathscr{R}^\#_{(U_0,\mathfrak{C})} \to M \otimes_B \mathscr{R}^\#_{(U_0,\mathfrak{C})} \otimes_B \Omega^1_{B/A} \right)$$

over $\mathscr{R}^\#_{(S_0,\mathfrak{S})}$, for $i = 0,1$ and $\# \in \{+, \emptyset\}$. On the other hand, if we set $A_0 = A/(x_d) = K\langle x_1, \ldots, x_{d-1}\rangle^\dagger$ and $B_0 = B/(x_d)$, then base changing along $B \to B_0$ gives an overconvergent ∇-module M_0 over B_0. Further base changing to relative Robba rings, we can consider the cohomology groups

$$\mathbf{R}^i g_*(M_0 \otimes_{B_0} \mathscr{R}^{x_d\#}_{B_0}) = H^i\left(M_0 \otimes \mathscr{R}^{x_d\#}_{B_0} \to M_0 \otimes_{B_0} \mathscr{R}^{x_d\#}_{B_0} \otimes_{B_0} \Omega^1_{B_0/A_0} \right)$$

over $\mathscr{R}^{x_d\#}_{A_0}$ for $i = 0,1$ and $\# \in \{+, \emptyset\}$.

Theorem 6.2. *For $i = 0,1$ and $\# \in \{+, \emptyset\}$ there is an isomorphism*

$$\mathbf{R}^i f_*(M \otimes_B \mathscr{R}^\#_{(U_0,\mathfrak{C})}) \cong \mathbf{R}^i g_*(M_0 \otimes_{B_0} \mathscr{R}^{x_d\#}_{B_0})$$

of $\mathscr{R}^\#_{(S_0,\mathfrak{S})} = \mathscr{R}^{x_d\#}_{A_0}$-modules with connection.

Proof. Let $\mathfrak{D} = \mathfrak{C}_0 \times_{\mathscr{V}} \mathbb{P}^1_{\mathscr{V}}$. Then there exists a modification of frames $(U_0, \overline{C}'_0, \mathfrak{D}') \to (U_0, \overline{C}_0, \mathfrak{D})$ and a smooth proper morphism of frames

$$(U_0, C'_0, \mathfrak{D}') \to (S_0, \overline{S}_0, \mathfrak{S}).$$

extending the obvious rational map $\mathfrak{D} \dashrightarrow \mathfrak{S}$. We consider the fibre product diagram of frames

$$\begin{tikzcd} & (U_0, \overline{C}''_0, \mathfrak{C}\times_{\mathfrak{S}}\mathfrak{D}') \arrow[dl, "g'"'] \arrow[dr, "f'"] & & \\ (U_0,\overline{C}_0,\mathfrak{C}) \arrow[dr, "f"'] & & (U_0,\overline{C}'_0,\mathfrak{D}') \arrow[dl, "g"] \arrow[r] & (U_0,C_0,\mathfrak{D}) \\ & (S_0,\overline{S}_0,\mathfrak{S}). & & \end{tikzcd}$$

If we denote by E_0 the restriction of E to U_0, then E_0 has a realisations $E_{0,\mathfrak{C}}$ on $]\overline{C}_0[_{\mathfrak{C}}$, $E_{0,\mathfrak{D}}$ on $]\overline{C}_0[_{\mathfrak{D}}$, $E_{0,\mathfrak{D}'}$ on $]\overline{C}'_0[_{\mathfrak{D}'}$ and $E_{0,\mathfrak{C}\times_{\mathfrak{S}}\mathfrak{D}'}$ on $]\overline{C}''_0[_{\mathfrak{C}\times_{\mathfrak{S}}\mathfrak{D}'}$. Essentially by definition, we have identifications

$$\mathbf{R}^i f_*(M\otimes_B \mathscr{R}^+_{(U_0,\mathfrak{C})}) = H^i\left(\Gamma\left(]\overline{C}_0[_{\mathfrak{C}}, E_{0,\mathfrak{C}}\otimes_{\mathscr{O}_{\mathfrak{C}}}\Omega^\bullet_{\mathfrak{C}_K/\mathfrak{S}_K}\right)\right)$$

and

$$\mathbf{R}^i g_*(M_0\otimes_{B_0}\mathscr{R}^{x_d+}_{B_0}) = H^i\left(\Gamma\left(]\overline{C}_0[_{\mathfrak{D}}, E_{0,\mathfrak{D}}\otimes_{\mathscr{O}_{\mathfrak{D}}}\Omega^\bullet_{\mathfrak{D}_K/\mathfrak{S}_K}\right)\right).$$

The strong fibration theorem tells us that

$$\begin{aligned}\mathbf{R}\Gamma\left(]\overline{C}_0[_{\mathfrak{C}}, E_{0,\mathfrak{C}}\otimes_{\mathscr{O}_{\mathfrak{C}}}\Omega^*_{\mathfrak{C}_K/\mathfrak{S}_K}\right) &\cong \mathbf{R}\Gamma\left(]\overline{C}''_0[_{\mathfrak{C}\times_{\mathfrak{S}}\mathfrak{D}'}, E_{0,\mathfrak{C}\times_{\mathfrak{S}}\mathfrak{D}'}\otimes_{\mathscr{O}_{\mathfrak{C}\times_{\mathfrak{S}}\mathfrak{D}'}}\Omega^*_{\mathfrak{C}\times_{\mathfrak{S}}\mathfrak{D}'/\mathfrak{S}_K}\right)\\ &\cong \mathbf{R}\Gamma\left(]\overline{C}'_0[_{\mathfrak{D}'}, E_{0,\mathfrak{D}'}\otimes_{\mathscr{O}_{\mathfrak{D}'}}\Omega^*_{\mathfrak{D}'_K/\mathfrak{S}_K}\right)\cong \mathbf{R}\Gamma\left(]\overline{C}_0[_{\mathfrak{D}}, E_{0,\mathfrak{D}}\otimes_{\mathscr{O}_{\mathfrak{D}}}\Omega^*_{\mathfrak{D}_K/\mathfrak{S}_K}\right),\end{aligned}$$

thus to obtain the claim for $\# = +$ it suffices to apply Proposition 6.1 (as well as the obvious analogue for coherent $j^\dagger_{U_0}\mathscr{O}_{]\overline{C}_0[_{\mathfrak{D}}}$-modules).

To see the claim for $\# = \emptyset$, let V be a strict neighbourhood of $]\overline{C}\setminus\overline{C}_0[_{\mathfrak{C}}$ inside $\mathfrak{C}_K$. Then we have

$$\mathbf{R}g'_*\left(E_{0,\mathfrak{C}\times_{\mathfrak{S}}\mathfrak{D}'}\otimes_{\mathscr{O}_{\mathfrak{C}\times_{\mathfrak{S}}\mathfrak{D}'}}\Omega^*_{\mathfrak{C}_K\times_{\mathfrak{S}_K}\mathfrak{D}'_K/\mathfrak{S}_K}\right)\Big|_{V\cap]\overline{C}_0[_{\mathfrak{C}}}\cong\left(E_{0,\mathfrak{C}}\otimes_{\mathscr{O}_{\mathfrak{C}}}\Omega^*_{\mathfrak{C}_K/\mathfrak{S}_K}\right)\Big|_{V\cap]\overline{C}_0[_{\mathfrak{C}}}$$

again by the strong fibration theorem. Now applying Proposition 6.1 we can deduce that taking the colimit over all such V gives an isomorphism

$$\mathbf{R}^i f_*(M\otimes_B\mathscr{R}_{(U_0,\mathfrak{C})})\cong \mathrm{colim}_V\mathbf{R}^i\Gamma\left(g'^{-1}(V)\cap]\overline{C}''_0[_{\mathfrak{C}\times_{\mathfrak{S}}\mathfrak{D}'}, E_{0,\mathfrak{C}\times_{\mathfrak{S}}\mathfrak{D}'}\otimes_{\mathscr{O}_{\mathfrak{C}\times_{\mathfrak{S}}\mathfrak{D}'}}\Omega^*_{\mathfrak{C}_K\times_{\mathfrak{S}_K}\mathfrak{D}'_K/\mathfrak{S}_K}\right).$$

