Noisy Patterns

Bridging the Gap between Stochastics and Dynamics

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The moving power of mathematical invention
is not reasoning but imagination
– August De Morgan

Shut up and calculate
– Richard Feynman

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1 Introduction

Let us go back a few short moments in time, to the moment you decided to pick up this thesis to read it (or to just browse through, wondering if all those scribbles actually mean something). What happened between the moment you thought of picking it up and the moment your arm started to move? A small electric pulse travelled through your musculocutaneous nerve to bring the signal from your head to your arm. From our experience, we know that this always works in a healthy person, which is a small miracle if one thinks about the challenges such a pulse meets on its way. It has to hop from node of Ranvier to node of Ranvier, jumping over myelin sheaths, triggering ion pumps all along the way to boost the signal. Our intuition tells us that the signal propagation must be robust, as we know that the route from the brain to the muscle is not a perfectly straight and smooth highway. Not every cell the pulse encounters on the way is identical, there will be random external fluctuations in the potentials, or, in other words, the route will be noisy. Despite these challenges, the pulse will make it to the end.

Here we hit the core of the problems discussed in this thesis: How can a pulse, or more generally a pattern, travel in a noisy environment. Hence the title ‘Noisy Patterns’. Whenever I explain this story to non-mathematicians, the follow-up question is often ‘so you work together with biologists?’ The question to that answer goes along the line ‘not yet, but maybe, somewhere, once, in the future.’ The goal of this thesis is not to study pulses in human nerve cells, our goal is much more basic, but not less challenging! Our goal is the following: Can we develop the mathematical tools and machinery needed to rigorously study travelling waves in stochastic reaction-diffusion equations, in the same way we can study and understand their deterministic counterparts?

Over the past four years, we set several steps in this direction, resulting in multiple papers that form the core of this thesis. These results are certainly not the first results on stochastic reaction-diffusion equations, but what sets our results apart is the fact that they are firmly rooted in dynamical systems theory, or to be more specific, rooted in the language of travelling waves familiar to the nonlinear waves family that Leiden is a part of. This thesis will not contain any new theorems on the existence of solutions nor on the existence of invariant measures. No, the questions we will ask ourselves are the same as our community has always asked: can we show the existence of travelling waves, can we compute their shape and speed and can we study their stability? The answers to these questions are spread out over this thesis, but informally, we can formulate the following theorem:

Theorem. *Under certain technical assumptions, travelling waves or pulses in bistable reaction-diffusion equations persist on exponentially long timescales when the equation is forced by a small multiplicative noise term. The average wave speed and shape are close to the deterministic values and can be expanded systematically around the deterministic wave using Taylor series.*

We now give two main examples of stochastic partial differential equations (SPDEs) to which the theorem above applies. The first example is the stochastic Nagumo[1] equation,

$$u_t = \rho u_{xx} + u(1-u)(u-a) + \sigma u(1-u)\,\xi(x,t), \qquad x \in \mathbb{R},\, t \in \mathbb{R}^+, \tag{1.0.1}$$

and the second example is the two-component FitzHugh-Nagumo (FHN) equation

$$\begin{aligned} u_t &= \rho_1 u_{xx} + u(1-u)(u-a) - v + \sigma u\,\xi(x,t) \\ v_t &= \rho_2 v_{xx} + \varepsilon(u - \gamma v) \end{aligned} \tag{1.0.2}$$

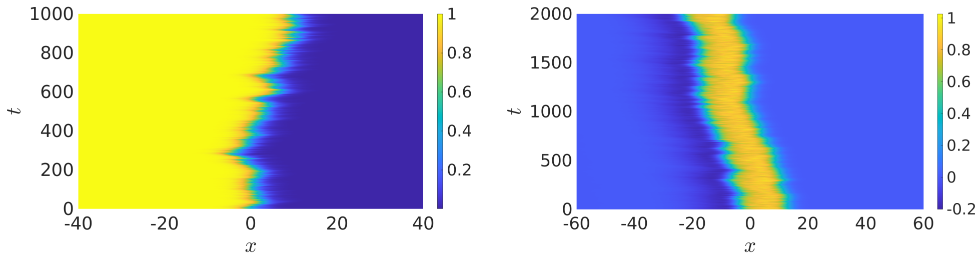
for positive $\rho_1, \rho_2, \varepsilon, \gamma$ and $a \in (0,1)$. Here ξ is a Gaussian process with

$$\begin{aligned} E[\xi(x,t)] &= 0, \\ E[\xi(x,t)\xi(x',t')] &= \delta(t-t')q(x-x'). \end{aligned} \tag{1.0.3}$$

The last line means that the noise is uncorrelated in time and correlated in space with correlation function q, typically a Gaussian. The parameter σ indicates the noise strength and will be assumed to be small. For both equations, the existence of a stable travelling wave/pulse is known in the deterministic case [60, 113] and for the Nagumo equation the profile and speed can be computed explicitly, see Chapter 2. Remark that the noise term is multiplicative, i.e. the noise term ξ is multiplied by a function depending on u. Furthermore, note that these functions are chosen in such a way that the noise disappears in the background states of the wave (0 and 1 for the Nagumo equation and 0 for the FHN-equation). This is an assumption we will make throughout this thesis. There are two main advantages to this assumption, a technical and practical one. From a technical point of view, this type of multiplicative noise ensures that deviations from the travelling wave disappear at $\pm\infty$, hence making the deviations integrable. From a more practical viewpoint, additive noise can cause the wave to disappear or cause new waves to form [79]. This makes the study of a single travelling wave significantly more difficult.

We illustrate the practical questions we will answer in this thesis by showing two realizations of the equations above in Figure 1.1. For $\sigma = 0$, the waves are perfectly frozen in the $x - ct$ frame where c is the speed of the wave, but this is clearly not the case for $\sigma \neq 0$. Our main goal is to understand these pictures, which leads to the following natural questions:

[1] Nagumo is just one of many names attached to this equation. It is also known as the Allen-Cahn or Huxley equation [100], while in biology it is referred to as an extended Fisher-Kolmogorov equation with strong Allee effect [13] and in chemistry the term Schlögl model is used. The Nagumo equation is a specific example of a bistable reaction-diffusion equation.



(a) (b)

Figure 1.1: Figure (a) shows a realization of equation (1.0.1) in the Stratonovich interpretation for $\rho = 1$, $a = 0.25$ and $\sigma = 0.5$, in a frame that moves with the deterministic speed. A clear drift to the right is visible. Figure (b) shows a realization of equation (1.0.2) for $\rho_1 = 1$, $\rho_2 = 0.01$, $a = 0.1$, $\varepsilon = 0.01$, $\gamma = 5$ and $\sigma = 0.1$ again in a frame that moves with the deterministic speed. This time a drift to the left is visible.

- Where is the pulse going? Is it drifting in a certain direction, and with which speed?
- What is the average shape of the pulse?
- How long can the pulse survive before the noise destroys it?

In order to answer these questions rigorously, we first need tools to understand equations such as (1.0.1) and (1.0.2). What is this object ξ? And how would we define a solution to these equations? We will introduce the necessary concepts to understand these equations in the next section. In the rest of this introduction, we will give a short introduction to the classic deterministic theory on travelling waves, followed by an overview of what is know for stochastic travelling waves. Then we will discuss the main difficulties in understanding stochastic travelling waves, which will explain why this thesis has become so heavy. We will end this introduction with a reading guide and an outlook.

1.1 Infinite dimensional stochastic integration

In the mathematical literature, equations such as (1.0.1) are often written[2] as

$$dU = [\rho U_{xx} + f(U)]dt + \sigma g(U) dW_t^Q. \tag{1.1.1}$$

This notation is a shorthand for

$$U(t) = U(0) + \int_0^t \rho U_{xx}(s) + f(U(s))ds + \sigma \int_0^t g(U(s))dW_s^Q. \tag{1.1.2}$$

[2] We will write deterministic equations and stochastic equations from the physics literature with small letters, but equations in Itô or Stratonovich interpretation always with capitals.

The question that arises immediately is 'what is W_t^Q and how is it related to $\xi(x,t)$?' To answer this question we need the concept of a stochastic integral in an infinite dimensional Hilbert space, but that is not an easy task. The fact that this subject is hard, even if one understands ordinary Itô integrals, explains why Chapters 2 and 3 are written for a single Brownian motion. We first had to study the travelling waves in a simpler framework before we could lift it to the full infinite dimensional case. The integration theory that we will outline below is not new, but readable introductions are scarce. A classic work on the subject is the book[3] by Da Prato and Zabchyk [92], and a more detailed study of Gaussian processes, the construction of the stochastic integrals and applications to SPDEs can be found in [77, 93]. A slightly lighter introduction to start with is [23]. An extensive survey of different approaches to stochastic integrals is given in [29] and a description of the specific type of noise that will become important in Chapter 4 can be found in [90]. It is also worthwhile to study Martin Hairer's lecture notes on SPDEs [44].

1.1.1 One dimensional Itô integral

The subject of stochastic integration theory basically evolves around the following question: Suppose I have a Brownian motion, or Wiener process, β_t and some function $f : [0,T] \times \Omega \to \mathbb{R}$. Under which conditions on f can we define

$$I(t) = \int_0^t f(s) d\beta_s \tag{1.1.3}$$

and what are the properties of $I(t)$? First we should notice is that β, f and I are all functions from $[0,T] \times \Omega$ to $\mathbb{R}$. This means that in principle we should write $I(t,\omega)$ and $f(s,\omega)$, but we will suppress the ω-dependence to avoid the notation becoming to cluttered as is common in the literature. There are multiple ways to define the stochastic integral above, most notably the Itô integral and the Stratonovich integral. Throughout this thesis we will use the Itô interpretation of the integral as it has better properties from a mathematical viewpoint, but the Stratonovich interpretation is often more physically relevant [110]. We will not construct the Itô integral, as there are many decent books on this subject [88], but we will record the most important results here.

Theorem 1.1.1. *Take a Brownian motion β_t on $[0,T]$ together with its natural filtration $\mathcal{F}_t$. Then, for every $f \in L^2([0,T] \times \Omega, \mathbb{R})$ that is adapted to $\mathcal{F}_t$, the Itô integral*

$$I(t) = \int_0^t f(s) d\beta_s \tag{1.1.4}$$

is well defined in $L^2([0,T] \times \Omega, \mathbb{R})$. Furthermore, $I(t)$ is a martingale with respect to $\mathcal{F}_t$, meaning that for all $0 \leq s \leq t \leq T$ we have

$$E[I(t)|\mathcal{F}_s] = I(s) \tag{1.1.5}$$

[3] Be sure to get the updated second edition.

and $I(t)$ satisfies the Itô isometry:

$$E\left[\left(\int_0^t f(s)d\beta_s\right)^2\right] = E\left[\int_0^t f(s)^2 ds\right]. \tag{1.1.6}$$

The martingale property and the Itô isometry will later turn out to be essential tools in studying the properties of stochastic travelling waves. The first because it ensures us that the average of all the stochastic integrals we encounter in this thesis is always 0, and the second because it will allow us to compute the average of the norm of solutions to our SPDEs.

1.1.2 Infinite dimensional Wiener processes

When dealing with SPDEs, we must choose in which space the driving noise of the equation lives and, to begin simple, we start with $\mathbb{R}$, as in the previous paragraph. For a function $f : [0,T] \times \Omega \to H$ for some separable Hilbert space H, we can easily define

$$I(t) = \int_0^t f(s)d\beta_s \tag{1.1.7}$$

by mixing the Itô integral with the Bochner integral. $I(t)$ is still a martingale (in H) and the Itô isometry now becomes

$$E\left[\|\int_0^t f(s)d\beta_s\|_H^2\right] = E\left[\int_0^t \|f(s)\|_H^2 ds\right]. \tag{1.1.8}$$

We simply replaced the Euclidean norm in $\mathbb{R}$ with the norm on H. This is the type of integral we use in Chapter 2 and 3. It allowed us to develop our methods without being too distracted by complicated computations and more abstract integration theory.

The main problem is that in this framework we only deal with noise in time, not in space, while this is often relevant in applications. In order to build a concept of a stochastic integral where the noise depends on space and time, one needs an equivalent of Brownian motion in a function space. The theory we explain here following [93] works in general for separable Hilbert spaces, but we will for clarity stick with $L^2(\mathbb{R})$ or $L^2(D)$ for $D \subset \mathbb{R}$.

We now wish to define an equivalent of β_t on $L^2(D)$. This process, which we will call a Q-Wiener process W_t^Q, should be completely decorrelated in time (i.e. white in time) but can have correlation in space (i.e. coloured in space), where the operator $Q : L^2(D) \to L^2(D)$ in the notation W_t^Q describes the correlation. We call Q the covariance operator. Mathematically speaking, this means that W_t^Q should, for all $v, w \in L^2(D)$, satisfy

$$E[\langle W_t^Q, v\rangle_{L^2(D)}\langle W_s^Q, w\rangle_{L^2(D)}] = s \wedge t\langle Qv, w\rangle_{L^2(D)}. \tag{1.1.9}$$

The question is if we can find operators Q for which this process exists in $L^2(D)$. To investigate this question, we make the strong assumption that Q on $L^2(D)$ is Hilbert-Schmidt, linear, symmetric, and positive semi-definite[4]. Then we know that Q is

[4] Note that linear, symmetric and positive semi-definite are essential for Q to make sense as a covariance operator.

compact and of trace class and there is an orthonormal basis (e_k) in $L^2(D)$ such that $Qe_k = \lambda_k e_k$ and $\mathrm{Tr}(Q) = \sum_k \lambda_k < \infty$ [94, Ch. 6]. This allows us to define (i.e. we can prove that it converges) the following process in $L^2(D)$:

$$W_t^Q = \sum_{k=0}^{\infty} \sqrt{\lambda_k} e_k \beta_k(t), \tag{1.1.10}$$

where all the $\beta_k(t)$ are i.i.d. standard Brownian motions. On the other hand, we call a process W_t^Q a Gaussian process in $L^2(D)$ with mean 0 and covariance Q when for all $w, v \in L^2(D)$ we have

$$E[\langle W_t^Q, v\rangle_{L^2(D)} \langle W_s^Q, w\rangle_{L^2(D)}] = s \wedge t \langle Qv, w\rangle_{L^2(D)}. \tag{1.1.11}$$

We can check that W_t^Q as defined in (1.1.10) is a Gaussian process by simply plugging the basis expansion into the equation above and using the fact that $E[\beta_m(t)\beta_n(s)] = t \wedge s\, \delta_{mn}$. The converse is also true. By [93, Prop. 2.1.10] we know that for every Q of trace class, the process defined by (1.1.11) can be represented by a direct sum as in (1.1.10). Hence, all Gaussian processes on $L^2(D)$ are characterized by a covariance operator with finite trace and we have an explicit representation. Therefore, the definition of the integral of f over a Q-Wiener process is now straightforward:

$$I(t) = \int_0^t f(s) dW_s^Q = \sum_{k=0}^{\infty} \sqrt{\lambda_k} \int_0^t f(s) e_k d\beta_k(s). \tag{1.1.12}$$

The stochastic integral can hence be constructed as an infinite sum of well understood 1D stochastic integrals. Following [93], we can again show that $I(t)$ is a martingale and satisfies the following version of the Itô isometry:

$$E\left[\| \int_0^t f(s) dW_s^Q \|_{L^2(D)}^2 \right] = E \sum_k \lambda_k \left[\int_0^t \|f(s) e_k\|_{L^2(D)}^2 ds \right]. \tag{1.1.13}$$

This shows us that the integral is defined for a wide range of functions, for example the integral is well defined for $f = 1$, even though this is not an $L^2(D)$ function when $D = \mathbb{R}$. We can define the class of allowed integrands more precisely by rewriting the Itô isometry in the following way:

$$\begin{aligned} E\left[\| \int_0^t f(s) dW_s^Q \|_{L^2(D)}^2 \right] &= E\left[\sum_k \int_0^t \|f(s) \sqrt{Q} e_k\|_{L^2(D)}^2 ds \right] \\ &:= E \int_0^t \|f(s)\sqrt{Q}\|_{HS(L^2(D))}^2 ds. \end{aligned} \tag{1.1.14}$$

Here $\|\cdot\|_{HS(L^2(D))}$ is the Hilbert-Schmidt operator norm on $L^2(D)$. In other words, the integral is well defined when $f(s)\sqrt{Q}$ is a Hilbert-Schmidt operator on $L^2(D)$ and $\|f(\cdot)\sqrt{Q}\|_{HS(L^2(D))}$ is square integrable.

Characterizing Q Now suppose Q has an integration kernel $q(x, y)$, i.e. for $v \in L^2(D)$ we have

$$Qv(x) = \int_D q(x, y)v(y)dy. \tag{1.1.15}$$

Then it must hold that $q \in L^2(D \times D)$ [94, Ch. 6]. When we (formally) replace v and w by $\delta(\cdot - x)$ and $\delta(\cdot - y)$ in the definition (1.1.11) of a Gaussian process, we find that

$$E[\langle W_t^Q, \delta(\cdot - x)\rangle_{L^2(D)} \langle W_s^Q, \delta(\cdot - y)\rangle_{L^2(D)}] = s \wedge t q(x, y). \tag{1.1.16}$$

In applications, this definition of the covariance is often written as

$$\langle dW^Q(x, t) dW^Q(y, s)\rangle = \delta(t - s)q(x, y). \tag{1.1.17}$$

Therefore, we need to make a distinction between the covariance Q (the covariance operator) and the covariance q (the kernel of the covariance operator). There is a clear reason why the applied literature chooses to work with the definition above instead of going for the more abstract definition via Gaussian processes on Hilbert spaces. The function q describes the correlation between two points in space and therefore has a clear interpretation and could in principle be measured. However, equation (1.1.17) will never be found in this form in the applied literature, as q is always chosen to be translation invariant, i.e. $q(x, y) = q(x - y)$. Hence, in applications it is assumed that the correlation depends only on the distance between two points.

A translation invariant kernel can only be an element of $L^2(D \times D)$ when D is finite. Therefore: A Gaussian process in $L^2(\mathbb{R})$ has a covariance operator Q with a kernel that is **not** translation invariant. The conclusion is that in order to model translation invariant noise on $\mathbb{R}$, and hence model physically relevant problems such as our two main examples (1.0.1) and (1.0.2), we need to study a bigger class of stochastic processes. More precisely, we need to study processes with a covariance operator that is not necessarily of trace class. These type of processes are known as a cylindrical Gaussian process. This implies that we need to study processes in larger spaces then $L^2(\mathbb{R})$, i.e. in distribution spaces or any type of abstract completion of $L^2(\mathbb{R})$.

1.1.3 Cylindrical Wiener processes and cylindrical integrals

In the previous section, we first defined Q to be an operator of trace class, constructed a Wiener process W_t^Q and then constructed the stochastic integral. The integrability condition, $f\sqrt{Q} \in L^2([0, T]; HS(L^2(\mathbb{R})))$, is then in general easy to check because Q itself is already of trace class. But what if we drop the assumption that Q is of trace class? The Itô isometry still makes sense under the same integrability condition, as long as we write $\sqrt{Q}e_k$ for $\sqrt{\lambda_k}e_k$ because an orthonormal basis for Q does not necessarily exist anymore. However, the construction of W_t^Q does not make sense without the basis representation for Q, nor does the stochastic integral.

A rather straightforward way to solve this problem would be the following procedure: Choose an explicit distribution space that contains $L^2(\mathbb{R})$ such that you can show that the sum (1.1.10) does converge, repeat the process of constructing and classifying the Gaussian process, constructing the integral, and then project the integral back into

$L^2(\mathbb{R})$. Such an approach can be found in [90], allowing for the explicit construction of a class of translation invariant processes on $L^2(\mathbb{R})$. However, we do not have to follow this procedure, as it turns out that the stochastic integral does not depend on the choice of completion of $L^2(\mathbb{R})$ we take, and hence we do not need to specify this abstract space. To make this precise, pick a non-negative symmetric operator $Q \in \mathcal{L}(L^2(\mathbb{R}), L^2(\mathbb{R}))$, possibly not of trace class. We then write[5]

$$L^2_Q(\mathbb{R}) = Q^{1/2}(L^2(\mathbb{R})), \tag{1.1.18}$$

which is again a separable Hilbert space with inner product

$$\langle v, w\rangle_{L^2_Q(\mathbb{R})} = \langle Q^{-1/2}v, Q^{-1/2}v\rangle_{L^2(\mathbb{R})}. \tag{1.1.19}$$

Furthermore, when (e_k) is an orthonormal basis of $L^2(\mathbb{R})$, $(\sqrt{Q}e_k)$ is a basis for $L^2_Q(\mathbb{R})$. Next, we define a Hilbert space $L^2_{\text{ext}}(\mathbb{R}) \supset L^2(\mathbb{R})$ such that the embedding $L^2_Q(\mathbb{R}) \subset L^2_{\text{ext}}(\mathbb{R})$ a Hilbert-Schmidt operator from $L^2_Q(\mathbb{R})$ to $L^2_{\text{ext}}(\mathbb{R})$. Such a larger Hilbert space always exists [93, §2.5], but is not necessarily unique. This extension space is the key ingredient that allows our noise process to be rigorously constructed. Following [66, eq. (2)], we introduce the sum below in $L^2_Q(\mathbb{R})$:

$$W^Q_{t,n} = \sum_{k=0}^{n} \sqrt{Q}e_k\beta_k(t). \tag{1.1.20}$$

This sum converges for $n \to \infty$ in $L^2_{\text{ext}}(\mathbb{R})$ for every $t \geq 0$. We will refer to this limiting process W^Q_t as a cylindrical Q-Wiener process in $L^2(\mathbb{R})$, but it is a regular Wiener process in $L^2_{\text{ext}}(\mathbb{R})$. Note that we call the process W^Q_t a cylindrical process in $L^2(\mathbb{R})$, even though it does not attain values in $L^2(\mathbb{R})$.

The stochastic integral over this cylindrical process can now be defined as the standard Itô integral over the Wiener process in $L^2_{\text{ext}}(\mathbb{R})$, and then be pulled back into $L^2(\mathbb{R})$. The important fact here is that the resulting integral in $L^2(\mathbb{R})$ does not depend on the choice of $L^2_{\text{ext}}(\mathbb{R})$! The take-home message in this section is therefore as follows: Even though W^Q_t and the integral over this process need to be defined in abstract spaces, in daily practice we can ignore this and treat the cylindrical integral as an ordinary integral, as long as we realise that we cannot simplify Qe_k to $\lambda_k e_k$. A more detailed explanation of this subject can be found in §4.5.1

Some remarks on conventions Now suppose we have the Itô integral

$$\int_0^t f(s)dW^Q_s. \tag{1.1.21}$$

In the previous paragraph, we learned that the integral is well defined when $\|f\sqrt{Q}\|_{HS(L^2(\mathbb{R}))}$ is finite. In the literature, this is often written as $\|f\|_{L^0_2(L^2(\mathbb{R}))} < \infty$, which is defined as

$$L^0_2(L^2(\mathbb{R})) = L_2(L^2_0(\mathbb{R}), L^2(\mathbb{R})) := HS(L^2_Q(\mathbb{R}), L^2(\mathbb{R})). \tag{1.1.22}$$

[5] In the literature, the pair $(L^2_Q(\mathbb{R}), L^2(\mathbb{R}))$ is often denoted as (U_0, U), but in our setting this might be confusing with the solution $U(t)$.

The space $L_2(L^2(\mathbb{R}), L^2(\mathbb{R}))$ is the standard space of Hilbert-Schmidt operators, so the superscript 0 indicates the weighting of the norm with $\sqrt{Q}$.

There are many more ways in the literature to write down the condition $\|f\|^2_{L_2^0} < \infty$. Often, integral (1.1.21) is understood as

$$\int_0^t f(s)dW_s^Q = \int_0^t \tilde{f}(s)dW_s, \tag{1.1.23}$$

where W_t is the standard cylindrical Wiener process ($Q = I$) and $\tilde{f} = f\sqrt{Q}$ has to be an element of $L_2(L^2(\mathbb{R}), L^2(\mathbb{R}))$, which is the same requirement f being an element of $L_2^0(L^2(\mathbb{R}))$. Note that in the literature, stochastic terms are often introduced as BdW_t, i.e. a HS-operator B multiplied by spacetime white noise. The integral $\int_0^t BdW_s$ is a B^2-Wiener process, so BdW_t can be understood as $dW_t^{B^2}$, but B is still called the covariance [70]. Analogous, sometimes (see e.g. [72]) q is chosen to be the kernel of $\sqrt{Q}$. If we denote the kernel of $\sqrt{Q}$ by $p(x, y)$ we can relate q and p by

$$q(x, y) = \int_{\mathbb{R}} p(x, z)p(z, y)dz, \tag{1.1.24}$$

or, in the case of translational invariant kernels, simply as

$$q = p * p. \tag{1.1.25}$$

1.1.4 Stochastic PDEs

With the theory of the previous section, we can finally understand equations such as (1.0.1) and (1.0.2) in a rigorous framework. For example, the stochastic Nagumo equation (1.0.1)

$$u_t = \rho u_{xx} + f(u) + \sigma g(u)\xi(x, t), \tag{1.1.26}$$

can now be understood to be the following equation in Itô interpretation

$$dU = [\rho U_{xx} + f(U)]dt + \sigma g(U)dW_t^Q, \tag{1.1.27}$$

or, formulated in Stratonovich interpretation, to be

$$dU = [\rho U_{xx} + f(U)]dt + \sigma g(U) \circ dW_t^Q. \tag{1.1.28}$$

The Stratonovich equation can again be written in Itô formulation [108]:

$$dU = [\rho U_{xx} + f(U) + \frac{\sigma^2}{2} q(0)g'(U)g(U)]dt + \sigma g(U)dW_t^Q. \tag{1.1.29}$$

Note that W_t^Q is now a cylindrical Q-Wiener process in $L^2(\mathbb{R})$ with translation invariant kernel $q(x - y)$:

$$Qv(x) = \int_{\mathbb{R}} q(x - y)v(y)dy. \tag{1.1.30}$$

Hence, whenever one comes across an equation such as (1.1.26), it is important to realise that it is actually a pre-equation in the sense that we first must agree what we mean by the notation. If the equation is interpreted in Stratonovich sense, we can generally agree that equation (1.1.26) means (1.1.29), but note that in applications it is often not mentioned which interpretation is used, see e.g. [3].

Another way to give meaning to equation (1.1.26) is to replace ξ by a smooth approximation, solve the equation, and then take the limit back to the nowhere differentiable ξ. It can then be shown that the limiting solution is a solution of (1.1.29), and these kind of results are known as Wong-Zakai theorems [45]. Unfortunately, this also implies that it is not obvious how equation (1.1.26) should be interpreted in the case of space-time white noise (i.e. $q(x) = \delta(x)$), and indeed, interpreting space-time white noise in Stratonovich sense needs the significantly more abstract theory of Martin Hairer's regularity structures [45].

For equations in Itô interpretation such as (1.1.27), existence results are well established by now. For the existence results in §2.4, we apply theorems from [77, 93]. The specific shape of the assumptions for these theorems explains the technical setup in §2.2. However, using the techniques for the stability proofs, we can in the future also directly prove existence results without resorting to the variational framework from [77, 93].

1.2 Reaction-Diffusion equations and travelling waves

The basic equation this thesis evolves around is given by the following reaction-diffusion equation (RDE):

$$u_t = \rho u_{xx} + f(u). \tag{1.2.1}$$

We call a wave profile Φ_0 and a speed c_0 a travelling wave[6] solution when $\Phi_0(x - c_0 t)$ is a solution to the RDE above. This is equivalent to demanding that Φ_0 is a stationary solution of the RDE written in the travelling wave coordinate $\xi = x - c_0 t$:

$$u_t = \rho u_{\xi\xi} + c_0 u_\xi + f(u). \tag{1.2.2}$$

Hence, the pair (Φ_0, c_0) is a travelling wave solution when

$$\rho \Phi_0'' + c_0 \Phi_0' + f(\Phi_0) = 0, \qquad \Phi_0(-\infty) = u^-,\ \Phi_0(\infty) = u^+, \tag{1.2.3}$$

where u^+ and u^- are zeros of f. We now need to ask the following question: Does this equation have a solution, and if yes, is it unique? Of course, this question depends heavily on the function f. For the deterministic version of equations (1.0.1) and (1.0.2), the solution exists and is unique [60, 113], while for another important equation, the Fisher-KPP equation, there are infinitely many pairs (Φ, c). Over the past decades, many more nonlinearities have been studied, see for an overview e.g. [65, 99].

In order to investigate whether or not the travelling wave, if it exists, is stable, we follow the classic approach. We assume that the solution u in the ξ frame can be split

[6] Throughout this thesis we shall always write a subscript 0 for the deterministic wave.

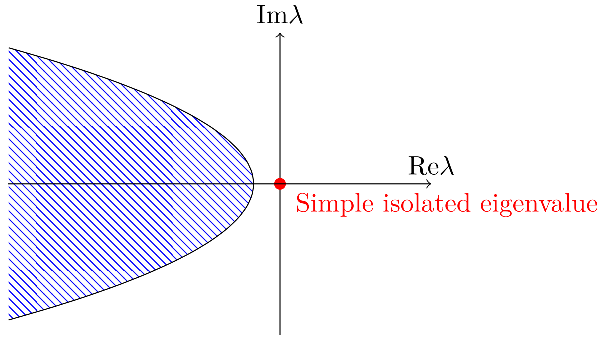


Figure 1.2: Sketch of the spectrum of the linear operator (1.2.5).

as $u(t) = \Phi_0 + v(t)$, which results in the following equation for $v(t)$:

$$v_t = \underbrace{\rho\Phi_0'' + c_0\Phi_0' + f(\Phi_0)}_{=0} + \mathcal{L}_{\text{tw}}v + N(v), \tag{1.2.4}$$

where the linear operator $\mathcal{L}_{\text{tw}} : H^2(\mathbb{R}) \to L^2(\mathbb{R})$ is given by

$$\mathcal{L}_{\text{tw}}v = \rho\partial_{\xi\xi}v + c_0\partial_\xi v + f'(\Phi_0)v \tag{1.2.5}$$

and the nonlinearity N by

$$N(v) = f(\Phi_0 + v) - f(\Phi_0) - f'(\Phi_0)v. \tag{1.2.6}$$

In other words, we can say that (Φ_0, c_0) is chosen such that $v(t) = 0$ is a solution of equation (1.2.4). Furthermore, the travelling wave is a stable solution of the RDE when $v(t) = 0$ is a stable solution of (1.2.4).

We will now give a short overview of the main techniques we use to prove stability that will be extended to the stochastic equations in the rest of this thesis. These techniques, which we will refer to as phase tracking throughout this thesis, were developed by Zumbrun and Howard in the 90s [118], mainly for the purpose of studying a much harder problem, shock waves. Therefore, a full proof of the stability of travelling waves in bistable RDEs using their techniques is not readily available in the literature. We will write down the main Ansatz and do the calculations to describe the outline of the proof, but the technical details are omitted.

