The homotopy exact sequence for overconvergent isocrystals
joint with Ambrus Pál

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1 Introduction

2 Pro-algebraic fundamental groups

3 Overconvergent isocrystals

4 Proof of $p$-adic HES
Let $F \to E \to B$ be a topological fibre bundle.
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$$\ldots \to \pi_{n+1}(B) \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to \ldots$$

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**Question**

What is the analogue of the in algebraic geometry?
Basic questions

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1. What is the algebraic analogue of a fibre bundle?
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Basic answer: a smooth and proper morphism $X \to S$ of schemes.
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Example

If $S/\mathbb{C}$ is a variety, and $X \to S$ is smooth and proper, then $X(\mathbb{C}) \to S(\mathbb{C})$ is a topological fibre bundle.
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More sophisticated answer: a morphism $X \rightarrow S$ admitting a proper hypercover $X_\bullet \rightarrow X$ and a compactification $X_\bullet \rightarrow \overline{X}_\bullet$ such that $\overline{X}_\bullet \rightarrow S$ is smooth and proper and $\overline{X}_\bullet \setminus X_\bullet$ is a relative NCD.
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**Basic Setup**

$f : X \to S$ is a smooth and projective morphism of varieties over a field $k$, with geometrically connected base and fibres. Fix $x \in X(k)$ and set $s = f(x)$. 

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More sophisticated answer: a morphism $X \to S$ admitting a proper hypercover $X_\bullet \to X$ and a compactification $X_\bullet \to \overline{X}_\bullet$ such that $\overline{X}_\bullet \to S$ is smooth and proper and $\overline{X}_\bullet \setminus X_\bullet$ is a relative NCD.

Basic Setup

$f : X \to S$ is a smooth and projective morphism of varieties over a field $k$, with geometrically connected base and fibres. Fix $x \in X(k)$ and set $s = f(x)$.

Thus we expect to see a right exact sequence

$$\pi_1(X_s, x) \to \pi_1(X, x) \to \pi_1(S, s) \to 1$$

of fundamental groups (whatever they are!).
The étale fundamental group

If $Y$ is a normal, connected, Noetherian scheme, and $\bar{y} \to Y$ is a geometric point, then Grothendieck defined the étale fundamental group $\pi_1^{\text{ét}}(Y, \bar{y})$. 
The étale fundamental group

If $Y$ is a normal, connected, Noetherian scheme, and $\bar{y} \to Y$ is a geometric point, then Grothendieck defined the étale fundamental group $\pi^\text{ét}_1(Y, \bar{y})$. It is uniquely characterised by the existence of an equivalence of categories

$$F\text{Ét}(Y) \cong \pi^\text{ét}_1(Y, \bar{y})\text{-FSet}$$

between finite étale covers of $Y$ and finite (discrete) $\pi^\text{ét}_1(Y, \bar{y})$-sets, such that the forgetful functor

$$\pi^\text{ét}_1(Y, \bar{y})\text{-FSet} \to \text{FSet}$$

corresponds to the ‘fibre over $\bar{y}$’ functor

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**Theorem (Grothendieck)**

*Assume the Basic Setup, and let $\bar{x} \to x$ be a geometric point over $x$, with corresponding geometric point $\bar{s}$ over $s$. Then the sequence

$$\pi^\text{ét}_1(X_{\bar{s}}, \bar{x}) \to \pi^\text{ét}_1(X, \bar{x}) \to \pi^\text{ét}_1(S, \bar{s}) \to 1$$

of pro-finite fundamental groups is exact.*
Introduction

Pro-algebraic fundamental groups

Overconvergent isocrystals

Proof of $p$-adic HES
Tannakian duality

Basic idea is that a pro-algebraic group $G$ can be reconstructed from its category of representations $\text{Rep}(G)$. 
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**Definition**

A Tannakian category over a field $F$ is a $F$-linear, abelian, rigid tensor category $\mathcal{T}$, such that:

1. $\text{End}(1) = F$;
2. There exists a faithful, exact, $F$-linear tensor functor $\omega: \mathcal{T} \to \text{Vec}_F'$ for some field extension $F'/F$.

Such a functor is called a fibre functor. If we can choose $F' = F$ then we say that $\mathcal{T}$ is neutral Tannakian.