An entirely similar argument (and again using the strong fibration theorem to pass between $\mathfrak{D}$ and $\mathfrak{D}'$) allows us to write

$$\mathbf{R}^i g_*(M_0\otimes_{B_0}\mathscr{R}^{x_d}_{B_0})\cong \mathrm{colim}_W\mathbf{R}^i\Gamma\left(f'^{-1}(W)\cap]\overline{C}''_0[_{\mathfrak{C}\times_{\mathfrak{S}}\mathfrak{D}'}, E_{0,\mathfrak{C}\times_{\mathfrak{S}}\mathfrak{D}'}\otimes_{\mathscr{O}_{\mathfrak{C}\times_{\mathfrak{S}}\mathfrak{D}'}}\Omega^*_{\mathfrak{C}_K\times_{\mathfrak{S}_K}\mathfrak{D}'_K/\mathfrak{S}_K}\right)$$

where now the colimit is over all strict neighbourhoods W of $]\overline{C}'\setminus\overline{C}'_0[_{\mathfrak{D}'}$ inside $\mathfrak{D}'_K$. Thus to complete the proof, we need to show that the two different families

$$g'^{-1}(V)\cap]\overline{C}''_0[_{\mathfrak{C}\times_{\mathfrak{S}}\mathfrak{D}'}\quad\text{and}\quad f'^{-1}(W)\cap]\overline{C}''_0[_{\mathfrak{C}\times_{\mathfrak{S}}\mathfrak{D}'}$$

of open subsets of $]\overline{C}''_0[_{\mathfrak{C}\times_{\mathfrak{S}}\mathfrak{D}'}$ are cofinal with each other. But given such a V, we know that $f'(g'^{-1}(V))$ has to be a strict neighbourhood of $]\overline{C}'\setminus\overline{C}'_0[_{\mathfrak{D}'}$ inside $\mathfrak{D}'_K$, and

$$g'^{-1}(V)\cap]\overline{C}''_0[_{\mathfrak{C}\times_{\mathfrak{S}}\mathfrak{D}'}\subset f'^{-1}(f'(g'^{-1}(V)))\cap]\overline{C}''_0[_{\mathfrak{C}\times_{\mathfrak{S}}\mathfrak{D}'}$$

Reversing the roles of f' and g' we obtain the result we want. $\square$

We can also compare these with the higher direct images of M_0 along the morphism $f_0 : U_0 \to S_0$, in other words with the cohomology groups

$$\mathbf{R}^i f_{0*} M_0 := H^i\left(M_0 \to M_0 \otimes_{B_0} \Omega^1_{B_0/A_0}\right),$$

as follows. Let L_0 denote the completed fraction field of A_0.

Theorem 6.3. *Assume that M_0 satisfies the equivalent conditions of Theorem 3.7. Then the natural base change maps*

$$\left(\mathbf{R}^i f_{0*} M_0\right) \otimes_{A_0} \mathscr{R}^{x_d+}_{A_0} \to \mathbf{R}^i g_*\left(M_0 \otimes_{B_0} \mathscr{R}^{x_d+}_{B_0}\right)$$

$$\left(\mathbf{R}^i f_{0*} M_0\right) \otimes_{A_0} \mathscr{R}^{x_d}_{A_0} \to \mathbf{R}^i g_*\left(M_0 \otimes_{B_0} \mathscr{R}^{x_d}_{B_0}\right)$$

are isomorphisms.

Proof. We will give the proof for $M_0 \otimes_{B_0} \mathscr{R}^{x_d}_{B_0}$, the proof for $M_0 \otimes_{B_0} \mathscr{R}^{x_d+}_{B_0}$ being essentially the same. First of all, as usual, since the claim is straightforward for modules of the form $N \otimes_{A_0} B_0$, we can successively replace M_0 by the cokernel of

$$\mathbf{R}^0 f_{0*} M_0 \otimes_{A_0} B_0 \hookrightarrow M_0,$$

and reduce to consider the case when $\mathbf{R}^0 f_{0*} M_0 = 0$.

As a coherent B_0-module, we can put a family of partially defined norms $\|\cdot\|_\lambda$ on M_0, coming from affinoid norms on fringe algebras of B_0 arising from a fixed presentation $K\langle x_1, \ldots, x_n\rangle^\dagger \to B_0$, the same is true for $M_0 \otimes_{B_0} \Omega^1_{B_0/A_0}$. Concretely, we can then describe $M_0 \otimes_{B_0} \mathscr{R}^{x_d}_{B_0}$ as the set of formal series

$$\sum_i m_i x_d^i$$

with $m_i \in M_0$, satisfying the following convergence condition:

- there exists some $\eta < 1$, such that for all $\eta < \rho < 1$, there exists some λ such that $\|m_i\|_\lambda$ exists for all i and

$$\|m_i\|_\lambda \rho^i \to 0 \quad \text{as} \quad i \to \pm\infty.$$

We equip this with the obvious $\mathscr{R}^{x_d}_{B_0}$-module structure, there is of course an entirely similar description of $M_0 \otimes_{B_0} \mathscr{R}^{x_d}_{B_0} \otimes_{B_0} \Omega^1_{B_0/A_0}$. Let

$$\nabla : M_0 \to M_0 \otimes_{B_0} \Omega^1_{B_0/A_0}$$

be the A_0-linear connection on M_0.

Claim. The map ∇ is strict, and $\mathrm{im}(\nabla)$ admits a topological complement.

Proof of Claim. First of all, since $\mathbf{R}^1 f_{0*} M_0 \hookrightarrow \mathbf{R}^1 f_{0*} M_{0,L_0}$ and the latter is separated by [Ked06a, 8.4.5] we deduce that $\mathbf{R}^1 f_{0*} M_0$ is separated, hence ∇ has closed image. Now since $\mathbf{R}^1 f_{0*} M_0$ is free (as it is finitely generated projective over $K\langle x_1, \ldots, x_d\rangle^\dagger$) we can find a topological complement to $\mathrm{im}(\nabla)$ simply by lifting a basis of $\mathbf{R}^1 f_{0*} M_0$. □

Since ∇ is in particular continuous, if $\sum_i m_i x_d^i$ is a convergent series in $M_0 \otimes_{B_0} \mathscr{R}^{x_d}_{B_0}$ then so is $\sum_i \nabla(m_i) x_d^i$, and the map

$$\nabla : M_0 \otimes_{B_0} \mathscr{R}^{x_d}_{B_0} \to M_0 \otimes_{B_0} \mathscr{R}^{x_d}_{B_0} \otimes_{B_0} \Omega^1_{B_0/A_0}$$

is given by $\sum_i m_i x_d^i \mapsto \sum_i \nabla(m_i) x_d^i$. It is then clear that the kernel of ∇ on $M_0 \otimes_{B_0} \mathscr{R}^{x_d}_{B_0}$ is zero, which proves the base change claim for $\mathbf{R}^0 f_{0*} M_0$. To deal with the $\mathbf{R}^1 f_{0*} M_0$ case, choose elements $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ in

$M_0 \otimes_{B_0} \Omega^1_{B_0/A_0}$ lifting a basis of $\mathbf{R}^1 f_{0*} M_0$. Since $\mathbf{R}^0 f_* M_0 = 0$, every $m \in M_0 \otimes_{B_0} \Omega^1_{B_0/A_0}$ can be written *uniquely* as

$$m = \nabla(n) + \sum_j \alpha_j \boldsymbol{e}_j$$

for elements $n \in M_0$ and $\alpha_j \in A_0$. Thus given any $\sum_i m_i x_d^i \in M_0 \otimes_{B_0} \mathscr{R}^{x_d}_{B_0} \otimes_{B_0} \Omega^1_{B_0/A_0}$ we can write each $m_i = \nabla(n_i) + \sum_j \alpha_{ij} \boldsymbol{e}_j$, and since the $\boldsymbol{e}_j$ generate a topological complement to $\mathrm{im}(\nabla)$ inside $M_0 \otimes_{B_0} \Omega^1_{B_0/A_0}$, we can check that the sums

$$\sum_i n_i x_d^i, \quad \text{and} \quad \sum_i \alpha_{ij} x_d^i, \; 1 \le j \le n$$

converge in $M_0 \otimes_{B_0} \mathscr{R}^{x_d}_{B_0}$. Therefore we can write

$$\sum_i m_i x_d^i = \nabla\left(\sum_i n_i x_d^i\right) + \sum_j \left(\sum_i \alpha_{ij} x_d^i\right) \boldsymbol{e}_j,$$

for unique elements $\sum_i \alpha_{ij} x_d^i \in \mathscr{R}^{x_d}_{A_0}$ and $\sum_i n_i x_d^i \in M_0 \otimes_{B_0} \mathscr{R}^{x_d}_{B_0}$, and this implies that the $\boldsymbol{e}_i$ also form a basis for $\mathbf{R}^1 g_* \left(M_0 \otimes_{B_0} \mathscr{R}_{B_0}\right)$ as a $\mathscr{R}^{x_d}_{A_0}$-module. □

