$\ell$-independence over local function fields

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Motivation

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Fundamental groups

\[ k = \text{field}, \ k^s = \text{separable closure}, \ G_k = \text{Gal}(k^s/k) \]
\[ X/k \text{ variety (separated scheme of finite type)} \]
\[ \ell \neq \text{char}(k) \text{ prime} \]

\[ H^i_{\ell}(X) := H^i_{\text{ét}}(X_{k^s}, \mathbb{Q}_\ell) \]
\[ \rho_\ell : G_k \to \text{GL}(H^i_{\ell}(X)) \]

Question

How does \( \rho_\ell \) depend on \( \ell \)? Is it ‘independent of \( \ell \)’ in some sense?
Example (Deligne)

Suppose that $k = \mathbb{F}_q$ is finite, and that $X/k$ is smooth and proper. Then for all $n \in \mathbb{Z}$

$$\text{Tr} (\text{Frob}_k^n \mid H^i_\ell (X))$$

is in $\mathbb{Q}$ and is independent of $\ell \neq p$.

Can also phrase this as follows: let $W_k \subset G_k$ consist of integral powers of Frob$_k$. Then $\forall \ell, \ell' \neq p$, and any alg. closed field $\Omega \supset \mathbb{Q}_\ell, \mathbb{Q}_{\ell'}$, \( (\rho_\ell \mid W_k)^{ss} \otimes \Omega \cong (\rho_{\ell'} \mid W_k)^{ss} \otimes \Omega \)

Remark

Conjecturally $\left( \rho_\ell \mid W_k \right)^{ss} = \left( \rho_\ell \mid W_k \right)$.
In general, should exist an abelian category $\mathcal{M}_k,\mathbb{Q}$ of (rational) mixed motives over $k$, cohomology groups $H^i_{\text{mot}}(X) \in \mathcal{M}_k,\mathbb{Q}$ and realisation functors

$$- \otimes \mathbb{Q}_\ell : \mathcal{M}_k,\mathbb{Q} \to \text{Rep}_{\mathbb{Q}_\ell}(G_k)$$

for all $\ell \neq \text{char}(k)$ such that

$$H^i_{\text{mot}}(X) \otimes \mathbb{Q}_\ell \cong H^i_\ell(X)$$

**Example**

Can construct a category of 1-motives $\mathcal{M}_k^{\leq 1},\mathbb{Q}$ ‘by hand’ independence results for curves and abelian varieties.
Now take $F$ a local field with finite residue field $k$, $\ell \neq \text{char}(k)$.

**Theorem (Grothendieck)**

*Every $\ell$-adic representation of $G_F$ is quasi-unipotent.*

Can use this to construct

$$\text{WD} : \text{Rep}_{\mathbb{Q}_\ell}(G_F) \to \text{Rep}_{\mathbb{Q}_\ell}(\text{WD}_F)$$

with target the category of Weil–Deligne representations. These are continuous representations

$$\rho : W_F \to \text{GL}(V)$$

of the Weil group (for the discrete topology on $V$) together with a nilpotent map $N : V \to V(1)$. 
Conjecture (Fontaine $C_{\text{WD}}(X, i)$)

$X/F$ variety, $i \geq 0$. Then for any $\ell, \ell' \neq p$ and any alg. closed field $\Omega \supset \mathbb{Q}_\ell, \mathbb{Q}_{\ell'}$ we have

$$\text{WD}(H^i_\ell(X)) \otimes \Omega \cong \text{WD}(H^i_{\ell'}(X)) \otimes \Omega$$

as object of $\text{WD}_\Omega(W_F)$.

Conjecture (Fontaine $C_{\text{WD}}(X, i)_{\text{faible}}$)

Same, but replacing $\text{WD}(H^i_\ell(X))$ with $\text{WD}(H^i_\ell(X))^{F-ss}$.