Spectral Properties The structure of the linear operator $\mathcal{L}_{\text{tw}}$ as introduced in equation (1.2.5) is essential to understand the dynamics of the travelling wave. Especially, the shape of the spectrum is important to us. We always assume that the linear operator has a spectrum as depicted by Figure 1.2. This means that we always have an isolated eigenvalue in 0, corresponding to the translation invariance of the system. Indeed, a direct calculation shows that $\mathcal{L}_{\text{tw}}\Phi_0' = 0$ so Φ_0' is the eigenfunction corresponding to the zero eigenvalue. We assume that the rest of the spectrum is in the left half plane, bounded away from the imaginary axis. This leads to a so called 'spectral gap' between the zero eigenvalue and the rest of the spectrum. The last important property of the spectrum is the fact that $\mathcal{L}_{\mathrm{tw}}$ is a sectorial operator. Using the techniques in [65], we can compute that the linearizations for equations (1.0.1) and (1.0.2) indeed have this structure.

From these properties we draw two important conclusions. First, the linear operator generates an analytic semigroup $S(t)$ [80], but more importantly, the semigroup can be split in two parts. The contour integral around the spectrum that defines $S(t)$ can be deformed into a contour integral around 0 and a contour integral around the rest of the spectrum. The integral around 0 results in the operator P, a projection operator onto Φ_0' given by $P(v) = \langle v, \psi_{\mathrm{tw}}\rangle_{L^2(\mathbb{R})}\Phi_0'$ for any $v \in L^2(\mathbb{R})$ with ψ_{tw} the normalized adjoint eigenfunction of Φ_0'. This means that ψ_{tw} is such that $\mathcal{L}_{\mathrm{tw}}^*\psi_{\mathrm{tw}} = 0$, $\langle \Phi_0', \psi_{\mathrm{tw}}\rangle_{L^2(\mathbb{R})} = 1$, where the formal adjoint $\mathcal{L}_{\mathrm{tw}}^*$ is defined as

$$\mathcal{L}_{\mathrm{tw}}^* v = \rho\partial_{\xi\xi}v - c_0\partial_\xi v + f'(\Phi_0)v. \tag{1.2.7}$$

This has the following important consequence. Whenever a function $v \in L^2(\mathbb{R})$ is orthogonal to ψ_{tw} we find the following bound for the semigroup:

$$\|S(t)v\|_{L^2(\mathbb{R})} = \|S(t)(I-P)v\|_{L^2(\mathbb{R})} \leq Me^{-\beta t}\|v\|_{L^2(\mathbb{R})} \tag{1.2.8}$$

for some $M \geq 1$ and a $\beta > 0$ such that $-\beta$ lies in the spectral gap. All these computations can be made rigorous using [80].

Phase tracking Phase tracking is the key ingredient of this thesis. Essentially, we must be able to define a position for the travelling wave. There are many ways to define the location of the travelling wave, but it can be shown that they all result in the same qualitative behaviour [117]. The phase we choose is the one most practical for technical reasons, but also has an intuitive interpretation. We begin by introducing an Ansatz of the form

$$u\big(\cdot + \gamma(t), t\big) = \Phi_0(\cdot) + v(\cdot, t), \tag{1.2.9}$$

in which $\gamma(t)$ is the phase of u. We now demand that the evolution of the phase is governed by the ODE

$$\gamma(t) = \gamma_0 + c_0 t + \int_0^t a\big(v(s)\big)ds \tag{1.2.10}$$

for some (nonlinear) functional $a : L^2(\mathbb{R}) \to \mathbb{R}$ that we are still free to choose. The resulting equation for v is then given by

$$v_t(t) = \mathcal{L}_{\mathrm{tw}}v(t) + N\big(v(t)\big) + a\big(v(t)\big)\partial_\xi\left[\Phi_0 + v(t)\right], \tag{1.2.11}$$

which is an extended version of equation (1.2.4). This can be recast into the mild form

$$v(t) = S(t)v_0 + \int_0^t S(t-s)\Big[N\big(v(s)\big) + a\big(v(s)\big)\partial_\xi\left[\Phi_0 + v(s)\right]\Big]ds, \tag{1.2.12}$$

where $S(t)$ is the semigroup generated by $\mathcal{L}_{\text{tw}}$, as discussed in the previous paragraph. To apply exponential bounds such as (1.2.8) to the semigroup S, we must avoid the neutral non-decaying part of the semigroup. In order to force the integrand to be orthogonal to the zero eigenspace, we recall that the integrand must be orthogonal to ψ_{tw} and choose

$$a(v) = -\frac{\langle N(v), \psi_{\text{tw}} \rangle_{L^2(\mathbb{R})}}{\langle \partial_\xi(\Phi_0 + v), \psi_{\text{tw}} \rangle_{L^2(\mathbb{R})}}. \tag{1.2.13}$$

This choice, together with an aptly chosen γ_0, defines the phase $\gamma(t)$ and ensures that $v(t)$ is orthogonal to ψ_{tw} for all time. By a standard bootstrapping procedure one can now establish the limits $\|v(t)\|_{L^2(\mathbb{R})} \to 0$ and $t^{-1}\gamma(t) \to c_0$ for $t \to \infty$, provided that the initial condition v_0 is sufficiently small. This allows us to conclude that the travelling wave is orbitally stable. By orbitally stable we mean that we have shown that perturbations around the wave die out, but they can induce a phase shift, so the solution converges to a shifted version of the wave we perturbed around.

At this point the definition for $\gamma(t)$ is rather technical. There is also a more intuitive interpretation to this definition. We need to control the growth in the direction of Φ_0'. Another way to do this is by just fixing the position of u. We can define the position of $u(\cdot, t)$ by the number $a(v(t))$ such that $\langle u(\cdot + c_0 t + a(v(t)), t) - \Phi_0, h \rangle_{L^2(\mathbb{R})} = 0$ for some reference function h. In other words, $\langle v(t), h \rangle_{L^2(\mathbb{R})} = 0$. This implies

$$\begin{aligned} 0 &= \frac{\partial}{\partial t} \langle v, h \rangle_{L^2(\mathbb{R})} \\ &= \langle \mathcal{L}_{\text{tw}} v + a(v) \partial_\xi (v + \Phi_0) + N(v), h \rangle_{L^2(\mathbb{R})} \\ &= \langle v, \mathcal{L}_{\text{tw}}^* h \rangle_{L^2(\mathbb{R})} + a(v) \langle \partial_\xi (v + \Phi_0), h \rangle_{L^2(\mathbb{R})} + \langle N(v), h \rangle_{L^2(\mathbb{R})}. \end{aligned} \tag{1.2.14}$$

Now to make sure that a is quadratic in v we have to choose $h = \psi_{\text{tw}}$ and we find

$$a(v) = -\frac{\langle N(v), \psi_{\text{tw}} \rangle_{L^2(\mathbb{R})}}{1 - \langle v, \psi_{\text{tw}}' \rangle_{L^2(\mathbb{R})}}. \tag{1.2.15}$$

and this is exactly the same as we found before. Therefore, one can think of $\gamma(t)$ not just as a way to fix the position, but especially as **the** position that conveniently removes deviations, both constant and linear in v, from $c_0 t$.

Remark 1. *Suppose for the moment that we do use ψ_{tw} to define the position, but we do not know yet what (Φ_0, c_0) is. The equation for $a(v)$ would then become*

$$a(v) = -\frac{\langle \rho \Phi_0'' + c_0 \Phi_0' + f(\Phi_0), \psi_{\text{tw}} \rangle_{L^2(\mathbb{R})} + \langle v, \mathcal{L}_{\text{tw}}^* \psi_{\text{tw}} \rangle_{L^2(\mathbb{R})} + \langle N(v), \psi_{\text{tw}} \rangle_{L^2(\mathbb{R})}}{1 - \langle v, \psi_{\text{tw}}' \rangle_{L^2(\mathbb{R})}}. \tag{1.2.16}$$

Hence, we could define (Φ_0, c_0) as the pair that ensures $a(0) = 0$. In the deterministic case this seems a rather cumbersome way to define (Φ_0, c_0), but in the stochastic case we a priori fix the position (be orthogonal to ψ_{tw}) and only after we did the computations decide what (Φ_σ, c_σ) must be, and then this definition becomes useful, see equation (2.2.46).

1.3 Stochastic travelling waves

Although there are not many results to be found in the mathematical literature on waves in stochastic bistable RDEs, the subject has a longer history in the physics and chemistry literature. Results for stochastic waves propagating into an unstable state (i.e. Fisher-KPP type dynamics) go back to 1989 [84] and results for bistable equations can already be found in 1991 [102]. In both cases, the driving motivation for studying stochastic waves was the desire to include thermal fluctuations into the system. In the second half of the 90s the subject of stochastic waves picked up speed and was centred around the group of García-Ojalvo in Barcelona, Schimansky-Geier in Berlin and Van Saarloos in Leiden. This resulted in a book called 'Noise in spatially extended systems' [39], which was published in 1999. In this book, basic techniques to study SPDEs are explained, and a range of specific examples is shown such as the Belousov-Zhabotinsky system, Ginzburg-Landau, Swift-Hohenberg and the bistable RDE. From a physics perspective, one could conclude that the subject has reached a certain maturity. However, mathematicians will agree with the authors that "many aspects need to be established in a more rigorous way." In this thesis we will develop techniques to study a large class of equations, including the equation studied [39, §6.1] for a limited set of parameters. The rest of the book is still open for every mathematician to explore, either by extending our techniques or by developing new ones.

1.3.1 Stochastic waves in the physics literature

The stochastic Nagumo equation as studied in [39] has the following form:

$$u_t = \rho u_{xx} + f(u) + \sigma g(u)\xi(x,t), \tag{1.3.1}$$

in which ξ is a Gaussian process that is white in time and coloured in space, i.e. it has the following properties:

$$\begin{aligned} E[\xi(x,t)] &= 0, \\ E[\xi(x,t)\xi(x',t')] &= \delta(t-t')q(x-x'). \end{aligned} \tag{1.3.2}$$

Here, q is a symmetric function describing the correlation in space. Typically, one can think of a spiked Gaussian. In this form, the equation is often referred to as a Langevin equation. In §1.1 we extensively discussed how this equation can be interpreted as an SPDE. In order to understand the observed wandering of the wavefront as in Figure 1.1, the following Ansatz is made. A phase $z(t)$ is introduced together with an unknown stochastic process $\Delta(t)$ and an average speed $\bar{c}$ such that

$$z(t) = z_0 + \bar{c}t - \Delta(t), \tag{1.3.3}$$

and then $u(t)$ is studied in the $x - z(t)$ frame. The equation is interpreted in the Stratonovich sense, so the classic chain rule still applies and the same computations as for the deterministic wave can be used. However, the Stratonovich integral over $\sigma g(u)\xi$ has an effective drift term of $\sigma^2/2q(0)g'(u)g(u)$ at lowest order, so the travelling wave equation at $\mathcal{O}(\sigma^2)$ becomes

$$\rho\Phi'' + \bar{c}\,\Phi' + f(\Phi) + \frac{\sigma^2}{2}q(0)g'(\Phi)g(\Phi) = 0, \tag{1.3.4}$$

which is an $\mathcal{O}(\sigma^2)$ perturbation of the deterministic equation (1.2.3). Now the key assumption under this approximation is that influence of the effective drift term on the front shape and the influence of the stochastic term $\Delta(t)$ on the front position can be separated at $\mathcal{O}(\sigma^2)$. In other words, the authors postulate that the $\Delta(t)$ dependence in $g(u(\cdot - \bar{c}t + \Delta(t), t))\xi$ does not influence the effective drift term at $\mathcal{O}(\sigma^2)$. Now this is not true as we will find out later in this thesis, but this does not take away the fact that equation (1.3.4) is a good approximation to the stochastic wave and this equation describes the numerical results quite accurately. See §4.3 for a more in depth comparison of equation (1.3.4) with our results.

Next, we would like to derive an equation for $\Delta(t)$. This can, at lowest order, be achieved by finding the $\mathcal{O}(\sigma)$ level of the equation for $u(x - z(t), t)$ by assuming that $\Delta(t)$ is of $\mathcal{O}(\sigma)$ and then tune $\Delta(t)$ such that the equation becomes solvable. This results in the following equation for $\Delta(t)$:

$$\dot{\Delta}(t) = \sigma \frac{\langle g(\Phi_0)\xi, \psi_{\mathrm{tw}} \rangle_{L^2(\mathbb{R})}}{\langle \Phi_0', \psi_{\mathrm{tw}} \rangle_{L^2(\mathbb{R})}}. \tag{1.3.5}$$

To characterize the wandering of the waves, a diffusion coefficient D is introduced:

$$D = \frac{E[\Delta(t)^2]}{2t}. \tag{1.3.6}$$

This coefficient can be computed using the equation for $\dot{\Delta}(t)$. As we will see in §4.2.4, this diffusion coefficient indeed describes the wandering of the wave at lowest order in σ. The diffusion coefficient is an important observable of the wave as it describes how, at lowest order, a single realization of the wave will wander about the deterministic speed. In recent work on understanding gene drift in populations, this coefficient again played an important role [13]. Also in the field of stochastic neural field equations this coefficient has been studied. See also §4.2.4 for a more detailed comparison of the drift coefficient with other results in the literature.

1.3.2 Stochastic waves in the mathematical literature

We now give a short overlook of the results in the mathematical literature. In the case of the monostable stochastic F-KPP equation, many of the main issues such as wave speed and existence of an invariant measure have been worked out by the group around Mueller. Especially, computing the equivalent of equation (1.3.6) and proving rigorously that the expansion in σ turned out to be a hard problem [85].

The first results on the bistable equation come from 2012 and are mainly numerical [79]. This result set the stage for further research, as it describes for equation (1.3.1) how the average wave speed and shape depend on the interpretation (Itô versus Stratonovich), on additive versus multiplicative noise, on the shape q of the correlation and on the intensity σ of the noise. Any successful attempt to study waves in the Nagumo equation must be able to reproduce their findings.

In the following years, Stannat and coworkers developed techniques to study travelling waves in bistable RDEs and neural field equations [72–74, 104, 105]. Building on this results Inglis and MacLaurin [57] developed techniques to study travelling waves in neural field equations. These works indeed present stability results for stochastic travelling waves, and certainly helped us gain insight into the important techniques needed for the subject. However, there are some technical and practical limitations. First, one needs an immediate contractivity condition, meaning that $M = 1$ in (1.2.8). With some effort, this can be shown for the Nagumo equation [105], but this is unknown for the FitzHugh-Nagumo equation.

Another major technical issue is that the covariance of the driving noise is assumed to be of trace class. In §1.1, we explained that this assumption destroys the translation invariance of the equation, and the physically relevant equation (1.3.1) is not included in these works. From a more practical viewpoint, we note that these approaches do not give an answer to some of our main questions, such as 'what is the average wave speed and profile' and 'how does the dynamics evolve over long timescales.' See also §2.1 for a more technical discussion of their methods.

A more recent formal approach came from Cartwright and Gottwald [19]. Using a collective coordinate approach, they came to the same stochastic wave as we will find in §2.2.4 for the Nagumo equation forced by a one dimensional Brownian motion. A major advantage of this approach is that it gives a quite direct way to compute the stochastic travelling wave, without any rigorous setup needed. However, up to now, their approach depends on the exact structure of the Nagumo equation, so it would be interesting to see if their approaches can be generalized to other equations. See for a more extensive discussion on stochastic travelling waves also the review by Kuehn [69].

1.3.3 Stochastic phase tracking

Our approach to studying stochastic travelling waves will be to extend the phase tracking approach from §1.2. Upon adding a stochastic term to the equation for $v(t)$, we directly see that we cannot choose $a(v)$ in such a way that the whole equation becomes orthogonal to ψ_{tw}. What we need is an extra degree of freedom to ensure that also the stochastic part of the equation can be chosen orthogonal to ψ_{tw}. Hence, we propose to replace the splitting

$$u(\cdot + \gamma(t), t) = v(t) + \Phi_0 \tag{1.3.7}$$

where $\gamma(t)$ satisfies the ODE

$$\gamma(t) = \gamma_0 + c_0 t + \int_0^t a(v(s))ds, \tag{1.3.8}$$

by the splitting

$$U(\cdot + \Gamma(t), t) = V(t) + \Phi_\sigma \tag{1.3.9}$$

where $\Gamma(t)$ satisfies the SDE

$$\Gamma(t) = \Gamma_0 + c_\sigma + \int_0^t a(V(s))ds + \sigma \int_0^t b(V(s))dW_s^Q. \tag{1.3.10}$$

Here we introduced a yet unknown stochastic travelling wave (Φ_σ, c_σ) and a function b that maps to a Hilbert-Schmidt operator from $L^2_Q(\mathbb{R})$ to $\mathbb{R}$. This will be made precise later, but for now, it is important to realise that, when we compute the equation for $V(t)$ using Itô calculus, the stochastic term for $V(t)$ will depend on b giving us the freedom to choose $V(t)$ orthogonal to ψ_{tw}. The advantage of splitting $U(t)$ this way is that we now, just as in the deterministic case, fully describe the dynamics of the solution along the family of travelling waves. Therefore, $\Gamma(t)$ gives us a meaningful way of defining the average speed of the wave. See §4.2.2 for an extensive discussion on choosing a, b, the wave (Φ_σ, c_σ) and what it means for the dynamics of the travelling wave.

1.4 Main difficulties

The first main difficulty is the following: If I define

$$V(t) = U(\cdot + \Gamma(t), t) - \Phi_\sigma \tag{1.4.1}$$

as we proposed in the previous section, what equation does $V(t)$ solve? Itô calculus is not suitable to directly calculate the time derivative of $U(\cdot + \Gamma(t), t)$. In §2.5 (for a single Brownian motion) and §4.5.3 (for a cylindrical Q-Wiener process) we explain how to derive an equation for $V(t)$. Especially computing the Itô correction term that describes the interaction between the stochastic terms from $U(t)$ and $\Gamma(t)$ is very subtle.

Unfortunately, the Itô calculus causes several delicate technical complications that are not observed in the deterministic setting. The first major issue comes from the Itô Isometry. In the deterministic setting, a semigroup can be used to lift a function from $L^2(\mathbb{R})$ to $H^1(\mathbb{R})$ at the cost of an integrable singularity: $\|S(t)\|_{\mathcal{L}(L^2,H^1)} \sim t^{-1/2}$. However, for stochastic integrals we run into the following problem when we apply the Itô Isometry:

$$E\|\int_0^t S(t-s)B(s)dW_s^Q\|_{H^1}^2 = E\int_0^t \|S(t-s)B(s)\|_{HS(L_Q^2,H^1)}^2 ds. \tag{1.4.2}$$

If we wish to estimate B on the right hand side in $L^2(\mathbb{R})$, we have to square the $(t-s)^{-1/2}$-singularity, which is unintegrable. This precludes us from estimating $E\|V(s)\|_{H^1}$ directly, but we are forced to study the integrated $H^1(\mathbb{R})$-norm. This extra integral gives us the possibility to get rid of the singularity, which is still anything but easy; see the sections on nonlinear stability §2.9, §3.5 and §5.5.

A second major complication is that stochastic phase-shifts lead to extra nonlinear diffusive terms. By contrast, deterministic phase-shifts such as (1.2.10) lead to extra convective terms, which are of lower order and hence less dangerous. As a consequence, we encounter quasi-linear equations in our analysis that do not immediately fit into a semigroup framework. Luckily, this extra nonlinear term is a scalar so we solve this problem by using a suitable stochastic time-transform to scale out this extra term.

However, scaling only works when the term in front of ∂_{xx} is a scalar, which is clearly not the case for equations with different diffusion coefficients such as (1.0.2). This causes the main divide between Chapter 2 and 3. In Chapter 2 we have to enforce that all diffusion coefficients are equal, excluding the FitzHugh-Nagumo equation. We solve this issue in Chapter 3 by scaling each component of the SPDE for V separately and carefully studying the off-diagonal elements of the semigroup.

A third major difficulty is proving results on long timescales. In the deterministic case, stability is a binary question: given an initial perturbation from the wave, you either converge back to (a translate of) the wave or not. In the stochastic case this is not true. At every moment in time the wave is pushed by the noise term, hence the probability of making a large excursion from the wave, i.e. becoming unstable, grows in time. The major question is if we can understand how fast this probability grows. Numerically it is straightforward to show that the supremum of deviations from the travelling wave grows logarithmically in time, but proving this cost us many grey hairs. A proof of the logarithmic growth bound can be found in Chapter 5.

1.5 Reading guide

One hardly ever reads a math paper from beginning to end in a linear fashion. Especially a thesis comprised of N papers ($N > 1$) should also not be attacked head on. The papers are presented here in a chronological way (of writing that is, not of publication) which makes sense because each new paper builds on the previous one. However, in the time that past between the first and the fourth paper, our capability to give a bird's-eye view on the subject grew, as well as our capability to write cleaner proofs. Therefore, Chapter 4 is a good place to start, as it directly describes the physically relevant equations such as (1.3.1) and gives an overview of our procedures and results. If you have just an hour to spare on this thesis, start with §4.1-§4.4.

After (or before?) Chapter 4 the nitty-gritty details start. Chapter 2 is the most detailed of all. This chapter is the starting point for all the other chapters. We deal with questions such as existence of solutions to the SPDE, existence of solutions to the SDE that describes the phase, existence and uniqueness of the stochastic travelling wave, we show how to compute the stochastic phase shift and how to compute the stochastic time transform needed for the stability analysis. After these preparatory computations, we prove the stability results.

There are two main restrictions on the results in Chapter 2. First, we needed that the diffusion coefficients were the same in all components. Lifting this strong restriction in order to study FitzHugh-Nagumo equations such as (1.0.2) is the main purpose of Chapter 3. Many of the details from the first chapter are not repeated and we focus on the techniques necessary to prove the results. The second restriction in Chapter 2 (which also applies to Chapter 3) is the fact that the equations are forced by a one-dimensional Brownian motion, and therefore these chapters can not directly deal with equations such as (1.0.1) and (1.0.2). As mentioned before, this was a deliberate choice because we had not yet developed the intuition needed to work with cylindrical Q-Wiener processes.

In Chapter 4 we develop the necessary tools to work with translation invariant processes, but also extensively discuss perturbation methods to approximate the stochastic wave and make comparisons with numerical approximations. These numerical results however also indicate that our results remain valid on exponentially long timescales which is a significantly stronger claim than what we prove in Chapters 2-4. Improving our results to include the exponentially long bounds is the subject of Chapter 5. It can be read directly after Chapter 4, but understanding the stability analysis in §2.9 will serve as a good basis to appreciate the delicacies in the proofs in Chapter 5. Studying the stability analysis for the FitzHugh-Nagumo equation in §3.5 also explains why we chose to set up the proof in Chapter 5 only for equations of Nagumo type.

Notation It is important to note that the four chapters to follow do not have a uniform notation and spelling. This introduction and Chapters 4 and 5 are written in British English while the other chapters are written in American English, although this separation is not always very strict. In Chapter 2, we write A_* for the operator $\rho\partial_{xx}$ to highlight the fact that we treat the operator as an operator from $H^1(\mathbb{R}) \to H^{-1}(\mathbb{R})$, in stead of $H^2(\mathbb{R}) \to L^2(\mathbb{R})$. Also note that in Chapters 2 and 3, Q is not the covariance operator, but defined as $Q = I - P$ where P is the projection onto the 0-eigenspace as defined in §1.2. As Chapters 2 and 3 are written for a single Brownian motion, there is within the chapters no chance of confusion.

1.6 Outlook

In this thesis we develop techniques for quite a broad class of RDEs, but our structural assumptions rule out two major classes of equations. In the first place, our techniques heavily depend on the spectral gap in the spectrum of the linear operator (see Figure 1.2). While this holds true for bistable equations, the essential spectrum typically touches the imaginary axis for waves in monostable equations such as the stochastic Fischer-KPP equation [85]:

$$du = [u_{xx} + u(1-u)]dt + \sigma\sqrt{u(1-u)}dW_t^Q \tag{1.6.1}$$

For the deterministic equation many waves exist, but compact initial conditions always converge to the wave with speed 2 [100]. Note that the multiplicative term $\sqrt{u(1-u)}$ is not Lipschitz in 0 and 1, which makes analysis much harder. In the stochastic setting, it was shown in [85], using a completely different class of techniques than ours, that the speed 2 is perturbed into

$$2 - \pi^2|\log(\sigma^2)|^{-2} + \mathcal{O}(\log|\log(\varepsilon)|\log(\varepsilon)|^{-3}). \tag{1.6.2}$$

This odd looking result, known as the Brunet-Derrida conjecture [17], deviates significantly from the perturbation results we will encounter in this thesis. It would be interesting to see if we could derive the same results. We could, for example, pose this equation in a weighted space to open up the spectral gap again. However, this would exclude perturbations with a heavy tail which might essentially contribute to the result above. Another option would be to split the semigroup not in an exponentially decaying and a nondecaying part (which is now impossible) but, inspired by [117], split the semigroup in an exponentially decaying part, a polynomially decaying part and a nondecaying part. This would probably require an extensive extension of our phase tracking Ansatz (1.3.10).

Another obstruction in our techniques is the fact that our methods rely on the smoothening properties of the semigroup. For example, we cannot yet directly deal with the classical FHN-equation (equation (1.0.2) with $\rho_2 = 0$) or with neural field equations. We expect for the FHN-equation that our perturbation results still hold for $\rho_2 \to 0$, but all our bounds will blow up.

Of course, there are many more RDEs that are well studied in the deterministic case but their stochastic counterparts not so much. Korteweg-de Vries, Gray Scott, Klausmeier, Burgers, to just name a few. This list shows that in the coming years many more exciting results can be expected!

2

Reaction-Diffusion Equations with Scalar Noise

We consider reaction-diffusion equations that are stochastically forced by a small multiplicative noise term. We show that spectrally stable traveling wave solutions to the deterministic system retain their orbital stability if the amplitude of the noise is sufficiently small. By applying a stochastic phase-shift together with a time-transform, we obtain a semilinear SPDE that describes the fluctuations from the primary wave. We subsequently develop a semigroup approach to handle the nonlinear stability question in a fashion that is closely related to modern deterministic methods.

2.1 Introduction

In this chapter[1] we consider stochastically perturbed versions of a class of reaction-diffusion equations that includes the bistable Nagumo equation

$$u_t = u_{xx} + f_{\text{cub}}(u) \tag{2.1.1}$$

and the FitzHugh-Nagumo equation

$$\begin{aligned} u_t &= u_{xx} + f_{\text{cub}}(u) - v \\ v_t &= v_{xx} + \varrho[u - \gamma v]. \end{aligned} \tag{2.1.2}$$

Here we take $\varrho > 0$, $\gamma > 0$ and consider the standard bistable nonlinearity

$$f_{\text{cub}}(u) = u(1-u)(u-a). \tag{2.1.3}$$

It is well-known [34, 100] that (2.1.1) admits spectrally stable traveling front solutions

$$u(x,t) = \frac{1}{2}\big[1 + \tanh(\frac{1}{4}\sqrt{2}(x - ct))\big] \tag{2.1.4}$$

[1] The content of this chapter has been published as *C.H.S. Hamster, H.J. Hupkes; Stability of Traveling Waves for Reaction-Diffusion Equations with Multiplicative Noise* in SIADS, see [48].

that travel with speed

$$c = \sqrt{2}\left(a - \frac{1}{2}\right). \tag{2.1.5}$$

In addition, the existence of traveling pulse solutions to (2.1.2) with $0 < \varrho \ll 1$ was established recently [22] using variational methods. Using the Maslov index, a proof for the spectral stability of these waves has recently been obtained in [24, 25].

Our main results show that these spectrally stable wave solutions survive in a suitable sense upon adding a small pointwise multiplicative noise term to the underlying PDE. This noise term is assumed to be globally Lipschitz and must vanish for the asymptotic values of the waves. For example, our results cover the scalar Stochastic Partial Differential Equation (SPDE)

$$dU = \left[U_{xx} + f_{\mathrm{cub}}(U)\right]dt + \sigma\chi(U)U(1-U)d\beta_t \tag{2.1.6}$$

together with the two-component SPDE

$$\begin{aligned} dU &= \left[U_{xx} + f_{\mathrm{cub}}(U) - V\right]dt + \sigma\chi(U)U(1-U)d\beta_t, \\ dV &= \left[V_{xx} + \varrho\big(U - \gamma V\big)\right]dt + \sigma(U - \gamma V)d\beta_t, \end{aligned} \tag{2.1.7}$$

both for small $|\sigma|$, in which (β_t) is a Brownian motion and $\chi(U)$ is a cut-off function with $\chi(U) = 1$ for $|U| \le 2$. The presence of this cut-off is required to enforce the global Lipschitz-smoothness of the noise term. In this regime, one can think of (2.1.6) and (2.1.7) as versions of the PDEs (2.1.1)-(2.1.2) where the parameters a and ϱ are replaced by $a + \sigma\dot{\beta}_t$ respectively $\varrho + \sigma\dot{\beta}_t$.

Many additional multi-component reaction-diffusion PDEs such as the Gray-Scott [76], Rinzel-Keller [97], Tonnelier-Gerstner [107] and Lotka-Volterra systems [56] are also known to admit spectrally stable traveling waves in the equal-diffusion setting [40, 46, 115]. This allows our results to be applied to these waves after appropriately truncating the deterministic nonlinearities (in regimes that are far away from the interesting dynamics).

Such cut-offs are not necessary when considering equal-diffusion three-component FitzHugh-Nagumo-type systems such as those studied in [89, 109]. Such equations were first used by Purwins to study the formation of patterns during gas discharges [101]. However, in the equal-diffusion setting there is at present only numerical evidence to suggest that spectrally stable waves exist for the underlying deterministic equation. Analytical approaches to prove such facts typically use methods from singular perturbation theory, but these often require the diffusive length scales to be strictly separated.

Noisy patterns Stochastic forcing of PDEs has become an important tool for modelers in a large number of fields, ranging from medical applications such as neuroscience [15, 16] and cardiology [116] to finance [30] and meteorology [35]. While a rather general existence theory for solutions to SPDEs has been developed over the past decades [23, 42, 92, 93], the study of patterns such as stripes, spots and waves in such systems is less well-developed.

Preliminary results for specific equations such as Ginzburg-Landau [14, 37] and Swift-Hohenberg [71] are available. Kühn and Gowda [43] analyzed both these equations in the linear regime before the onset of the Turing bifurcation. They obtained scaling laws for the natural covariance operators that can be used as early-warning signs to predict the appearance of patterns.

In addition, several numerical studies have been initiated to study the impact of noise on patterns, see e.g. [79, 103, 112]. The results in [79] relating to (2.1.6) are particularly interesting from our perspective. Indeed, they clearly show that traveling wave solutions persist under the stochastic forcing, but the speed decreases linearly in σ^2 and the wave becomes steeper.

Rigorous results concerning the impact of stochastic forcing on deterministic waves are still relatively scarce. However, some important contributions have already been made, focusing on two important issues that need to be addressed. The first of these is that one needs to identify appropriate mechanisms to identify the phase, speed and shape of a stochastic wave. The second issue is that one needs to control the influence of the nonlinear terms by using the decay properties of the linear terms.

Phase tracking An appealing intuitive idea is to define the phase $\vartheta(u)$ of a solution profile u relative to the deterministic traveling wave Φ by writing

$$\vartheta(u) = \mathrm{argmin}_{\vartheta \in \mathbb{R}} \| u - \Phi(\cdot + \vartheta) \|_{L^2}, \tag{2.1.8}$$

which picks the closest translate of Φ. Inspired by this idea, Stannat [104, 105] obtained orbital stability results for a class of systems including (2.1.6) by appending an ODE to track the position of the wave. This is done via a gradient-descent technique, whereby the phase is updated continuously in the direction that lowers the norm in (2.1.8). A slight drawback of this method is that the phase is always lagging in a certain sense. In particular, it is not immediately clear how to define a stochastic speed and relate it with its deterministic counterpart.