7. Base change for $\mathbf{R}^1 f_*$

We now have the necessary tools to prove the $\mathbf{R}^1 f_* M$ case of Theorem 3.7. Let the setup and notation be as in §3.1, thus we have a good affine curve $f : U \to S$ over a smooth, affine, connected base, $\bar{f} : C \to S$ a good compactification and

$$\bar{f} : (C, \overline{C}, \mathfrak{C}) \to (S, \overline{S}, \mathfrak{S})$$

a morphism of frames extending $\bar{f}$ as in Lemma 3.4. Let $A \to B$ be the associated morphism of K-dagger algebras, and E an overconvergent isocrystal on U/K with realisation M over B.

Theorem 7.1. *Assume that M is F-able, and has constant total irregularity $\mathrm{Irr}^{\mathrm{tot}}_M$. Then $\mathbf{R}^1 f_* M$ is finitely generated over A, and for any closed point $s : A \to K'$ the base change map*

$$\mathbf{R}^1 f_* M \otimes_A K' \to H^1(M_s)$$

is an isomorphism.

Remark 7.2. As in Remark 5.2, it follows that formation of $\mathbf{R}^1 f_* M$ commutes with arbitrary base change $A \to A'$ of MW-type K-dagger algebras, and any Frobenius structure on M induces one on $\mathbf{R}^1 f_* M$.

The key case to consider will be when we have $\mathbf{R}^0 f_* M = 0$.

Theorem 7.3. *Assume that M is F-able. Suppose that for all closed points $s : A \to K'$ we have*

$$H^0(M_s) = 0, \;\; \dim_{K'} H^1(M_s) = m$$

for some non-negative integer m, independent of s. Then $\mathbf{R}^1 f_* M$ is locally free of rank m, and for any closed point $s : A \to K'$ the base change map

$$\mathbf{R}^1 f_* M \otimes_A K' \to H^1(M_s)$$

is an isomorphism.

Theorem 7.3 ⇒ Theorem 7.1. Given a general M as in Theorem 7.1 we have by Theorem 5.1 an injection

$$\mathbf{R}^0 f_* M \otimes_A B \to M$$

of F-able ∇-modules. Since relatively constant ∇-modules have trivial irregularity, we deduce that the hypotheses of Theorem 7.1 remain true for the cokernel of this injection. Moreover, it follows easily from the projection formula that the conclusions of Theorem 7.1 hold for the ∇-module $\mathbf{R}^0 f_* M \otimes_A B$. Appealing to the five lemma and iterating, we may therefore reduce to considering the case when $\mathbf{R}^0 f_* M = 0$. Theorem 5.1 then implies that $H^0(M_s) = 0$ for all such s, and hence [CM01, Corollaire 5.0-12] implies that $\dim_{K'} H^1(M_s)$ is constant. Thus we may apply Theorem 7.3. □

The situation here is the opposite to the one we had in §5 - the hard part is showing finiteness of $\mathbf{R}^1 f_* M$, the base change claim will then follow relatively easily. To prove Theorem 7.3 we will proceed by induction on the dimension $d = \dim A$, the case $d = 0$ amounting to finiteness of rigid cohomology with coefficients for smooth curves [Ked06a, §6]. The main structure of the proof will be essentially geometric, working on the 'weak formal scheme' $\mathrm{Spf}(A^{\mathrm{int}})$. Theorem 3.5 tells us that $\mathbf{R}^1 f_* M$ becomes coherent on some open subspace $U \subset \mathrm{Spf}(A^{\mathrm{int}})$, and we will use constancy of $\dim_{K'} H^1$ together with the induction hypothesis to successively enlarge the open set U. A basic form of the argument we will use can be found in the proof of the following lemma, and was already used in the proof of Proposition 2.11.

Lemma 7.4. *Assumptions as in Theorem 7.3. For any dagger localisation $A \to A'$ the map*

$$\mathbf{R}^1 f_* M \to \mathbf{R}^1 f_* M_{A'}$$

is injective.

Proof. Write ∇ for the A-linear connection on M, and for any dagger localisation $A \to A''$, write $B_{A''} = B \otimes_A^\dagger A''$. We need to show that if $m \in M_{A'}$ is such that $\nabla(m) \in M \otimes_B \Omega^1_{B/A}$, then in fact $m \in M$. By the dagger analogue of Tate's acyclicity theorem, the question is local on A in the sense that it suffices to produce a dagger open cover $\{A \to A_i\}$ such that $m \in M_{A_i}$ for all i.

Suppose therefore that we have some collection of dagger localisations $\mathscr{C} = \{A \to A_i\}$ (not necessarily covering A) such that $m \in M_{A_i}$ for all i. A non-empty such $\mathscr{C}$ exists by hypothesis. Let $U_{\mathscr{C}}$ denote the union of the images of the induced open immersions $U_i \to S$ on the reductions. We shall show that if $U_{\mathscr{C}} \neq S$, then we can find another dagger localisation $A \to A''$ such that $m \in M_{A''}$, and adding A'' to $\mathscr{C}$ enlarges $U_{\mathscr{C}}$. The result will then follow by Noetherian induction.

Now, if $U_{\mathscr{C}} \neq S$, then after possibly enlarging K, which is harmless, we may choose a smooth k-rational point z on the (reduced) complement of $U_{\mathscr{C}}$ in S. Localising around z we may by [Ked05, Theorem 1] pick a finite étale map $(x_1, \ldots, x_d) : S \to \mathbb{A}^d_k$ such that $S \setminus U_{\mathscr{C}}$ maps into the hyperplane $\{x_d = 0\}$. It suffices to produce some A'' as above such that $\mathrm{Spec}\,(A''_0) \subset S$ contains z. Since modifications induce isomorphisms on the level of dagger algebras, we may apply Lemma 1.8 and replace $(S, \overline{S}, \mathfrak{S})$ by a modification in order to extend the given finite étale map $S \to \mathbb{A}^d_k$ to a proper, étale, Cartesian morphism of frames

$$(S, \overline{S}, \mathfrak{S}) \to (\mathbb{A}^d_k, \mathbb{P}^d_k, \widehat{\mathbb{P}}^d_{\mathscr{V}}).$$

Pushing forward along this morphism we may therefore assume that

$$(S, \overline{S}, \mathfrak{S}) = (\mathbb{A}^d_k, \mathbb{P}^d_k, \widehat{\mathbb{P}}^d_{\mathscr{V}})$$

and that $U_{\mathscr{C}} \supset \{x_d \neq 0\}$ as open subschemes of $\mathbb{A}^d_k$. Thus we have $A' = K\langle x_1, \ldots, x_d, x_d^{-1}\rangle^\dagger$, and we are now in a position to apply the results of §6 above; we will freely use the notations introduced there. We have a Cartesian diagram of rings

$$\begin{tikzcd} B \arrow[r] \arrow[d] & \mathscr{R}^+_{(U_0,\mathfrak{C})} \arrow[d] \\ B\langle x_d^{-1}\rangle^\dagger \arrow[r] & \mathscr{R}_{(U_0,\mathfrak{C})} \end{tikzcd}$$

and an F-able ∇-module M over B, with constant total irregularity. We need to show that the induced map

$$\mathbf{R}^1 f_* M \to \mathbf{R}^1 f_* \left(M_{A'}\right)$$

is injective. It therefore suffices to show that

$$\mathbf{R}^0 f_* \left(\frac{M_{A'}}{M}\right) = 0,$$

and since the above square is Cartesian (and M is projective), that

$$\mathbf{R}^0 f_* \left(M \otimes_B \frac{\mathscr{R}_{(U_0,\mathfrak{C})}}{\mathscr{R}^+_{(U_0,\mathfrak{C})}}\right) = 0.$$