**Theorem (Saavedra)**

Let $\mathcal{T}$ be a neutral Tannakian category over $F$, with fibre functor $\omega: \mathcal{T} \to \text{Vec}_F$.

Then there exists a unique pro-algebraic group $G = G(\mathcal{T}, \omega)$ over $F$, and an equivalence $\text{Rep}(G) \cong \mathcal{T}$ which identifies $\omega: \mathcal{T} \to \text{Vec}_F$ with the forgetful functor $\text{Rep}(G) \to \text{Vec}_F$.
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Let $Y$ be a normal, connected, Noetherian scheme, and $\text{Loc}^\text{ét}_{Q_\ell}(Y)$ the category of lisse $Q_\ell$-sheaves on $Y_{\text{ét}}$. Then $\text{Loc}^\text{ét}_{Q_\ell}(Y)$ is neutral Tannakian over $Q_\ell$, and any geometric point $\bar{y} \to Y$ provides a fibre functor

$$\text{Loc}^\text{ét}_{Q_\ell}(Y) \to \text{Vec}_{Q_\ell}$$

$$\mathcal{F} \mapsto \mathcal{F}_{\bar{y}}.$$ 

The corresponding fundamental group $\pi_1^\text{ét}(Y, \bar{y})_{Q_\ell}$ is the $Q_\ell$-pro-algebraic completion of $\pi_1^\text{ét}(Y, \bar{y})$. This is ‘well-behaved’ only when $\ell$ is invertible on $Y$. 

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The corresponding fundamental group $\pi_{1}^{\text{ét}}(Y, \bar{y})_{\mathbb{Q}_{\ell}}$ is the $\mathbb{Q}_{\ell}$-pro-algebraic completion of $\pi_{1}^{\text{ét}}(Y, \bar{y})$. This is ‘well-behaved’ only when $\ell$ is invertible on $Y$.

2. $Y/k$ a smooth, geometrically connected variety over a field $k$ of characteristic 0. Then the category $\text{MIC}(Y/k)$ of vector bundles with integrable connection on $Y$ is Tannakian over $k$. If there exists a rational point $y \in Y(k)$ it is moreover neutral Tannakian, and

$$y^{*} : \text{MIC}(Y/k) \to \text{Vec}_{k}$$

is a fibre functor. This gives rise to the de Rham fundamental group $\pi_{1}^{\text{dR}}(Y, y)$. 

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The homotopy exact sequence for overconvergent isocrystals
More generally, if $Y/k$ is a smooth, geometrically connected variety over any field $k$, then the category $\text{Strat}(X/k)$ of $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules is Tannakian over $k$. If there exists a rational point $y \in Y(k)$, then it is moreover neutral Tannakian, and

$$y^* : \text{Strat}(Y/k) \to \text{Vec}_k$$

is a fibre functor. This gives rise to the stratified fundamental group $\pi_1^{\text{strat}}(Y, y)$. If $\text{char}(k) = 0$ then $\pi_1^{\text{dR}}(Y, y) = \pi_1^{\text{strat}}(Y, y)$. 
More generally, if $Y/k$ is a smooth, geometrically connected variety over any field $k$, then the category $\text{Strat}(X/k)$ of $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules is Tannakian over $k$. If there exists a rational point $y \in Y(k)$, then it is moreover neutral Tannakian, and

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If $K$ is a complete, valued field of characteristic 0, and $Y/K$ is a smooth, geometrically connected analytic variety, then the category $\text{MIC}(Y/K)$ of analytic vector bundles with integrable connection on $Y$ is Tannakian over $K$. If $y \in Y(K)$ is a rational point, then it is neutral Tannakian, and

$$y^* : \text{MIC}(Y/K) \to \text{Vec}_K$$

is a fibre functor. The corresponding fundamental group is denoted $\pi_1^{dR}(Y, y)$. 

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Assume the Basic Setup.
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1. If \( \ell \neq \text{char}(k) \) then exactness of

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follows from ‘right exactness’ of the pro-algebraic completion functor.
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2. If $\text{char}(k) = 0$ and the base $S$ is smooth, then exactness of

$$\pi_1^{\text{dR}}(X_s, x) \to \pi_1^{\text{dR}}(X, x) \to \pi_1^{\text{dR}}(S, s) \to 1$$

can be deduced over $\mathbb{C}$ (more or less) using the Riemann–Hilbert correspondence. In general, it was proved using transcendental methods by Zhang.
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3. Again, if the base $S$ is smooth, but now $\text{char}(k)$ is arbitrary, then exactness of

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was proved by dos Santos. This gives a proof of exactness for $\pi_1^{\text{dR}}$ in char 0 using only algebraic methods.
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was proved by dos Santos. This gives a proof of exactness for $\pi^\text{dR}_1$ in char 0 using only algebraic methods.