This gradient-descent approach has been extended to neural field equations with additive noise [72, 74]. In order to clarify the dynamic effects caused by the noise, the authors employed a perturbative approach and expanded the phase of the wave and the shape of the perturbations in powers of the noise strength σ. By taking the infinite update-speed limit, the authors were able to eliminate the phase lag mentioned above. At lowest order they roughly recovered the diffusive wandering of the phase that was predicted by Bressloff and Webber [16]. This perturbative expansion can be maintained on finite time intervals, which increase in length to infinity as the noise size σ is decreased. However, one needs separate control on the deviations of the phase and the shape from the deterministic wave, which are both required to stay small.

Inglis and MacLaurin take a directer approach in [57] by using a stochastic differential equation for the phase that forces (2.1.8) to hold. For equations with additive noise, they obtain results that allow waves to be tracked over finite time intervals. As above, this tracking time increases to infinity as $\sigma \downarrow 0$. The main issue here is that global minima do not necessarily behave in a continuous fashion. This means that (2.1.8) can become multi-valued at times, leading to sudden jumps of the phase. However, under a (restrictive) technical condition the extension of the tracking time can be performed uniformly in σ.

Nonlinear effects In order to control the nonlinear terms over long time intervals one needs the linear flow to admit suitable decay properties. We write $S(t)$ for the semigroup generated by the linear operator $\mathcal{L}_{\mathrm{tw}}$ associated to the linearization of the PDEs above around their traveling wave Φ. A direct consequence of the translational invariance is that $\mathcal{L}_{\mathrm{tw}}\Phi' = 0$ and hence $S(t)\Phi' = \Phi'$ for all $t > 0$. In order to isolate this neutral mode, we write P for the spectral projection onto Φ', together with its complement $Q = I - P$. Assuming a standard spectral gap condition on the remainder of the spectrum of $\mathcal{L}_{\mathrm{tw}}$, one can subsequently obtain the estimate

$$\|S(t)Q\|_{L^2 \to L^2} \leq Me^{-\beta t} \tag{2.1.9}$$

for some constants $\beta > 0$ and $M \geq 1$; see, for example, [113, Lem. 5.1.2].

The common feature in all the approaches described above is that they require the identity $M = 1$ to hold. In this special case the linear flow is immediately contractive in the direction orthogonal to the translational eigenfunction. This identity certainly holds if one can obtain an estimate of the form

$$\langle \mathcal{L}_{\mathrm{tw}} v, v \rangle \leq -\beta \|v\|_{H^1}^2 + \kappa \|Pv\|^2 \tag{2.1.10}$$

for some $\kappa > 0$, since one can then use the commutation property $PS(t) = S(t)P$ to compute

$$\frac{d}{dt}\|S(t)Qv\|_{L^2}^2 = 2\langle \mathcal{L}_{\mathrm{tw}} S(t)Qv, S(t)Qv\rangle_{L^2} \leq -2\beta \|S(t)Qv\|_{H^1}^2 \leq -2\beta\|S(t)Qv\|_{L^2}^2. \tag{2.1.11}$$

In the deterministic case, coercive estimates of this type can be used to obtain similar differential inequalities for the L^2-norm of perturbations from the phase-adjusted traveling wave. Using the Itô formula this can be generalized to the stochastic case [72, 74, 104, 105], allowing stability estimates to be obtained that do not need any control over the H^1-norm of these perturbations. The approach developed in [57] proceeds directly from (2.1.9) using a renormalisation method. Similar L^2-stability results can be obtained in this fashion, again crucially using the fact that $M = 1$; see [57, (6.15)].

In light of the discussion above, a considerable effort is underway to identify systems for which the immediate contractivity condition $M = 1$ indeed holds. This has been explicitly verified for the Nagumo PDE (2.1.6) and several classes of one-component systems [74, 104, 105]. However, these computations are very delicate and typically proceed on an ad-hoc basis. For example, it is unclear (and doubtful) whether such a condition holds for the FitzHugh-Nagumo PDE (2.1.7). We refer to [111, §1] for an informative discussion on this issue.

In the case $M > 1$ the semigroup is still eventually contractive on the range of Q, but it can cause transient dynamics that grow on short timescales. Such dynamics play an important role and need to be tracked over temporal intervals of intermediate length. In this case the nonlinearities cannot be immediately dominated by the linear terms as above. To control these terms it is hence crucial to understand the H^1-norm of perturbations, which poses some challenging regularity issues in the stochastic setting.

Semigroup approach In this chapter we take a step towards harnessing the power of modern deterministic nonlinear stability techniques for use in the stochastic setting. In particular, inspired by the informative expository paper [117], we abandon any attempt to describe the phase of the wave via a priori geometric conditions. Instead, we initiate a semigroup approach based on the stochastic variation of constants formula. This leads to a stochastic evolution equation for the phase that follows naturally from technical considerations. More specifically, we use the phase to neutralize the dangerous non-decaying terms in our evolution equation. Our tracking mechanism is robust and allows us to focus solely on the behavior of the perturbation from the phase-shifted wave. This allows us to track solutions up to the point where this perturbation becomes too large as a result of the stochastic forcing, which resembles an Ornstein-Uhlenbeck process and hence is unbounded almost certainly. In particular, we do not need to impose restrictions on the size of the phaseshift as in [68, 72].

The first main advantage of our approach is that it provides orbital stability results without requiring the immediate contractivity condition described above. Indeed, we are able to track the H^1-norm of perturbations and not merely the L^2-norm, which allows us to have $M > 1$ in (2.1.9). This significantly broadens the class of systems that can be understood and aligns the relevant spectral assumptions with those that are traditionally used in deterministic settings.

The second main advantage is that we are (in some sense) able to isolate the drift-like contributions to the shape and speed of the wave that are caused by the noise term. This becomes fully visible in our analysis of (2.1.6), where the noise term is specially tailored to the deterministic wave Φ in the sense that it is proportional to the neutral mode Φ'. In this case we are able to obtain an exponential stability result for a modified waveprofile Φ_σ that propagates with a modified speed c_σ and exists for all positive time. This allows us to rigorously understand the changes to the waveprofile and speed that were numerically observed for (2.1.6) in [79]. In general, if the $\mathbb{R}^n$-orbit of the traveling wave of an n-component reaction-diffusion equation contains no self-intersections, our results allow special forcing terms to be constructed for which the modified waves remain exponentially stable.

However, the need to use stochastic calculus causes several delicate technical complications that are not observed in the deterministic setting. For example, the Itô Isometry is based on L^2-norms. At times, this forces us to square the natural semigroup decay rates, which leads to short-term regularity issues. Indeed, the heat semigroup $S(t)$ behaves as $\|S(t)\|_{\mathcal{L}(L^2;H^1)} \sim t^{-1/2}$, which is in $L^1(0,1)$ but not in $L^2(0,1)$. This precludes us from obtaining supremum control on the H^1-norm of our solutions. Instead, we obtain bounds on square integrals of the H^1-norm. For this reason, we need to carefully track how the cubic behavior of $f_{\mathrm{cub}}(u)$ propagates through our arguments.

A second major complication is that stochastic phase-shifts lead to extra nonlinear diffusive terms. By contrast, deterministic phase-shifts lead to extra convective terms, which are of lower order and hence less dangerous. As a consequence, we encounter quasi-linear equations in our analysis that do not immediately fit into a semigroup framework. We solve this problem by using a suitable stochastic time-transform to scale out the extra diffusive terms. The fact that we need the diffusion coefficients in (2.1.2) to be identical is a direct consequence of this procedure.

Outlook Let us emphasize that we view the present chapter merely as a proof-of-concept result for a pure semigroup-based approach. For example, in the following chapter we show how the severe restriction on the diffusion coefficients of (2.1.2) can be removed by exploiting the block-structure of the semigroup.

In addition, our results here use the variational framework developed by Liu and Röckner [77] in order to ensure that our SPDE has a well-defined global weak solution. In future work, we intend to replace this procedure by constructing local mild solutions directly based on fixed-point arguments.

Finally, we are interested in more delicate spectral stability scenarios, which allow one or more branches of essential spectrum to touch the imaginary axis in a quadratic tangency. Situations of this type are encountered when analyzing the two-dimensional stability of traveling planar waves [8, 53, 54, 64] or when studying viscous shocks in the context of conservation laws [6, 7, 82].

Organization This chapter is organized as follows. We formulate our phase-tracking mechanism and state our main results in §2.2. In §2.3 we obtain preliminary estimates on our nonlinearities, which are used in §2.4 to fit our coupled SPDE into the theory outlined in [77, 93]. This guarantees that our SPDE has well-defined solutions, to which we apply a stochastic phase-shift in §2.5 followed by a stochastic time-transform in §2.6. These steps lead to a stochastic variation of constants formula.

In §2.7 we develop two fixed-point arguments that capture the modifications to the waveprofile and speed that arise from the stochastic forcing. These modifications allow us to obtain suitable estimates on the nonlinearities in the variation of constants formula in §2.8, which allow us to pursue a nonlinear-stability argument in §2.9.

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2.2 Main results

In this chapter we are interested in the stability of traveling wave solutions to SPDEs of the form

$$dU = \big[A_* U + f(U)\big]dt + \sigma g(U) d\beta_t. \tag{2.2.1}$$

Here we take $U = U(x,t) \in \mathbb{R}^n$ with $x \in \mathbb{R}$ and $t \geq 0$.

In §2.2.1 we formulate several conditions on the nonlinearity f and the diffusion operator A_*, which imply that in the deterministic case $\sigma = 0$ the system (2.2.1) has a variational structure and admits a spectrally stable traveling wave solution. In §2.2.2 we impose several standard conditions on the noise term in (2.2.1), which guarantee that (2.2.1) is covered by the variational framework developed in [77]. In addition, we couple an extra SDE to our SPDE that will serve as a phase-tracking mechanism. Finally, in §2.2.3 and §2.2.4 we formulate and discuss our main results concerning the impact of the noise term on the deterministic traveling wave solutions.

2.2.1 Deterministic setup

We start here by stating our conditions on the form of A_* and f. These conditions require A_* to be a diffusion operator with identical diffusion coefficients and restrict the growth-rate of f to be at most cubic.

(HA) For any $u \in C^2(\mathbb{R};\mathbb{R}^n)$ we have $A_* u = \rho I_n u_{xx}$, in which $\rho > 0$ and I_n is the $n \times n$-identity matrix.

(Hf) We have $f \in C^3(\mathbb{R}^n;\mathbb{R}^n)$ and there exist $u_\pm \in \mathbb{R}^n$ for which $f(u_-) = f(u_+) = 0$. In addition, there exists a constant $K_f > 0$ so that the bound

$$\big|D^3 f(u)\big| \le K_f \tag{2.2.2}$$

holds for all $u \in \mathbb{R}^n$.

We now demand that the deterministic part of (2.2.1) has a traveling wave solution that connects the two equilibria $u_\pm$ (which are allowed to be equal). This traveling wave should approach these equilibria at an exponential rate.

(HTw) There exists a waveprofile $\Phi_0 \in C^2(\mathbb{R};\mathbb{R}^n)$ and a wavespeed $c_0 \in \mathbb{R}$ so that the function

$$u(x,t) = \Phi_0(x - c_0 t) \tag{2.2.3}$$

satisfies the deterministic PDE

$$u_t = A_* u + f(u) \tag{2.2.4}$$

for all $(x,t) \in \mathbb{R} \times \mathbb{R}$. In addition, there is a constant $K > 0$ together with exponents $\nu_\pm > 0$ so that the bound

$$|\Phi_0(\xi) - u_-| + |\Phi_0'(\xi)| \le K e^{-\nu_- |\xi|} \tag{2.2.5}$$

holds for all $\xi \le 0$, while the bound

$$|\Phi_0(\xi) - u_+| + |\Phi_0'(\xi)| \le K e^{-\nu_+ |\xi|} \tag{2.2.6}$$

holds for all $\xi \ge 0$.

Throughout this chapter, we will use the shorthands

$$L^2 = L^2(\mathbb{R};\mathbb{R}^n), \quad H^1 = H^1(\mathbb{R};\mathbb{R}^n), \quad H^2 = H^2(\mathbb{R};\mathbb{R}^n). \tag{2.2.7}$$

Linearizing the deterministic PDE (2.2.4) around the traveling wave (Φ_0, c_0), we obtain the linear operator

$$\mathcal{L}_{\mathrm{tw}} : H^2 \to L^2 \tag{2.2.8}$$

that acts as

$$[\mathcal{L}_{\mathrm{tw}} v](\xi) = [A_* v](\xi) + c_0 v'(\xi) + Df\big(\Phi_0(\xi)\big) v(\xi). \tag{2.2.9}$$

The formal adjoint

$$\mathcal{L}_{\mathrm{tw}}^{\mathrm{adj}} : H^2 \to L^2 \tag{2.2.10}$$

of this operator acts as

$$[\mathcal{L}_{\mathrm{tw}}^{\mathrm{adj}} w](\xi) = [A_* w](\xi) - c_0 w'(\xi) + Df\big(\Phi_0(\xi)\big)^T w(\xi). \tag{2.2.11}$$

Indeed, one easily verifies that

$$\langle \mathcal{L}_{\mathrm{tw}} v, w \rangle_{L^2} = \langle v, \mathcal{L}_{\mathrm{tw}}^{\mathrm{adj}} w \rangle_{L^2} \tag{2.2.12}$$

whenever $(v, w) \in H^2 \times H^2$. Here $\langle \cdot, \cdot \rangle_{L^2}$ denotes the standard inner-product on L^2.

We now impose a standard spectral stability condition on the wave. In particular, we require that the standard translational eigenvalue at zero is a simple eigenvalue. In addition, the remainder of the spectrum of $\mathcal{L}_{\mathrm{tw}}$ must be strictly bounded to the left of the imaginary axis.

(HS) There exists $\beta > 0$ so that the operator $\mathcal{L}_{\mathrm{tw}} - \lambda$ is invertible for all $\lambda \in \mathbb{C} \setminus \{0\}$ that have $\Re\lambda \geq -2\beta$, while $\mathcal{L}_{\mathrm{tw}}$ is a Fredholm operator with index zero. In addition, we have the identities

$$\mathrm{Ker}(\mathcal{L}_{\mathrm{tw}}) = \mathrm{span}\{\Phi_0'\}, \quad \mathrm{Ker}(\mathcal{L}_{\mathrm{tw}}^{\mathrm{adj}}) = \mathrm{span}\{\psi_{\mathrm{tw}}\} \tag{2.2.13}$$

for some $\psi_{\mathrm{tw}} \in H^2$ that has

$$\langle \Phi_0', \psi_{\mathrm{tw}} \rangle_{L^2} = 1. \tag{2.2.14}$$

We conclude by imposing a standard monotonicity condition on f, which ensures that the SPDE (2.2.1) fits into the variational framework of [77]. We remark here that we view this condition purely as a technical convenience, since it guarantees that solutions to (2.2.1) do not blow up. However, it does not a play a key role in the heart of our computations, where we restrict our attention to solutions that remain small in some sense.

(HVar) There exists $K_{\mathrm{var}} > 0$ so that the one-sided inequality

$$\langle f(u_A) - f(u_B), u_A - u_B \rangle_{\mathbb{R}^n} \leq K_{\mathrm{var}} |u_A - u_B|^2 \tag{2.2.15}$$

holds for all pairs $(u_A, u_B) \in \mathbb{R}^n \times \mathbb{R}^n$.

2.2.2 Stochastic setup

Our first condition here states that the noise term in (2.2.1) is driven by a standard Brownian motion. Let us emphasize that we made this choice purely to enhance the readability of our arguments. Indeed, our results can easily[2] be generalized to the situation where the noise is driven by cylindrical Q-Wiener processes.

(Hβ) The process $(\beta_t)_{t \geq 0}$ is a Brownian motion with respect to the complete filtered probability space

$$\Big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}\Big). \tag{2.2.16}$$

[2] The authors of Chapter 4 do not agree with this statement.

We require the function Dg to be globally Lipschitz and uniformly bounded. While the former condition is essential in our analysis to ensure that our cut-offs only depend on L^2-norms, the latter condition is only used to fit (2.2.1) into the framework of [77].

(Hg) We have $g \in C^2(\mathbb{R}^n;\mathbb{R}^n)$ with $g(u_-) = g(u_+) = 0$. In addition, there is $K_g > 0$ so that

$$|Dg(u)| \leq K_g \tag{2.2.17}$$

holds for all $u \in \mathbb{R}^n$, while

$$|g(u_A) - g(u_B)| + |Dg(u_A) - Dg(u_B)| \leq K_g\,|u_A - u_B| \tag{2.2.18}$$

holds for all pairs $(u_A, u_B) \in \mathbb{R}^n \times \mathbb{R}^n$.

We remark here that it is advantageous to view SPDEs as evolutions on Hilbert spaces, since powerful tools are available in this setting. However, in the case where $u_- \neq u_+$, the waveprofile Φ_0 does not lie in the natural statespace L^2. In order to circumvent this problem, we use Φ_0 as a reference function that connects u_- to u_+, allowing us to measure deviations from this function in the Hilbert spaces H^1 and L^2.

In order to highlight this dual role and prevent any confusion, we introduce the duplicate notation

$$\Phi_{\mathrm{ref}} = \Phi_0 \tag{2.2.19}$$

and emphasize the fact that Φ_{ref} remains fixed in the original frame, unlike the wave-solution (2.2.3). We also introduce the sets

$$\mathcal{U}_{L^2} = \Phi_{\mathrm{ref}} + L^2, \quad \mathcal{U}_{H^1} = \Phi_{\mathrm{ref}} + H^1, \quad \mathcal{U}_{H^2} = \Phi_{\mathrm{ref}} + H^2, \tag{2.2.20}$$

which we will use as the relevant state-spaces to capture the solutions U to (2.2.1).

We now set out to append a phase-tracking SDE to (2.2.1). In the deterministic case, we would couple the PDE to a phase-shift γ that solves an ODE of the form

$$\dot{\gamma}(t) = c_0 + \mathcal{O}\Big(U(t) - \Phi_0\big(\,\cdot - \gamma(t)\big)\Big). \tag{2.2.21}$$

By tuning the forcing function it is possible to remove the non-decaying terms in the original PDE, which act in the direction of $\Phi_0'\big(\cdot - \gamma(t)\big)$. This allows a nonlinear stability argument to be closed; see e.g. [117].

In this chapter we extend this procedure by introducing a phase-shift Γ that experiences the stochastic forcing

$$d\Gamma = \Big[c_\sigma + \mathcal{O}\Big(U(t) - \Phi_\sigma\big(\,\cdot - \Gamma(t)\big)\Big)\Big]dt + \mathcal{O}(\sigma)d\beta_t. \tag{2.2.22}$$

By choosing the function Φ_σ, the scalar c_σ and the two forcing functions in an appropriate fashion, the dangerous neutral terms can be eliminated from the original SPDE. These are hence purely technical considerations, but in §2.2.4 we discuss how these choices can be related to quantities that are interesting from an applied point of view.

In order to define our forcing functions in a fashion that is globally Lipschitz continuous, we introduce the constant

$$K_{\mathrm{ip}} = \big[\|g(\Phi_0)\|_{L^2} + 2K_g\big]\|\psi_{\mathrm{tw}}\|_{L^2}. \tag{2.2.23}$$

In addition, we pick two C^∞-smooth non-decreasing cut-off functions

$$\chi_{\mathrm{low}} : \mathbb{R} \to [\frac{1}{4}, \infty), \quad \chi_{\mathrm{high}} : \mathbb{R} \to [-K_{\mathrm{ip}} - 1, K_{\mathrm{ip}} + 1] \tag{2.2.24}$$

that satisfy the identities

$$\chi_{\mathrm{low}}(\vartheta) = \frac{1}{4} \text{ for } \vartheta \leq \frac{1}{4}, \quad \chi_{\mathrm{low}}(\vartheta) = \vartheta \text{ for } \vartheta \geq \frac{1}{2}, \tag{2.2.25}$$

together with

$$\chi_{\mathrm{high}}(\vartheta) = \vartheta \text{ for } |\vartheta| \leq K_{\mathrm{ip}}, \quad \chi_{\mathrm{high}}(\vartheta) = \mathrm{sign}(\vartheta)\big[K_{\mathrm{ip}} + 1\big] \text{ for } |\vartheta| \geq K_{\mathrm{ip}} + 1. \tag{2.2.26}$$

For any $u \in \mathcal{U}_{H^1}$ and $\psi \in H^1$, this allows us to introduce the functions

$$\begin{array}{rcl} b(u, \psi) & = & -\Big[\chi_{\mathrm{low}}\big(\langle \partial_\xi u, \psi \rangle_{L^2}\big)\Big]^{-1} \chi_{\mathrm{high}}\big(\langle g(u), \psi \rangle_{L^2}\big), \\ \kappa_\sigma(u, \psi) & = & 1 + \dfrac{1}{2\rho}\sigma^2 b(u, \psi)^2. \end{array} \tag{2.2.27}$$

In addition, for any $u \in \mathcal{U}_{H^1}$, $c \in \mathbb{R}$ and $\psi \in H^1$ we define the expression

$$\mathcal{J}_\sigma(u, c, \psi) \quad = \quad \kappa_\sigma(u, \psi)^{-1}\Big[f(u) + cu' + \sigma^2 b(u, \psi)\partial_\xi[g(u)]\Big], \tag{2.2.28}$$

while for any $u \in \mathcal{U}_{H^1}$, $c \in \mathbb{R}$ and $\psi \in H^2$ we write

$$a_\sigma(u, c, \psi) \quad = \quad -\kappa_\sigma(u, \psi)\Big[\chi_{\mathrm{low}}\big(\langle \partial_\xi u, \psi \rangle_{L^2}\big)\Big]^{-1}\Big[\langle u, A_* \psi \rangle_{L^2} + \langle \mathcal{J}_\sigma(u, c, \psi), \psi \rangle_{L^2}\Big]. \tag{2.2.29}$$

Finally, we introduce the right-shift operators

$$[T_\gamma u](\xi) = u(\xi - \gamma) \tag{2.2.30}$$

that act on any function $u : \mathbb{R} \to \mathbb{R}^n$.

With these ingredients in hand, we are ready to introduce the main SPDE that we analyze in this chapter. We formally write this SPDE as the skew-coupled system[3]

$$\begin{array}{rcl} dU & = & \big[A_* U + f(U)\big]dt + \sigma g(U) d\beta_t, \\ d\Gamma & = & \big[c + a_\sigma\big(U, c, T_\Gamma \psi_{\mathrm{tw}}\big)\big]dt + \sigma b\big(U, T_\Gamma \psi_{\mathrm{tw}}\big)\, d\beta_t, \end{array} \tag{2.2.31}$$

noting that we seek solutions with $\big(U(t), \Gamma(t)\big) \in \mathcal{U}_{H^1} \times \mathbb{R}$. Observe that the first equation is the same as (2.2.1).

In order to make this precise, we introduce the spaces

$$\begin{array}{rcl} \mathcal{N}^2\big([0, T]; (\mathcal{F}_t); \mathcal{H}\big) & = & \{X \in L^2\big([0, T] \times \Omega; dt \otimes \mathbb{P}; \mathcal{H}\big) : \\ & & \qquad X \text{ has a } (\mathcal{F}_t)\text{-progressively measurable version}\}, \end{array} \tag{2.2.32}$$

[3] Note here that formally $b(U, T_\Gamma \psi_{\mathrm{tw}})$ is a multiplication operator from $\mathbb{R} \to \mathbb{R}$, hence a number. If we generalize β_t to a cylindrical $Q-$Wiener process on a space H then the term involving b becomes a functional from H to $\mathbb{R}$, see Chapter 4.

where we allow $\mathcal{H} \in \{\mathbb{R}, L^2, H^1\}$. We note that we follow the convention of [93, 95] here by requiring progressive measurability instead of the usual stronger notion of predictability. Since we are exclusively dealing with Brownian motions, this choice suffices to construct stochastic integrals.

Our first result clarifies what we mean by a solution to (2.2.31). We note that (i) and (ii) in Proposition 2.2.1 imply that (X, Γ) is an $L^2 \times \mathbb{R}$-valued continuous $(\mathcal{F}_t)$-adapted process. We remark that in the integral equation (2.2.42) we interpret the diffusion operator A_* as an element of $\mathcal{L}(H^1; H^{-1})$, where H^{-1} is the dual of H^1 under the standard embeddings

$$H^1 \hookrightarrow L^2 \cong \big[L^2\big]^* \hookrightarrow H^{-1} = \big[H^1\big]^*. \tag{2.2.33}$$

We note that the set (H^1, L^2, H^{-1}) is commonly referred to as a Gelfand triple; see e.g. [32, §5.9] for a more detailed explanation. For $(v, w) \in H^{-1} \times H^1$ we write $\langle v, w\rangle_{H^{-1};H^1}$ to refer to the duality pairing between H^1 and H^{-1}. If in fact $v \in L^2$, then we have

$$\langle v, w\rangle_{H^{-1};H^1} = \langle v, w\rangle_{L^2}. \tag{2.2.34}$$

Proposition 2.2.1 (see §2.4)**.** *Suppose that (HA), (Hf), (HVar), (HTw), (HS), (Hg) and (Hβ) are all satisfied and fix $T > 0$, $c \in \mathbb{R}$ and $0 \le \sigma \le 1$. In addition, pick an initial condition*

$$(X_0, \Gamma_0) \in L^2 \times \mathbb{R}. \tag{2.2.35}$$

Then there are maps

$$X : [0, T] \times \Omega \to L^2, \quad \Gamma : [0, T] \times \Omega \to \mathbb{R} \tag{2.2.36}$$

that satisfy the following properties.

(i) For almost all $\omega \in \Omega$, the map

$$t \mapsto \big(X(t, \omega), \Gamma(t, \omega)\big) \tag{2.2.37}$$

is of class $C([0, T]; L^2 \times \mathbb{R})$.

(ii) For all $t \in [0, T]$, the map

$$\omega \mapsto \big(X(t, \omega), \Gamma(t, \omega)\big) \in L^2 \times \mathbb{R} \tag{2.2.38}$$

is $(\mathcal{F}_t)$-measurable.

(iii) We have the inclusion

$$X \in L^6\big(\Omega, \mathbb{P}; C([0, T]; L^2)\big), \tag{2.2.39}$$

together with

$$\begin{aligned} X &\in \mathcal{N}^2\big([0, T]; (\mathcal{F}_t); H^1\big), \\ \Gamma &\in \mathcal{N}^2\big([0, T]; (\mathcal{F}_t); \mathbb{R}\big) \end{aligned} \tag{2.2.40}$$

and

$$\begin{aligned} g(X + \Phi_{\mathrm{ref}}) &\in \mathcal{N}^2\big([0, T]; (\mathcal{F}_t); L^2\big), \\ b\big(X + \Phi_{\mathrm{ref}}, T_\Gamma \psi_{\mathrm{tw}}\big) &\in \mathcal{N}^2\big([0, T]; (\mathcal{F}_t); \mathbb{R}\big). \end{aligned} \tag{2.2.41}$$

(iv) *For almost all $\omega \in \Omega$, the identities*

$$\begin{aligned} X(t) &= X_0 + \int_0^t A_*[X(s) + \Phi_{\mathrm{ref}}]\, ds + \int_0^t f\big(X(s) + \Phi_{\mathrm{ref}}\big)\, ds \\ &\quad + \sigma \int_0^t g\big(X(s) + \Phi_{\mathrm{ref}}\big) d\beta_s \end{aligned} \tag{2.2.42}$$

and

$$\begin{aligned} \Gamma(t) &= \Gamma_0 + \int_0^t \big[c + a_\sigma\big(X(s) + \Phi_{\mathrm{ref}}, c, T_{\Gamma(s)}\psi_{\mathrm{tw}}\big)\big]\, ds \\ &\quad + \sigma \int_0^t b\big(X(s) + \Phi_{\mathrm{ref}}, T_{\Gamma(s)}\psi_{\mathrm{tw}}\big) d\beta_s \end{aligned} \tag{2.2.43}$$

hold[4] *for all* $0 \le t \le T$.

(v) *Suppose that the pair* $(\tilde{X}, \tilde{\Gamma}) : [0, T] \times \Omega \to L^2 \times \mathbb{R}$ *also satisfies (i)-(iv). Then for almost all* $\omega \in \Omega$*, we have*

$$(\tilde{X}, \tilde{\Gamma})(t) = (X, \Gamma)(t) \quad \textit{for all } 0 \le t \le T. \tag{2.2.44}$$

2.2.3 Wave stability

By inserting the traveling wave Ansatz (2.2.3) into the deterministic PDE (2.2.4), we observe that

$$A_*\Phi_0 + \mathcal{J}_0(\Phi_0, c_0, \psi_{\mathrm{tw}}) = 0, \tag{2.2.45}$$

which means that $a_0(\Phi_0, c_0, \psi_{\mathrm{tw}}) = 0$. Our first result here shows that this can be extended into a branch of profiles and speeds for which

$$a_\sigma(\Phi_\sigma, c_\sigma, \psi_{\mathrm{tw}}) = 0. \tag{2.2.46}$$

Roughly speaking, this means that the adjusted phase $\Gamma(t) - ct$ will (instantaneously) feel only stochastic forcing if one takes $c = c_\sigma$ and $U(t) = T_{\Gamma(t)}\Phi_\sigma$ in (2.2.31).

Proposition 2.2.2 (see §2.7)**.** *Suppose that* (HA)*,* (Hf)*,* (HTw)*,* (HS) *and* (Hg) *are all satisfied and pick a sufficiently large constant* $K > 0$*. Then there exists* $\delta_\sigma > 0$ *so that for every* $0 \le \sigma \le \delta_\sigma$*, there is a unique pair*

$$(\Phi_\sigma, c_\sigma) \in \mathcal{U}_{H^2} \times \mathbb{R} \tag{2.2.47}$$

that satisfies the system

$$A_*\Phi_\sigma + \mathcal{J}_\sigma\big(\Phi_\sigma, c_\sigma, \psi_{\mathrm{tw}}\big) = 0 \tag{2.2.48}$$

and admits the bound

$$\|\Phi_\sigma - \Phi_0\|_{H^2} + |c_\sigma - c_0| \le K\sigma^2. \tag{2.2.49}$$

[4] Note that this equation initially only holds as an identity in H^{-1}. Inclusion (2.2.39) makes that we can interpret the integrals in L^2. We have $X \in \mathcal{N}^2\big([0, T]; (\mathcal{F}_t); H^1\big)$ but this does not mean that $X(t) \in H^1$ pointwise.

We are interested in solutions to (2.2.31) with an initial condition for U that is close to Φ_σ. We will use the remaining degree of freedom to pick the initial phase Γ in such a way that the orthogonality condition described in the following result is enforced.