By the induction hypothesis we know that M_0 satisfies both conditions of Theorem 3.7. Thus we may apply Theorems 6.2 and 6.3 to deduce that

$$\mathbf{R}^0 f_* \left(M \otimes_B \mathscr{R}_{(U_0,\mathfrak{C})}\right) = 0$$

and

$$\mathbf{R}^1 f_* \left(M \otimes_B \mathscr{R}^+_{(U_0,\mathfrak{C})}\right) \otimes_{\mathscr{R}^+_{(U_0,\mathfrak{C})}} \mathscr{R}_{(U_0,\mathfrak{C})} \overset{\sim}{\to} \mathbf{R}^1 f_* \left(M \otimes_B \mathscr{R}_{(U_0,\mathfrak{C})}\right),$$

in particular

$$\mathbf{R}^1 f_* \left(M \otimes_B \mathscr{R}^+_{(U_0,\mathfrak{C})}\right) \hookrightarrow \mathbf{R}^1 f_* \left(M \otimes_B \mathscr{R}_{(U_0,\mathfrak{C})}\right).$$

We can now use the long exact sequence associated to

$$0 \to M \otimes_B \mathscr{R}^+_{(U_0,\mathfrak{C})} \to M \otimes_B \mathscr{R}_{(U_0,\mathfrak{C})} \to M \otimes_B \frac{\mathscr{R}_{(U_0,\mathfrak{C})}}{\mathscr{R}^+_{(U_0,\mathfrak{C})}} \to 0$$

to conclude. □

Lemma 7.5. *Hypotheses as in Theorem 7.3. If there exists an open cover $\{A \to A_i\}$ such that the conclusions of Theorem 7.3 hold for each M_{A_i}, then they hold for M.*

Proof. Choose a dagger localisation $A \to A'$ such that the conclusions of Theorem 3.5 hold for $M_{A'}$. By shrinking A' we may assume that $A_i \to A'$ for all i, and hence that $A_{ij} : A_i \otimes^\dagger_A A_j \hookrightarrow A'$ for all i, j. By comparing with closed points of A', we therefore have base change isomorphisms

$$\mathbf{R}^1 f_* M_{A_i} \otimes_{A_i} A' \overset{\sim}{\to} \mathbf{R}^1 f_* M_{A'}$$

which we can use to embed $\mathbf{R}^1 f_* M_{A_i}$ and $\mathbf{R}^1 f_* M_{A_i} \otimes_{A_i} A_{ij}$ inside $\mathbf{R}^1 f_* M'$. Let N_{ij} denote the sum of $\mathbf{R}^1 f_* M_{A_i} \otimes_{A_i} A_{ij}$ and $\mathbf{R}^1 f_* M_{A_j} \otimes_{A_j} A_{ij}$ inside $\mathbf{R}^1 f_* M'$, this is therefore an overconvergent ∇-module over A_{ij}. Since

$$N_{ij} \hookrightarrow \mathbf{R}^1 f_* M_{A'}$$

we can deduce by applying [Ked07a, Proposition 5.3.1] to the kernel of

$$N_{ij} \otimes_{A_{ij}} A' \to \mathbf{R}^1 f_* M_{A'}$$

that in fact the map $N_{ij} \otimes_{A_{ij}} A' \to \mathbf{R}^1 f_* M_{A'}$ remains injective. We thus find that

$$\left(\mathbf{R}^1 f_* M_{A_i} \otimes_{A_i} A_{ij} + \mathbf{R}^1 f_* M_{A_j} \otimes_{A_j} A_{ij}\right) \otimes_{A_{ij}} B \xrightarrow{\sim} \mathbf{R}^1 f_* M_B,$$

and in particular, the cokernel of the natural map

$$\mathbf{R}^1 f_* M_{A_i} \otimes_{A_i} A_{ij} \to \mathbf{R}^1 f_* M_{A_i} \otimes_{A_i} A_{ij} + \mathbf{R}^1 f_* M_{A_j} \otimes_{A_j} A_{ij}$$

has to vanish after applying $- \otimes_{A_{ij}} B$. Since this cokernel is an overconvergent ∇-module, it must already be zero over A_{ij}, and thus we deduce that

$$\mathbf{R}^1 f_* M_{A_i} \otimes_{A_i} A_{ij} = \mathbf{R}^1 f_* M_{A_j} \otimes_{A_j} A_{ij}$$

inside $\mathbf{R}^1 f_* M$. Hence by descent for coherent sheaves on dagger spaces, the intersection $N = \bigcap_i \mathbf{R}^1 f_* M_{A_i}$ of all the $\mathbf{R}^1 f_* M_{A_i}$ inside $\mathbf{R}^1 f_* M$ is a locally free A-module of rank m whose base change to A_i is exactly $\mathbf{R}^1 f_* M_{A_i}$. By Lemma 7.4 above, $\mathbf{R}^1 f_* M$ naturally embeds into N: it is thus finite, and hence a ∇-module, locally free of rank $r \leq m$. However, since the base change map

$$\mathbf{R}^1 f_* M_A \otimes_A K' \to H^1(M_s)$$

associated to any closed point s is surjective (which is immediate from the definitions), we deduce that in fact $r = m$, $\mathbf{R}^1 f_* M = N$, and the base change map along closed points $A \to K'$, as well as to each A_i, is an isomorphism. □

We need one more lemma before we can prove Theorem 7.3, which in a way is the key result making the whole approach work.

Lemma 7.6. *Let N_1, N_2 be finite projective modules of rank m over $K\langle x_1, \ldots, x_d, x_d^{-1}\rangle^\dagger$ and $\mathscr{R}^{x_d+}_{K\langle x_1,\ldots,x_{d-1}\rangle^\dagger}$ respectively, and let*

$$\alpha : N_1 \otimes_{K\langle x_1,\ldots,x_d,x_d^{-1}\rangle^\dagger} \mathscr{R}^{x_d}_{K\langle x_1,\ldots,x_{d-1}\rangle^\dagger} \xrightarrow{\sim} N_2 \otimes_{\mathscr{R}^{x_d+}_{K\langle x_1,\ldots,x_{d-1}\rangle^\dagger}} \mathscr{R}^{x_d}_{K\langle x_1,\ldots,x_{d-1}\rangle^\dagger}$$

be an $\mathscr{R}^{x_d}_{K\langle x_1,\ldots,x_{d-1}\rangle^\dagger}$-linear isomorphism. Then the intersection of N_1 and $\alpha^{-1}(N_2)$ inside $N_1 \otimes_{K\langle x_1,\ldots,x_d,x_d^{-1}\rangle^\dagger} \mathscr{R}^{x_d}_{K\langle x_1,\ldots,x_{d-1}\rangle^\dagger}$ is a finite projective $K\langle x_1, \ldots, x_d\rangle^\dagger$-module of rank m.

Remark 7.7. It is in order to be able to apply this key lemma that makes it vital to work with overconvergent, rather than just convergent, relative cohomology groups.