4. We’ll come back to this one!
1. Introduction

2. Pro-algebraic fundamental groups

3. Overconvergent isocrystals

4. Proof of $p$-adic HES
Now suppose $k$ is perfect and $\text{char}(k) = p > 0$. A $p$-adic analogue of $\text{MIC}(Y/k)$ or $\text{Strat}(Y/k)$ is given by the category of overconvergent isocrystals.
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To describe this category, let $\mathcal{V}$ be a complete DVR with residue field $k$ and fraction field $K$ of characteristic 0. Assume that $Y$ is smooth, and that there exists a projective formal scheme $\mathcal{Y}$ over $\mathcal{V}$ and an open embedding $Y \hookrightarrow \mathcal{Y}_k$ such that:
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- $\mathcal{Y}$ is smooth over $\mathcal{V}$ in a neighbourhood of $Y$;
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**Definition**

Let us call such an embedding $Y \hookrightarrow \mathcal{Y}$ a ‘good embedding’.
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$$sp : \mathcal{Y}_K \to \mathcal{Y}_k$$

be the ‘reduction mod $p$ map’, so, again locally, the tube

$$\mathcal{Y} := sp^{-1}(Y)$$

is defined by $\{|t| \geq 1\}$. 
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be the ‘reduction mod $p$ map’, so, again locally, the tube

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is defined by $\{|t| \geq 1\}$. We can therefore consider the ‘strict neighbourhoods’

$\mathcal{Y}[: \subset V_\lambda \subset \mathcal{Y}_K$ defined locally by $\{|t| \geq \lambda\}$ for $\lambda \to 1^-$.
If $\lambda$ is close enough to 1, then the $V_\lambda$ are smooth over $K$, and by definition

$$\text{Isoc}^\dagger(Y/K) \subset 2\text{-colim}_\lambda \text{MIC}(V_\lambda)$$

is a full subcategory defined by certain convergence conditions on the Taylor series.
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is a full subcategory defined by certain convergence conditions on the Taylor series. This doesn’t depend on any of the choices involved.
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**Theorem (Crew)**

*If \( Y/k \) is geometrically connected, \( \text{Isoc}^\dagger(Y/K) \) is Tannakian over \( K \). If \( y \in Y(k) \) is a rational point, then it is neutral Tannakian, and*

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y^*: \text{Isoc}^\dagger(Y/K) \to \text{Vec}_K
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*is a fibre functor.*
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We define the overconvergent fundamental group $\pi_{1}^{\dagger}(Y, y)$ to be the associated pro-algebraic group over $K$. 

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Assume the Basic Setup, with ground field $k$ perfect of characteristic $p > 0$, and smooth base $S$. 
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**Theorem (L., Pál)**

The sequence

$$\pi^\dagger_1(X_s, x) \to \pi^\dagger_1(X, x) \to \pi^\dagger_1(S, s) \to 1$$

of pro-algebraic groups is exact.
In particular, this implies a weak form of the Lefschetz hyperplane theorem for $p$-adic fundamental groups.
In particular, this implies a weak form of the Lefschetz hyperplane theorem for $p$-adic fundamental groups.

**Corollary**

Let $X$ be smooth, projective and geometrically connected, $Y \subset X$ a hyperplane section of dimension $\geq 1$ and $y \in Y(k)$. Then the induced map

$$\pi^\dagger_1(Y, y) \to \pi^\dagger_1(X, x)$$

is surjective.
In particular, this implies a weak form of the Lefschetz hyperplane theorem for $p$-adic fundamental groups.

**Corollary**

Let $X$ be smooth, projective and geometrically connected, $Y \subset X$ a hyperplane section of dimension $\geq 1$ and $y \in Y(k)$. Then the induced map

$$\pi_1^\dagger(Y, y) \to \pi_1^\dagger(X, x)$$

is surjective.