Proposition 2.2.3 (see §2.7). *Suppose that (HA), (Hf), (HTw), (HS) and (Hg) are all satisfied. Then there exist constants $\delta_0 > 0$, $\delta_\sigma > 0$ and $K > 0$ so that the following holds true. For every $0 \le \sigma \le \delta_\sigma$ and any $u_0 \in \mathcal{U}_{L^2}$ that has*

$$\|u_0 - \Phi_\sigma\|_{L^2} < \delta_0, \tag{2.2.50}$$

there exists $\gamma_0 \in \mathbb{R}$ for which the function

$$v_{\gamma_0} = T_{-\gamma_0}[u_0] - \Phi_\sigma \tag{2.2.51}$$

satisfies the identity

$$\langle v_{\gamma_0}, \psi_{\mathrm{tw}} \rangle_{L^2} = 0 \tag{2.2.52}$$

together with the bound

$$|\gamma_0| + \|v_{\gamma_0}\|_{L^2} \le K \|u_0 - \Phi_\sigma\|_{L^2}. \tag{2.2.53}$$

If in fact $u_0 \in \mathcal{U}_{H^1}$, then we also have the estimate

$$|\gamma_0| + \|v_{\gamma_0}\|_{H^1} \le K \|u_0 - \Phi_\sigma\|_{H^1}. \tag{2.2.54}$$

Let us now pick any $u_0 \in \mathcal{U}_{H^1}$ for which (2.2.50) holds. We write (X_{u_0}, Γ_{u_0}) for the process described in Proposition 2.2.1 with the initial condition

$$(X_0, \Gamma_0) = \big(u_0 - \Phi_{\mathrm{ref}}, \gamma_0\big), \tag{2.2.55}$$

in which γ_0 is the initial phase defined in Proposition 2.2.3. We then define the process

$$V_{u_0}(t) = T_{-\Gamma_{u_0}(t)}\big[X_{u_0}(t) + \Phi_{\mathrm{ref}}\big] - \Phi_\sigma, \tag{2.2.56}$$

which can be thought of as the deviation of the solution U of (2.2.31) from the stochastic wave Φ_σ shifted to the position $\Gamma_{u_0}(t)$.

In order to measure the size of the perturbation, we pick $\varepsilon > 0$ and introduce the scalar function

$$N_{\varepsilon;u_0}(t) = \|V_{u_0}(t)\|_{L^2}^2 + \int_0^t e^{-\varepsilon(t-s)} \|V_{u_0}(s)\|_{H^1}^2 \, ds. \tag{2.2.57}$$

For each $T > 0$ we now define a probability

$$p_\varepsilon(T, \eta, u_0) = P\Big(\sup_{0 \le t \le T} N_{\varepsilon;u_0}(t) > \eta \Big). \tag{2.2.58}$$

Our first main result shows that the probability that $N_{\varepsilon;u_0}$ remains small on timescales of order σ^{-2} can be pushed arbitrarily close to one by restricting the strength of the noise and the size of the initial perturbation.

2

Theorem 2.2.4 (see §2.9)**.** *Suppose that (HA), (Hf), $(HVar)$, (HTw), (HS), (Hg) and $(H\beta)$ are all satisfied and pick sufficiently small constants $\varepsilon > 0$, $\delta_0 > 0$, $\delta_\eta > 0$ and $\delta_\sigma > 0$. Then there exists a constant $K > 0$ so that for every $T > 1$, any $0 \le \sigma \le \delta_\sigma T^{-1/2}$, any $u_0 \in \mathcal{U}_{H^1}$ that satisfies (2.2.50) and any $0 < \eta \le \delta_\eta$, we have the inequality*

$$p_\varepsilon(T, \eta, u_0) \le \eta^{-1} K \Big[\|u_0 - \Phi_\sigma\|_{H^1}^2 + \sigma^2 T \Big]. \tag{2.2.59}$$

Our second main result concerns the special case where the noise pushes the stochastic wave Φ_σ in a rigid fashion. This is the case when

$$g(\Phi_0) = \vartheta_0 \Phi_0' \tag{2.2.60}$$

for some proportionality constant $\vartheta_0 \in \mathbb{R}$. It is easy to verify that (2.2.60) with $\vartheta_0 = -\sqrt{2}$ holds for (2.1.6).

In this setting we expect the perturbation V to decay exponentially on timescales of order σ^{-2} with large probability. In order to formalize this, we pick small constants $\varepsilon > 0$ and $\alpha > 0$ and introduce the scalar function

$$N_{\varepsilon,\alpha;u_0}(t) = e^{\alpha t} \|V_{u_0}(t)\|_{L^2}^2 + \int_0^t e^{-\varepsilon(t-s)} e^{\alpha s} \|V_{u_0}(s)\|_{H^1}^2 \, ds, \tag{2.2.61}$$

together with the associated probabilities

$$p_{\varepsilon,\alpha}(T, \eta, u_0) = P\Big(\sup_{0 \le t \le T} N_{\varepsilon,\alpha;u_0}(t) > \eta \Big). \tag{2.2.62}$$

Theorem 2.2.5 (see §2.9)**.** *Suppose that (HA), (Hf), $(HVar)$, (HTw), (HS), (Hg) and $(H\beta)$ are all satisfied . Suppose furthermore that (2.2.60) holds and pick sufficiently small constants $\varepsilon > 0$, $\delta_0 > 0$, $\alpha > 0$, $\delta_\eta > 0$ and $\delta_\sigma > 0$. Then there exists a constant $K > 0$ so that for any $T > 1$, every $0 \le \sigma \le \delta_\sigma T^{-1/2}$, any $u_0 \in \mathcal{U}_{H^1}$ that satisfies (2.2.50) and any $0 < \eta \le \delta_\eta$, we have the inequality*

$$p_{\varepsilon,\alpha}(T, \eta, u_0) \le \eta^{-1} K \|u_0 - \Phi_\sigma\|_{H^1}^2. \tag{2.2.63}$$

2.2.4 Interpretation

In §2.5 we show that the pair $(V, \Gamma) = (V_{u_0}, \Gamma_{u_0})$ defined in §2.2.3 satisfies the SPDE

$$\begin{aligned} dV &= \mathcal{R}_\sigma(V)\, dt + \sigma \mathcal{S}_\sigma(V)\, d\beta_t, \\ d\Gamma &= \big[c_\sigma + a_\sigma\big(\Phi_\sigma + V, c_\sigma, \psi_{\mathrm{tw}}\big) \big]\, dt + \sigma b\big(\Phi_\sigma + V, \psi_{\mathrm{tw}}\big)\, d\beta_t, \end{aligned} \tag{2.2.64}$$

in which the nonlinearities satisfy the identities

$$a_\sigma\big(\Phi_\sigma, c_\sigma, \psi_{\mathrm{tw}}\big) = 0, \quad \mathcal{R}_\sigma(0) = 0, \quad \mathcal{S}_\sigma(0) = g(\Phi_\sigma) + b(\Phi_\sigma)\Phi_\sigma', \tag{2.2.65}$$

together with the asymptotics

$$D_1 a_\sigma\big(\Phi_\sigma, c_\sigma, \psi_{\mathrm{tw}}\big) = \mathcal{O}(\sigma^2), \quad D\mathcal{R}_\sigma(0) = \mathcal{O}(\sigma^2). \tag{2.2.66}$$

For our discussion here we take $V(0) = 0$ and $\Gamma(0) = 0$, which corresponds with the initial condition condition $U(0) = \Phi_\sigma$ for the original system (2.2.1).

The identities (2.2.65) imply that $V(t)$ and $\Gamma(t) - c_\sigma t$ experience no deterministic forcing at $t = 0$. We now briefly discuss the consequences of this observation on the behaviour of (2.2.64) in the two regimes covered by Theorems 2.2.4 and 2.2.5.

Exponential stability Our results are easiest to interpret in the special case

$$g(\Phi_0) = \vartheta_0 \Phi_0', \tag{2.2.67}$$

where Theorem 2.2.5 applies. Remarkably, the modified profiles and speeds (Φ_σ, c_σ) can be computed explicitly in this setting.[5]

Proposition 2.2.6 (see §2.7)**.** *Consider the setting of Proposition 2.2.2 and suppose that (2.2.60) holds. Then for all sufficiently small* $0 \le \sigma \le \delta_\sigma$ *we have the identities*

$$\begin{aligned} \Phi_\sigma(\xi) &= \Phi_0\Big(\big[1 + \frac{1}{2\rho}\sigma^2\vartheta_0^2\big]^{1/2}\xi\Big), \\ c_\sigma &= \big[1 + \frac{1}{2\rho}\sigma^2\vartheta_0^2\big]^{-1/2} c_0, \end{aligned} \tag{2.2.68}$$

together with

$$g(\Phi_\sigma) = \big[1 + \frac{1}{2\rho}\sigma^2\vartheta_0^2\big]^{-1/2}\vartheta_0\Phi_\sigma' = -b\big(\Phi_\sigma, \psi_{\mathrm{tw}}\big)\Phi_\sigma'. \tag{2.2.69}$$

A direct consequence of (2.2.69) is that the identity

$$\mathcal{S}_\sigma(0) = 0 \tag{2.2.70}$$

can be added to the list (2.2.65). In particular, we obtain the explicit solution

$$\begin{aligned} (V, \Gamma) &= \Big(0, c_\sigma t + \sigma b\big(\Phi_\sigma, \psi_{\mathrm{tw}}\big)\beta_t\Big) \\ &= \Big(0, c_\sigma t - \sigma\big[1 + \frac{1}{2\rho}\sigma^2\vartheta_0^2\big]^{-1/2}\vartheta_0\beta_t\Big) \end{aligned} \tag{2.2.71}$$

for the system (2.2.64). This corresponds to the solution

$$U(t) = \Phi_\sigma(\cdot + \Gamma(t)) \tag{2.2.72}$$

for (2.2.31), which exists for all $t \ge 0$.

We hence see that the shape Φ_σ of the stochastic profile remains fixed, while the phase $\Gamma(t)$ of the wave performs a scaled Brownian motion around the position $c_\sigma t$. Since the identities (2.2.68) imply that the waveprofile is steepened while the speed is slowed down, our results indeed confirm the numerical observations from [79] that were discussed in §2.1.

Any small perturbation in the V component will decay exponentially fast with high probability on account of Theorem 2.2.5. Intuitively, the leading order behavior for V resembles a geometric Brownian motion, as the noise term is proportional to V while the deterministic forcing leads to exponential decay. In particular, we expect that our approach can keep track of the wave for timescales that are far longer than the $\mathcal{O}(\sigma^{-2})$ bounds stated in our results.

[5] Similar formula can be obtained by following the formal approach in [19], which appeared during the revision phase of this paper.

Orbital stability In general we have $\mathcal{S}_\sigma(0) \neq 0$, which prevents us from solving (2.2.64) explicitly. Indeed, Theorem 2.2.4 states that $V(t)$ will remain small with high probability, but the stochastic forcing will preclude it from converging to zero. However, our construction does guarantee that $\langle V(t), \psi_{\mathrm{tw}} \rangle_{L^2} = 0$ as long as V stays small. Since $\langle \Phi_0', \psi_{\mathrm{tw}} \rangle_{L^2} = 1$, this still allows us to interpret $\Gamma(t)$ as the position of the wave. In particular, if the expression $t^{-1}\Gamma(t)$ converges in a suitable sense as $t \to \infty$ then it is natural to use this limit as a proxy for the notion of a wavespeed.

In order to explore this, we introduce the formal expansion

$$V(t) = \sigma V_\sigma^{(1)}(t) + \mathcal{O}(\sigma^2) \tag{2.2.73}$$

and use the mild formulation developed in §2.6 to obtain

$$V_\sigma^{(1)}(t) = \int_0^t S(t-s)\mathcal{S}_\sigma(0)\, d\beta_s. \tag{2.2.74}$$

Here S denotes the semigroup generated by $\mathcal{L}_{\mathrm{tw}}$, which by construction decays exponentially when applied to $\mathcal{S}_\sigma(0)$. In particular, for any bilinear map $B : H^1 \times H^1 \to \mathbb{R}$ we can use the Itô isometry to obtain

$$\begin{aligned} EB\big[V_\sigma^{(1)}(t), V_\sigma^{(1)}(t)\big] &= \int_0^t B\big[S(t-s)\mathcal{S}_\sigma(0), S(t-s)\mathcal{S}_\sigma(0)\big]\, ds \\ &= \int_0^t B\big[S(s)\mathcal{S}_\sigma(0), S(s)\mathcal{S}_\sigma(0)\big]\, ds, \end{aligned} \tag{2.2.75}$$

which converges in the limit $t \to \infty$.

Introducing the formal expansion

$$\Gamma(t) = c_\sigma t + \sigma\Gamma_\sigma^{(1)}(t) + \sigma^2\Gamma_\sigma^{(2)}(t) + \mathcal{O}(\sigma^3), \tag{2.2.76}$$

the first bound in (2.2.66) implies that

$$\Gamma_\sigma^{(1)}(t) \quad = \quad b(\Phi_\sigma, \psi_{\mathrm{tw}})\beta_t \tag{2.2.77}$$

together with

$$\begin{aligned} \Gamma_\sigma^{(2)}(t) &= \frac{1}{2}\int_0^t D_1^2 a_\sigma\big(\Phi_\sigma, c_\sigma, \psi_{\mathrm{tw}}\big)\big[V_\sigma^{(1)}(s), V_\sigma^{(1)}(s)\big]\, ds \\ &\quad + D_1 b\big(\Phi_\sigma, \psi_{\mathrm{tw}}\big)\left[\int_0^t V_\sigma^{(1)}(s)\, d\beta_s\right]. \end{aligned} \tag{2.2.78}$$

Since $EV_\sigma^{(1)}(t) = 0$ we obtain

$$E\Gamma_\sigma^{(1)}(t) = 0 \tag{2.2.79}$$

together with

$$\begin{aligned} E\Gamma_\sigma^{(2)}(t) &= \frac{1}{2}\int_0^t \int_0^s D_1^2 a_\sigma\big(\Phi_\sigma, c_\sigma, \psi_{\mathrm{tw}}\big)\big[S(s')\mathcal{S}_\sigma(0), S(s')\mathcal{S}_\sigma(0)\big]\, ds'\, ds \\ &= \frac{1}{2}\int_0^t (t-s) D_1^2 a_\sigma\big(\Phi_\sigma, c_\sigma, \psi_{\mathrm{tw}}\big)\big[S(s)\mathcal{S}_\sigma(0), S(s)\mathcal{S}_\sigma(0)\big]\, ds. \end{aligned} \tag{2.2.80}$$

Upon writing

$$c_{\infty}^{(2)} = c_{\sigma} + \frac{\sigma^2}{2} \int_0^{\infty} D_1^2 a_{\sigma}\big(\Phi_{\sigma}, c_{\sigma}, \psi_{\mathrm{tw}}\big)\big[S(s)\mathcal{S}_{\sigma}(0), S(s)\mathcal{S}_{\sigma}(0)\big]\, ds, \tag{2.2.81}$$

we hence conjecture that the expected limiting value of the wavespeed behaves as $c_{\infty}^{(2)} + \mathcal{O}(\sigma^3)$. Since $c_{\sigma} = c_0 + \mathcal{O}(\sigma^2)$ this would mean that the stochastic contributions to the wavespeed are second order in σ.

We remark that computations of this kind resemble the multi-scale approach initiated by Lang in [72] and Stannat and Krüger in [68]. However, our approach does allow us to consider limiting expressions such as (2.2.81), for which one needs the exponential decay of the semigroup. Indeed, (2.2.74) resembles a mean-reverting Ornstein-Uhlenbeck process, which has a variation that can be globally bounded in time, despite the fact that the individual paths are unbounded.

As above, we expect to be able to track the wave for timescales that are longer than the $\mathcal{O}(\sigma^{-2})$ bounds stated in our results. The key issue is that the mild version of the Burkholder-Davis-Gundy inequality that we use is not able to fully incorporate the mean-reverting effects of the semigroup. We emphasize that even the standard scalar Ornstein-Uhlenbeck process requires sophisticated probabilistic machinery to uncover statistics concerning the behavior of the running supremum [2, 96]. We explore these issues in more detail in the following chapters. For the moment however, we note that our initial numerical experiments seem to confirm that the expression (2.2.81) indeed captures the leading order stochastic correction to the wavespeed.

2.3 Preliminary estimates

In this section we derive several preliminary estimates for the functions f, g, $\mathcal{J}_0$, b and κ_{σ}. We will write the arguments $(u, \bar{c}) \in \mathcal{U}_{H^1} \times \mathbb{R}$ as

$$u = \Phi + v, \quad \bar{c} = c + d, \tag{2.3.1}$$

in which we take $(\Phi, c) \in \mathcal{U}_{H^1} \times \mathbb{R}$ and $(v, d) \in H^1 \times \mathbb{R}$. We do not restrict ourselves to the case where $(\Phi, c) = (\Phi_0, c_0)$, but impose the following condition.

(hPar) The conditions (HTw) and (HS) hold and the pair $(\Phi, c) \in \mathcal{U}_{H^1} \times \mathbb{R}$ satisfies the bounds

$$\|\Phi - \Phi_0\|_{H^1} \leq \min\{1, [4\|\psi_{\mathrm{tw}}\|_{L^2}]^{-1}\}, \quad |c - c_0| \leq 1. \tag{2.3.2}$$

In §2.3.1 we obtain global and Lipschitz bounds for the functions f and g. These bounds are subsequently used in §2.3.2 to analyze the auxiliary functions $\mathcal{J}_0$, b and κ_{σ}. Throughout this chapter we use the convention that all numbered constants appearing in proofs are strictly positive and have the same dependencies as the constants appearing in the statement of the result.

2.3.1 Bounds on f and g

The conditions (Hf) and (Hg) allow us to obtain standard cubic bounds on f and globally Lipschitz bounds on g. We also consider expressions of the form $\partial_{\xi} g(u)$, which give rise to quadratic bounds.

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Lemma 2.3.1. *Suppose that (Hf) and (hPar) are satisfied. Then there exists a constant $K > 0$, which does not depend on the pair (Φ, c), so that the following holds true. For any $v \in H^1$ and $\psi \in H^1$ we have the bounds*

$$\begin{array}{lcl} \|f(\Phi + v)\|_{L^2} & \leq & K\big[1 + \|v\|_{H^1}^2 \|v\|_{L^2}\big], \\ |\langle f(\Phi + v), \psi \rangle_{L^2}| & \leq & K\big[1 + \|v\|_{H^1} \|v\|_{L^2}^2\big] \|\psi\|_{H^1}, \end{array} \tag{2.3.3}$$

while for any set of pairs $(v_A, v_B) \in H^1 \times H^1$ and $(\psi_A, \psi_B) \in H^1 \times H^1$, the expressions

$$\begin{array}{lcl} \Delta_{AB} f & = & f(\Phi + v_A) - f(\Phi + v_B), \\ \Delta_{AB}\langle f, \cdot \rangle_{L^2} & = & \langle f(\Phi + v_A), \psi_A \rangle_{L^2} - \langle f(\Phi + v_B), \psi_B \rangle_{L^2} \end{array} \tag{2.3.4}$$

satisfy the estimates

$$\begin{array}{lcl} \|\Delta_{AB} f\|_{L^2} & \leq & K\|v_A - v_B\|_{L^2} \\ & & +K\Big(\|v_A\|_{H^1}\|v_A\|_{L^2} + \|v_B\|_{H^1}\|v_B\|_{L^2}\Big)\|v_A - v_B\|_{H^1}, \\ |\Delta_{AB}\langle f, \cdot \rangle_{L^2}| & \leq & K\|v_A - v_B\|_{L^2}\|\psi_A\|_{H^1} \\ & & +K\|v_A - v_B\|_{H^1}\big(\|v_A\|_{L^2}^2 + \|v_B\|_{L^2}^2\big)\|\psi_A\|_{H^1} \\ & & +K\Big[1 + \|v_B\|_{H^1}\|v_B\|_{L^2}^2\Big]\|\psi_A - \psi_B\|_{H^1}. \end{array} \tag{2.3.5}$$

Proof. Exploiting (Hf) we obtain

$$\big|D^2 f(u)\big| \leq C_1[1 + |u|], \tag{2.3.6}$$

together with

$$|Df(u)| \leq C_1[1 + |u|^2] \tag{2.3.7}$$

for all $u \in \mathbb{R}^n$. In particular, (hPar) yields the pointwise Lipschitz bound

$$|f(\Phi + v_A) - f(\Phi + v_B)| \leq C_2[1 + |v_A|^2 + |v_B|^2]\,|v_A - v_B|\,. \tag{2.3.8}$$

Using the Sobolev embedding $\|\cdot\|_\infty \leq C_3\|\cdot\|_{H^1}$ this immediately implies the first estimate in (2.3.5). Applying this estimate with $v_A = 0$ and $v_B = \Phi_0 - \Phi$ we find

$$\begin{array}{lcl} \|f(\Phi)\|_{L^2} & \leq & \|f(\Phi_0)\|_{L^2} + \|f(\Phi) - f(\Phi_0)\|_{L^2} \\ & \leq & C_4. \end{array} \tag{2.3.9}$$

Exploiting

$$\|f(\Phi + v)\|_{L^2} \leq \|f(\Phi)\|_{L^2} + \|f(\Phi + v) - f(\Phi)\|_{L^2}, \tag{2.3.10}$$

we hence obtain

$$\|f(\Phi + v)\|_{L^2} \quad \leq \quad C_5\big[1 + \|v\|_{L^2} + \|v\|_{H^1}^2\|v\|_{L^2}\big]. \tag{2.3.11}$$

The first estimate in (2.3.3) now follows by noting that $\|v\|_{L^2} \leq \|v\|_{H^1}^2\|v\|_{L^2}$ for $\|v\|_{L^2} \geq 1$.

Turning to the inner products, (2.3.8) allows us to compute

$$|\langle f(\Phi + v_A) - f(\Phi + v_B), \psi_A \rangle_{L^2}| \leq C_2 \|v_A - v_B\|_{L^2} \|\psi_A\|_{L^2} + C_2 \big[\|v_A\|_{L^2}^2 + \|v_B\|_{L^2}^2 \big] \|v_A - v_B\|_{H^1} \|\psi_A\|_{H^1}. \tag{2.3.12}$$

Exploiting (2.3.9), the second estimate in (2.3.3) hence follows from the bound

$$|\langle f(\Phi + v), \psi \rangle_{L^2}| \quad \leq \quad |\langle f(\Phi), \psi \rangle_{L^2}| + |\langle f(\Phi + v) - f(\Phi), \psi \rangle_{L^2}|, \tag{2.3.13}$$

using a similar observation as above to eliminate the $\|v\|_{L^2}\|\psi\|_{L^2}$ term. Finally, the second estimate in (2.3.5) can be obtained by applying (2.3.12) and (2.3.3) to the splitting

$$|\langle f(\Phi + v_A), \psi_A \rangle_{L^2} - \langle f(\Phi + v_B), \psi_B \rangle_{L^2}| \quad \leq \quad |\langle f(\Phi + v_A) - f(\Phi + v_B), \psi_A \rangle_{L^2}| + |\langle f(\Phi + v_B), \psi_A - \psi_B \rangle_{L^2}|. \tag{2.3.14}$$

□

Lemma 2.3.2. *Suppose that (Hg) and (hPar) are satisfied. Then there exists a constant $K > 0$, which does not depend on the pair (Φ, c), so that the following holds true. For any $v \in H^1$ we have the bounds*

$$\begin{aligned} \|g(\Phi + v)\|_{L^2} &\leq \|g(\Phi_0)\|_{L^2} + K_g(1 + \|v\|_{L^2}) \\ &\leq K[1 + \|v\|_{L^2}], \\ \|\partial_\xi g(\Phi + v)\|_{L^2} &\leq K[1 + \|v\|_{H^1}], \end{aligned} \tag{2.3.15}$$

while for any pair $(v_A, v_B) \in H^1 \times H^1$ we have the estimates

$$\begin{aligned} \|g(\Phi + v_A) - g(\Phi + v_B)\|_{L^2} &\leq K\|v_A - v_B\|_{L^2}, \\ \|\partial_\xi[g(\Phi + v_A) - g(\Phi + v_B)]\|_{L^2} &\leq K\big[1 + \|v_A\|_{H^1}\big]\|v_A - v_B\|_{H^1}. \end{aligned} \tag{2.3.16}$$

Proof. The Lipschitz estimate on g implies that

$$\|g(\Phi + v_A) - g(\Phi + v_B)\|_{L^2} \leq K_g \|v_A - v_B\|_{L^2}. \tag{2.3.17}$$

Applying this inequality with $v_A = v$ and $v_B = \Phi_0 - \Phi$ we obtain

$$\|g(\Phi + v)\|_{L^2} \quad \leq \quad \|g(\Phi_0)\|_{L^2} + K_g\big[\|\Phi - \Phi_0\|_{L^2} + \|v\|_{L^2}\big], \tag{2.3.18}$$

which in view of (hPar) yields the first line of (2.3.15).

The uniform bound

$$|Dg(\Phi + v)| \leq K_g \tag{2.3.19}$$

together with the identity

$$\partial_\xi g(\Phi + v) = Dg(\Phi + v)\big(\Phi' + v'\big) \tag{2.3.20}$$

immediately imply the second estimate in (2.3.15). Finally, using

$$|Dg(\Phi + v) - Dg(\Phi + w)| \le K_g\,|v - w| \tag{2.3.21}$$

and the identity

$$\begin{aligned}\partial_\xi\big[g(\Phi + v_A) - g(\Phi + v_B)\big] \quad &= \quad \big[Dg(\Phi + v_A) - Dg(\Phi + v_B)\big]\big(\Phi' + v_A'\big) \\ &\quad +Dg(\Phi + v_B)\big(v_A' - v_B'\big),\end{aligned} \tag{2.3.22}$$

we obtain

$$\begin{aligned}\|\partial_\xi\big[g(\Phi + v_A) - g(\Phi + v_B)\big]\|_{L^2} \quad &\le \quad K_g\|v_A - v_B\|_\infty\big[\|\Phi'\|_{L^2} + \|v_A'\|_{L^2}\big] \\ &\quad +K_g\|v_A' - v_B'\|_{L^2}.\end{aligned} \tag{2.3.23}$$

The second estimate in (2.3.16) now follows easily. □

Lemma 2.3.3. *Suppose that (Hg) and (hPar) are satisfied. Then there exists a constant $K > 0$, which does not depend on the pair (Φ, c), so that the following holds true. For any $v \in H^1$ and $\psi \in H^1$ we have the bounds*

$$\begin{aligned}|\langle g(\Phi + v), \psi\rangle_{L^2}| \quad &\le \quad K[1 + \|v\|_{L^2}]\|\psi\|_{L^2}, \\ |\langle \partial_\xi g(\Phi + v), \psi\rangle_{L^2}| \quad &\le \quad K[1 + \|v\|_{L^2}]\|\psi\|_{H^1},\end{aligned} \tag{2.3.24}$$

while for any set of pairs $(v_A, v_B) \in H^1 \times H^1$ and $(\psi_A, \psi_B) \in H^1 \times H^1$ we have the estimates

$$\begin{aligned}|\langle g(\Phi + v_A), \psi_A\rangle_{L^2} - \langle g(\Phi + v_B), \psi_B\rangle_{L^2}| \le& K\|v_A - v_B\|_{L^2}\|\psi_A\|_{L^2} \\ &+ K[1 + \|v_B\|_{L^2}]\|\psi_A - \psi_B\|_{L^2}, \\ |\langle \partial_\xi[g(\Phi + v_A)], \psi_A\rangle_{L^2} - \langle \partial_\xi[g(\Phi + v_B)], \psi_B\rangle_{L^2}| \le& K\|v_A - v_B\|_{L^2}\|\psi_A\|_{H^1} \\ &+ K[1 + \|v_B\|_{L^2}]\|\psi_A - \psi_B\|_{H^1}.\end{aligned} \tag{2.3.25}$$

Proof. The estimates (2.3.24) follow immediately from the bound $\|g(\Phi + v)\|_{L^2} \le K\big[1 + \|v\|_{L^2}\big]$. The first bound in (2.3.25) can be obtained from Lemma 2.3.2 by noting that

$$\begin{aligned}|\langle g(\Phi + v_A), \psi_A\rangle_{L^2} - \langle g(\Phi + v_B), \psi_B\rangle_{L^2}| \quad &\le \quad |\langle g(\Phi + v_A) - g(\Phi + v_B), \psi_A\rangle_{L^2}| \\ &\quad + |\langle g(\Phi + v_B), \psi_A - \psi_B\rangle_{L^2}|.\end{aligned} \tag{2.3.26}$$

The final bound can be obtained by transferring the derivative to the functions ψ_A and ψ_B. □

2.3.2 Bounds on $\mathcal{J}_0$, b and κ_σ

We are now ready to obtain global and Lipschitz bounds on the functions $\mathcal{J}_0$, b and κ_σ. In addition, we show that it suffices to impose an a priori bound on $\|v\|_{L^2}$ in order to avoid hitting the cut-offs in the definition of b. This is crucial for the estimates in §2.9, where we have uniform control on $\|v\|_{L^2}$, but only an integrated form of control on $\|v\|_{H^1}$.

Lemma 2.3.4. *Suppose that (Hf) and (hPar) are satisfied. Then there exists a constant $K > 0$, which does not depend on the pair (Φ, c), so that the following holds true. For any $(v, d) \in H^1 \times \mathbb{R}$ and $\psi \in H^1$ we have the bounds*[6]

$$\begin{array}{lll} \|\mathcal{J}_0(\Phi + v, c + d)\|_{L^2} & \leq & K(1 + |d|)\big[1 + \|v\|_{H^1} + \|v\|_{H^1}^2\|v\|_{L^2}\big], \\ |\langle \mathcal{J}_0(\Phi + v, c + d), \psi\rangle_{L^2}| & \leq & K(1 + |d|)\big[1 + \|v\|_{H^1}\|v\|_{L^2}^2\big]\|\psi\|_{H^1}. \end{array} \tag{2.3.27}$$

In addition, for any set of pairs $(v_A, v_B) \in H^1 \times H^1$, $(d_A, d_B) \in \mathbb{R} \times \mathbb{R}$ and $(\psi_A, \psi_B) \in H^1 \times H^1$, the expressions

$$\begin{array}{lll} \Delta_{AB}\mathcal{J}_0 & = & \mathcal{J}_0(\Phi + v_A, c + d_A) - \mathcal{J}_0(\Phi + v_B, c + d_B), \\ \Delta_{AB}\langle \mathcal{J}_0, \cdot\rangle_{L^2} & = & \langle \mathcal{J}_0(\Phi + v_A, c + d_A), \psi_A\rangle_{L^2} - \langle \mathcal{J}_0(\Phi + v_B, c + d_B), \psi_B\rangle_{L^2} \end{array} \tag{2.3.28}$$

satisfy the estimates

$$\begin{array}{lll} \|\Delta_{AB}\mathcal{J}_0\|_{L^2} & \leq & K[\|v_A\|_{H^1}\|v_A\|_{L^2} + \|v_B\|_{H^1}\|v_B\|_{L^2}]\|v_A - v_B\|_{H^1} \\ & & \quad +[1 + \|v_A\|_{H^1}]\,|d_A - d_B| \\ & & \quad +K(1 + |d_B|)\|v_A - v_B\|_{H^1}, \\ |\Delta_{AB}\langle \mathcal{J}_0, \cdot\rangle_{L^2}| & \leq & K[\|v_A\|_{L^2}^2 + \|v_B\|_{L^2}^2]\|v_A - v_B\|_{H^1}\|\psi_A\|_{H^1} \\ & & \quad +[1 + \|v_A\|_{L^2}]\,|d_A - d_B|\,\|\psi_A\|_{H^1} \\ & & \quad +K(1 + |d_B|)\|v_A - v_B\|_{L^2}\|\psi_A\|_{H^1} \\ & & \quad +K(1 + |d_B|)\big[1 + \|v_B\|_{H^1}\|v_B\|_{L^2}^2\big]\|\psi_A - \psi_B\|_{H^1}. \end{array} \tag{2.3.29}$$

Proof. We first note that the terms in (2.3.3)-(2.3.5) can be absorbed in (2.3.27)-(2.3.29), so it suffices to study the function

$$\mathcal{J}_{0;II}(u, \overline{c}) = \overline{c}u'. \tag{2.3.30}$$

Recalling that (hPar) implies

$$|c| + \|\Phi'\|_{L^2} \leq C_1, \tag{2.3.31}$$

we find

$$\|\mathcal{J}_{0;II}(\Phi + v, c + d)\|_{L^2} \leq C_2(1 + |d|)(1 + \|v\|_{H^1}), \tag{2.3.32}$$

together with

$$|\langle \mathcal{J}_{0;II}(\Phi + v, c + d), \psi\rangle_{L^2}| \leq C_2(1 + |d|)(1 + \|v\|_{L^2})\|\psi\|_{H^1}, \tag{2.3.33}$$

which can be absorbed in (2.3.27).