Proof. First choose $\lambda > 1$ close enough to 1 that there exists a locally free sheaf $\mathscr{E}_\lambda$ on the rigid analytic space

$$U_\lambda := \left\{ |x_i| \leq \lambda,\ |x_d| \geq \lambda^{-1} \right\}$$

whose module of global sections tensored with $K\langle x_1, \ldots, x_d, x_d^{-1}\rangle^\dagger$ is precisely N_1. Next choose $\lambda^{-1} < \rho < 1$ and $1 < \eta_\rho < \lambda$ close enough to 1 such that there exists a locally free sheaf $\mathscr{E}_\rho$ on

$$U_\rho := \left\{ |x_i| \leq \eta_\rho,\ |x_d| \leq \rho \right\}$$

whose module of global sections tensored with $\cup_{\eta>1} K\langle \eta^{-1}x_1, \ldots, \eta^{-1}x_{d-1}, \rho^{-1}x_d\rangle$ coincides with $N_2 \otimes \cup_{\eta>1} K\langle \eta^{-1}x_1, \ldots, \eta^{-1}x_{d-1}, \rho^{-1}x_d\rangle$. After possibly increasing λ, the isomorphism α is defined over

$$\cap_{\lambda^{-1} \leq \rho' < 1} \cup_{\eta>1} K\langle \eta^{-1}x_1, \ldots, \eta^{-1}x_{d-1}, \lambda x_d^{-1}, \rho'^{-1}x_d\rangle$$

and hence (after possibly decreasing η_ρ) induces an isomorphism

$$\mathscr{E}_\lambda|_{U_\lambda\cap U_\rho} \xrightarrow{\sim} \mathscr{E}_\rho|_{U_\lambda\cap U_\rho}$$

of locally free sheaves on

$$U_\lambda \cap U_\rho = \left\{ |x_i| \le \eta_\rho,\ \lambda^{-1} \le |x_d| \le \rho \right\}.$$

Thus $\mathscr{E}_\lambda$ and $\mathscr{E}_\rho$ glue to give a locally free sheaf $\mathscr{E}$ on

$$U_\lambda \cup U_\rho = \{ |x_i| \le \lambda \}.$$

Set $N = \Gamma(U_\lambda \cup U_\rho, \mathscr{E}) \otimes K\langle x_1,\ldots,x_d\rangle^\dagger$. This is a then a finite projective (and therefore free) module over $K\langle x_1,\ldots,x_d\rangle^\dagger$ such that $N_1 = N\otimes K\langle x_1,\ldots,x_d,x_d^{-1}\rangle^\dagger$ and $N_2 = N\otimes \mathscr{R}^{x_d+}_{K\langle x_1,\ldots,x_{d-1}\rangle^\dagger}$. The result then follows from the fact that the digram

$$\begin{tikzcd}
K\langle x_1,\ldots,x_d\rangle^\dagger \arrow[r] \arrow[d] & \mathscr{R}^{x_d+}_{K\langle x_1,\ldots,x_{d-1}\rangle^\dagger} \arrow[d] \\
K\langle x_1,\ldots,x_d,x_d^1\rangle^\dagger \arrow[r] & \mathscr{R}^{x_d}_{K\langle x_1,\ldots,x_{d-1}\rangle^\dagger}
\end{tikzcd}$$

of rings is Cartesian. □

Proof of Theorem 7.3. The claim we are trying to prove, i.e. finite generation of $\mathbf{R}^1 f_* M$ and commutation with base change to closed points, is local on A by Lemma 7.5, and we can now argue entirely similarly to the proof of Lemma 7.4 above.

In other words, we know by Theorem 3.5 and Lemma 3.9 that after making a localisation $A \to A'$ the higher direct image $\mathbf{R}^1 f_* M$ becomes locally free and commutes with base change to closed points. By extending K and using Noetherian induction, it suffices to show that the same also holds over some dagger localisation of A containing the residue disc of a given smooth rational point of the complement $S\setminus \mathrm{Spec}\left(\overline{A}'\right)$.

Localising around this point, applying [Ked05, Theorem 1] and lifting, and using Lemma 7.5 above, we can reduce to the case when $A = K\langle x_1,\ldots,x_d\rangle^\dagger$ and $A' = K\langle x_1,\ldots,x_d,x_d^{-1}\rangle^\dagger$. Again, we are now in a position to apply the results from §6 above, and we will freely use the notation introduced there. Consider the commutative diagram

$$\begin{tikzcd}
\mathbf{R}^1 f_* M \arrow[r] \arrow[d] & \mathbf{R}^1 f_*\left(M\otimes_B \mathscr{R}^+_{(U_0,\mathfrak{C})}\right) \arrow[d] \\
\mathbf{R}^1 f_* M_{A'} \arrow[r] & \mathbf{R}^1 f_*\left(M\otimes_B \mathscr{R}_{(U_0,\mathfrak{C})}\right).
\end{tikzcd}$$

The induction hypothesis together with Theorems 6.2 and 6.3 imply that

$$\mathbf{R}^1 f_*\left(M\otimes_B \mathscr{R}^+_{(U_0,\mathfrak{C})}\right) \text{ and } \mathbf{R}^1 f_*\left(M\otimes_B \mathscr{R}_{(U_0,\mathfrak{C})}\right)$$

are finite projective of rank m over $\mathscr{R}^{x_d+}_{K\langle x_1,\ldots,x_{d-1}\rangle^\dagger}$ and $\mathscr{R}^{x_d}_{K\langle x_1,\ldots,x_{d-1}\rangle^\dagger}$ respectively, and that the base change map

$$\mathbf{R}^1 f_*\left(M\otimes_B \mathscr{R}^+_{(U_0,\mathfrak{C})}\right) \otimes_{\mathscr{R}^{x_d+}_{K\langle x_1,\ldots,x_{d-1}\rangle^\dagger}} \mathscr{R}^{x_d}_{K\langle x_1,\ldots,x_{d-1}\rangle^\dagger} \to \mathbf{R}^1 f_*\left(M\otimes_B \mathscr{R}_{(U_0,\mathfrak{C})}\right)$$

is an isomorphism. We claim that the base change map

$$\mathbf{R}^1 f_* M_{A'} \otimes_{A'} \mathscr{R}^{x_d}_{K\langle x_1,\ldots,x_{d-1}\rangle^\dagger} \to \mathbf{R}^1 f_*\left(M\otimes \mathscr{R}_{(U_0,\mathfrak{C})}\right)$$

is an isomorphism. Indeed, since both sides are projective of the same rank, it suffices to prove that it is surjective. To see this surjectivity, first note that since the map

$$\mathbf{R}^1 f_* M \to \mathbf{R}^1 f_{0*} M_0$$

is surjective, the image of

$$\mathbf{R}^1 f_* M \otimes_A \mathscr{R}^{x_d+}_{K\langle x_1,\ldots,x_{d-1}\rangle^\dagger} \to \mathbf{R}^1 f_* \left(M \otimes_B \mathscr{R}^{+}_{(U_0,\mathfrak{C})} \right)$$

surjects onto $\mathbf{R}^1 f_{0*} M_0$ via the natural quotient map $x_d \mapsto 0$. Since ∇-modules are rigid, it follows that in fact

$$\mathbf{R}^1 f_* M \otimes_A \mathscr{R}^{x_d+}_{K\langle x_1,\ldots,x_{d-1}\rangle^\dagger} \to \mathbf{R}^1 f_* \left(M \otimes_B \mathscr{R}^{+}_{(U_0,\mathfrak{C})} \right)$$

is surjective, hence so is

$$\mathbf{R}^1 f_* M_{A'} \otimes_{A'} \mathscr{R}^{x_d}_{K\langle x_1,\ldots,x_{d-1}\rangle^\dagger} \to \mathbf{R}^1 f_* \left(M \otimes_B \mathscr{R}_{(U_0,\mathfrak{C})} \right)$$

simply by some diagram chasing. Finally, by Lemmas 7.4 and 7.6 we can deduce that $\mathbf{R}^1 f_* M$ is contained in finite projective module of rank m, it is therefore finite over $K\langle x_1,\ldots,x_d\rangle^\dagger$, and, since it admits an integrable connection, projective of rank $\leq m$. Since the base change map

$$\mathbf{R}^1 f_* M \otimes_A K' \to H^1(M_s)$$

to each closed point of A is surjective, we can conclude in fact $\mathbf{R}^1 f_* M$ is in fact of rank $= m$ and that every such base change map is an isomorphism. $\square$

Applying Remarks 5.2 and 7.2, this completes the proof of Theorem 3.7.