**Proof.**

Put $Y$ into a Lefschetz pencil $\widetilde{X} \to \mathbb{P}^1_k$ with a section $\mathbb{P}^1_k \to \widetilde{X}$, where $\widetilde{X} \to X$ is a blowup. Now apply the HES over the smooth locus of $\widetilde{X} \to \mathbb{P}^1_k$. 

□
We can also use the HES to compare $\pi_1^\dagger$ with $\pi_1^{\text{ét}}$. 
We can also use the HES to compare $\pi_1^\dagger$ with $\pi_1^{\text{ét}}$. So assume that $k = \bar{k}$, and that $X/k$ is smooth, projective and connected. Fix $x \in X(k)$. Then we have a natural map

$$\pi_1^\dagger(X, x) \to \pi_1^{\text{ét}}(X, x)$$

induced by sending a finite étale cover $f : Y \to X$ to $f_* \mathcal{O}_Y^\dagger \in \text{Isoc}^\dagger(X/K)$.
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Since $\pi_1^{\text{ét}}(X, x)$ is pro-finite this has to factor through the component group

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**Corollary**

_This induces an isomorphism_ $\pi_0(\pi_1^\dagger(X, x)) \cong \pi_1^\text{ét}(X, x).$
Applications (contd.)

We can also use the HES to compare $\pi_1^\dagger$ with $\pi_1^{\text{ét}}$. So assume that $k = \bar{k}$, and that $X/k$ is smooth, projective and connected. Fix $x \in X(k)$. Then we have a natural map

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**Corollary**

*This induces an isomorphism* $\pi_0(\pi_1^\dagger(X, x)) \cong \pi_1^{\text{ét}}(X, x)$.

**Proof.**

We want to show that any $E \in \text{Isoc}^\dagger(X/K)$ with finite monodromy group is trivialised by a finite étale cover of $X$. By a result of Crew, it suffices to show that $E$ admits a Frobenius structure. Using the Lefschetz theorem, this can be reduced to the case of curves, where in fact it suffices to show that $E$ can be trivialised by a finite *separable* map. We can now argue by lifting to characteristic 0.
1 Introduction

2 Pro-algebraic fundamental groups

3 Overconvergent isocrystals

4 Proof of $p$-adic HES
Two main steps:
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1. Prove that for a smooth and projective morphism $W \rightarrow V$ of smooth analytic varieties over $K$, with geometrically connected fibres and base, the homotopy sequence

$$\pi_{1}^{\text{dR}}(W_v, w) \rightarrow \pi_{1}^{\text{dR}}(W, w) \rightarrow \pi_{1}^{\text{dR}}(V, v) \rightarrow 1$$

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2. Show how to reduce the algebraic result for isocrystals over $k$ to the analytic result for vector bundles with integrable connection.
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is exact.

2. Show how to reduce the algebraic result for isocrystals over $k$ to the analytic result for vector bundles with integrable connection.

The first can be achieved by transporting dos Santos’ methods from algebraic geometry to analytic geometry. I will focus on explaining the second.
Since $G$ can be recovered from $\text{Rep}(G)$, it is natural to ask if we can phrase exactness of a sequence

$$K \xrightarrow{a} G \xrightarrow{b} H \xrightarrow{} 1$$

in terms of the associated categories of representations.
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**Theorem (Esnault, Hai, Sun)**

Let

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be a sequence of pro-algebraic groups, such that $b \circ a$ is trivial and $b$ is surjective. Then the sequence is exact iff the following three conditions hold.
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1. For any $V \in \text{Rep}(G)$, $a^*(V)$ is trivial if and only if $V \cong b^*(W)$ for some $W \in \text{Rep}(H)$. 
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1. For any $V \in \text{Rep}(G)$, $a^*(V)$ is trivial if and only if $V \cong b^*(W)$ for some $W \in \text{Rep}(H)$.
2. If $V \in \text{Rep}(G)$, and $U_0 \subset a^*(V)$ is the largest trivial sub-object, then there exists some $V_0 \subset V$ such that $a^*(V_0) = U_0$. 