In addition, writing

$$\begin{array}{lll} \Delta_{AB}\mathcal{J}_{0;II} & = & \mathcal{J}_{0;II}(\Phi + v_A, c + d_A) - \mathcal{J}_{0;II}(\Phi + v_B, c + d_B), \\ \Delta_{AB}\langle \mathcal{J}_{0;II}, \cdot\rangle_{L^2} & = & \langle \mathcal{J}_{0;II}(\Phi + v_A, c + d_A), \psi_A\rangle_{L^2} \\ & & \quad -\langle \mathcal{J}_{0;II}(\Phi + v_B, c + d_B), \psi_B\rangle_{L^2}, \end{array} \tag{2.3.34}$$

[6] We are dropping the third argument of $\mathcal{J}_0$ here since it is irrelevant when $\sigma = 0$.

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we compute

$$\Delta_{AB}\mathcal{J}_{0;II} = (d_A - d_B)(\Phi' + v_A') + (c + d_B)(v_A' - v_B'). \tag{2.3.35}$$

This yields

$$\|\Delta_{AB}\mathcal{J}_{0;II}\|_{L^2} \le C_3\,|d_A - d_B|\,(1 + \|v_A\|_{H^1}) + C_3(1 + |d_B|)\|v_A - v_B\|_{H^1}, \tag{2.3.36}$$

which establishes the first estimate in (2.3.29).

In a similar fashion, we obtain

$$|\langle \Delta_{AB}\mathcal{J}_{0;II}, \psi\rangle_{L^2}| \le C_3\,|d_A - d_B|\,(1 + \|v_A\|_{L^2})\|\psi\|_{H^1} + C_3(1 + |d_B|)\|v_A - v_B\|_{L^2}\|\psi\|_{H^1}. \tag{2.3.37}$$

The remaining estimate now follows from the inequality

$$\begin{aligned} |\Delta_{AB}\langle \mathcal{J}_{0;II}, \cdot\rangle_{L^2}| \quad &\le \quad |\langle \Delta_{AB}\mathcal{J}_{0;II}, \psi_A\rangle_{L^2}| \\ &\qquad + |\langle \mathcal{J}_{0;II}(\Phi + v_B, c + d_B), \psi_A - \psi_B\rangle_{L^2}|\,. \end{aligned} \tag{2.3.38}$$

□

Lemma 2.3.5. *Assume that (hPar) is satisfied. Then there exists a constant $K > 0$, which does not depend on the pair (Φ, c), so that the following holds true. For any $v \in H^1$ and $\psi \in H^1$ we have the bound*

$$|\langle \partial_\xi(\Phi + v), \psi\rangle_{L^2}| \le K\big[1 + \|v\|_{L^2}\big]\|\psi\|_{H^1}, \tag{2.3.39}$$

while for any set of pairs $(v_A, v_B) \in H^1 \times H^1$ and $(\psi_A, \psi_B) \in H^1 \times H^1$ we have the estimate

$$\begin{aligned} |\langle \partial_\xi[\Phi + v_A], \psi_A\rangle_{L^2} - \langle \partial_\xi[\Phi + v_B], \psi_B\rangle_{L^2}| \quad &\le \quad K\|v_A - v_B\|_{L^2}\|\psi_A\|_{H^1} \\ &\qquad + K\big[1 + \|v_B\|_{L^2}\big]\|\psi_A - \psi_B\|_{H^1}. \end{aligned} \tag{2.3.40}$$

Proof. The desired bounds follow from the identity

$$|\langle \partial_\xi(\Phi + v), \psi\rangle_{L^2}| = |\langle \Phi + v, \partial_\xi\psi\rangle_{L^2}|\,, \tag{2.3.41}$$

together with the estimate

$$\begin{aligned} |\langle \partial_\xi[\Phi + v_A], \psi_A\rangle_{L^2} - \langle \partial_\xi[\Phi + v_B], \psi_B\rangle_{L^2}| \quad &\le \quad |\langle v_A - v_B, \partial_\xi\psi_A\rangle_{L^2}| \\ &\qquad + |\langle \partial_\xi\Phi, \psi_A - \psi_B\rangle_{L^2}| \\ &\qquad + |\langle v_B, \partial_\xi[\psi_A - \psi_B]\rangle_{L^2}|\,. \end{aligned} \tag{2.3.42}$$

□

Lemma 2.3.6. *Suppose that (Hg) and (hPar) are satisfied. Then there exist constants $K_b > 0$ and $K > 0$, which do not depend on the pair (Φ, c), so that the following holds true. For any $v \in H^1$ and $\psi \in H^1$ we have the bound*

$$|b(\Phi + v, \psi)| \quad \le \quad K_b, \tag{2.3.43}$$

while for any set of pairs $(v_A, v_B) \in H^1 \times H^1$ *and* $(\psi_A, \psi_B) \in H^1 \times H^1$ *we have the estimate*

$$\begin{aligned} |b(\Phi + v_A, \psi_A) - b(\Phi + v_B, \psi_B)| \quad \leq \quad & K\|v_A - v_B\|_{L^2}\|\psi_A\|_{H^1} \\ & +K\big[1 + \|v_B\|_{L^2}\big]\|\psi_A - \psi_B\|_{H^1}. \end{aligned} \tag{2.3.44}$$

Proof. The uniform bound (2.3.43) follows directly from the properties of the cut-off functions. Upon introducing the function

$$\tilde{b}(x, y) = -\chi_{\mathrm{low}}(x)^{-1}\chi_{\mathrm{high}}(y), \tag{2.3.45}$$

the global Lipschitz smoothness of the cut-off functions implies that

$$\big|\tilde{b}(x_A, y_A) - \tilde{b}(x_B, y_B)\big| \leq C_1\big[\,|x_B - x_A| + |y_B - y_A|\,\big]. \tag{2.3.46}$$

Using the identity

$$b(u, \psi) = \tilde{b}\Big(\langle \partial_\xi u, \psi\rangle_{L^2}, \langle g(u), \psi\rangle_{L^2}\Big), \tag{2.3.47}$$

the desired bound (2.3.44) follows from Lemmas 2.3.3 and 2.3.5. □

Lemma 2.3.7. *Assume that (Hg) and (hPar) are satisfied. Then for any* $v \in H^1$ *that has*

$$\|v\|_{L^2} \leq \min\{1, [4\|\psi_{\mathrm{tw}}\|_{H^1}]^{-1}\}, \tag{2.3.48}$$

we have the identity

$$b(\Phi + v, \psi_{\mathrm{tw}}) = -\big[\langle \partial_\xi[\Phi + v], \psi_{\mathrm{tw}}\rangle_{L^2}\big]^{-1}\langle g(\Phi + v), \psi_{\mathrm{tw}}\rangle_{L^2}. \tag{2.3.49}$$

Proof. Using (2.3.15) and recalling the definition (2.2.23), we find that

$$|\langle g(\Phi + v), \psi_{\mathrm{tw}}\rangle_{L^2}| \leq \Big[\|g(\Phi_0)\|_{L^2} + 2K_g\Big]\|\psi_{\mathrm{tw}}\|_{L^2} = K_{\mathrm{ip}}. \tag{2.3.50}$$

In addition, we note that (hPar) and the normalisation (2.2.14) imply that

$$\langle \partial_\xi \Phi, \psi_{\mathrm{tw}}\rangle_{L^2} = \langle \partial_\xi \Phi_0, \psi_{\mathrm{tw}}\rangle_{L^2} + \langle \partial_\xi[\Phi - \Phi_0], \psi_{\mathrm{tw}}\rangle_{L^2} \geq 1 - \frac{1}{4} = \frac{3}{4}. \tag{2.3.51}$$

This allows us to estimate

$$\langle \partial_\xi(\Phi + v), \psi_{\mathrm{tw}}\rangle_{L^2} \geq \frac{3}{4} - \langle v, \partial_\xi \psi_{\mathrm{tw}}\rangle_{L^2} \geq \frac{3}{4} - \frac{1}{4} = \frac{1}{2}, \tag{2.3.52}$$

which shows that the cut-off functions do not modify their arguments. □

Lemma 2.3.8. *Suppose that (Hg) and (hPar) are satisfied. Then there exists a constant* $K_\kappa > 0$*, which does not depend on the pair* (Φ, c)*, so that for any* $0 \leq \sigma \leq 1$*, any* $v \in H^1$ *and any* $\psi \in H^1$*, we have the bound*

$$|\kappa_\sigma(\Phi + v, \psi)| + \big|\kappa_\sigma(\Phi + v, \psi)^{-1}\big| + \Big|\kappa_\sigma(\Phi + v, \psi)^{-1/2}\Big| \leq K_\kappa. \tag{2.3.53}$$

Proof. This follows directly from the bound

$$1 \leq \kappa_\sigma(\Phi + v, \psi) \leq 1 + \frac{1}{2\rho}\sigma^2 K_b^2. \tag{2.3.54}$$

□

In order to state our final result, we introduce the functions

$$\begin{aligned} \nu_\sigma^{(1)}(u,\psi) &= \kappa_\sigma(u,\psi) - 1, \\ \nu_\sigma^{(-1)}(u,\psi) &= \kappa_\sigma(u,\psi)^{-1} - 1, \\ \nu_\sigma^{(-1/2)}(u,\psi) &= \kappa_\sigma(u,\psi)^{-1/2} - 1, \end{aligned} \tag{2.3.55}$$

which isolate the σ-dependence in κ_σ.

Lemma 2.3.9. *Suppose that (Hg) and (hPar) are satisfied and pick $\vartheta \in \{-1, -\frac{1}{2}, 1\}$. Then there exist constants $K_\nu > 0$ and $K > 0$, which do not depend on the pair (Φ, c), so that the following holds true. For any $0 \leq \sigma \leq 1$, any $v \in H^1$ and any $\psi \in H^1$ we have the bound*

$$\left|\nu_\sigma^{(\vartheta)}(\Phi + v, \psi)\right| \leq \sigma^2 K_\nu, \tag{2.3.56}$$

while for any $0 \leq \sigma \leq 1$ and any set of pairs $(v_A, v_B) \in H^1 \times H^1$ and $(\psi_A, \psi_B) \in H^1 \times H^1$, we have the estimate

$$\begin{aligned} \left|\nu_\sigma^{(\vartheta)}(\Phi + v_A, \psi_A) - \nu_\sigma^{(\vartheta)}(\Phi + v_B, \psi_B)\right| \leq\; & K\sigma^2 \|v_A - v_B\|_{L^2} \|\psi_A\|_{H^1} \\ & + K\sigma^2 \big[1 + \|v_B\|_{L^2}\big] \|\psi_A - \psi_B\|_{H^1}. \end{aligned} \tag{2.3.57}$$

Proof. As a preparation, we observe that for any $x \geq 0$ and $y \geq 0$ we have the inequality

$$\left|\frac{1}{1+x} - \frac{1}{1+y}\right| = \frac{|y-x|}{(1+x)(1+y)} \leq |y - x|, \tag{2.3.58}$$

together with

$$\left|\frac{1}{\sqrt{1+x}} - \frac{1}{\sqrt{1+y}}\right| = \frac{|y-x|}{\sqrt{(1+x)(1+y)}(\sqrt{1+x} + \sqrt{1+y})} \leq \frac{1}{2}|y - x|. \tag{2.3.59}$$

Applying these bounds with $y = 0$, we obtain

$$\left|\nu_\sigma^{(\vartheta)}(\Phi + v, \psi)\right| \leq \frac{1}{2\rho}\sigma^2 \left|b(\Phi + v, \psi)\right|^2 \leq \frac{1}{2\rho}\sigma^2 K_b^2, \tag{2.3.60}$$

which yields (2.3.56). In addition, we may compute

$$\begin{aligned} \left|\nu_\sigma^{(\vartheta)}(\Phi + v_A, \psi_A) - \nu_\sigma^{(\vartheta)}(\Phi + v_B, \psi_B)\right| &\leq \frac{\sigma^2}{2\rho}\left|b(\Phi + v_A, \psi_A)^2 - b(\Phi + v_B, \psi_B)^2\right| \\ &= \frac{\sigma^2}{2\rho}\left|b(\Phi + v_A, \psi_A) + b(\Phi + v_B, \psi_B)\right| \\ &\quad \times \left|b(\Phi + v_A, \psi_A) - b(\Phi + v_B, \psi_B)\right|. \end{aligned} \tag{2.3.61}$$

In particular, the bounds (2.3.57) follow from Lemma 2.3.6. □

2.4 Variational solution

In this section we set out to establish Proposition 2.2.1. Our strategy is to fit the first component of (2.2.31) into the framework of [77]. Indeed, the conditions (H1)-(H4) in [77] are explicitly verified in Lemma 2.4.1 below. The second line of (2.2.31) can subsequently be treated as an SDE for Γ with random coefficients. In Lemma 2.4.3 below we show that this SDE fits into the framework that was developed in [93, Chapter 3] to handle such equations.

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Lemma 2.4.1. *Suppose that (HA), (Hf), (HTw), (HS), (HVar) and (Hg) are all satisfied. Then there exist constants $K > 0$ and $\vartheta > 0$ so that the following properties hold true.*

(i) For any triplet $(v_A, v_B, v) \in H^1 \times H^1 \times H^1$, the map

$$s \mapsto \langle A_*[v_A + sv_B], v\rangle_{H^{-1};H^1} + \langle f(\Phi_{\mathrm{ref}} + v_A + sv_B), v\rangle_{L^2} \tag{2.4.1}$$

is continuous.

(ii) For every pair $(v_A, v_B) \in H^1 \times H^1$, we have the inequality

$$\begin{aligned} K\|v_A - v_B\|_{L^2}^2 \geq\ & 2\langle A_*(v_A - v_B), v_A - v_B\rangle_{H^{-1};H^1} \\ & +2\langle f(\Phi_{\mathrm{ref}} + v_A) - f(\Phi_{\mathrm{ref}} + v_B), v_A - v_B\rangle_{L^2} \\ & +\|g(\Phi_{\mathrm{ref}} + v_A) - g(\Phi_{\mathrm{ref}} + v_B)\|_{L^2}^2. \end{aligned} \tag{2.4.2}$$

(iii) For any $v \in H^1$ we have the inequality

$$2\langle A_*v, v\rangle_{H^{-1};H^1} + 2\langle f(\Phi_{\mathrm{ref}} + v), v\rangle_{L^2} + \|g(\Phi_{\mathrm{ref}} + v)\|_{L^2}^2 + \vartheta\|v\|_{H^1}^2 \leq K\big[1 + \|v\|_{L^2}^2\big]. \tag{2.4.3}$$

(iv) For any $v \in H^1$ we have the bound

$$\|A_*v\|_{H^{-1}}^2 + \|f(\Phi_{\mathrm{ref}} + v)\|_{H^{-1}}^2 \leq K\big[1 + \|v\|_{H^1}^2\big]\big[1 + \|v\|_{L^2}^4\big]. \tag{2.4.4}$$

Proof. Item (i) follows from the linearity of A_* and the Lipschitz bound (2.3.5). In addition, writing

$$\mathcal{I} = \langle f(\Phi_{\mathrm{ref}} + v_A) - f(\Phi_{\mathrm{ref}} + v_B), v_A - v_B\rangle_{L^2}, \tag{2.4.5}$$

(HVar) implies the one-sided inequality

$$\begin{aligned} \mathcal{I} &= \langle f(\Phi_{\mathrm{ref}} + v_A) - f(\Phi_{\mathrm{ref}} + v_B), \Phi_{\mathrm{ref}} + v_A - (\Phi_{\mathrm{ref}} + v_B)\rangle_{L^2} \\ &\leq C_1\|v_A - v_B\|_{L^2}^2. \end{aligned} \tag{2.4.6}$$

Item (ii) hence follows from the Lipschitz bound (2.3.16) together with the bound

$$\langle A_*v, v\rangle_{H^{-1},H^1} \leq -\rho\|v\|_{H^1}^2. \tag{2.4.7}$$

A second consequence of (HVar) is that

$$\begin{array}{rcl}\langle f(\Phi_{\mathrm{ref}}+v),v\rangle_{L^2} & = & \langle f(\Phi_{\mathrm{ref}}+v)-f(\Phi_{\mathrm{ref}}),(\Phi_{\mathrm{ref}}+v)-\Phi_{\mathrm{ref}}\rangle_{L^2} \\ & & +\langle f(\Phi_{\mathrm{ref}}),v\rangle_{L^2} \\ & \le & C_1\|v\|_{L^2}^2+\|f(\Phi_{\mathrm{ref}})\|_{L^2}\|v\|_{L^2} \\ & \le & C_2\big[1+\|v\|_{L^2}^2\big].\end{array} \tag{2.4.8}$$

In particular, we may obtain (iii) by combining (2.4.7) with (2.3.15).

Finally, for any $v \in H^1$ and $\psi \in H^1$ we may use (2.3.3) to compute

$$\begin{array}{rcl}\langle f(\Phi_{\mathrm{ref}}+v),\psi\rangle_{H^{-1};H^1} & = & \langle f(\Phi_{\mathrm{ref}}+v),\psi\rangle_{L^2} \\ & \le & C_3\Big[1+\|v\|_{H^1}\|v\|_{L^2}^2\Big]\|\psi\|_{H^1}.\end{array} \tag{2.4.9}$$

In other words, we see that

$$\|f(\Phi_{\mathrm{ref}}+v)\|_{H^{-1}} \le C_3\big[1+\|v\|_{H^1}\|v\|_{L^2}^2\big] \le C_3(1+\|v\|_{H^1})(1+\|v\|_{L^2}^2), \tag{2.4.10}$$

which yields (iv). □

Lemma 2.4.2. *Suppose that (HA), (Hf), (Hg) and (hPar) are all satisfied. Then there exists a constant $K>0$, which does not depend on the pair (Φ,c), so that the following properties hold true for any $0\le\sigma\le 1$.*

(i) For any $v\in H^1$ and any $\psi\in H^2$ with $\|\psi\|_{H^2}\le 2\|\psi_{\mathrm{tw}}\|_{H^2}$, we have the bound

$$|a_\sigma(\Phi+v,c,\psi)| \le K\Big[1+\|v\|_{H^1}\|v\|_{L^2}^2\Big]. \tag{2.4.11}$$

(ii) For any $v\in H^1$ and any pair $(\psi_A,\psi_B)\in H^2\times H^2$ for which $\|\psi_A\|_{H^2}\le 2\|\psi_{\mathrm{tw}}\|_{H^2}$ and $\|\psi_B\|_{H^2}\le 2\|\psi_{\mathrm{tw}}\|_{H^2}$, the difference

$$\Delta_{AB}a_\sigma = a_\sigma(\Phi+v,c,\psi_A)-a_\sigma(\Phi+v,c,\psi_B) \tag{2.4.12}$$

satisfies the bound

$$|\Delta_{AB}a_\sigma| \le K\Big[1+\|v\|_{H^1}(1+\|v\|_{L^2}^3)\Big]\|\psi_A-\psi_B\|_{H^1}. \tag{2.4.13}$$

Proof. We first compute

$$\begin{array}{rcl}\kappa_\sigma(u,\psi)\langle\mathcal{J}_\sigma(u,c,\psi),\psi\rangle_{L^2} & = & \langle f(u)+cu'+\sigma^2 b(u,\psi)\partial_\xi[g(u)],\psi\rangle_{L^2} \\ & = & \langle\mathcal{J}_0(u,c),\psi\rangle_{L^2}+\sigma^2 b(u,\psi)\langle\partial_\xi[g(u)],\psi\rangle_{L^2}.\end{array} \tag{2.4.14}$$

Upon defining

$$\begin{array}{rcl}\mathcal{E}_I(u,c,\psi) & = & \langle\mathcal{J}_0(u,c),\psi\rangle_{L^2}, \\ \mathcal{E}_{II}(u,\psi) & = & \sigma^2 b(u,\psi)\langle\partial_\xi g(u),\psi\rangle_{L^2}, \\ \mathcal{E}_{III}(u,\psi) & = & \kappa_\sigma(u,\psi)\langle u,A_*\psi\rangle_{L^2},\end{array} \tag{2.4.15}$$

we hence see that

$$a_\sigma(u,c,\psi) = -\Big[\chi_{\mathrm{low}}\big(\langle \partial_\xi u, \psi\rangle_{L^2}\big)\Big]^{-1}\big[\mathcal{E}_I(u,c,\psi) + \mathcal{E}_{II}(u,\psi) + \mathcal{E}_{III}(u,\psi)\big]. \quad (2.4.16)$$

For $\# \in \{I, II, III\}$, we define

$$\Delta_{AB}\mathcal{E}_\# = \mathcal{E}_\#(\Phi + v, c, \psi_A) - \mathcal{E}_\#(\Phi + v, c, \psi_B). \quad (2.4.17)$$

We note that Lemmas 2.3.3, 2.3.4 and 2.3.6 yield the bounds

$$\begin{array}{lcl} |\mathcal{E}_I(\Phi + v, c, \psi)| & \leq & C_1\big[1 + \|v\|_{H^1}\|v\|_{L^2}^2\big], \\ |\mathcal{E}_{II}(\Phi + v, \psi)| & \leq & C_1\big[1 + \|v\|_{L^2}\big], \end{array} \quad (2.4.18)$$

together with

$$\begin{array}{lcl} |\Delta_{AB}\mathcal{E}_I| & \leq & C_1\big[1 + \|v\|_{H^1}\|v\|_{L^2}^2\big]\|\psi_A - \psi_B\|_{H^1}, \\ |\Delta_{AB}\mathcal{E}_{II}| & \leq & C_1[1 + \|v\|_{L^2}]^2\|\psi_A - \psi_B\|_{H^1} \\ & & +C_1[1 + \|v\|_{L^2}]\|\psi_A - \psi_B\|_{H^1}. \end{array} \quad (2.4.19)$$

A direct estimate using the a-priori bound on $\|\psi\|_{H^2}$ and (2.3.53) yields

$$\begin{array}{lcl} |\mathcal{E}_{III}(\Phi + v, \psi)| & \leq & K_\kappa\big[\,|\langle \Phi, A_*\psi\rangle_{L^2}| + |\langle v, A_*\psi\rangle_{L^2}|\,\big] \\ & \leq & C_2\big[1 + \|v\|_{L^2}\big]. \end{array} \quad (2.4.20)$$

By transferring one of the derivatives in A_*, we also obtain using Lemma 2.3.9 the bound

$$\begin{array}{lcl} |\Delta\mathcal{E}_{III}| & \leq & |\kappa_\sigma(\Phi + v, \psi_A) - \kappa_\sigma(\Phi + v, \psi_B)|\,|\langle \Phi + v, A_*\psi_A\rangle_{L^2}| \\ & & + |\kappa_\sigma(\Phi + v, \psi_B)|\,|\langle \Phi + v, A_*[\psi_A - \psi_B]\rangle_{L^2}| \\ & \leq & C_3(1 + \|v\|_{L^2})^2\|\psi_A - \psi_B\|_{H^1} \\ & & +C_3\big[1 + \|v\|_{H^1}\big]\|\psi_A - \psi_B\|_{H^1}. \end{array} \quad (2.4.21)$$

Upon writing

$$\begin{array}{lcl} \mathcal{E}(u, c, \psi) & = & \mathcal{E}_I(u, c, \psi) + \mathcal{E}_{II}(u, \psi) + \mathcal{E}_{II}(u, \psi), \\ \Delta_{AB}\mathcal{E} & = & \Delta_{AB}\mathcal{E}_I + \Delta_{AB}\mathcal{E}_{II} + \Delta_{AB}\mathcal{E}_{III}, \end{array} \quad (2.4.22)$$

we hence conclude that

$$\begin{array}{lcl} |\mathcal{E}(\Phi + v, c, \psi)| & \leq & C_4\big[1 + \|v\|_{H^1}\|v\|_{L^2}^2\big], \\ |\Delta_{AB}\mathcal{E}| & \leq & C_4\big[1 + \|v\|_{H^1}\big]\big[1 + \|v\|_{L^2}^2\big]\|\psi_A - \psi_B\|_{H^1}. \end{array} \quad (2.4.23)$$

Item (i) follows immediately from the first bound, since $\chi_{\mathrm{low}}(\cdot)^{-1}$ is globally bounded. To obtain (ii), we compute

$$\begin{aligned}|\Delta_{AB} a_\sigma| \;&\leq\; C_5\, |\langle \partial_\xi(\Phi + v), \psi_A\rangle_{L^2} - \langle \partial_\xi(\Phi + v), \psi_B\rangle_{L^2}|\, |\mathcal{E}(\Phi + v, \psi_A)| \\ &\qquad + C_5\, |\Delta_{AB}\mathcal{E}| \\ &\leq\; C_6[1 + \|v\|_{L^2}]\big[1 + \|v\|_{H^1}\|v\|_{L^2}^2\big]\,\|\psi_A - \psi_B\|_{H^1} \\ &\qquad + C_6\big[1 + \|v\|_{H^1}\big]\big[1 + \|v\|_{L^2}^2\big]\,\|\psi_A - \psi_B\|_{H^1} \\ &\leq\; C_7\Big[1 + \|v\|_{H^1}(1 + \|v\|_{L^2}^3)\Big]\,\|\psi_A - \psi_B\|_{H^1},\end{aligned} \tag{2.4.24}$$

in which we used several estimates of the form

$$\|v\|_{L^2} \leq C_8\big[1 + \|v\|_{L^2}^4\big] \leq C_8\big[1 + \|v\|_{H^1}\|v\|_{L^2}^3\big]. \tag{2.4.25}$$

□

Upon introducing the shorthands

$$\begin{aligned} p(v, \gamma) \;&=\; c + a_\sigma(\Phi_{\mathrm{ref}} + v, c, T_\gamma \psi_{\mathrm{tw}}), \\ q(v, \gamma) \;&=\; b(\Phi_{\mathrm{ref}} + v, T_\gamma \psi_{\mathrm{tw}}), \end{aligned} \tag{2.4.26}$$

the second line of (2.2.31) can be written as

$$d\Gamma = p\big(X(t), \Gamma(t)\big)dt + \sigma q\big(X(t), \Gamma(t)\big)d\beta_t. \tag{2.4.27}$$

Taking the view-point that $X(t) = X(t, \omega)$ is known upon picking $\omega \in \Omega$, (2.4.27) can be viewed as an SDE with random coefficients. Our next result relates directly to the conditions of [93, Thm. 3.1.1], which is specially tailored for equations of this type.

Lemma 2.4.3. *Suppose that (HA), (Hf), (HTw), (HS) and (Hg) are all satisfied and fix $c \in \mathbb{R}$ together with $0 \leq \sigma \leq 1$. Then there exists $K > 0$ so that the following properties are satisfied.*

(i) For any $v \in H^1$ and any pair $(\gamma_A, \gamma_B) \in \mathbb{R}^2$, we have the inequality

$$\begin{aligned} K\big[1 + \|v\|_{H^1}(1 + \|v\|_{L^2}^3)\big]\,|\gamma_A - \gamma_B|^2 \;&\geq\; 2\big[\gamma_A - \gamma_B\big]\big[p(v, \gamma_A) - p(v, \gamma_B)\big] \\ &\qquad + |q(v, \gamma_A) - q(v, \gamma_B)|^2. \end{aligned} \tag{2.4.28}$$

(ii) For any $v \in H^1$ and $\gamma \in \mathbb{R}$, we have the inequality

$$2\gamma p(v, \gamma) + |q(v, \gamma)|^2 \leq K\big[1 + \|v\|_{H^1}\|v\|_{L^2}^2\big]\big[1 + \gamma^2\big]. \tag{2.4.29}$$

(iii) For any $v \in H^1$ and $\gamma \in \mathbb{R}$, we have the bound

$$|p(v, \gamma)| + |q(v, \gamma)|^2 \leq K\big[1 + \|v\|_{H^1}\|v\|_{L^2}^2\big]. \tag{2.4.30}$$

Proof. The exponential decay of ψ'_{tw} and ψ''_{tw} implies that

$$\|T_{\gamma_A}\psi_{\mathrm{tw}} - T_{\gamma_B}\psi_{\mathrm{tw}}\|_{H^1} \le C_1 \left|\gamma_A - \gamma_B\right|. \tag{2.4.31}$$

Using Lemmas 2.3.6 and 2.4.2, we hence find the bounds

$$\begin{aligned} |p(v,\gamma)| &\le C_2\big[1 + \|v\|_{H^1}\|v\|_{L^2}^2\big], \\ |q(v,\gamma)| &\le K_b, \end{aligned} \tag{2.4.32}$$

together with

$$\begin{aligned} |p(v,\gamma_A) - p(v,\gamma_B)| &\le C_3\big[1 + \|v\|_{H^1}(1 + \|v\|_{L^2}^3)\big]\left|\gamma_A - \gamma_B\right|, \\ |q(v,\gamma_A) - q(v,\gamma_B)| &\le C_3[1 + \|v\|_{L^2}]\left|\gamma_A - \gamma_B\right|. \end{aligned} \tag{2.4.33}$$

Items (i), (ii) and (iii) can now be verified directly. □

Proof of Prop. 2.2.1. The existence of the $dt \otimes \mathbb{P}$ version of X that is $(\mathcal{F}_t)$-progressively measurable as a map into H^1, follows from [93, Ex. 4.2.3].

We remark that the conditions (H1) through (H4) appearing in [77] correspond directly with items (i)-(iv) of Lemma 2.4.1. In particular, we may apply the main result from this paper with $\alpha = 2$ and $\beta = 4$ to verify the remaining statements concerning X.