8. Partially overconvergent cohomology

Theorem 3.7 is a statement concerning relative Monsky–Washnitzer cohomology, and as such only applies when the base variety S is smooth and affine. In order to be able to deduce a statement for arbitrary bases S, we will need to 'complete along the base' A and prove a similar result for partially (vertically) overconvergent cohomology. We will consider the general setup as in 3.1. Thus $f: U \to S$ is a good curve over a smooth, affine, connected base, and $A \to B$ is a map of MW-type K-dagger algebras lifting f, coming from a morphism of frames

$$(C, \overline{C}, \mathfrak{C}) \to (S, \overline{S}, \mathfrak{S})$$

as in Lemma 3.4. We will let M be a ∇-module over B, obtained as the realisation of an overconvergent isocrystal on U/K. Write

$$A = \mathrm{colim}_\lambda A_\lambda \quad \text{and} \quad B = \mathrm{colim}_\lambda B_\lambda$$

as colimits of smooth affinoid algebras over K, such that $A_\lambda \to B_\lambda$ for all λ. Let $\widehat{A}$ denote the completion of A with respect to the supremum norm, and set

$$B_{\widehat{A}} := \mathrm{colim}_\lambda \widehat{A} \widehat{\otimes}_{A_\lambda} B_\lambda,$$

this is 'relative dagger algebra' over $\widehat{A}$. Set $M_{\widehat{A}} = M \otimes_B B_{\widehat{A}}$ and define $\mathbf{R}^i f_* M_{\widehat{A}}$ to be the cohomology groups of the complex

$$M_{\widehat{A}} \xrightarrow{\nabla} M_{\widehat{A}} \otimes_B \Omega^1_{B/A}.$$

Our base change result is then the following.

Theorem 8.1. *Assume that M is F-able, and the equivalent conditions of Theorem 3.7 hold for M. Then the base change map*

$$\mathbf{R}^i f_* M \otimes_A \widehat{A} \to \mathbf{R}^i f_* M_{\widehat{A}}$$

is an isomorphism for $i = 0, 1$.

We will need some preliminaries on topological modules over the Banach ring $\widehat{A}$. This ring is a Tate ring in the sense of Huber [Hub96, §1.1] that is also separated, complete, reduced, and admits a Noetherian ring of definition $\widehat{A}^{\mathrm{int}} \subset \widehat{A}$, consisting of elements of supremum norm ≤ 1.

Definition 8.2. A topological $\widehat{A}$-module N is said to be *locally convex* if there exists a neighbourhood base of 0 consisting of $\widehat{A}^{\mathrm{int}}$-lattices in N.

For clarity, we will sometimes refer to a topology being locally convex rel. $\widehat{A}$. As with the case of vector spaces over a non-archimedean field, a locally convex topology on an $\widehat{A}$-module N is determined by its collection of open $\widehat{A}^{\mathrm{int}}$-lattices. Locally convex topologies are exactly those which can be defined using a collection of norms on N, all of which are compatible with the supremum norm on $\widehat{A}$.

Example 8.3.
(1) Any $\widehat{A}$-module N admits a finest locally convex topology, for which all $\widehat{A}^{\mathrm{int}}$-lattices are open. We will call this the strong topology on N. If N is finitely generated, then this is the quotient topology arising from any surjection $\widehat{A}^{\oplus n} \to N$, and N is separated and complete with respect to this topology.
(2) Be warned that even finitely generated $\widehat{A}$-modules may admit distinct locally convex topologies. For example, there is a locally convex topology on $K\langle x \rangle$ (as a free module over itself) for which a basis of open lattices is given by $\Lambda_{n,m} = p^n \mathcal{V}\langle x \rangle + x^m K\langle x \rangle$, for $n, m \in \mathbb{Z}_{\geq 0}$. This is strictly weaker than the strong topology, and $K\langle x \rangle$ is not complete with respect to this topology. Its completion is $K[\![x]\!]$, endowed with the direct product topology via $K[\![x]\!] \cong \prod_{i=1}^{\infty} K$. In particular a separated, locally convex $\widehat{A}$-module may contain a dense, finitely generated proper submodule.

We can avoid the somewhat pathological behaviour of the second example by comparing with the situation over the completed fraction field L of $\widehat{A}$, over which any finitely generated module has a unique separated, locally convex topology.

Lemma 8.4. *Let N be a finite projective $\widehat{A}$-module, and L the completed fraction field of $\widehat{A}$. Then the strong topology on N is the subspace topology coming from the inclusion*

$$N \hookrightarrow N \otimes_{\widehat{A}} L.$$

Proof. By choosing an isomorphism $N \oplus P \cong \widehat{A}^{\oplus n}$ we immediately reduce to the case $N = \widehat{A}^{\oplus n}$. In this case, both topologies are induced by the canonical norm

$$\|(a_1, \ldots, a_n)\| := \max_{1 \leq i \leq n} \|a_i\|_{\sup}$$

on $\widehat{A}^{\oplus n}$ coming from the supremum norm on $\widehat{A}$. $\square$

Proof of Theorem 8.1. First note that the equivalent conditions of Theorem 3.7 are preserved under passing from M to either the submodule $\mathbf{R}^0 f_* M \otimes_A B$ or the quotient of M by this submodule. Thus using induction and the five lemma it suffices to consider the two cases when $\mathbf{R}^0 f_* M = 0$ and when $M = N \otimes_A B$ for some (φ, ∇)-module N over A. The case when $M = N \otimes_A B$ is easily handled by the projection formula, we will concentrate on the latter.

Let L denote the completed fraction field of A. By Remark 3.8 we know that $\mathbf{R}^0 f_* M = 0 \Rightarrow \mathbf{R}^0 f_* M_L = 0$ and hence $\mathbf{R}^0 f_* M_{\widehat{A}} \hookrightarrow \mathbf{R}^0 f_* M_L = 0$, this gives the base change claim for $\mathbf{R}^0 f_* M$. To prove the base change claim for $\mathbf{R}^1 f_* M$, we give $B_{\widehat{A}}$ the (locally convex rel. $\widehat{A}$) inductive limit topology coming from the affinoid topology on each $\widehat{A} \otimes_{A_\lambda} B_\lambda$. This then induces a locally convex (rel. $\widehat{A}$) topology on any finitely generated $B_{\widehat{A}}$-module, such as $M_{\widehat{A}}$ or $M_{\widehat{A}} \otimes_B \Omega^1_{B/A}$. We equip $\mathbf{R}^1 f_* M_{\widehat{A}}$ with the quotient topology via

$$M_{\widehat{A}} \to M_{\widehat{A}} \otimes_B \Omega^1_{B/A},$$

which makes it a (potentially non-separated) locally convex $\widehat{A}$-module. We can play the same game with M_L, to obtain a locally convex topology on the finite dimensional L-vector space $\mathbf{R}^1 f_* M_L$, which in fact *is* separated by [Ked06a, Proposition 8.4.5]. The natural map

$$\mathbf{R}^1 f_* M_{\widehat{A}} \to \mathbf{R}^1 f_* M_L$$

is then continuous. Since the map

$$\mathbf{R}^1 f_* M \otimes_A L \to \mathbf{R}^1 f_* M_L$$

is an isomorphism (by Remark 3.8), it follows that

$$\mathbf{R}^1 f_* M \otimes_A \widehat{A} \to \mathbf{R}^1 f_* M_{\widehat{A}}$$

is injective, and since $M \otimes_A \widehat{A} \to M_{\widehat{A}}$ has dense image, so does

$$\mathbf{R}^1 f_* M \otimes_A \widehat{A} \to \mathbf{R}^1 f_* M_{\widehat{A}}.$$

Let Q be the maximal separated quotient of $\mathbf{R}^1 f_* M_{\widehat{A}}$, i.e. the quotient by the closure of $\{0\}$. Then we have a factorisation

$$\begin{tikzcd} \mathbf{R}^1 f_* M \otimes_A \widehat{A} \arrow[r] & \mathbf{R}^1 f_* M_{\widehat{A}} \arrow[r] \arrow[d] & \mathbf{R}^1 f_* M_L \\ & Q \arrow[ur] & \end{tikzcd}$$

and the map $\mathbf{R}^1 f_* M \otimes_A \widehat{A} \to Q$ is also injective with dense image. Now using Lemma 8.4 together with continuity of the map $Q \to \mathbf{R}^1 f_* M_L$, we can see that the topology on $\mathbf{R}^1 f_* M \otimes_A \widehat{A}$ induced by the inclusion

$$\mathbf{R}^1 f_* M \otimes_A \widehat{A} \hookrightarrow Q$$

has to be finer than the strong topology, it is therefore equal to the strong topology. Hence $\mathbf{R}^1 f_* M \otimes_A \widehat{A}$ is complete with respect to this topology, and thus has closed image in Q; therefore $\mathbf{R}^1 f_* M \otimes_A \widehat{A} \xrightarrow{\sim} Q$.