For any family of Weil–Deligne representations $\{E_\ell\}_{\ell \in \mathcal{P}}$ we will say that they are (weakly) independence of $\ell$ if they satisfy the above conjecture.
If $V$ is a Weil–Deligne representation, $\exists$! increasing filtration $M_\bullet V$ such that

$$N^k : \text{Gr}_k^M V \sim \to \text{Gr}_{-k}^M V(k)$$

**Lemma (Deligne)**

A family $\{E_\ell\}_{\ell \in \mathbb{P}}$ of Weil–Deligne representations is weakly independent of $\ell$ iff $\forall k \in \mathbb{Z}$

$$\text{Tr}(\cdot | \text{Gr}_k^M E_\ell) : \mathcal{W}_F \to \mathbb{Q}_\ell$$

takes values in $\mathbb{Q}$ and is independent of $\ell$.

Today: concentrate on the case when $F \cong k((t))$ is a local field of equicharacteristic.

1. How to extend these conjectures to include $\ell = p$?
2. Prove them when $X/F$ is smooth and proper.
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$k = \text{finite field, characteristic } p, \ F \cong k((t)), \ K = W(k)[1/p],\ 
\sigma = \text{Frobenius. For } \ell \neq p \text{ the functor}

$$WD : \text{Rep}_{\mathbb{Q}_\ell}(G_F) \to \text{Rep}_{\mathbb{Q}_\ell}(WD_F)$$

arises from the $\ell$-adic local monodromy theorem \( \rightsquigarrow \) want to replace this with the $p$-adic monodromy theorem.

Definition

The Robba ring $\mathcal{R}$ over $K$ is the ring of analytic functions over $K$ convergent on some half-open annulus $\{\eta \leq |t| < 1\}$.

Have a Frobenius $\sigma : \mathcal{R} \to \mathcal{R}$ and a derivation $\partial_t : \mathcal{R} \to \mathcal{R} \rightsquigarrow$ notion of a $(\varphi, \nabla)$-module over $\mathcal{R}$. Denote the category $\mathbf{M} \Phi_{\nabla, \mathcal{R}}$. 
Theorem (André, Mebkhout, Kedlaya)

Every $(\varphi, \nabla)$-module $M$ over $\mathcal{R}$ is quasi-unipotent.

The theorem means that after making a finite separable extension of $F = k((t))$, and formally adjoining $\log t$, $M$ admits a basis of horizontal sections.

Corollary

Let $K^{\text{un}}$ denote the maximal unramified extension of $K$. Then there exists an exact, faithful functor

\[ \text{M} \Phi_{\mathcal{R}} \rightarrow \text{Rep}_{K^{\text{un}}}(\text{WD}_F) \]
So what we want a theory of $p$-adic cohomology landing in the category $\mathcal{M}_{\Phi_\nabla}$, modelled on rigid/crystalline cohomology.

Set

$$\mathcal{E} := \hat{\mathcal{W}}[t][t^{-1}][1/p]$$

this is a complete DVF with residue field $F = k((t))$.  
$\Rightarrow$ rigid cohomology for varieties over $F$ is a functor

$$H^*_\text{rig}(-/\mathcal{E}) : \text{Var}_F \to \mathcal{M}_{\Phi_\nabla}$$

to $(\varphi, \nabla)$-modules over $\mathcal{E}$. 
Note that we can write

\[ \mathcal{R} = \left\{ \sum_{i \in \mathbb{Z}} a_i t^i \left| \begin{array}{c}
\forall \rho < 1, \quad |a_i| \rho^i \to 0 \text{ as } i \to \infty \\
\exists \lambda < 1 \text{ s.t. } |a_i| \lambda^i \to 0 \text{ as } i \to -\infty
\end{array} \right. \right\} \]

\[ \mathcal{E} = \left\{ \sum_{i \in \mathbb{Z}} a_i t^i \left| \begin{array}{c}
\sup_i |a_i| < \infty \\
|a_i| \to 0 \text{ as } i \to -\infty
\end{array} \right. \right\} \]

So that \( \mathcal{E} \not\subseteq \mathcal{R} \) and \( \mathcal{R} \not\subseteq \mathcal{E} \).