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3. If $U \in \text{Rep}(K)$ is a sub-quotient of an object in the essential image of $a^*$, then it is a sub-object of such an object.
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In practise, (1) and (2) are rather straightforward to check, but (3) almost impossible.
What happens if we drop condition (3)?
Weak exactness

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**Definition**

We say that a sequence

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of pro-algebraic groups is weakly exact if $b \circ a$ is trivial, $b$ is surjective, and the normal closure of $a(K)$ is ker $b$. 
Weak exactness

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**Proposition**

Let

\[ K \xrightarrow{a} G \xrightarrow{b} H \rightarrow 1 \]

be a sequence of pro-algebraic groups, such that \( b \circ a \) is trivial, and \( b \) is surjective. Then the sequence is weakly exact iff the following two conditions hold.

1. For any \( V \in \text{Rep}(G) \), \( a^*(V) \) is trivial if and only if \( V \cong b^*(W) \) for some \( W \in \text{Rep}(H) \).
2. If \( V \in \text{Rep}(G) \), and \( U_0 \subset a^*(V) \) is the largest trivial sub-object, then there exists some \( V_0 \subset V \) such that \( a^*(V_0) = U_0 \).
In geometric situations, (1) and (2) essentially boil down to the existence of a well-behaved push-forward functor.
Geometric push-forwards

In geometric situations, (1) and (2) essentially boil down to the existence of a well-behaved push-forward functor.

Proposition

Assume the Basic Setup, with $k$ perfect of characteristic $p > 0$, and $S$ smooth. Then there exists a push-forward functor

$$f_* : \text{Isoc}^\dagger(X/K) \to \text{Isoc}^\dagger(S/K)$$

right adjoint to $f^*$, such that

$$s^* f_* E \cong H^0_{\text{rig}}(X_s/K, E|_{X_s})$$

for all $E \in \text{Isoc}^\dagger(X/K)$. 
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for all $E \in \text{Isoc}^\dagger(X/K)$.

**Corollary**

The sequence

$$\pi_1^\dagger(X_s, x) \to \pi_1^\dagger(X, x) \to \pi_1^\dagger(S, s) \to 1$$

is weakly exact.
We can now use this to reduce the proof of the HES to the case of curves.
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We can now use this to reduce the proof of the HES to the case of curves. Take $f : X \to S$ as in the Basic Setup, and and fix $X \hookrightarrow \mathbb{P}^n_S$. Let $d$ be the relative dimension, and assume that $d \geq 2$. Let $\tilde{S} = \mathbb{P}^n_S$ be the dual projective space, and set

$$\tilde{X} := \{(x, H) \in X \times_S \mathbb{P}^n_S | x \in H\} \subset X \times_S \tilde{S}.$$
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Let $U \subset \tilde{S}$ be the smooth locus of

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and $\tilde{X}_U$ the base change.
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The homotopy exact sequence for overconvergent isocrystals
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Let $U \subset \tilde{S}$ be the smooth locus of $\tilde{f} : \tilde{X} \to \tilde{S}$ and $\tilde{X}_U$ the base change. Lift $x$ to a rational point $\tilde{x} \in \tilde{X}_U$, and set $\tilde{s} = \tilde{f}(\tilde{x})$. Then we have a commutative diagram

$$
\begin{array}{ccc}
\tilde{X}_s & \longrightarrow & \tilde{X}_U \\
\downarrow & & \downarrow \\
\tilde{X} & \longrightarrow & \tilde{S} \\
\downarrow & & \downarrow \\
X_s & \longrightarrow & X \longrightarrow S
\end{array}
$$

where $(\tilde{X}_U, \tilde{x}) \to (U, \tilde{s})$ is as in the Basic Setup, but with relative dimension $d - 1$. 

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The homotopy exact sequence for overconvergent isocrystals
We therefore have the diagram

\[
\begin{array}{ccc}
\pi_1^\dagger(\tilde{X}_\tilde{s}, \tilde{x}) & \rightarrow & \pi_1^\dagger(\tilde{X}_U, \tilde{x}) \\
\downarrow & & \downarrow \\
\pi_1^\dagger(\tilde{X}_s, \tilde{x}) & \rightarrow & \pi_1^\dagger(\tilde{X}, \tilde{x}) \\
\downarrow & & \downarrow \\
\pi_1^\dagger(X_s, x) & \rightarrow & \pi_1^\dagger(X, x) \\
\rightarrow & & \rightarrow \\
\pi_1^\dagger(U, \tilde{s}) & \rightarrow & \pi_1^\dagger(U, s) \\
\rightarrow & & \rightarrow \\
1 & & 1
\end{array}
\]