One more application of Lemma 8.4 tells us that the topology on Q must also be the strong topology (since it maps continuously into $\mathbf{R}^1 f_* M_L$), and hence the map $\mathbf{R}^1 f_* M \otimes_A \widehat{A} \to Q$ is in fact a homeomorphism. In other words, the exact sequence

$$0 \to \overline{\{0\}} \to \mathbf{R}^1 f_* M_{\widehat{A}} \to Q \to 0$$

admits a topological splitting, which implies that in fact $\overline{\{0\}} = \{0\}$ and $\mathbf{R}^1 f_* M \otimes_A \widehat{A} \xrightarrow{\sim} \mathbf{R}^1 f_* M_{\widehat{A}}$. □

We will use the above theorem in a slightly different, and more geometric, form. Let $\mathfrak{S}^\circ$ denote the open formal subscheme of $\mathfrak{S}$ whose underlying topological space is S, and let $\mathfrak{C}^\circ$ be the inverse image of $\mathfrak{S}^\circ$ under $\bar{f} : \mathfrak{C} \to \mathfrak{S}$. Let

$$\mathrm{sp} : \mathfrak{C}^\circ_K \to \mathfrak{C}^\circ$$

be the specialisation map, and

$$\mathscr{O}_{\mathfrak{C}^\circ,\mathbb{Q}}({}^\dagger C\setminus U)=\mathrm{sp}_*j_U^\dagger\mathscr{O}_{\mathfrak{C}^\circ_K}$$

the sheaf of functions on $\mathfrak{C}^\circ$ with overconvergent singularities along $C\setminus U$. If we realise E on $\mathfrak{C}^\circ_K$ and pushforward along the specialisation map we obtain a coherent $\mathscr{O}_{\mathfrak{C}^\circ,\mathbb{Q}}({}^\dagger C\setminus U)$-module $\mathrm{sp}_*E_{\mathfrak{C}^\circ}$ together with an integrable connection.

Corollary 8.5. *With assumptions as in Theorem 8.1, the relative de Rham cohomology sheaves*

$$\mathbf{R}^i\bar{f}_*\left(\mathrm{sp}_*E_{\mathfrak{C}^\circ}\otimes_{\mathscr{O}_{\mathfrak{C}^\circ}}\Omega^*_{\mathfrak{C}^\circ/\mathfrak{S}^\circ}\right)$$

are coherent $\mathscr{O}_{\mathfrak{S}^\circ,\mathbb{Q}}$-modules.

9. Local acyclicity via arithmetic $\mathscr{D}^\dagger$-modules

We are now ready to prove our second local acyclicity result for smooth relative curves. This will be valid over not necessarily smooth bases S, but will involve the additional assumption that the resiude field k is perfect; we will assume this from now on. Fix an arbitrary k-variety S, let $f:U\to S$ be a good curve, and $E\in\mathrm{Isoc}_F^\dagger(U/K)$. Then for any geometric point $\bar{s}\to S$ over a point $s\in S$ we can pullback E to get an overconvergent isocrystal $E_{\bar{s}}$ on $U_{\bar{s}}$ over $K(\bar{s}):=K\otimes_{W(k)}W(k(\bar{s}))$. If we let $C_{\bar{s}}$ denote the smooth compactification of $U_{\bar{s}}$, then for every point $x\in C_{\bar{s}}\setminus U_{\bar{s}}$ we can apply the construction of [Cre98, §7] to pullback $E_{\bar{s}}$ to a punctured formal neighbourhood of x in $C_{\bar{s}}$ to obtain an overconvergent ∇-module M_x over a copy of the Robba ring $\mathscr{R}_{K(\bar{s}),x}$ at x.

Definition 9.1. We define the Swan conductor of $E_{\bar{s}}$ at x to be the irregularity of the overconvergent ∇-module M_x,

$$\mathrm{Sw}_x(E_{\bar{s}}):=\mathrm{Irr}(M_x).$$

We define the total dimension of $E_{\bar{s}}$ at x to be

$$\dim\mathrm{tot}_x(E_{\bar{s}}):=\mathrm{Sw}_x(E_{\bar{s}})+\mathrm{rank}\,E_{\bar{s}}$$

and finally set

$$\varphi_E(\bar{s}):=\sum_{x\in C_{\bar{s}}\setminus U_{\bar{s}}}\dim\mathrm{tot}_x(E_{\bar{s}}).$$

The positive integer $\varphi_E(\bar{s})$ only depends on the point $s\in S$ lying under $\bar{s}$, we thus obtain a function

$$\varphi_E:S\to\mathbb{Z}_{\geq 0}.$$

The following is our second partial p-adic analogue of [Lau81, Corollary 2.1.2].

Theorem 9.2. *Let $f:U\to S$ be a good curve over a k-variety S, and $E\in\mathrm{Isoc}_F^\dagger(U/K)$.*

(1) *The function $\varphi_E:S\to\mathbb{Z}_{\geq 0}$ is constructible and lower semi-continuous.*
(2) *$f_!\rho(E)\in D^{b,\dagger}_{\mathrm{isoc},F}(S/K)$ if and only if φ_E is locally constant on S.*

Remark 9.3. (1) This result is weaker than [Lau81, Corollary 2.1.2] in that we assume the complement $C\setminus U$ is finite étale over S, whereas in [Lau81] it is only required to be finite flat. A formalism of vanishing and nearby cycles in p-adic cohomology is developed in [Abe18], it would be interesting to see whether Abe's machinery can be used to prove a p-adic version of the more general result.

(2) If the equivalent conditions of Theorem 9.2(2) are satisfied, then the constructible cohomology sheaves

$$\mathbf{R}^i f_! E := {}^c\mathscr{H}^i(f_!\rho(E)) \in \mathrm{Isoc}_F^\dagger(S/K)$$

are of formation commuting with arbitrary base change $S' \to S$. If in addition S is smooth over k, then the constructible cohomology sheaves

$$\mathbf{R}^i f_* E := {}^c\mathscr{H}^i(f_+\rho(E)) \in \mathrm{Isoc}_F^\dagger(S/K)$$

are also overconvergent isocrystals, and of formation commuting with base change $S' \to S$ of *smooth* varieties. Moreover, in this case we have perfect pairings

$$\mathbf{R}^i f_* E \otimes \mathbf{R}^{2-i} f_! E \to \mathscr{O}_{S/K}^\dagger(-1)$$

of overconvergent isocrystals, which are compatible with any given Frobenius structure on E. Thus even in the smooth case, Theorem 9.2 gives us slightly more than Theorem 3.7.

(3) It seems likely that 'being an iterated extension of objects which admit a Frobenius structure' is an unnecessarily strong condition on the isocrystal E. It would be interesting to investigate exactly what the right condition is, in the absence of Frobenius structures, to ensure finiteness of cohomology. This condition should be vacuous in the proper case.

Let us first consider Theorem 9.2(1), which boils down to two claims:

(1) there exists an open subset $U \subset S$ such that φ_E is constant on U (applied recursively to the complement of U in S and so forth);

(2) if $\eta, s \in S$ and $s \in \overline{\{\eta\}}$ then $\varphi_E(s) \leq \varphi_E(\eta)$.

Note that if $a : S' \to S$ is any morphism, and $s' \in S$, then $\varphi_{a^*E}(s') = \varphi_E(a(s'))$. Hence to prove the first of these we may replace S by any flat morphism $a : S' \to S$, and for the second we may replace S by an alteration followed by the inclusion of an open affine containing s (and hence η). In either case, we can assume that S smooth and affine. Working one connected component at a time, we may assume S connected.