Finally, we note that items (i)-(iii) of Lemma 2.4.3 allow us to apply [93, Thm. 3.1.1], provided that the function

$$t \mapsto \big[1 + \|X(t)\|_{H^1}(1 + \|X(t)\|_{L^2}^3)\big] \tag{2.4.34}$$

is integrable on $[0,T]$ for almost all $\omega \in \Omega$. This however follows directly from the inclusions

$$X \in L^6\big(\Omega, \mathbb{P}; C([0,T]; L^2)\big) \cap \mathcal{N}^2\big([0,T]; (\mathcal{F}_t); H^1\big), \tag{2.4.35}$$

allowing us to verify the statements concerning Γ. The remaining inclusions (2.2.41) follow directly from the bounds in Lemmas 2.3.2 and 2.3.6. □

2.5 The stochastic phase-shift

In this section we consider the process (X,Γ) described in Proposition 2.2.1 and define the new process

$$V(t) = T_{-\Gamma(t)}[X(t) + \Phi_{\mathrm{ref}}] - \Phi \tag{2.5.1}$$

for some $\Phi \in \mathcal{U}_{H^1}$. In addition, we introduce the nonlinearity

$$\begin{aligned} \mathcal{R}_{\sigma;\Phi,c}(v) &= \kappa_\sigma(\Phi + v, \psi_{\mathrm{tw}})A_*[\Phi + v] \\ &\quad + f(\Phi + v) + \sigma^2 b(\Phi + v, \psi_{\mathrm{tw}})\partial_\xi[g(\Phi + v)] \\ &\quad + \Big[c + a_\sigma\big(\Phi + v, c, \psi_{\mathrm{tw}}\big)\Big][\Phi' + v'] \\ &= \kappa_\sigma(\Phi + v, \psi_{\mathrm{tw}})\Big[A_*[\Phi + v] + \mathcal{J}_\sigma(\Phi + v, c, \psi_{\mathrm{tw}})\Big] \\ &\quad + a_\sigma\big(\Phi + v, c, \psi_{\mathrm{tw}}\big)[\Phi' + v'], \end{aligned} \tag{2.5.2}$$

together with

$$\mathcal{S}_{\Phi}(v) \quad = \quad g(\Phi + v) + b(\Phi + v, \psi_{\mathrm{tw}})[\Phi' + v']. \tag{2.5.3}$$

Our main result states that the shifted process V can be interpreted as a weak solution to the SPDE

$$dV \quad = \quad \mathcal{R}_{\sigma;\Phi,c}(V)\, dt + \sigma \mathcal{S}_{\Phi}(V) d\beta_t. \tag{2.5.4}$$

Proposition 2.5.1. *Consider the setting of Proposition 2.2.1 and suppose that (hPar) is satisfied. Then the map*

$$V : [0,T] \times \Omega \to L^2 \tag{2.5.5}$$

defined by (2.5.1) satisfies the following properties.

(i) For almost all $\omega \in \Omega$, the map $t \mapsto V(t,\omega)$ is of class $C([0,T]; L^2)$.

(ii) For all $t \in [0,T]$, the map $\omega \mapsto V(t,\omega) \in L^2$ is $(\mathcal{F}_t)$-measurable.

(iii) We have the inclusion

$$V \quad \in \quad \mathcal{N}^2\big([0,T]; (\mathcal{F}_t); H^1\big) \tag{2.5.6}$$

together with

$$\mathcal{S}_{\Phi}(V) \quad \in \quad \mathcal{N}^2\big([0,T]; (\mathcal{F}_t); L^2\big). \tag{2.5.7}$$

(iv) For almost all $\omega \in \Omega$, we have the inclusion

$$\mathcal{R}_{\sigma;\Phi,c}\big(V(\cdot,\omega)\big) \in L^1([0,T]; H^{-1}). \tag{2.5.8}$$

(v) For almost all $\omega \in \Omega$, the identity

$$V(t) \quad = \quad V(0) + \int_0^t \mathcal{R}_{\sigma;\Phi,c}\big(V(s)\big)\, ds + \sigma \int_0^t \mathcal{S}_{\Phi}\big(V(s)\big) d\beta_s \tag{2.5.9}$$

holds for all $0 \leq t \leq T$.

Taking derivatives of translation operators typically requires extra regularity of the underlying function, which prevents us from applying an Itô formula directly to (2.5.1). In order to circumvent this technical issue, we pick a test function $\zeta \in C_c^{\infty}(\mathbb{R}; \mathbb{R}^n)$ and consider the two maps

$$\phi_{1;\zeta} : H^{-1} \times \mathbb{R} \to \mathbb{R}, \qquad \phi_{2;\zeta} : \mathbb{R} \to \mathbb{R} \tag{2.5.10}$$

that act as

$$\begin{aligned} \phi_{1;\zeta}\big(x, \gamma\big) \quad &= \quad \langle x, T_{\gamma}\zeta \rangle_{H^{-1};H^1}, \\ \phi_{2;\zeta}\big(\gamma\big) \quad &= \quad \langle T_{-\gamma}\Phi_{\mathrm{ref}} - \Phi, \zeta \rangle_{H^{-1};H^1} \\ &= \quad \langle T_{-\gamma}\Phi_{\mathrm{ref}} - \Phi, \zeta \rangle_{L^2}. \end{aligned} \tag{2.5.11}$$

These two maps do have sufficient smoothness for our purposes here.

Lemma 2.5.2. *Consider the setting of Proposition 2.2.1. Then for almost all* $\omega \in \Omega$ *the identity*

$$\begin{aligned}
\phi_{1;\zeta}\big(X(t), \Gamma(t)\big) = & \\
& \phi_{1;\zeta}\big(X(0), \Gamma(0)\big) \\
& + \int_0^t \langle A_*[X(s) + \Phi_{\mathrm{ref}}] + f(X(s) + \Phi_{\mathrm{ref}}), T_{\Gamma(s)}\zeta\rangle_{H^{-1};H^1} \, ds \\
& - \int_0^t \big[c + a_\sigma\big(X(s) + \Phi_{\mathrm{ref}}, c, T_{\Gamma(s)}\psi_{\mathrm{tw}}\big)\big] \langle X(s), T_{\Gamma(s)}\zeta'\rangle_{L^2} \, ds \\
& - \frac{1}{2}\sigma^2 \int_0^t 2b\big(X(s) + \Phi_{\mathrm{ref}}, T_{\Gamma(s)}\psi_{\mathrm{tw}}\big) \langle g(X(s) + \Phi_{\mathrm{ref}}), T_{\Gamma(s)}\zeta'\rangle_{L^2} \, ds \\
& + \frac{1}{2}\sigma^2 \int_0^t b\big(X(s) + \Phi_{\mathrm{ref}}, T_{\Gamma(s)}\psi_{\mathrm{tw}}\big)^2 \langle X(s), T_{\Gamma(s)}\zeta''\rangle_{L^2} \, ds \\
& + \sigma \int_0^t \langle g(X(s) + \Phi_{\mathrm{ref}}), T_{\Gamma(s)}\zeta\rangle_{L^2} \, d\beta_s \\
& - \sigma \int_0^t b\big(X(s) + \Phi_{\mathrm{ref}}, T_{\Gamma(s)}\psi_{\mathrm{tw}}\big) \langle X(s), T_{\Gamma(s)}\zeta'\rangle_{L^2} \, d\beta_s
\end{aligned} \tag{2.5.12}$$

holds for all $0 \leq t \leq T$.

Proof. We note that $\phi_{1;\zeta}$ is C^2-smooth, with derivatives given by

$$D\phi_{1;\zeta}(x, \gamma)[y, \beta] = \langle y, T_\gamma\zeta\rangle_{H^{-1};H^1} - \beta\langle x, T_\gamma\zeta'\rangle_{H^{-1};H^1}, \tag{2.5.13}$$

together with

$$D^2\phi_{1;\zeta}(x, \gamma)[y, \beta][y, \beta] = -2\beta\langle y, T_\gamma\zeta'\rangle_{H^{-1};H^1} + \beta^2\langle x, T_\gamma\zeta''\rangle_{H^{-1};H^1}. \tag{2.5.14}$$

Applying a standard Itô formula such as [27, Thm. 1] with $S = I$, the result readily follows. □

Lemma 2.5.3. *Consider the setting of Proposition 2.2.1. Then for almost all* $\omega \in \Omega$ *the identity*

$$\begin{aligned}
\phi_{2;\zeta}\big(\Gamma(t)\big) \quad = \quad & \phi_{2;\zeta}(\Gamma(0)) \\
& - \int_0^t \big[c + a_\sigma\big(X(s) + \Phi_{\mathrm{ref}}, c, T_{\Gamma(s)}\psi_{\mathrm{tw}}\big)\big] \langle \Phi_{\mathrm{ref}}, T_{\Gamma(s)}\zeta'\rangle_{L^2} \, ds \\
& + \frac{1}{2}\sigma^2 \int_0^t b\big(X(s) + \Phi_{\mathrm{ref}}, T_{\Gamma(s)}\psi_{\mathrm{tw}}\big)^2 \langle \Phi_{\mathrm{ref}}, T_{\Gamma(s)}\zeta''\rangle_{L^2} \, ds \\
& - \sigma \int_0^t b\big(X(s) + \Phi_{\mathrm{ref}}, T_{\Gamma(s)}\psi_{\mathrm{tw}}\big) \langle \Phi_{\mathrm{ref}}, T_{\Gamma(s)}\zeta'\rangle_{L^2} \, d\beta_s
\end{aligned} \tag{2.5.15}$$

holds for all $0 \leq t \leq T$.

Proof. We note that $\phi_{2;\zeta}$ is C^2-smooth, with derivatives given by

$$D\phi_{2;\zeta}(\gamma)[\beta] = -\beta\langle \Phi_{\mathrm{ref}}, T_\gamma \zeta' \rangle_{L^2}, \tag{2.5.16}$$

together with

$$D^2\phi_{2;\zeta}(\gamma)[\beta][\beta] = \beta^2\langle \Phi_{\mathrm{ref}}, T_\gamma \zeta'' \rangle_{L^2}. \tag{2.5.17}$$

The result again follows from the Itô formula. □

Corollary 2.5.4. *Consider the setting of Proposition 2.2.1, suppose that (hPar) is satisfied and pick a test-function $\zeta \in C_c^\infty(\mathbb{R};\mathbb{R}^n)$. Then for almost all $\omega \in \Omega$, the map V defined by (2.5.1) satisfies the identity*

$$\langle V(t), \zeta\rangle_{L^2} = \langle V(0), \zeta\rangle_{L^2} + \int_0^t \langle \mathcal{R}_{\sigma;\Phi,c}\big(V(s)\big), \zeta\rangle_{H^{-1};H^1} ds + \sigma \int_0^t \langle \mathcal{S}_\Phi\big(V(s)\big), \zeta\rangle_{L^2} d\beta_s \tag{2.5.18}$$

for all $0 \le t \le T$.

Proof. For any $\gamma \in \mathbb{R}$, we have the identities

$$a_\sigma(u, c, T_\gamma\psi) = a_\sigma(T_{-\gamma}u, c, \psi), \quad b(u, T_\gamma\psi) = b(T_{-\gamma}u, \psi), \tag{2.5.19}$$

together with the commutation relations

$$T_\gamma f(u) = f(T_\gamma u), \quad T_\gamma g(u) = g(T_\gamma u), \quad T_\gamma A_* u = A_* T_\gamma u. \tag{2.5.20}$$

By construction, we also have

$$\langle V(t), \zeta\rangle_{L^2} = \phi_{1;\zeta}\big(X(t), \Gamma(t)\big) + \phi_{2;\zeta}\big(\Gamma(t)\big), \tag{2.5.21}$$

together with

$$T_{-\Gamma(s)}[X(s) + \Phi_{\mathrm{ref}}] = \Phi + V(s). \tag{2.5.22}$$

The derivatives in (2.5.12) and (2.5.15) can now be transferred from ζ to yield (2.5.18).

We emphasize that the identity

$$\frac{1}{2}\sigma^2 b(\Phi + V(s), \psi_{\mathrm{tw}})^2[X'' + \Phi_{\mathrm{ref}}''] = \frac{1}{2\rho}\sigma^2 b(\Phi + V(s), \psi_{\mathrm{tw}})^2 A_*[X(s) + \Phi_{\mathrm{ref}}] \tag{2.5.23}$$

is a crucial ingredient in this computation. This is where we use the requirement in (HA) that all the diffusion coefficients in A_* are equal. □

Proof of Proposition 2.5.1. Items (i) and (ii) follow immediately from items (i) and (ii) of Proposition 2.2.1. Turning to (iii), notice first that we have the isometry

$$\|T_\gamma x\|_{H^1} = \|x\|_{H^1}. \tag{2.5.24}$$

Observe in addition that

$$\|T_\gamma \Phi_{\mathrm{ref}} - \Phi\|_{H^1} \le \|T_\gamma \Phi_{\mathrm{ref}} - \Phi_{\mathrm{ref}}\|_{H^1} + \|\Phi_{\mathrm{ref}} - \Phi\|_{H^1} \le C_1\big[1 + |\gamma|\big], \tag{2.5.25}$$

since Φ'_{ref} and Φ''_{ref} decay exponentially. In particular, the inclusion (2.5.6) follows from the corresponding inclusions (2.2.40) for the pair (X, Γ). The second inclusion (2.5.7) now follows immediately from the bounds in Lemmas 2.3.2 and 2.3.6.

Using Lemmas 2.3.1, 2.3.2, 2.3.6 and 2.4.2, we obtain the bound

$$\begin{aligned} \|\mathcal{R}_{\sigma;\Phi,c}(v)\|_{H^{-1}} \quad \leq \quad & C_2 K_\kappa \big[1 + \|v\|_{H^1}\big] \\ & + C_2 \big[1 + \|v\|_{H^1}^2 \|v\|_{L^2}\big] \\ & + C_2 \sigma^2 K_b \big[1 + \|v\|_{H^1}\big] \\ & + \big[1 + \|v\|_{H^1} \|v\|_{L^2}^2\big] \big[1 + \|v\|_{H^1}\big]. \end{aligned} \tag{2.5.26}$$

Since items (i) and (iii) imply that

$$\sup_{0 \leq t \leq T} \|V(t,\omega)\|_{L^2} + \int_0^T \|V(t,\omega)\|_{H^1}^2 \, dt < \infty \tag{2.5.27}$$

for almost all $\omega \in \Omega$, item (iv) follows from the standard bound

$$\int_0^T \|V(t,\omega)\|_{H^1} \, dt \leq \sqrt{T} \Big[\int_0^T \|V(t,\omega)\|_{H^1}^2 \, dt \Big]^{1/2}. \tag{2.5.28}$$

Finally, we note that items (iii) and (iv) imply that the integrals in (2.5.9) are well-defined. In view of Corollary 2.5.4, we can apply a standard diagonalisation argument involving the separability of L^2 and the density of test-functions to conclude that (v) holds. □

2.6 The stochastic time transform

We note that (2.5.9) can be interpreted as a quasi-linear equation due to the presence of the $\kappa_\sigma A_*$ term. In this section we transform our problem to a semilinear form by rescaling time, using the fact that κ_σ is a scalar. In addition, we investigate the impact of this transformation on the probabilities (2.2.62).

Recalling the map V defined in Proposition 2.5.1, we write

$$\tau_\Phi(t,\omega) = \int_0^t \kappa_\sigma\big(\Phi + V(s,\omega), \psi_{\mathrm{tw}}\big) \, ds. \tag{2.6.1}$$

Using Lemma 2.3.8 we see that $t \mapsto \tau_\Phi(t)$ is a continuous strictly increasing $(\mathcal{F}_t)$-adapted process that satisfies

$$t \leq \tau_\Phi(t) \leq K_\kappa t \tag{2.6.2}$$

for $0 \leq t \leq T$. In particular, we can define a map

$$t_\Phi : [0,T] \times \Omega \to [0,T] \tag{2.6.3}$$

for which

$$\tau_\Phi(t_\Phi(\tau,\omega),\omega) = \tau. \tag{2.6.4}$$

We now introduce the time-transformed map

$$\overline{V} : [0,T] \times \Omega \to L^2 \tag{2.6.5}$$

that acts as

$$\overline{V}(\tau,\omega) \quad = \quad V\big(t_\Phi(\tau,\omega),\omega\big). \tag{2.6.6}$$

Before stating our main results, we first investigate the effects of this transformation on the terms appearing in (2.5.9).

Lemma 2.6.1. *Consider the setting of Proposition 2.2.1 and suppose that (hPar) is satisfied. Then the map t_Φ defined in (2.6.3) satisfies the following properties.*

(i) For every $0 \le \tau \le T$, the random variable $\omega \mapsto t_\Phi(\tau,\omega)$ is an $(\mathcal{F}_t)$-stopping time.

(ii) The map $\tau \mapsto t_\Phi(\tau,\omega)$ is continuous and strictly increasing for all $\omega \in \Omega$.

(iii) For any $0 \le \tau \le T$ and $\omega \in \Omega$ we have the bounds

$$K_\kappa^{-1}\tau \le t_\Phi(\tau,\omega) \le \tau. \tag{2.6.7}$$

(iv) For every $0 \le t \le T$, the identity

$$t_\Phi(\tau_\Phi(t,\omega),\omega) = t \tag{2.6.8}$$

holds on the set $\{\omega : \tau_\Phi(t,\omega) \le T\}$.

Proof. On account of the identity

$$\{\omega : t_\Phi(\tau,\omega) \le t\} = \{\omega : \tau_\Phi(t,\omega) \ge \tau\} \tag{2.6.9}$$

and the fact that the latter set is in $\mathcal{F}_t$, we may conclude that $t_\Phi(\tau)$ is an $(\mathcal{F}_t)$-stopping time. The remaining properties follow directly from (2.6.2)-(2.6.4). □

Lemma 2.6.2. *Consider the setting of Proposition 2.2.1, recall the maps $(t_\Phi, \overline{V})$ defined by (2.6.3) and (2.6.6) and suppose that (hPar) is satisfied. Then there exists a filtration $(\overline{\mathcal{F}}_\tau)_{\tau\ge 0}$ together with a $(\overline{\mathcal{F}}_\tau)$-Brownian motion $(\overline{\beta}_\tau)_{\tau \ge 0}$ so that for any $H \in \mathcal{N}^2([0,T];(\mathcal{F}_t);L^2)$, the process*

$$\overline{H}(\tau,\omega) = H\big(t_\Phi(\tau,\omega),\omega\big) \tag{2.6.10}$$

satisfies the following properties.

(i) We have the inclusion

$$\overline{H} \in \mathcal{N}^2([0,T];(\overline{\mathcal{F}}_\tau);L^2), \tag{2.6.11}$$

together with the bound

$$E\int_0^T \|\overline{H}(\tau)\|_{L^2}^2 \, d\tau \le K_\kappa E \int_0^T \|H(t)\|_{L^2}^2 \, dt. \tag{2.6.12}$$

(ii) For almost all $\omega \in \Omega$, the identity

$$\int_0^{t_\Phi(\tau)} H(s)\, d\beta_s = \int_0^{\tau} \overline{H}(\tau')\kappa_\sigma(\Phi + \overline{V}(\tau'), \psi_{\mathrm{tw}})^{-1/2}\, d\overline{\beta}_{\tau'} \tag{2.6.13}$$

holds for all $0 \le \tau \le T$.

Proof. Following [59, §1.2.3], we write

$$\overline{\mathcal{F}}_\tau = \{A \in \cup_{t \ge 0}\mathcal{F}_t : A \cap \{t_\Phi(\tau) \le t\} \in \mathcal{F}_t \text{ for all } t \ge 0\}. \tag{2.6.14}$$

The fact that $\overline{H}$ is $(\overline{\mathcal{F}}_\tau)$-progressively measurable can be established following the proof of [58, Lem. 10.8(c)]. In addition, we note that for almost all $\omega \in \Omega$ the path

$$t \mapsto \|V(t,\omega)\|_{L^2}^2 \tag{2.6.15}$$

is in $L^1([0,T])$, which allows us to apply the deterministic substitution rule to obtain

$$\int_0^{t_\Phi(\tau)} \|V(s)\|_{L^2}^2\, ds = \int_0^{\tau} \|V\big(t_\Phi(\tau')\big)\|_{L^2}^2 \partial_{\tau'} t_\Phi(\tau')\, d\tau'. \tag{2.6.16}$$

We now note that

$$\begin{aligned} \partial_\tau t_\Phi(\tau) &= \big[\partial_t \tau_\Phi\big(t_\Phi(\tau)\big)\big]^{-1} \\ &= \kappa_\sigma\big(\Phi + V\big(t_\Phi(\tau)\big), \psi_{\mathrm{tw}}\big)^{-1} \\ &= \kappa_\sigma\big(\Phi + \overline{V}(\tau), \psi_{\mathrm{tw}}\big)^{-1}. \end{aligned} \tag{2.6.17}$$

In particular, we see that

$$|\partial_\tau t_\Phi(\tau)| \ge K_\kappa^{-1} \tag{2.6.18}$$

and hence

$$\int_0^{\tau} \|V\big(t_\Phi(\tau')\big)\|_{L^2}^2\, d\tau' \le K_\kappa \int_0^{t_\Phi(\tau)} \|V(s)\|_{L^2}^2\, ds. \tag{2.6.19}$$

The bound (2.6.12) now follows from $t_\Phi(T,\omega) \le T$.

To obtain (ii), we introduce the Brownian-motion $(\overline{\beta}_\tau)_{\tau \ge 0}$ that is given by

$$\overline{\beta}_\tau = \int_0^{\tau} \frac{1}{\sqrt{\partial_{\tau'} t_\Phi(\tau')}}\, d\beta_{t_\Phi(\tau')}. \tag{2.6.20}$$

For any test-function $\zeta \in C_c^\infty(\mathbb{R};\mathbb{R}^n)$ and $0 \le t \le T$, the proof of [59, Lem. 5.1.3.5] implies that for almost all $\omega \in \Omega$ the identity

$$\begin{aligned} \int_0^{t_\Phi(\tau)} \langle H(s), \zeta\rangle_{L^2} d\beta_s &= \int_0^{\tau} \langle H\big(t_\Phi(\tau')\big), \zeta\rangle_{L^2} \sqrt{\partial_{\tau'} t_\Phi(\tau')}\, d\overline{\beta}_{\tau'} \\ &= \int_0^{\tau} \langle \overline{H}(\tau'), \zeta\rangle_{L^2} \kappa_\sigma\big(\Phi + \overline{V}(\tau'), \psi_{\mathrm{tw}}\big)^{-1/2}\, d\overline{\beta}_{\tau'} \end{aligned} \tag{2.6.21}$$

holds for all $0 \le \tau \le T$. Since (i) and (ii) together imply that the right-hand side of (2.6.13) is well-defined as a stochastic-integral, a standard diagonalisation argument involving the separability of L^2 shows that both sides must be equal for almost all $\omega \in \Omega$. □

In order to formulate the time-transformed SPDE, we introduce the nonlinearity

$$\begin{aligned}\overline{\mathcal{R}}_{\sigma;\Phi,c}(v) &= \kappa_\sigma(\Phi+v,\psi_{\mathrm{tw}})^{-1}\mathcal{R}_{\sigma;\Phi,c}(v) - \mathcal{L}_{\mathrm{tw}}v \\ &= A_*[\Phi+v] + \mathcal{J}_\sigma(\Phi+v,c,\psi_{\mathrm{tw}}) \\ &\quad + \kappa_\sigma(\Phi+v,\psi_{\mathrm{tw}})^{-1}a(\Phi+v,c,\psi_{\mathrm{tw}})[\Phi'+v'] - \mathcal{L}_{\mathrm{tw}}v,\end{aligned} \tag{2.6.22}$$

together with

$$\begin{aligned}\overline{\mathcal{S}}_{\sigma;\Phi}(v) &= \kappa_\sigma(\Phi+v,\psi_{\mathrm{tw}})^{-1/2}\mathcal{S}_\Phi(v) \\ &= \kappa_\sigma(\Phi+v,\psi_{\mathrm{tw}})^{-1/2}\Big[g(\Phi+v) + b(\Phi+v,\psi_{\mathrm{tw}})[\Phi'+v']\Big].\end{aligned} \tag{2.6.23}$$

Proposition 2.6.3. *Consider the setting of Proposition 2.2.1 and suppose that (hPar) is satisfied. Then the map*

$$\overline{V} : [0,T]\times\Omega \to L^2 \tag{2.6.24}$$

defined by the transformations (2.5.1) and (2.6.6) satisfies the following properties.

(i) For almost all $\omega\in\Omega$, the map $\tau\mapsto\overline{V}(\tau;\omega)$ is of class $C\big([0,T];L^2\big)$.

(ii) For all $\tau\in[0,T]$, the map $\omega\mapsto\overline{V}(\tau,\omega)$ is $(\overline{\mathcal{F}}_\tau)$-measurable.

(iii) We have the inclusion

$$\overline{V}\in\mathcal{N}^2\big([0,T];(\overline{\mathcal{F}})_\tau;H^1\big), \tag{2.6.25}$$

together with

$$\overline{\mathcal{S}}_{\sigma;\Phi}(\overline{V})\in\mathcal{N}^2\big([0,T];(\overline{\mathcal{F}})_\tau;L^2\big). \tag{2.6.26}$$

(iv) For almost all $\omega\in\Omega$, we have the inclusion

$$\overline{\mathcal{R}}_{\sigma;\Phi,c}\big(\overline{V}(\cdot,\omega)\big)\in L^1([0,T];L^2). \tag{2.6.27}$$

(v) For almost all $\omega\in\Omega$, the identity

$$\begin{aligned}\overline{V}(\tau) &= \overline{V}(0) + \int_0^\tau \Big[\mathcal{L}_{\mathrm{tw}}\overline{V}(\tau') + \overline{\mathcal{R}}_{\sigma;\Phi,c}\big(\overline{V}(\tau')\big)\Big]\,d\tau' \\ &\quad + \sigma\int_0^\tau \overline{\mathcal{S}}_{\sigma;\Phi}\big(\overline{V}(\tau')\big)\,d\beta_{\tau'}\end{aligned} \tag{2.6.28}$$

holds for all $0\le t\le T$.

(vi) For almost all $\omega\in\Omega$, the identity

$$\begin{aligned}\overline{V}(\tau) &= S(\tau)\overline{V}(0) + \int_0^\tau S(\tau-\tau')\overline{\mathcal{R}}_{\sigma;\Phi,c}\big(\overline{V}(\tau')\big)\,d\tau' \\ &\quad + \sigma\int_0^\tau S(\tau-\tau')\overline{\mathcal{S}}_{\sigma;\Phi}\big(\overline{V}(\tau')\big)\,d\beta_{\tau'}\end{aligned} \tag{2.6.29}$$

holds for all $\tau\in[0,T]$, in which

$$S:[0,\infty)\to\mathcal{L}(L^2;L^2) \tag{2.6.30}$$

denotes the analytic semigroup generated by $\mathcal{L}_{\mathrm{tw}}$.

Proof. Items (i)-(iii) follow by applying (i) of Lemma 2.6.2 to the maps V, $\partial_\xi V$ and using the definition (2.6.23). Item (iv) can be obtained from the computation (2.5.26), noting that the $A_* v$ contribution is no longer present.

Item (v) can be obtained by applying the stochastic time-transform (2.6.13) and the deterministic time-transform

$$\int_0^{t_\Phi(\tau)} \mathcal{R}_{\sigma;\Phi,c}\big(V(s)\big)\, ds \quad = \quad \int_0^{\tau} \overline{\mathcal{R}}_{\sigma;\Phi,c}\big(\overline{V}(\tau')\big)\big[\kappa_\sigma\big(\Phi + \overline{V}(\tau'), \psi_{\mathrm{tw}}\big)\big]^{-1}\, d\tau' \tag{2.6.31}$$

to the integral equation (2.5.9).

Turning to (vi), we note that A_* generates a standard diagonal heat-semigroup, which is obviously analytic. Noting that

$$\mathcal{L}_{\mathrm{tw}} - A_* \in \mathcal{L}(H^1; L^2) \tag{2.6.32}$$

and recalling the interpolation estimate

$$\|v\|_{H^1} \le C_1 \|v\|_{H^2}^{1/2} \|v\|_{L^2}^{1/2}, \tag{2.6.33}$$

we may apply [80, Prop. 3.2.2(iii)] to conclude that also $\mathcal{L}_{\mathrm{tw}}$ generates an analytic semigroup. We may now apply [92, Prop. 6.3] and the computation in the proof of [80, Prop. 4.1.4] to conclude the integral identity (2.6.29). □

We now introduce the scalar functions

$$\begin{aligned} N_{\varepsilon,\alpha}(t) &= e^{\alpha t}\|V(t)\|_{L^2}^2 + \int_0^t e^{-\varepsilon(t-s)} e^{\alpha s}\|V(s)\|_{H^1}^2\, ds, \\ \overline{N}_{\varepsilon,\alpha}(\tau) &= e^{\alpha\tau}\|\overline{V}(\tau)\|_{L^2}^2 + \int_0^\tau e^{-\varepsilon(\tau-\tau')} e^{\alpha\tau'}\|\overline{V}(\tau')\|_{H^1}^2\, d\tau', \end{aligned} \tag{2.6.34}$$

together with the associated probabilities

$$\begin{aligned} p_{\varepsilon,\alpha}(T,\eta) &= P\Big(\sup_{0\le t\le T} N_{\varepsilon,\alpha}(t) > \eta\Big), \\ \overline{p}_{\varepsilon,\alpha}(T,\eta) &= P\Big(\sup_{0\le \tau\le T} \overline{N}_{\varepsilon,\alpha}(\tau) > \eta\Big). \end{aligned} \tag{2.6.35}$$

Our second main result shows that these two sets of probabilities can be effectively compared with each other.

Proposition 2.6.4. *Consider the setting of Proposition 2.2.1 and recall the maps* V *and* $\overline{V}$ *defined by (2.5.1) and (2.6.6). Then we have the bound*

$$p_{\varepsilon,\alpha}(T,\eta) \le \overline{p}_{K_\kappa^{-1}\varepsilon,\alpha}\big(K_\kappa T, K_\kappa^{-1}\eta\big). \tag{2.6.36}$$

Proof. We note that

$$e^{\alpha t}\|V(t)\|_{L^2} = e^{\alpha t}\|\overline{V}\big(\tau_\Phi(t)\big)\|_{L^2} \le e^{\alpha \tau_\Phi(t)}\|\overline{V}\big(\tau_\Phi(t)\big)\|_{L^2}, \tag{2.6.37}$$

which implies that

$$\sup_{0\le t\le T} e^{\alpha t}\|V(t)\|_{L^2}^2 \le \sup_{0\le \tau\le K_\kappa T} e^{\alpha\tau}\|\overline{V}(\tau)\|_{L^2}^2. \tag{2.6.38}$$

In addition, we compute

$$\begin{aligned}\int_0^t e^{-\varepsilon(t-s)} e^{\alpha s} \|V(s)\|_{H^1}^2 \, ds \\ = \int_0^{\tau_\Phi(t)} e^{-\varepsilon(t - t_\Phi(\tau'))} e^{\alpha t_\Phi(\tau')} \|\overline{V}(\tau')\|_{H^1}^2 \kappa_\sigma\big(\Phi + \overline{V}(\tau'), \psi_{\mathrm{tw}}\big)^{-1} \, d\tau'.\end{aligned} \tag{2.6.39}$$

Using (2.6.18) we obtain the estimate

$$t - t_\Phi(\tau') = t_\Phi\big(\tau_\Phi(t)\big) - t_\Phi(\tau') = \int_{\tau'}^{\tau_\Phi(t)} \partial_{\tau''} t_\Phi(\tau'') \, d\tau'' \geq K_\kappa^{-1} \left|\tau_\Phi(t) - \tau'\right|, \tag{2.6.40}$$

which yields

$$\int_0^t e^{-\varepsilon(t-s)} e^{\alpha s} \|V(s)\|_{H^1}^2 \, ds \leq K_\kappa \int_0^{\tau_\Phi(t)} e^{-K_\kappa^{-1}\varepsilon(\tau_\Phi(t) - \tau')} e^{\alpha \tau'} \|\overline{V}(\tau')\|_{H^1}^2 \, d\tau'. \tag{2.6.41}$$

In particular, we conclude that

$$\sup_{0 \leq t \leq T} \int_0^t e^{-\varepsilon(t-s)} e^{\alpha s} \|V(s)\|_{H^1}^2 \, ds \leq \sup_{0 \leq \tau \leq K_\kappa T} K_\kappa \int_0^\tau e^{-K_\kappa^{-1}\varepsilon(\tau - \tau')} e^{\alpha \tau'} \|\overline{V}(\tau')\|_{H^1}^2 \, d\tau'. \tag{2.6.42}$$

This yields the implication

$$\sup_{0 \leq \tau \leq K_\kappa T} \overline{N}_{K_\kappa^{-1}\varepsilon, \alpha}(\tau) \leq K_\kappa^{-1} \eta \Rightarrow \sup_{0 \leq t \leq T} N_{\varepsilon, \alpha}(t) \leq \eta, \tag{2.6.43}$$

from which the desired inequality immediately follows. □

2.7 The stochastic wave

In this section we set out to construct the branch of modified waves (Φ_σ, c_σ) and analyze the phase condition

$$\langle T_{-\gamma_0}[u_0] - \Phi_\sigma, \psi_{\mathrm{tw}} \rangle_{L^2} = 0 \tag{2.7.1}$$

for $u_0 \approx \Phi_\sigma$. In particular, we establish Propositions 2.2.2, 2.2.3 and 2.2.6.