and

\[
\begin{array}{ccc}
\pi_1^\dagger(\tilde{S}, \tilde{s}) & \rightarrow & \pi_1^\dagger(\tilde{S}, s) \\
\downarrow & & \downarrow \\
\pi_1^\dagger(X_s, x) & \rightarrow & \pi_1^\dagger(X, x) \\
\rightarrow & & \rightarrow \\
\pi_1^\dagger(S, s) & \rightarrow & \pi_1^\dagger(S, s) \\
\rightarrow & & \rightarrow \\
1 & & 1
\end{array}
\]

of fundamental groups.
We therefore have the diagram

\[
\begin{array}{cccccc}
\pi_1^\dagger(\tilde{X}_s, \tilde{x}) & \rightarrow & \pi_1^\dagger(\tilde{X}_U, \tilde{x}) & \rightarrow & \pi_1^\dagger(U, \tilde{s}) & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \\
\pi_1^\dagger(\tilde{X}_s, \tilde{x}) & \rightarrow & \pi_1^\dagger(\tilde{X}, \tilde{x}) & \rightarrow & \pi_1^\dagger(\tilde{S}, \tilde{s}) & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \\
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of fundamental groups.

**Lemma**

*The normal closure of the image of \( \pi_1^\dagger(\tilde{X}_s, \tilde{x}) \rightarrow \pi_1^\dagger(X_s, x) \) is the whole of \( \pi_1^\dagger(X_s, x) \).*
We therefore have the diagram

\[
\begin{array}{cccccc}
\pi_1^\dagger(\tilde{X}_s, \tilde{x}) & \rightarrow & \pi_1^\dagger(\tilde{X}_U, \tilde{x}) & \rightarrow & \pi_1^\dagger(U, \tilde{s}) & \rightarrow & 1 \\
\pi_1^\dagger(\tilde{X}_s, \tilde{x}) & \rightarrow & \pi_1^\dagger(\tilde{X}, \tilde{x}) & \rightarrow & \pi_1^\dagger(\tilde{S}, \tilde{s}) & \rightarrow & 1 \\
\pi_1^\dagger(X_s, x) & \rightarrow & \pi_1^\dagger(X, x) & \rightarrow & \pi_1^\dagger(S, s) & \rightarrow & 1 \\
\end{array}
\]

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**Lemma**

The normal closure of the image of \(\pi_1^\dagger(\tilde{X}_s, \tilde{x}) \rightarrow \pi_1^\dagger(X_s, x)\) is the whole of \(\pi_1^\dagger(X_s, x)\).

By some diagram chasing we can therefore deduce that if the homotopy sequence for \(\tilde{X}_U \rightarrow U\) is exact, then so is the homotopy sequence for \(X \rightarrow S\). By induction we may therefore assume that \(d = 1\).
Now assume the Basic Setup, with \( f \) of relative dimension 1 and \( S \) smooth. Suppose that \( U \subset S \) is a Zariski open containing \( s \).
Now assume the Basic Setup, with $f$ of relative dimension 1 and $S$ smooth. Suppose that $U \subset S$ is a Zariski open containing $s$. Then we have a diagram

$$
\begin{array}{ccc}
\pi_1^+(X_s, x) & \longrightarrow & \pi_1^+(X_U, x) \\
\downarrow & & \downarrow \\
\pi_1^+(X_s, x) & \longrightarrow & \pi_1^+(X, x) \end{array}
\begin{array}{ccc}
\longrightarrow & \longrightarrow & \longrightarrow \\
\downarrow & \downarrow & \downarrow \\
\pi_1^+(U, s) & \longrightarrow & \pi_1^+(S, s) \\
\longrightarrow & \longrightarrow & \longrightarrow \\
& & 1
\end{array}
$$