Now applying Lemma 3.3 we may assume that the good curve $f : U \to S$ is simple, and choose a simple compactification $\bar{f} : C \to S$. The first claim then follows from Corollary 3.10 together with the Grothendieck–Ogg–Shafarevich formula

$$\chi(U_{\bar{s}}, E_{\bar{s}}) = \chi(U_{\bar{s}}) \cdot \mathrm{rank}\, E_{\bar{s}} - \sum_{x \in C_{\bar{s}} \setminus U_{\bar{s}}} \mathrm{Sw}_x(E_{\bar{s}}),$$

see for example [CM01, Corollaire 5.0-12]. For the second, we may replace S by a suitable alteration of $\overline{\{\eta\}}$, and thus assume that η is the generic point of S. The claim then follows from Proposition 2.10.

To prove Theorem 9.2(2) we first suppose that $f_!\rho(E) \in D^{b,\dagger}_{\mathrm{isoc},F}(S/K)$; we must show that φ_E is locally constant. We may clearly assume that S is connected, and since we already know that φ_E is constructible, it suffices to show that it is constant on the set $|S|$ of closed point of S. If $i_s : s \to S$ is the inclusion of a closed point, inducing a Cartesian diagram

$$\begin{tikzcd} X_s \arrow[r, "i'_s"] \arrow[d, "f_s"'] & X \arrow[d, "f"] \\ s \arrow[r, "i_s"] & S, \end{tikzcd}$$

then we have that

$$i_s^+ f_! \rho(E) \cong f_{s!} i_s'^+ \rho(E) \cong f_{s!}\rho(i_s^* E).$$

Since $f_!\rho(E) \in D^{b,\dagger}_{\mathrm{isoc},F}(S/K)$ and i_s^+ is exact for the constructible t-structure, we deduce the existence of objects

$$\mathbf{R}^i f_! E := {}^c\mathscr{H}^i(f_!\rho(E)) \in \mathrm{Isoc}^\dagger_F(S/K)$$

such that $i_s^*\mathbf{R}^i f_! E \cong H^i_{c,\mathrm{rig}}(U_s/K(s), E_s)$ for all s. In particular, we can see that the *compactly supported* Euler characteristic

$$s \mapsto \chi_c(U_s, E_s) = \chi_c(U_{\bar{s}}, E_{\bar{s}})$$

is constant on $|S|$. Since $\mathrm{Sw}_x(E_{\bar{s}}^\vee) = \mathrm{Sw}_x(E_{\bar{s}})$, applying Poincaré duality and the p-adic Grothendieck–Ogg–Shafarevich formula tells us that

$$\begin{aligned}\chi_c(U_{\bar{s}}, E_{\bar{s}}) = \chi(U_{\bar{s}}, E_{\bar{s}}^\vee) &= \chi(U_{\bar{s}}) \cdot \mathrm{rank}\, E_{\bar{s}}^\vee - \sum_{x \in C_{\bar{s}} \setminus U_{\bar{s}}} \mathrm{Sw}_x(E_{\bar{s}}^\vee) \\ &= \chi(U_{\bar{s}}) \cdot \mathrm{rank}\, E_{\bar{s}} - \sum_{x \in C_{\bar{s}} \setminus U_{\bar{s}}} \mathrm{Sw}_x(E_{\bar{s}}).\end{aligned}$$

Thus constancy of $\chi_c(U_s, E_s)$ implies that of $\varphi_E(s)$, and we are done.

To prove the converse implication, then, let us assume that φ_E is constant, and take an alteration $a : S' \to S$ with S' smooth. Then we have a Cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{a'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{a} & S \end{array}$$

and isomorphisms

$$a^+ f_!\rho(E) \cong f'_! a'^+ \rho(E) \cong f'_!\rho(a'^* E).$$

Using [Abe18, Lemma 3.3] together with t-exactness of a^+, it therefore suffices to show that $f'_!\rho(a'^*E) \in D^{b,\dagger}_{\mathrm{isoc},F}(S'/K)$, and thus replacing S by S' we may assume that S is smooth. Since the question is local on S, we may also assume that it is affine and connected. Now by Poincaré duality

$$f_!\rho(E) \cong \mathbf{D}_S f_+ \rho(E^\vee)[-2\dim S],$$

it therefore suffices to show that $f_+\rho(E^\vee) \in D^{b,\dagger}_{\mathrm{isoc},F}(S/K)$. Since $\varphi_E = \varphi_{E^\vee}$ we may replace E by $E^\vee$, and since S is smooth we have $\rho(E) \cong \mathrm{sp}_+ E[-\dim S]$; we must therefore show that φ_E constant $\Rightarrow f_+\mathrm{sp}_+ E \in D^{b,\dagger}_{\mathrm{isoc},F}(S/K)$. Now let

$$\bar{f} : (C, \overline{C}, \mathfrak{C}) \to (S, \overline{S}, \mathfrak{S})$$

be a morphism of frames extending a good compactification of f as in Lemma 3.4. Let $\mathfrak{S}^\circ \subset \mathfrak{S}$ denote the open formal subscheme with underlying topological space S and let $\mathfrak{C}^\circ$ denote the fibre product of $\mathfrak{S}^\circ$ with $\mathfrak{C}$ over $\mathfrak{S}$. Let $\hat{E}$ denote the image of E inside the category $\mathrm{Isoc}_F((U,C)/K)$ of isocrystals on U overconvergent along C (which are extensions of isocrystals admitting Frobenius structures). Since the diagram

$$\begin{array}{ccccc} \mathrm{Isoc}^\dagger_F(U/K) & \xrightarrow{\mathrm{sp}_+} & D^{b,\dagger}_{\mathrm{hol},F}(U/K) & \xrightarrow{f_+} & D^{b,\dagger}_{\mathrm{hol},F}(S/K) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Isoc}_F((U,C)/K) & \xrightarrow{\mathrm{sp}_+} & D^{b}_{\mathrm{hol},F}((U,C)/K) & \xrightarrow{f_+} & D^{b}_{\mathrm{hol},F}(S/K) \end{array}$$

commutes up to natural isomorphism, it suffices by Lemma 1.11 to show that

$$f_+\mathrm{sp}_+\hat{E} \in D^b_{\mathrm{isoc}}(S/K).$$

By Lemma 1.12 the functor f_+ on the category of 'convergent' holonomic modules can be computed in terms of the realisation $\mathfrak{C}^\circ \to \mathfrak{S}^\circ$ via

$$\bar{f}_+ : D^b_{\mathrm{coh}}(\mathscr{D}^\dagger_{\mathfrak{C}^\circ,\mathbb{Q}}) \to D^b_{\mathrm{coh}}(\mathscr{D}^\dagger_{\mathfrak{S}^\circ,\mathbb{Q}}),$$

and moreover it suffices to show that the $\mathscr{O}_{\mathfrak{S}^\circ,\mathbb{Q}}$-modules underlying cohomology sheaves of $\bar{f}_+\mathrm{sp}_+\hat{E}$ are coherent. In this case the construction of the functor sp_+E is very simple. Explicitly, we realise E on $\mathfrak{C}^\circ_K$, and pushforward the resulting module with integrable connection $E_{\mathfrak{C}^\circ}$ via the specialisation map

$$\mathrm{sp} : \mathfrak{C}^\circ_K \to \mathfrak{C}^\circ.$$

This results in a coherent $\mathscr{O}_{\mathfrak{C}^\circ,\mathbb{Q}}({}^\dagger C \setminus U)$-module with integrable connection, which by [CT12, Theorem 2.3.15] extends to the structure of an overholonomic (and in particular, coherent) $\mathscr{D}^\dagger_{\mathfrak{C}^\circ,\mathbb{Q}}$-module (remember that E is F-able). This $\mathscr{D}^\dagger_{\mathfrak{C}^\circ,\mathbb{Q}}$-module is nothing other than $\mathrm{sp}_+\hat{E}$. Finally using [Ber02, (4.3.6.3)] we can identify

$$\bar{f}_+\mathrm{sp}_+\hat{E}[-1] \cong \mathbf{R}\bar{f}_*\left(\mathrm{sp}_*E_{\mathfrak{C}^\circ} \otimes_{\mathscr{O}_{\mathfrak{C}^\circ}} \Omega^*_{\mathfrak{C}^\circ/\mathfrak{S}^\circ}\right)$$

and therefore apply Corollary 8.5 to conclude.

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