A key role in our analysis is reserved for the function

$$\begin{aligned}\mathcal{M}_{\sigma;\Phi,c}(v,d) &= \mathcal{J}_\sigma(\Phi + v, c + d, \psi_{\mathrm{tw}}) - \mathcal{J}_0(\Phi, c) \\ &\quad - d\Phi_0' + [A_* - \mathcal{L}_{\mathrm{tw}}]v,\end{aligned} \tag{2.7.2}$$

defined for $(\Phi, c) \in \mathcal{U}_{H^1} \times \mathbb{R}$ and $(v, d) \in H^1 \times \mathbb{R}$. Indeed, we will construct a solution to

$$A_* \Phi_\sigma + \mathcal{J}_\sigma(\Phi_\sigma, c_\sigma, \psi_{\mathrm{tw}}) = 0 \tag{2.7.3}$$

by writing

$$\Phi_\sigma = \Phi_0 + v, \quad c_\sigma = c_0 + d. \tag{2.7.4}$$

Using the fact that the pair (Φ_0, c_0) is a solution to (2.7.3) for $\sigma = 0$, one readily verifies that the pair $(v, d) \in H^2 \times \mathbb{R}$ must satisfy the system

$$d\Phi_0' + \mathcal{L}_{\mathrm{tw}} v = -\mathcal{M}_{\sigma;\Phi_0,c_0}(v, d). \tag{2.7.5}$$

In addition, the function $\mathcal{M}_{\sigma;\Phi_\sigma,c_\sigma}$ will be used in §2.8 to obtain bounds on the nonlinearity $\overline{\mathcal{R}}_{\sigma;\Phi_\sigma,c_\sigma}$.

In §2.7.1 we obtain global and Lipschitz bounds on $\mathcal{M}_{\sigma;\Phi,c}$. These bounds are subsequently used in §2.7.2 to setup two fixed-point constructions that provide solutions to (2.7.1) and (2.7.3).

2

2.7.1 Bounds for $\mathcal{M}_\sigma$

In order to streamline our estimates, it is convenient to decompose the function $\mathcal{J}_\sigma$ as

$$\begin{aligned} \mathcal{J}_\sigma(u, \overline{c}, \psi_{\mathrm{tw}}) &= \kappa_\sigma(u, \psi_{\mathrm{tw}})^{-1}\Big[f(u) + \overline{c}u' + \sigma^2 b(u, \psi_{\mathrm{tw}})\partial_\xi[g(u)]\Big] \\ &= \mathcal{J}_0(u, \overline{c}) + \mathcal{E}_{\sigma;I}(u, \overline{c}) + \mathcal{E}_{\sigma;II}(u). \end{aligned} \tag{2.7.6}$$

Here we have introduced the function

$$\begin{aligned} \mathcal{E}_{\sigma;I}(u, \overline{c}) &= \nu_\sigma^{(-1)}(u, \psi_{\mathrm{tw}})[f(u) + \overline{c}u'] \\ &= \nu_\sigma^{(-1)}(u, \psi_{\mathrm{tw}})\mathcal{J}_0(u, \overline{c}), \end{aligned} \tag{2.7.7}$$

together with

$$\mathcal{E}_{\sigma;II}(u) = \sigma^2 \kappa_\sigma(u, \psi_{\mathrm{tw}})^{-1} b(u, \psi_{\mathrm{tw}})\partial_\xi[g(u)] \tag{2.7.8}$$

where ν_σ^{-1} is as defined in (2.3.55).
This decomposition allows us to rewrite (2.7.2) in the intermediate form

$$\mathcal{M}_{\sigma;\Phi,c}(v, d) = \mathcal{M}_{0;\Phi,c}(v, d) + \mathcal{E}_{\sigma;I}(\Phi + v, c + d) + \mathcal{E}_{\sigma;II}(\Phi + v). \tag{2.7.9}$$

We now make a final splitting

$$\begin{aligned} \mathcal{M}_{0;\Phi,c}(v, d) &= \mathcal{J}_0(\Phi + v, c + d) - \mathcal{J}_0(\Phi, c) - Df(\Phi_0)v - c_0 v' - d\Phi_0' \\ &= \mathcal{N}_{I;f,\Phi}(v) + \mathcal{N}_{II;\Phi,c}(v, d), \end{aligned} \tag{2.7.10}$$

in which we have introduced the function

$$\mathcal{N}_{I;f,\Phi}(v) = f(\Phi + v) - f(\Phi) - Df(\Phi)v, \tag{2.7.11}$$

together with

$$\mathcal{N}_{II;\Phi,c}(v, d) = dv' + \big[Df(\Phi) - Df(\Phi_0)\big]v + (c - c_0)v' + d[\Phi' - \Phi_0']. \tag{2.7.12}$$

We hence arrive at the convenient final expression

$$\mathcal{M}_{\sigma;\Phi,c}(v, d) = \mathcal{N}_{I;f,\Phi}(v) + \mathcal{N}_{II;\Phi,c}(v, d) + \mathcal{E}_{\sigma;I}(\Phi + v, c + d) + \mathcal{E}_{\sigma;II}(\Phi + v) \tag{2.7.13}$$

and set out to analyze each of these terms separately.

2

Lemma 2.7.1. *Suppose that (Hf) and (hPar) are satisfied. Then there exists $K > 0$ so that for any $v \in H^1$ we have the bound*

$$\|\mathcal{N}_{I;f,\Phi}(v)\|_{L^2} \leq K\big[1 + \|v\|_{H^1}\big]\|v\|_{H^1}\|v\|_{L^2}, \tag{2.7.14}$$

while for any pair $(v_A, v_B) \in H^1 \times H^1$ we have the estimates

$$\begin{aligned} \|\mathcal{N}_{I;f,\Phi}(v_A) - \mathcal{N}_{I;f,\Phi}(v_B)\|_{L^2} &\leq K\big[1 + \|v_A\|_{H^1} + \|v_B\|_{H^1}\big]\big[\|v_A\|_{H^1} + \|v_B\|_{H^1}\big] \\ &\quad \times \|v_A - v_B\|_{L^2}, \\ |\langle \mathcal{N}_{I;f,\Phi}(v_A) - \mathcal{N}_{I;f,\Phi}(v_B), \psi_{\mathrm{tw}}\rangle_{L^2}| &\leq K\big[1 + \|v_A\|_{H^1} + \|v_B\|_{H^1}\big]\big[\|v_A\|_{L^2} + \|v_B\|_{L^2}\big] \\ &\quad \times \|v_A - v_B\|_{L^2}. \end{aligned} \tag{2.7.15}$$

Proof. Using (2.3.6) and (hPar) we obtain the pointwise bound

$$|\mathcal{N}_{I;f,\Phi}(v)| \leq C_1[1 + |v|]\,|v|^2, \tag{2.7.16}$$

from which (2.7.14) easily follows. In addition, we may compute

$$\begin{aligned} \mathcal{N}_{I;f,\Phi}(v_A) - \mathcal{N}_{I;f,\Phi}(v_B) &= f(\Phi + v_A) - f(\Phi + v_B) - Df(\Phi + v_B)(v_A - v_B) \\ &\quad + \big(Df(\Phi + v_B) - Df(\Phi)\big)(v_A - v_B) \\ &= \mathcal{N}_{I;f,\Phi+v_B}(v_A - v_B) + \big(Df(\Phi + v_B) - Df(\Phi)\big)(v_A - v_B). \end{aligned} \tag{2.7.17}$$

Applying (2.3.6) and (hPar) a second time, we obtain the pointwise bound

$$\begin{aligned} |\mathcal{N}_{I;f,\Phi}(v_A) - \mathcal{N}_{I;f,\Phi}(v_B)| &\leq C_2[1 + |v_A| + |v_B|\,]\,|v_A - v_B|^2 \\ &\quad + C_2[1 + |v_B|]\,|v_B|\,|v_A - v_B| \\ &\leq C_3[1 + |v_A| + |v_B|\,]\big[\,|v_A| + |v_B|\,\big]\,|v_A - v_B|\,, \end{aligned} \tag{2.7.18}$$

from which the estimates in (2.7.15) can be readily obtained. □

Lemma 2.7.2. *Suppose that (Hf) and (hPar) are satisfied. Then there exists $K > 0$ so that for any $(v, d) \in H^1 \times \mathbb{R}$ we have the bound*

$$\|\mathcal{N}_{II;\Phi,c}(v, d)\|_{L^2} \leq K\big[\,|c - c_0| + \|\Phi - \Phi_0\|_{H^1} + |d|\,\big]\big[\|v\|_{H^1} + |d|\,\big], \tag{2.7.19}$$

while for any set of pairs $(v_A, v_B) \in H^1 \times H^1$ and $(d_A, d_B) \in \mathbb{R}^2$ the expression

$$\Delta_{AB}\mathcal{N}_{II;\Phi,c} = \mathcal{N}_{II;\Phi,c}(v_A, d_A) - \mathcal{N}_{II;\Phi,c}(v_B, d_B) \tag{2.7.20}$$

satisfies the estimates

$$\begin{aligned} \|\Delta_{AB}\mathcal{N}_{II;\Phi,c}\|_{L^2} &\leq K\big[\|v_A\|_{H^1} + |d_B| + \|\Phi - \Phi_0\|_{H^1} + |c - c_0|\,\big] \\ &\quad \times \big[\|v_A - v_B\|_{H^1} + |d_A - d_B|\,\big], \\ |\langle \Delta_{AB}\mathcal{N}_{II;\Phi,c}, \psi_{\mathrm{tw}}\rangle_{L^2}| &\leq K\big[\|v_A\|_{L^2} + |d_B| + \|\Phi - \Phi_0\|_{L^2} + |c - c_0|\,\big] \\ &\quad \times \big[\|v_A - v_B\|_{L^2} + |d_A - d_B|\,\big]. \end{aligned} \tag{2.7.21}$$

Proof. In view of (hPar), we obtain the pointwise bound

$$|\mathcal{N}_{II;\Phi,c}(v,d)| \leq \big[\,|d| + |c - c_0|\,\big]\,|v'| + C_1\,|\Phi - \Phi_0|\,|v| + |\Phi' - \Phi_0'|\,|d|\,, \tag{2.7.22}$$

from which (2.7.19) follows. In addition, we obtain the pointwise bound

$$\begin{aligned} |\Delta_{AB}\mathcal{N}_{II;\Phi,c}| \quad &\leq \quad |d_A - d_B|\,|v_A'| + \big[\,|d_B| + |c - c_0|\,\big]\,|v_A' - v_B'| \\ &\qquad + K\,|\Phi - \Phi_0|\,\big[\,|v_A - v_B|\,\big] + |\Phi' - \Phi_0'|\,|d_A - d_B| \end{aligned} \tag{2.7.23}$$

from which (2.7.21) follows. □

Lemma 2.7.3. *Suppose that (Hf), (Hg) and (hPar) are satisfied. Then there exists $K > 0$ so that for any $0 \leq \sigma \leq 1$ and $(v,d) \in H^1 \times \mathbb{R}$, we have the bound*

$$\|\mathcal{E}_{\sigma;I}(\Phi + v, c + d)\|_{L^2} \quad \leq \quad K\sigma^2(1 + |d|)\big[1 + \|v\|_{H^1} + \|v\|_{H^1}^2\|v\|_{L^2}\big], \tag{2.7.24}$$

while for any $0 \leq \sigma \leq 1$ and any set of pairs $(v_A, v_B) \in H^1 \times H^1$ and $(d_A, d_B) \in \mathbb{R}^2$, the expression

$$\Delta_{AB}\mathcal{E}_{\sigma;I} = \mathcal{E}_{\sigma;I}(\Phi + v_A, c + d_A) - \mathcal{E}_{\sigma;I}(\Phi + v_B, c + d_B) \tag{2.7.25}$$

satisfies the estimates

$$\begin{aligned} \|\Delta_{AB}\mathcal{E}_{\sigma;I}\|_{L^2} \quad &\leq \quad K\sigma^2(1 + |d_A|)\big[1 + \|v_A\|_{H^1} + \|v_A\|_{H^1}^2\|v_A\|_{L^2}\big]\|v_A - v_B\|_{L^2} \\ &\qquad + K\sigma^2\big[1 + |d_B| + \|v_A\|_{H^1}\|v_A\|_{L^2} + \|v_B\|_{H^1}\|v_B\|_{L^2}\big]\|v_A - v_B\|_{H^1} \\ &\qquad + K\sigma^2\big[1 + \|v_A\|_{H^1}\big]\,|d_A - d_B|\,, \end{aligned} \tag{2.7.26}$$

$$\begin{aligned} |\langle \Delta_{AB}\mathcal{E}_{\sigma;I}, \psi_{\mathrm{tw}}\rangle_{L^2}| \quad &\leq \quad K\sigma^2(1 + |d_A| + |d_B|)\big[1 + \|v_A\|_{H^1}\|v_A\|_{L^2}^2\big]\|v_A - v_B\|_{L^2} \\ &\qquad + K\sigma^2\big[\|v_A\|_{L^2}^2 + \|v_B\|_{L^2}^2\big]\|v_A - v_B\|_{H^1} \\ &\qquad + K\sigma^2[1 + \|v_A\|_{L^2}]\,|d_A - d_B|\,. \end{aligned} \tag{2.7.27}$$

Proof. The bound (2.7.24) follows directly from Lemmas 2.3.4 and 2.3.9. In addition, these results allow us to compute

$$\begin{aligned} \|\Delta_{AB}\mathcal{E}_{\sigma;I}\|_{L^2} \leq{} & \Big|\nu_\sigma^{(-1)}(\Phi + v_A, \psi_{\mathrm{tw}}) - \nu_\sigma^{(-1)}(\Phi + v_B, \psi_{\mathrm{tw}})\Big|\,\|\mathcal{J}_0(\Phi + v_A, c + d_A)\|_{L^2} \\ & + \Big|\nu_\sigma^{(-1)}(\Phi + v_B, \psi_{\mathrm{tw}})\Big|\,\|\mathcal{J}_0(\Phi + v_A, c + d_A) - \mathcal{J}_0(\Phi + v_B, c + d_B)\|_{L^2} \\ \leq{} & C_1\sigma^2\|v_A - v_B\|_{L^2}(1 + |d_A|)\big[1 + \|v_A\|_{H^1} + \|v_A\|_{H^1}^2\|v_A\|_{L^2}\big] \\ & + C_1\sigma^2\big[\|v_A\|_{H^1}\|v_A\|_{L^2} + \|v_B\|_{H^1}\|v_B\|_{L^2}\big]\|v_A - v_B\|_{H^1} \\ & + C_1\sigma^2[1 + \|v_A\|_{H^1}]\,|d_A - d_B| \\ & + C_1\sigma^2(1 + |d_B|)\|v_A - v_B\|_{H^1}, \end{aligned} \tag{2.7.28}$$

together with

$$\begin{aligned}
|\langle \Delta_{AB}\mathcal{E}_{\sigma;I}, \psi_{\mathrm{tw}}\rangle_{L^2}| \le & \\
& \Big|\nu_\sigma^{(-1)}(\Phi + v_A, \psi_{\mathrm{tw}}) - \nu_\sigma^{(-1)}(\Phi + v_B, \psi_{\mathrm{tw}})\Big| \, |\langle \mathcal{J}_0(\Phi + v_A, c + d_A), \psi_{\mathrm{tw}}\rangle_{L^2}| \\
& + \Big|\nu_\sigma^{(-1)}(\Phi + v_B, \psi_{\mathrm{tw}})\Big| \, |\langle \mathcal{J}_0(\Phi + v_A, c + d_A) - \mathcal{J}_0(\Phi + v_B, c + d_B), \psi_{\mathrm{tw}}\rangle_{L^2}| \\
& + C_2\sigma^2 \|v_A - v_B\|_{L^2}(1 + |d_A|)\big[1 + \|v_A\|_{H^1}\|v_A\|_{L^2}^2\big] \\
& + C_2\sigma^2 [\|v_A\|_{L^2}^2 + \|v_B\|_{L^2}^2]\|v_A - v_B\|_{H^1} \\
& + C_2\sigma^2[1 + \|v_A\|_{L^2}] \, |d_A - d_B| \\
& + C_2\sigma^2(1 + |d_B|)\|v_A - v_B\|_{L^2}.
\end{aligned} \tag{2.7.29}$$

These terms can all be absorbed by the expressions in (2.7.26) and (2.7.27). □

Lemma 2.7.4. *Suppose that (Hg) and (hPar) are satisfied. Then there exists $K > 0$ so that for any $0 \le \sigma \le 1$ and $v \in H^1$ we have the bound*

$$\|\mathcal{E}_{\sigma;II}(\Phi + v)\|_{L^2} \quad \le \quad K\sigma^2\big[1 + \|v\|_{H^1}\big], \tag{2.7.30}$$

while for any $0 \le \sigma \le 1$ and any pair $(v_A, v_B) \in H^1 \times H^1$ the expression

$$\Delta_{AB}\mathcal{E}_{\sigma;II} = \mathcal{E}_{\sigma;II}(\Phi + v_A) - \mathcal{E}_{\sigma;II}(\Phi + v_B) \tag{2.7.31}$$

satisfies the estimates

$$\begin{aligned}
\|\Delta_{AB}\mathcal{E}_{\sigma;II}\|_{L^2} \quad & \le \quad K\sigma^2\big[1 + \|v_A\|_{H^1}\big]\|v_A - v_B\|_{H^1}, \\
|\langle \Delta_{AB}\mathcal{E}_{\sigma;II}, \psi_{\mathrm{tw}}\rangle_{L^2}| \quad & \le \quad K\sigma^2\big[1 + \|v_A\|_{L^2}\big]\|v_A - v_B\|_{L^2}.
\end{aligned} \tag{2.7.32}$$

Proof. The bound (2.7.30) follows directly from Lemmas 2.3.2, 2.3.6 and 2.3.8. In addition, we may compute

$$\begin{aligned}
\|\Delta_{AB}\mathcal{E}_{\sigma;II}\|_{L^2} \le & \, \sigma^2 \Big|\nu_\sigma^{(-1)}(\Phi + v_A, \psi_{\mathrm{tw}}) - \nu_\sigma^{(-1)}(\Phi + v_B, \psi_{\mathrm{tw}})\Big| \, K_b\|\partial_\xi[g(\Phi + v_A)]\|_{L^2} \\
& + \sigma^2 K_\nu \, |b(\Phi + v_A, \psi_{\mathrm{tw}}) - b(\Phi + v_B, \psi_{\mathrm{tw}})| \, \|\partial_\xi[g(\Phi + v_A)]\|_{L^2} \\
& + \sigma^2 K_\kappa K_b \|\partial_\xi[g(\Phi + v_A) - g(\Phi + v_B)]\|_{L^2} \\
\le & \, C_1\sigma^2 \|v_A - v_B\|_{L^2}\big[1 + \|v_A\|_{H^1}\big] \\
& + C_1\sigma^2 \|v_A - v_B\|_{L^2}\big[1 + \|v_A\|_{H^1}\big] \\
& + C_1\sigma^2\big[1 + \|v_A\|_{H^1}\big]\|v_A - v_B\|_{H^1},
\end{aligned} \tag{2.7.33}$$

together with

$$\begin{aligned}
&|\langle \Delta_{AB}\mathcal{E}_{\sigma;II}, \psi_{\mathrm{tw}}\rangle_{L^2}| \\
&\le \sigma^2 \left| \nu_\sigma^{(-1)}(\Phi + v_A, \psi_{\mathrm{tw}}) - \nu_\sigma^{(-1)}(\Phi + v_B, \psi_{\mathrm{tw}}) \right| K_b \, |\langle \partial_\xi [g(\Phi + v_A)], \psi_{\mathrm{tw}}\rangle_{L^2}| \\
&\quad + \sigma^2 K_\nu \, |b(\Phi + v_A, \psi_{\mathrm{tw}}) - b(\Phi + v_B, \psi_{\mathrm{tw}})| \, |\langle \partial_\xi [g(\Phi + v_A)], \psi_{\mathrm{tw}}\rangle_{L^2}| \\
&\quad + \sigma^2 K_\nu K_b \, |\langle \partial_\xi [g(\Phi + v_A) - g(\Phi + v_B)], \psi_{\mathrm{tw}}\rangle_{L^2}| \\
&\le C_2 \sigma^2 \|v_A - v_B\|_{L^2} \big[1 + \|v_A\|_{L^2}\big] \\
&\quad + C_2 \sigma^2 \|v_A - v_B\|_{L^2} \big[1 + \|v_A\|_{L^2}\big] \\
&\quad + C_2 \sigma^2 \|v_A - v_B\|_{L^2}.
\end{aligned} \tag{2.7.34}$$

These expressions can be absorbed into the bounds (2.7.32). □

Corollary 2.7.5. *Suppose that (Hf), (Hg) and (hPar) are satisfied. Then there exists $K > 0$ so that the following holds true. For any $0 \le \sigma \le 1$ and any $(v, d) \in H^1 \times \mathbb{R}$ that has $|d| \le 1$, we have the estimate*

$$\begin{aligned}
\|\mathcal{M}_{\sigma;\Phi,c}(v, d)\|_{L^2} \le\ & K\big[1 + \|v\|_{H^1}\big] \|v\|_{H^1} \|v\|_{L^2} \\
& + K\big[\, |c - c_0| + \|\Phi - \Phi_0\|_{H^1} + |d| \,\big] \big[\|v\|_{H^1} + |d| \,\big] \\
& + K\sigma^2 \big[1 + \|v\|_{H^1}\big].
\end{aligned} \tag{2.7.35}$$

In addition, for any $0 \le \sigma \le 1$ and any set of pairs $(v_A, v_B) \in H^1 \times H^1$ and $(d_A, d_B) \in \mathbb{R}^2$ for which $|d_A| \le 1$ and $|d_B| \le 1$, the expression

$$\Delta_{AB}\mathcal{M}_{\sigma;\Phi,c} = \mathcal{M}_{\sigma;\Phi,c}(v_A, d_A) - \mathcal{M}_{\sigma;\Phi,c}(v_B, d_B) \tag{2.7.36}$$

satisfies the estimates

$$\begin{aligned}
\|\Delta_{AB}\mathcal{M}_{\sigma;\Phi,c}\|_{L^2} \le\ & K\big[1 + \|v_A\|_{H^1} + \|v_B\|_{H^1}\big] \big[\|v_A\|_{H^1} + \|v_B\|_{H^1} \big] \|v_A - v_B\|_{L^2} \\
& + K\big[\sigma^2 + \|v_A\|_{H^1} + |d_B| + \|\Phi - \Phi_0\|_{H^1} + |c - c_0| \,\big] \\
& \times \big[\|v_A - v_B\|_{H^1} + |d_A - d_B| \,\big] \\
& + K\sigma^2 \|v_A\|_{H^1}^2 \|v_A\|_{L^2} \|v_A - v_B\|_{L^2} \\
& + K\sigma^2 \Big[\|v_A\|_{H^1} \|v_A\|_{L^2} + \|v_B\|_{H^1} \|v_B\|_{L^2} \Big] \|v_A - v_B\|_{H^1},
\end{aligned} \tag{2.7.37}$$

2

$$
\begin{aligned}
|\langle \Delta_{AB} \mathcal{M}_{\sigma;\Phi,c}, \psi_{\mathrm{tw}} \rangle_{L^2}| \;\le\; & K\big[1 + \|v_A\|_{H^1} + \|v_B\|_{H^1}\big]\big[\|v_A\|_{L^2} + \|v_B\|_{L^2}\big]\|v_A - v_B\|_{L^2} \\
& + K\big[\sigma^2 + \|v_A\|_{L^2} + |d_B| + \|\Phi - \Phi_0\|_{L^2} + |c - c_0|\big] \\
& \times \big[\|v_A - v_B\|_{L^2} + |d_A - d_B|\big] \\
& + K\sigma^2 \|v_A\|_{H^1}\|v_A\|^2_{L^2}\|v_A - v_B\|_{L^2} \\
& + K\sigma^2 \Big[\|v_A\|^2_{L^2} + \|v_B\|^2_{L^2}\Big]\|v_A - v_B\|_{H^1}.
\end{aligned}
\tag{2.7.38}
$$

Proof. In view of the identity (2.7.13) it suffices to note that the terms (2.7.14), (2.7.19), (2.7.24) and (2.7.30) can be absorbed in (2.7.35), while the expressions (2.7.15), (2.7.21), (2.7.26), (2.7.27) and (2.7.32) can be absorbed in (2.7.37) and (2.7.38). □

Corollary 2.7.6. *Suppose that (Hf) and (Hg) are satisfied. Then there exists $K > 0$ so that the following holds true. For any $0 \le \sigma \le 1$ and any $(v, d) \in H^1 \times \mathbb{R}$ that has $\|v\|_{H^1} \le 1$ together with $|d| \le 1$, we have the estimate*

$$
\|\mathcal{M}_{\sigma;\Phi_0,c_0}(v, d)\|_{L^2} \quad \le \quad K\big[\|v\|_{L^2} + |d|\big]\big[\|v\|_{H^1} + |d|\big] + K\sigma^2. \tag{2.7.39}
$$

In addition, for any $0 \le \sigma \le 1$ and any set of pairs $(v_A, v_B) \in H^1 \times H^1$ and $(d_A, d_B) \in \mathbb{R}^2$ for which the bounds

$$
\|v_A\|_{H^1} \le 1, \quad |d_A| \le 1, \quad \|v_B\|_{H^1} \le 1, \quad |d_B| \le 1 \tag{2.7.40}
$$

hold, the expression

$$
\Delta_{AB} \mathcal{M}_{\sigma;\Phi_0,c_0} = \mathcal{M}_{\sigma;\Phi_0,c_0}(v_A, d_A) - \mathcal{M}_{\sigma;\Phi_0,c_0}(v_B, d_B) \tag{2.7.41}
$$

satisfies the estimate

$$
\|\Delta_{AB} \mathcal{M}_{\sigma;\Phi_0,c_0}\|_{L^2} \le K\big[\sigma^2 + \|v_A\|_{H^1} + \|v_B\|_{H^1} + |d_B|\big]\big[\|v_A - v_B\|_{H^1} + |d_A - d_B|\big]. \tag{2.7.42}
$$

Proof. These bounds can easily be obtained by simplifying the corresponding expressions from Corollary 2.7.5. □

2.7.2 Fixed-point constructions

As a final preparation before setting up our fixed-point problems, we need to control the higher order effects that arise when translating the adjoint eigenfunction ψ_{tw}. In particular, for any $\gamma \in \mathbb{R}$ we introduce the function

$$
\mathcal{N}_{\mathrm{tw}}(\gamma) = T_\gamma \psi_{\mathrm{tw}} - \psi_{\mathrm{tw}} + \gamma \psi'_{\mathrm{tw}} \tag{2.7.43}
$$

and obtain the following bounds.

Lemma 2.7.7. *Suppose that (HTw) and (HS) hold. Then there exists $K > 0$ so that for any $\gamma \in \mathbb{R}$ we have the bound*

$$\|\mathcal{N}_{\mathrm{tw}}(\gamma)\|_{L^2} \le K\gamma^2, \tag{2.7.44}$$

while for any pair $(\gamma_A, \gamma_B) \in \mathbb{R}^2$ we have the estimate

$$\|\mathcal{N}_{\mathrm{tw}}(\gamma_A) - \mathcal{N}_{\mathrm{tw}}(\gamma_B)\|_{L^2} \le K\big[\,|\gamma_A| + |\gamma_B|\,\big]\,|\gamma_A - \gamma_B|\,. \tag{2.7.45}$$

Proof. In view of (2.4.31), we have the a priori bound

$$\|\mathcal{N}_{\mathrm{tw}}(\gamma)\|_{L^2} \le C_1\big[1 + |\gamma|\,\big], \tag{2.7.46}$$

together with

$$\|\mathcal{N}_{\mathrm{tw}}(\gamma_A) - \mathcal{N}_{\mathrm{tw}}(\gamma_B)\|_{L^2} \le C_1\,|\gamma_A - \gamma_B|\,. \tag{2.7.47}$$

In particular, we can restrict our attention to the situation where $|\gamma| \le 1$ and $|\gamma_A| + |\gamma_B| \le 1$. In this case we obtain the pointwise bounds

$$|\mathcal{N}_{\mathrm{tw}}(\gamma)(\xi)| \le \frac{1}{2}\gamma^2 \sup_{\xi-1\le\xi'\le\xi+1} |\psi''_{\mathrm{tw}}(\xi')| \tag{2.7.48}$$

together with

$$|\mathcal{N}_{\mathrm{tw}}(\gamma_A)(\xi) - \mathcal{N}_{\mathrm{tw}}(\gamma_B)(\xi)| \le \Big[\sup_{\xi-1\le\xi'\le\xi+1} |\psi''_{\mathrm{tw}}(\xi')|\Big]\Big[\frac{1}{2}(\gamma_A-\gamma_B)^2 + |\gamma_B|\,|\gamma_A-\gamma_B|\Big]. \tag{2.7.49}$$

The desired bounds now follow from the exponential decay of ψ''_{tw}. □

Proof of Proposition 2.2.2. As a consequence of (HS), there exists a bounded linear map

$$\mathcal{L}_{\mathrm{inv}} : L^2 \to H^2 \times \mathbb{R} \tag{2.7.50}$$

so that for any $h \in L^2$, the pair $(v, d) = \mathcal{L}_{\mathrm{inv}}h$ is the unique solution in $H^2 \times \mathbb{R}$ to the problem

$$\mathcal{L}_{\mathrm{tw}}v = h - \Phi_0' d. \tag{2.7.51}$$

Indeed, we take $d = \langle h, \psi_{\mathrm{tw}}\rangle_{L^2}$, which in view of the normalization (2.2.14) ensures that the right-hand side of (2.7.51) is in the range of $\mathcal{L}_{\mathrm{tw}}$.