**Definition**

\[ \mathcal{E}^\dagger := \mathcal{E} \cap \mathcal{R} = \left\{ \sum_{i \in \mathbb{Z}} a_i t^i \left| \begin{array}{c}
\sup_i |a_i| < \infty \\
\exists \lambda < 1 \text{ s.t. } |a_i| \lambda^i \to 0 \text{ as } i \to -\infty
\end{array} \right. \right\} \]
\( \mathcal{E}^\dagger \) is a Henselian DVF with residue field \( F \), and we have

\[
\begin{array}{c}
\text{MΦ}^\triangledown_{\mathcal{E}^\dagger} \\
\text{MΦ}^\triangledown_{\mathcal{E}} \\
\text{MΦ}^\triangledown_R \\
\end{array}
\]

**Theorem (Kedlaya)**

The functor \( \text{MΦ}^\triangledown_{\mathcal{E}^\dagger} \rightarrow \text{MΦ}^\triangledown_{\mathcal{E}} \) is fully faithful, and if \( X \in \text{Var}_F \) is smooth and proper, \( H^i_{\text{rig}}(X/\mathcal{E}) \) is in the essential image.

Should think of \( \text{MΦ}^\triangledown_{\mathcal{E}^\dagger} \rightarrow \text{MΦ}^\triangledown_{\mathcal{E}} \) as analogous to the inclusion \( \text{Rep}^{\text{pst}}_{\mathbb{Q}_p}(G_K) \subset \text{Rep}_{\mathbb{Q}_p}(G_K) \).
Theorem (L., Pál)

Rigid cohomology descends to the bounded Robba ring \( \mathcal{E}^\dagger \), in other words \( \exists \) functor

\[
H^\ast_{\text{rig}}(-/\mathcal{E}^\dagger) : \text{Var}_F \rightarrow M\Phi^\nabla_{\mathcal{E}^\dagger}
\]

satisfying all the axioms of an ‘extended’ Weil cohomology theory, whose base change to \( \mathcal{E} \) is isomorphic to \( H^\ast_{\text{rig}}(-/\mathcal{E}^\dagger) \).

There also are versions with compact support, as well as support in a closed subscheme, and categories of coefficients \((F-)\text{Isoc}^\dagger(X/\mathcal{E}^\dagger)\) and \((F-)\text{Isoc}^\dagger(X/K)\) for this theory.
Corollary

Let $X/F$ be a variety, then we can define a $p$-adic Weil–Deligne representation $H_p^i(X)$ associated to $X$ via

$$H_{\text{rig}}^i(X/R) := H_{\text{rig}}^i(X/E^\dagger) \otimes R.$$ 

Hence we can extend Fontaine’s conjectures $C_{\text{WD}}(X,i)$ and $C_{\text{WD}}(X,i)_{\text{faible}}$ to include $\ell = p$.

Note that the $p$-adic Weil–Deligne representations are defined over $\mathbb{Q}_p' := K^\text{un}$. Set $\mathbb{Q}_\ell' = \mathbb{Q}_\ell$ if $\ell \neq p$. 
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Theorem (Chiarellotto, L.)

Let $X/F$ be smooth and proper, and $i \geq 0$. Then $C_{WD}(X, i)_{faible}$ holds.

The first key step consists of reducing to the following case.

Definition

Let $X/F$ be smooth and proper, with semistable reduction. We say that $F$ is globally defined if there exists a smooth curve $C$ over $k$, a rational point $c \in C(k)$ with $\overline{k(C)}_c \cong F$ and a proper, flat scheme $\mathcal{X} \to C$, smooth away from $c$ and semistable at $c$, such that $\mathcal{X} \times_C \text{Spec}(F) \cong X$. 
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Proposition

Suppose that $C_{\text{WD}}(X, i)_{\text{faible}}$ holds for all smooth and proper $F$-varieties which are semistable and globally defined. Then $C_{\text{WD}}(X, i)_{\text{faible}}$ holds for all smooth and proper $F$-varieties.

Ingredients:

1. Alterations
2. Weight-monodromy conjecture
3. Cohomological descent
4. ‘Spreading out’ lemma
5. Uniqueness of ‘geometric weight filtrations’

Lemma

Assume that $\mathcal{X} \to \text{Spec}(\mathcal{O}_F)$ is semistable, and $n \geq 1$. Then there exists a globally defined semistable scheme $\mathcal{Y} \to \text{Spec}(\mathcal{O}_F)$ such that $\mathcal{X} \otimes \mathcal{O}_F/t^n \cong \mathcal{Y} \otimes \mathcal{O}_F/t^n$. 
The proof of $C_{WD}(X, i)_{faible}$ for smooth and proper $F$-varieties therefore reduces to the following.