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Now assume the Basic Setup, with $f$ of relative dimension 1 and $S$ smooth. Suppose that $U \subset S$ is a Zariski open containing $s$. Then we have a diagram

\[
\begin{array}{c}
\pi_1^\dagger(X_s, x) \longrightarrow \pi_1^\dagger(X_U, x) \longrightarrow \pi_1^\dagger(U, s) \longrightarrow 1 \\
\downarrow \quad \quad \quad \quad \downarrow \quad \quad \quad \quad \downarrow \\
\pi_1^\dagger(X_s, x) \longrightarrow \pi_1^\dagger(X, x) \longrightarrow \pi_1^\dagger(S, s) \longrightarrow 1
\end{array}
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of fundamental groups. Using weak exactness, we can see that exactness of the homotopy sequence for $X_U \to U$ implies exactness of the homotopy sequence for $X \to S$. Hence we can assume that the base $S = \text{Spec}(A_0)$ is affine.
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\uparrow & & \uparrow & & \downarrow & & \\
\pi_1^\dagger(X_s, x) & \rightarrow & \pi_1^\dagger(X, x) & \rightarrow & \pi_1^\dagger(S, s) & \rightarrow & 1
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In particular, we can lift $S$ to a smooth affine $\mathcal{V}$-scheme $\text{Spec}(A)$, and the family $X \rightarrow S$ to a smooth projective family of curves over $\text{Spec}(A)$.
Thus there exist good embeddings $S \hookrightarrow \mathcal{G}$ and $X \hookrightarrow \mathcal{X}$ and a commutative, \textit{Cartesian} diagram

\[
\begin{array}{ccc}
X & \longrightarrow & \mathcal{X} \\
\downarrow & & \downarrow \\
S & \longrightarrow & \mathcal{G}
\end{array}
\]

such that the map $\mathcal{X} \rightarrow \mathcal{G}$ is smooth around $X$. 

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The homotopy exact sequence for overconvergent isocrystals
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If we now let $W_\lambda$ the associated ‘strict neighbourhoods’ of $]X[$ and $V_\lambda$ those of $]S[,$ then for $\lambda$ closed enough to 1 there are induced \textit{smooth and projective} maps $W_\lambda \rightarrow V_\lambda$. 

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Thus there exist good embeddings $S \hookrightarrow \mathcal{G}$ and $X \hookrightarrow \mathfrak{X}$ and a commutative, *Cartesian* diagram

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\begin{array}{ccc}
X & \longrightarrow & \mathfrak{X} \\
\downarrow & & \downarrow \\
S & \longrightarrow & \mathcal{G}
\end{array}
\]

such that the map $\mathfrak{X} \rightarrow \mathcal{G}$ is smooth around $X$. Let $\tilde{x}$ be a lift of $x$ to a $K$-point of $\mathfrak{X}$, and $\tilde{s}$ the image of $\tilde{x}$ in $\mathcal{G}$.

If we now let $W_\lambda$ the associated ‘strict neighbourhoods’ of $\mathfrak{X}$ and $V_\lambda$ those of $\mathcal{G}$, then for $\lambda$ closed enough to 1 there are induced *smooth and projective* maps $W_\lambda \to V_\lambda$. So by assumption there is an exact sequence

\[
\pi_1^{dR}(\mathfrak{X}_K, \tilde{s}, \tilde{x}) \to \pi_1^{dR}(W_\lambda, \tilde{x}) \to \pi_1^{dR}(V_\lambda, \tilde{s}) \to 1
\]

of pro-algebraic groups over $K$, for all $\lambda$ close enough to 1.
Since we definition we have

\[ \text{Isoc}^\dagger(X/K) \subset 2\text{-colim}_\lambda \text{MIC}(W_\lambda/K) \]
\[ \text{Isoc}^\dagger(S/K) \subset 2\text{-colim}_\lambda \text{MIC}(V_\lambda/K) \]
\[ \text{Isoc}^\dagger(X_s/K) \subset \text{MIC}(\mathfrak{X}_K, \tilde{s}/K) \]

stable by sub-quotients, we get a commutative diagram

\[
\begin{array}{ccccccccc}
\pi^\text{dR}_1(\mathfrak{X}_K, \tilde{s}, \tilde{x}) & \longrightarrow & \lim_{\lambda} \pi^\text{dR}_1(W_\lambda, \tilde{x}) & \longrightarrow & \lim_{\lambda} \pi^\text{dR}_1(V_\lambda, \tilde{s}) & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\pi^\dagger_1(X_s, x) & \longrightarrow & \pi^\dagger_1(X, x) & \longrightarrow & \pi^\dagger_1(S, s) & \longrightarrow & 1
\end{array}
\]

with exact top row. Again, some diagram chasing together with weak exactness lets us deduce exactness of the bottom row.
Thank-you!