**Theorem**

Let $C/k$ be a smooth curve $c \in C(k)$, $U = C \setminus c$, $F = \widehat{k(C)}_c$. Suppose that $\{\mathcal{F}_\ell\}_\ell$ is a collection of local systems on $U$, such that for all $u \in U$

$$\text{Tr}(\cdot | \mathcal{F}_\ell, \bar{u}) : \text{Frob}^Z_u \rightarrow \mathbb{Q}'_\ell$$

takes values in $\mathbb{Q}$ and is independent of $\ell$. Then for all $k \geq 0$

$$\text{Tr}(\cdot | \text{Gr}_k^M \mathcal{F}_\ell, \bar{c}) : \mathcal{W}_F \rightarrow \mathbb{Q}'_\ell$$

takes values in $\mathbb{Q}$ and is independent of $\ell$.

This was proved by Deligne for $\ell \neq p$; to include $\ell = p$ we use the theory of arithmetic $\mathcal{D}^\dagger$-modules.
What about mixed characteristic local fields? The weight monodromy conjecture is used in two key places, but the rest of the proof should work. So for smooth and proper varieties, $C(X, i)_{faible}$ would follow from the weight monodromy conjecture for $X$.

There are also results for proper varieties.

**Theorem**

Let $X/F$ be proper, and $k \in \mathbb{Z}$. Then

$$
\sum (-1)^i \text{Tr}(\cdot | \text{Gr}_k^M H^i_\ell(X)) : W_F \to \mathbb{Q}_\ell
$$

has values in $\mathbb{Q}$ and is independent of $\ell$. 
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What about other invariants such as homotopy groups?

**Example**

Unipotent $\pi_1$ is expected to be ‘motivic’ $\leadsto$ should have ‘$\ell$-independence’ results for this.

So let $X$ be a pointed variety over $F$.

**Definition**

For $\ell \neq p$ define $\pi_1^\ell(X)$ to be the $\mathbb{Q}_\ell$-pro-unipotent completion of $\pi_1^{\text{ét}}(X_F)$. This comes with an action of $G_F$. 
When $\ell = p$ need to use Tannakian methods.

**Definition**

We define $\pi_1^p(X)$ to be the Tannaka dual of the category $\mathcal{N}\text{Isoc}^\dagger(\mathcal{X}/\mathcal{E}^\dagger)$ of unipotent overconvergent isocrystals on $\mathcal{X}/\mathcal{E}^\dagger$.

Thus $\pi_1^p(X)$ is a (pro-unipotent) affine group scheme over $\mathcal{E}^\dagger$.

**Theorem (L.)**

The group scheme $\pi_1^p(X)$ has a canonical structure as a ‘non-abelian’ $(\varphi, \nabla)$-module over $\mathcal{E}^\dagger$. 
Let $\ell$ be any prime. Set $L_\ell := \text{Lie}(\pi_1^\ell(X))$, $\mathcal{U}_\ell := \mathcal{U}(L_\ell)$, $a_\ell :=$ augmentation ideal.

$$\Rightarrow \mathcal{U}_\ell / a_\ell^k \in \text{Rep}_{\mathbb{Q}_\ell}(G_F) \ (\ell \neq p)$$

$$\mathcal{U}_p / a_p^k \in \text{M}_{\Phi_E}^\nabla$$

Conjecture ($C_{WD}(X, \pi_1)$)

For all $k \geq 1$ the Weil Deligne representations associated to $\mathcal{U}_\ell / a_\ell^k$ are independent of $\ell$.

Over finite fields, can prove Frobenius semisimplicity for $\mathcal{U}_\ell / a_\ell^k$. 
Theorem (Chiarellotto, L.)

Assume that $X$ is smooth and proper over $F$ with semistable reduction. Then $C_{\text{WD}}(X, \pi_1)$ holds.

As before, we reduce to the ‘globally defined’ case, and then show that the $\mathcal{U}_\ell / \alpha_\ell^k$ can be ‘spread out’ to local systems on some global model $C$ of $F$.

Questions

1. Can we remove the semistable hypothesis?
2. Does the argument work for mixed characteristic local fields? (We know the weight-monodromy conjecture for $H^1$.)
Thank-you!