It now suffices to find a solution to the fixed-point problem

$$(v, d) = -\mathcal{L}_{\mathrm{inv}}\mathcal{M}_{\sigma;\Phi_0,c_0}(v, d). \tag{2.7.52}$$

Upon introducing the set

$$\mathcal{Z}_\Theta = \big\{(v, d) \in H^2 \times \mathbb{R} : \|v\|_{H^2} + |d| \le \min\{1, \Theta\sigma^2\}\big\} \subset H^2 \times \mathbb{R} \tag{2.7.53}$$

and applying Corollary 2.7.6, we see that for any $(v, d) \in \mathcal{Z}_\Theta$ we have

$$\|\mathcal{M}_{\sigma;\Phi_0,c_0}(v, d)\|_{L^2} \le K\big(\Theta^4\sigma^4 + \sigma^2\big) = K\sigma^2\big(\Theta^2\sigma^2 + 1\big), \tag{2.7.54}$$

while for any two pairs $(v_A, d_A) \in \mathcal{Z}_\Theta$ and $(v_B, d_B) \in \mathcal{Z}_\Theta$ we have

$$\|\mathcal{M}_{\sigma;\Phi_0,c_0}(v_A, d_A) - \mathcal{M}_{\sigma;\Phi_0,c_0}(v_B, d_B)\|_{L^2} \le K\sigma^2\big[1 + 2\Theta\big]\big[\|v_A - v_B\|_{H^1} + |d_A - d_B|\big]. \tag{2.7.55}$$

In particular, choosing Θ to be sufficiently large and $\delta_\sigma > 0$ to be sufficiently small, we see that the map $-\mathcal{L}_{\mathrm{inv}}\mathcal{M}_{\sigma;\Phi_0,c_0}$ is a contraction on $\mathcal{Z}_\Theta$ for all $0 \le \sigma \le \delta_\sigma$. □

2

Proof of Proposition 2.2.3. We first recall that

$$\langle \Phi_{\mathrm{ref}}, \psi'_{\mathrm{tw}} \rangle_{L^2} = -\langle \Phi'_{\mathrm{ref}}, \psi_{\mathrm{tw}} \rangle_{L^2} = -\langle \Phi'_0, \psi_{\mathrm{tw}} \rangle_{L^2} = -1. \tag{2.7.56}$$

Writing $u_0 = x_0 + \Phi_{\mathrm{ref}}$, this allows us to compute

$$\begin{aligned} \langle v_\gamma, \psi_{\mathrm{tw}} \rangle_{L^2} &= \langle x_0 + \Phi_{\mathrm{ref}}, T_\gamma \psi_{\mathrm{tw}} \rangle_{L^2} - \langle \Phi_\sigma, \psi_{\mathrm{tw}} \rangle_{L^2} \\ &= \langle x_0 + \Phi_{\mathrm{ref}}, \psi_{\mathrm{tw}} - \gamma \psi'_{\mathrm{tw}} + \mathcal{N}_{\mathrm{tw}}(\gamma) \rangle_{L^2} - \langle \Phi_\sigma, \psi_{\mathrm{tw}} \rangle_{L^2} \\ &= \gamma + \langle x_0 + \Phi_{\mathrm{ref}} - \Phi_\sigma, \psi_{\mathrm{tw}} \rangle_{L^2} + \mathcal{E}_\sigma(x_0, \gamma), \end{aligned} \tag{2.7.57}$$

in which we have introduced the expression

$$\mathcal{E}_\sigma(x_0, \gamma) = -\gamma \langle x_0, \psi'_{\mathrm{tw}} \rangle_{L^2} + \langle x_0 + \Phi_{\mathrm{ref}}, \mathcal{N}_{\mathrm{tw}}(\gamma) \rangle_{L^2}. \tag{2.7.58}$$

Using Lemma 2.7.7, we obtain the estimate

$$|\mathcal{E}_\sigma(x_0, \gamma)| \le C_1 \|x_0\|_{L^2} |\gamma| + C_1 \big[1 + \|x_0\|_{L^2}\big] \gamma^2, \tag{2.7.59}$$

together with the Lipschitz bound

$$\begin{aligned} \|\mathcal{E}_\sigma(x_0, \gamma_A) - \mathcal{E}_\sigma(x_0, \gamma_B)\|_{L^2} \le \; & C_2 \|x_0\|_{L^2} |\gamma_A - \gamma_B| \\ & + C_2 \big[1 + \|x_0\|_{L^2}\big] \big[\, |\gamma_A| + |\gamma_B| \,\big] \, |\gamma_A - \gamma_B| \, . \end{aligned} \tag{2.7.60}$$

In particular, upon choosing $\delta_{\mathrm{fix}} > 0$ to be sufficiently small and imposing the restriction

$$\|x_0\|_{L^2} + \|x_0 + \Phi_{\mathrm{ref}} - \Phi_\sigma\|_{L^2} < \delta_{\mathrm{fix}}, \tag{2.7.61}$$

we can define γ_0 as the unique solution to the fixed-point problem

$$-\gamma = \langle x_0 + \Phi_{\mathrm{ref}} - \Phi_\sigma, \psi_{\mathrm{tw}} \rangle_{L^2} + \mathcal{E}_\sigma(x_0, \gamma) \tag{2.7.62}$$

on the set

$$\Sigma_{x_0} = \{\gamma : |\gamma| \le 2 \|x_0 + \Phi_{\mathrm{ref}} - \Phi_\sigma\|_{L^2} \|\psi_{\mathrm{tw}}\|_{L^2}\}. \tag{2.7.63}$$

By choosing $\delta_\sigma > 0$ and $\delta_0 > 0$ to be sufficiently small, the bound (2.2.49) allows us to conclude that (2.7.61) is satisfied whenever (2.2.50) holds.

For any $\gamma \in \mathbb{R}$ we can compute

$$\begin{aligned} \|T_{-\gamma} \Phi_\sigma - \Phi_\sigma\|^2_{L^2} &= \int \big(\Phi_\sigma(\xi + \gamma) - \Phi_\sigma(\xi)\big)^2 \, d\xi \\ &= \int \big[\int_0^\gamma \Phi'_\sigma(\xi + s) \, ds \big]^2 \, d\xi \\ &\le \int |\gamma| \int_0^\gamma \Phi'_\sigma(\xi + s)^2 \, ds \, d\xi \\ &= |\gamma|^2 \int \Phi'_\sigma(\xi)^2 \, d\xi \\ &= |\gamma|^2 \, \|\Phi'_\sigma\|^2_{L^2}. \end{aligned} \tag{2.7.64}$$

In particular, we obtain the bound

$$\begin{aligned} \|v_{\gamma_0}\|_{L^2} &= \|x_0 + \Phi_{\mathrm{ref}} - T_{\gamma_0}\Phi_\sigma\|_{L^2} \\ &\leq \|x_0 + \Phi_{\mathrm{ref}} - \Phi_\sigma\|_{L^2} + \|T_{\gamma_0}\Phi_\sigma - \Phi_\sigma\|_{L^2} \\ &\leq \|x_0 + \Phi_{\mathrm{ref}} - \Phi_\sigma\|_{L^2} + C_3\,|\gamma_0|\,. \end{aligned} \tag{2.7.65}$$

The desired estimate (2.2.53) hence follows from $\gamma_0 \in \Sigma_{x_0}$. The final estimate (2.2.54) follows in a similar fashion, exploiting $\Phi_\sigma'' \in L^2$. □

Proof of Proposition 2.2.6. For convenience, we introduce the notation

$$\alpha_\sigma = \big[1 + \frac{1}{2\rho}\sigma^2\vartheta_0^2\big]^{1/2}. \tag{2.7.66}$$

Using the definitions (2.2.68) one easily verifies the identities

$$\Phi_\sigma'(\xi) = \alpha_\sigma\Phi_0'(\alpha_\sigma\xi), \quad \Phi_\sigma''(\xi) = \alpha_\sigma^2\Phi_0''(\alpha_\sigma\xi), \tag{2.7.67}$$

which yields

$$g\big(\Phi_\sigma(\xi)\big) = g\big(\Phi_0(\alpha_\sigma\xi)\big) = \vartheta_0\Phi_0'\big(\alpha_\sigma\xi\big) = \vartheta_0\alpha_\sigma^{-1}\Phi_\sigma'(\xi), \tag{2.7.68}$$

together with

$$f(\Phi_\sigma) + c_\sigma\Phi_\sigma' = -\alpha_\sigma^{-2}A_*\Phi_\sigma. \tag{2.7.69}$$

Since the cut-off functions in the definition of b act as the identity for small $\sigma \geq 0$, we obtain

$$\begin{aligned} b(\Phi_\sigma, \psi_{\mathrm{tw}}) &= -\vartheta_0\alpha_\sigma^{-1}, \\ \kappa_\sigma(\Phi_\sigma, \psi_{\mathrm{tw}}) &= 1 + \frac{1}{2\rho}\vartheta_0^2\alpha_\sigma^{-2}, \end{aligned} \tag{2.7.70}$$

which implies

$$\begin{aligned} \mathcal{J}_\sigma(\Phi_\sigma, c_\sigma, \psi_{\mathrm{tw}}) &= [1 + \frac{1}{2\rho}\sigma^2\vartheta_0^2\alpha_\sigma^{-2}]^{-1}\big[f(\Phi_\sigma) + c_\sigma\Phi_\sigma' - \sigma^2\vartheta_0^2\alpha_\sigma^{-2}\Phi_\sigma''\big] \\ &= -[1 + \frac{1}{2\rho}\sigma^2\vartheta_0^2\alpha_\sigma^{-2}]^{-1}\big[\alpha_\sigma^{-2}A_*\Phi_\sigma + \frac{\sigma^2}{\rho}\vartheta_0^2\alpha_\sigma^{-2}A_*\Phi_\sigma\big] \\ &= -[\alpha_\sigma^2 + \frac{1}{2\rho}\sigma^2\vartheta_0^2]^{-1}\big[1 + \frac{\sigma^2}{\rho}\vartheta_0^2\big]A_*\Phi_\sigma \\ &= -A_*\Phi_\sigma. \end{aligned} \tag{2.7.71}$$

The claims now follow from the uniqueness statement in Proposition 2.2.2. □

2.8 Bounds on mild nonlinearities

In this section we set out to obtain bounds on the nonlinearities $\overline{\mathcal{R}}_{\sigma;\Phi_\sigma,c_\sigma}$ and $\overline{\mathcal{S}}_{\sigma;\Phi_\sigma}$ defined in (2.6.22)-(2.6.23). In addition, we show that our choices (2.2.27) and (2.2.29) for a_σ and b prevent these nonlinearities from having a component in the subspace of L^2 on which the semigroup $S(t)$ does not decay, provided the cut-offs are not hit.

Our main result below shows that the construction of Φ_σ has eliminated all $\mathcal{O}(1)$-terms from the deterministic nonlinearity $\overline{\mathcal{R}}$, leaving only a small linear contribution together with the expected higher order terms. It is important to note here that these higher order terms depend at most quadratically on $\|v\|_{H^1}$, besides powers of $\|v\|_{L^2}$.

In general, the stochastic nonlinearity $\overline{\mathcal{S}}_{\sigma;\Phi_\sigma}$ will have an $\mathcal{O}(1)$-term, but we have an explicit expression for this contribution so we also discuss the case when this contribution disappears. In both cases, the higher order terms depend at most linearly on $\|v\|_{H^1}$.

2

Proposition 2.8.1. *Consider the setting of Proposition 2.2.2 and recall the definitions (2.6.22) and (2.6.23). Then there exists* $K > 0$ *so that for any* $0 \leq \sigma \leq \delta_\sigma$ *and any* $v \in H^1$*, the following properties hold true.*

(i) We have the bound

$$\|\overline{\mathcal{R}}_{\sigma;\Phi_\sigma,c_\sigma}(v)\|_{L^2} \quad \leq \quad K\sigma^2\|v\|_{H^1} + K\|v\|_{H^1}^2\big[1 + \|v\|_{L^2}^2 + \sigma^2\|v\|_{L^2}^3\big]. \tag{2.8.1}$$

(ii) We have the estimate

$$\|\overline{\mathcal{S}}_{\sigma;\Phi_\sigma}(v)\|_{L^2} \leq K\big[1 + \|v\|_{H^1}\big]. \tag{2.8.2}$$

(iii) If the inequality

$$\|v\|_{L^2} \leq \min\{1, [4\|\psi_{\mathrm{tw}}\|_{H^1}]^{-1}\} \tag{2.8.3}$$

holds, then we have the identities

$$\langle\overline{\mathcal{R}}_{\sigma;\Phi_\sigma,c_\sigma}(v), \psi_{\mathrm{tw}}\rangle_{L^2} = \langle\overline{\mathcal{S}}_{\sigma;\Phi_\sigma}(v), \psi_{\mathrm{tw}}\rangle_{L^2} = 0. \tag{2.8.4}$$

(iv) If the identity

$$g(\Phi_\sigma) = -b(\Phi_\sigma, \psi_{\mathrm{tw}})\Phi_\sigma' \tag{2.8.5}$$

holds, then we have the bound

$$\|\overline{\mathcal{S}}_{\sigma;\Phi_\sigma}(v)\|_{L^2} \leq K\|v\|_{H^1}. \tag{2.8.6}$$

In order to derive a compact expression for $\overline{\mathcal{R}}_{\sigma;\Phi_\sigma,c_\sigma}$, it is convenient to recall the definition (2.7.2) and introduce the function

$$\overline{\mathcal{R}}_{\sigma;I}(v) = \mathcal{M}_{\sigma;\Phi_\sigma,c_\sigma}(v,0) - \mathcal{M}_{\sigma;\Phi_\sigma,c_\sigma}(0,0). \tag{2.8.7}$$

We note that the bounds in Corollary 2.7.5 are directly applicable to this function.

Lemma 2.8.2. *Consider the setting of Proposition 2.2.2. Then for any* $0 \leq \sigma \leq \delta_\sigma$ *and* $v \in H^1$*, we have the identity*

$$\overline{\mathcal{R}}_{\sigma;\Phi_\sigma,c_\sigma}(v) = \overline{\mathcal{R}}_{\sigma;I}(v) - \Big[\chi_{\mathrm{low}}\big(\langle\partial_\xi[\Phi_\sigma + v], \psi_{\mathrm{tw}}\rangle_{L^2}\big)\Big]^{-1}\langle\overline{\mathcal{R}}_{\sigma;I}(v), \psi_{\mathrm{tw}}\rangle_{L^2}[\Phi_\sigma' + v']. \tag{2.8.8}$$

Proof. Inspecting (2.7.2) and using the defining property (2.2.48) for (Φ_σ, c_σ), we see that

$$-\mathcal{M}_{\sigma;\Phi_\sigma,c_\sigma}(0,0) = A_*\Phi_\sigma + \mathcal{J}_0(\Phi_\sigma, c_\sigma). \tag{2.8.9}$$

Applying (2.7.2) once more, we hence find

$$\begin{array}{lcl} \mathcal{J}_\sigma(\Phi_\sigma + v, c_\sigma, \psi_{\mathrm{tw}}) & = & \mathcal{J}_0(\Phi_\sigma, c_\sigma) + [\mathcal{L}_{\mathrm{tw}} - A_*]v + \mathcal{M}_{\sigma;\Phi_\sigma,c_\sigma}(v,0) \\ & = & [\mathcal{L}_{\mathrm{tw}} - A_*]v - A_*\Phi_\sigma + \mathcal{M}_{\sigma;\Phi_\sigma,c_\sigma}(v,0) - \mathcal{M}_{\sigma;\Phi_\sigma,c_\sigma}(0,0) \\ & = & [\mathcal{L}_{\mathrm{tw}} - A_*]v - A_*\Phi_\sigma + \overline{\mathcal{R}}_{\sigma;I}(v). \end{array} \tag{2.8.10}$$

Writing

$$\mathcal{I}_\sigma(v) = \langle \Phi_\sigma + v, A_*\psi_{\mathrm{tw}} \rangle_{L^2} + \langle \mathcal{J}_\sigma(\Phi_\sigma + v, c_\sigma, \psi_{\mathrm{tw}}), \psi_{\mathrm{tw}} \rangle_{L^2} \tag{2.8.11}$$

and using $\mathcal{L}_{\mathrm{tw}}^{\mathrm{adj}}\psi_{\mathrm{tw}} = 0$, we may compute

$$\begin{array}{lcl} \mathcal{I}_\sigma(v) & = & \langle \Phi_\sigma, A_*\psi_{\mathrm{tw}} \rangle_{L^2} + \langle v, [A_* - \mathcal{L}_{\mathrm{tw}}^{\mathrm{adj}}]\psi_{\mathrm{tw}} \rangle_{L^2} \\ & & \quad + \langle \mathcal{J}_\sigma(\Phi_\sigma + v, c_\sigma, \psi_{\mathrm{tw}}), \psi_{\mathrm{tw}} \rangle_{L^2} \\ & = & \langle A_*\Phi_\sigma, \psi_{\mathrm{tw}} \rangle_{L^2} + \langle [A_* - \mathcal{L}_{\mathrm{tw}}]v, \psi_{\mathrm{tw}} \rangle_{L^2} \\ & & \quad + \langle \mathcal{J}_\sigma(\Phi_\sigma + v, c_\sigma, \psi_{\mathrm{tw}}), \psi_{\mathrm{tw}} \rangle_{L^2} \\ & = & \langle \overline{\mathcal{R}}_{\sigma;I}(v), \psi_{\mathrm{tw}} \rangle_{L^2}. \end{array} \tag{2.8.12}$$

In view of the definition (2.2.29) for a_σ, we now obtain

$$\begin{array}{lcl} \dfrac{a_\sigma\big(\Phi_\sigma + v, c_\sigma, \psi_{\mathrm{tw}}\big)}{\kappa_\sigma(\Phi_\sigma + v, \psi_{\mathrm{tw}})} & = & -\Big[\chi_{\mathrm{low}}\big(\langle \partial_\xi[\Phi_\sigma + v], \psi_{\mathrm{tw}} \rangle_{L^2}\big)\Big]^{-1} \mathcal{I}_\sigma(v) \\ & = & -\Big[\chi_{\mathrm{low}}\big(\langle \partial_\xi[\Phi_\sigma + v], \psi_{\mathrm{tw}} \rangle_{L^2}\big)\Big]^{-1} \langle \overline{\mathcal{R}}_{\sigma;I}(v), \psi_{\mathrm{tw}} \rangle_{L^2}. \end{array} \tag{2.8.13}$$

In particular, the desired identity (2.8.8) follows directly from the definition (2.6.22). □

Lemma 2.8.3. *Consider the setting of Proposition 2.2.2. Then there exists $K > 0$ so that for any $v \in H^1$ and $0 \le \sigma \le \delta_\sigma$ we have the bound*

$$\|\overline{\mathcal{R}}_{\sigma;I}(v)\|_{L^2} \le K\sigma^2\|v\|_{H^1} + K\|v\|_{H^1}^2\big[1 + \|v\|_{L^2} + \sigma^2\|v\|_{L^2}^2\big], \tag{2.8.14}$$

together with

$$\big|\langle \overline{\mathcal{R}}_{\sigma;I}(v), \psi_{\mathrm{tw}} \rangle_{L^2}\big| \le K\|v\|_{L^2}\big[\sigma^2 + \|v\|_{L^2}\big] + K\|v\|_{H^1}\big[\|v\|_{L^2}^2 + \sigma^2\|v\|_{L^2}^3\big]. \tag{2.8.15}$$

Proof. Applying Corollary 2.7.5, we find

$$\begin{array}{lcl} \|\overline{\mathcal{R}}_{\sigma;I}(v)\|_{L^2} & \le & C_1\big[1 + \|v\|_{H^1}\big]\|v\|_{H^1}\|v\|_{L^2} \\ & & \quad + C_1\big[\sigma^2 + \|v\|_{H^1}\big]\|v\|_{H^1} \\ & & \quad + C_1\sigma^2\|v\|_{H^1}^2\|v\|_{L^2}\|v\|_{L^2} \\ & & \quad + C_1\sigma^2\|v\|_{H^1}\|v\|_{L^2}\|v\|_{H^1}, \end{array} \tag{2.8.16}$$

together with

$$\begin{array}{rcl}\big|\langle\overline{\mathcal{R}}_{\sigma;I}(v),\psi_{\mathrm{tw}}\rangle_{L^2}\big| & \leq & C_2\big[1+\|v\|_{H^1}\big]\|v\|_{L^2}\|v\|_{L^2} \\ & & +C_2\big[\sigma^2+\|v\|_{L^2}\big]\|v\|_{L^2} \\ & & +C_2\sigma^2\|v\|_{H^1}\|v\|_{L^2}^2\|v\|_{L^2} \\ & & +C_2\sigma^2\|v\|_{L^2}^2\|v\|_{H^1}.\end{array} \tag{2.8.17}$$

These expressions can be absorbed into (2.8.14) and (2.8.15). □

Lemma 2.8.4. *Consider the setting of Proposition 2.2.2. Then there exists $K>0$ so that for any $0\leq\sigma\leq\delta_\sigma$ and any $v\in H^1$ we have the bound*

$$\|\overline{\mathcal{S}}_{\sigma;\Phi_\sigma}(v)-\overline{\mathcal{S}}_{\sigma;\Phi_\sigma}(0)\|_{L^2}\leq K\|v\|_{H^1}. \tag{2.8.18}$$

Proof. Writing

$$\begin{array}{rcl}\mathcal{I} & = & \overline{\mathcal{S}}_{\sigma;\Phi_\sigma}(v)-\overline{\mathcal{S}}_{\sigma;\Phi_\sigma}(0) \\ & = & \kappa_\sigma(\Phi_\sigma+v,\psi_{\mathrm{tw}})^{-1/2}\Big[g(\Phi_\sigma+v)+b(\Phi_\sigma+v,\psi_{\mathrm{tw}})\partial_\xi[\Phi_\sigma+v]\Big] \\ & & -\kappa_\sigma(\Phi_\sigma,\psi_{\mathrm{tw}})^{-1/2}\Big[g(\Phi_\sigma)+b(\Phi_\sigma,\psi_{\mathrm{tw}})\partial_\xi[\Phi_\sigma]\Big]\end{array} \tag{2.8.19}$$

and using Lemmas 2.3.2, 2.3.6, 2.3.8 and 2.3.9, we compute

$$\begin{array}{rcl}\|\mathcal{I}\|_{L^2} & \leq & \Big|\nu_\sigma^{(-1/2)}(\Phi_\sigma+v,\psi_{\mathrm{tw}})-\nu_\sigma^{(-1/2)}(\Phi_\sigma,\psi_{\mathrm{tw}})\Big|\,\big[\|g(\Phi_\sigma)\|_{L^2}+K_b\|\Phi_\sigma'\|_{L^2}\big] \\ & & +K_\kappa\|g(\Phi_\sigma+v)-g(\Phi_\sigma)\|_{L^2} \\ & & +K_\kappa\,|b(\Phi_\sigma+v,\psi_{\mathrm{tw}})-b(\Phi_\sigma,\psi_{\mathrm{tw}})|\,\|\Phi_\sigma'\|_{L^2} \\ & & +K_\kappa K_b\|v'\|_{L^2}.\end{array} \tag{2.8.20}$$

Applying these results once more, we find

$$\begin{array}{rcl}\|\mathcal{I}\|_{L^2} & \leq & C_1\sigma^2\|v\|_{L^2}+C_1\|v\|_{L^2}+C_1\|v\|_{L^2}+C_1\|v\|_{H^1} \\ & \leq & C_2\|v\|_{H^1},\end{array} \tag{2.8.21}$$

as desired. □

Proof of Proposition 2.8.1. To obtain (i), we use (2.8.8) together with Lemma 2.8.3 to compute

$$\begin{array}{rcl}\|\overline{\mathcal{R}}_{\sigma;\Phi_\sigma,c_\sigma}(v)\|_{L^2} & \leq & \|\overline{\mathcal{R}}_{\sigma;I}(v)\|_{L^2}+C_1\,\big|\langle\overline{\mathcal{R}}_{\sigma;I}(v),\psi_{\mathrm{tw}}\rangle_{L^2}\big|\,\big[1+\|v\|_{H^1}\big] \\ & \leq & C_2\sigma^2\|v\|_{H^1}+C_2\|v\|_{H^1}^2\big[1+\|v\|_{L^2}+\sigma^2\|v\|_{L^2}^2\big] \\ & & +C_2\|v\|_{L^2}\big[\sigma^2+\|v\|_{L^2}\big]\big[1+\|v\|_{H^1}\big] \\ & & +C_2\|v\|_{H^1}\big[\|v\|_{L^2}^2+\sigma^2\|v\|_{L^2}^3\big]\big[1+\|v\|_{H^1}\big].\end{array} \tag{2.8.22}$$

These terms can all be absorbed into (2.8.1).

The bound (ii) follows directly from Lemma 2.8.4, using the estimate

$$\|\overline{\mathcal{S}}_{\sigma;\Phi_\sigma}(v)\|_{L^2} \leq \|\overline{\mathcal{S}}_{\sigma;\Phi_\sigma}(0)\|_{L^2} + \|\overline{\mathcal{S}}_{\sigma;\Phi_\sigma}(v) - \overline{\mathcal{S}}_{\sigma;\Phi_\sigma}(0)\|_{L^2} \tag{2.8.23}$$

and the a-priori bound

$$\|\overline{\mathcal{S}}_{\sigma;\Phi_\sigma}(0)\|_{L^2} \leq C_3. \tag{2.8.24}$$

The bound (iv) follows in the same fashion, since the condition (2.8.5) implies that

$$\overline{\mathcal{S}}_{\sigma;\Phi_\sigma}(0) = 0. \tag{2.8.25}$$

Finally, (iii) follows from the identities (2.8.8) and (2.3.49), using the proof of Lemma 2.3.7 to show that the cut-off function χ_{low} in (2.8.8) acts as the identity. □

2.9 Nonlinear stability of mild solutions

In this section we prove Theorems 2.2.4 and 2.2.5, providing an orbital and an exponential stability result for the stochastic waves (Φ_σ, c_σ) on timescales of order σ^{-2}. Recalling the function (2.6.34), our key statement is that $E \sup_t \overline{N}_{\varepsilon,\alpha}(t)$ can be bounded in terms of itself, the noise-strength σ and the initial condition $\|\overline{V}(0)\|_{H^1}^2$. This requires a number of technical regularity estimates, which we obtain in §2.9.2-2.9.3.

In order to prevent cumbersome notation and to highlight the broad applicability of our techniques here, we do not refer to the specific functions $\overline{V}$ and the specific nonlinearities $\overline{\mathcal{R}}_{\sigma;\Phi_\sigma,c_\sigma}$ here. Instead, we assume the following general condition concerning the form of our nonlinearities.

(hFB) We have $\|B_{\text{cn}}\|_{L^2} = K_{B;\text{cn}} < \infty$ and the maps

$$F_{\text{lin}} : H^1 \to L^2, \quad F_{\text{nl}} : H^1 \to L^2, \quad B_{\text{lin}} : H^1 \to L^2 \tag{2.9.1}$$

satisfy the bounds

$$\begin{aligned} \|F_{\text{lin}}(v)\|_{L^2} &\leq K_{F;\text{lin}}\|v\|_{H^1}, \\ \|F_{\text{nl}}(v)\|_{L^2} &\leq K_{F;\text{nl}}\|v\|_{H^1}^2(1 + \|v\|_{L^2}^m), \\ \|B_{\text{lin}}(v)\|_{L^2} &\leq K_{B;\text{lin}}\|v\|_{H^1} \end{aligned} \tag{2.9.2}$$

for some $m > 0$. In addition, there exists $\eta_0 > 0$ so that

$$\langle \sigma^2 F_{\text{lin}}(v) + F_{\text{nl}}(v), \psi_{\text{tw}} \rangle_{L^2} = 0, \quad \langle B_{\text{cn}} + B_{\text{lin}}(v), \psi_{\text{tw}} \rangle_{L^2} = 0 \tag{2.9.3}$$

whenever $\|v\|_{L^2} \leq \eta_0$.

Using the nonlinearities above, we can discuss the mild formulation of the SPDE that we are interested in. At present, we simply assume that a solution is a priori available, but one can also set out to construct such a solution directly.

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(hSol) For any $T > 0$, there exists a continuous $(\mathcal{F}_t)$-adapted process $V : \Omega \times [0,T] \to L^2$ for which we have the inclusions

$$V \in \mathcal{N}^2\big([0,T];(\mathcal{F}_t);H^1\big), \quad B_{\mathrm{lin}}(V) \in \mathcal{N}^2\big([0,T];(\mathcal{F}_t);L^2\big). \tag{2.9.4}$$

In addition, for almost all $\omega \in \Omega$ we have the inclusions

$$F_{\mathrm{lin}}\big(V(\cdot,\omega)\big) \in L^1([0,T];L^2), \quad F_{\mathrm{nl}}\big(V(\cdot,\omega)\big) \in L^1([0,T];L^2) \tag{2.9.5}$$

together with the identity

$$\begin{aligned} V(t) \;\; = \;\; & S(t)V(0) + \sigma^2 \int_0^t S(t-s)F_{\mathrm{lin}}\big(V(s)\big)\,ds + \int_0^t S(t-s)F_{\mathrm{nl}}\big(V(s)\big)\,ds \\ & + \sigma \int_0^t S(t-s)B_{\mathrm{cn}}\,d\beta_s + \sigma \int_0^t S(t-s)B_{\mathrm{lin}}\big(V(s)\big)\,d\beta_s, \end{aligned} \tag{2.9.6}$$

which holds for all $t \in [0,T]$. Finally, we have $\langle V(0), \psi_{\mathrm{tw}}\rangle_{L^2} = 0$.

For any $\varepsilon > 0$ and $\alpha \geq 0$, we recall the notation

$$N_{\varepsilon,\alpha}(t) = e^{\alpha t}\|V(t)\|_{L^2}^2 + \int_0^t e^{-\varepsilon(t-s)}e^{\alpha s}\|V(s)\|_{H^1}^2 ds. \tag{2.9.7}$$

For any $T > 0$ and $\eta > 0$, we introduce the $(\mathcal{F}_t)$-stopping time

$$\tau_{\varepsilon,\alpha}(T,\eta) = \inf\Big\{0 \leq t < T : N_{\varepsilon,\alpha}(t) > \eta\Big\}, \tag{2.9.8}$$

writing $\tau_{\varepsilon,\alpha}(T,\eta) = T$ if the set is empty. Our two main results here, which we establish in §2.9.3 provide bounds on the expectation of $\sup_{0\leq t \leq \tau_{\varepsilon,\alpha}(T,\eta)} N_{\varepsilon,\alpha}(t)$.

Proposition 2.9.1. *Assume that (HA), (HTw), (HS), (Hβ), (hSol) and (hFB) are satisfied. Pick a constant $0 < \varepsilon < \beta$, together with two sufficiently small constants $\delta_\eta > 0$ and $\delta_\sigma > 0$. Then there exists a constant $K > 0$ so that for any $T > 1$, any $0 < \eta \leq \delta_\eta$ and any $0 \leq \sigma \leq \delta_\sigma T^{-1/2}$ we have the bound*

$$E \sup_{0 \leq t \leq \tau_{\varepsilon,0}(T,\eta)} N_{\varepsilon,0}(t) \;\; \leq \;\; K\Big[\|V(0)\|_{H^1}^2 + \sigma^2 T\Big]. \tag{2.9.9}$$

Proposition 2.9.2. *Assume that (HA), (HTw), (HS), (Hβ), (hSol) and (hFB) are satisfied and that $B_{\mathrm{cn}} = 0$. Pick two constants $\varepsilon > 0$, $\alpha \geq 0$ for which $\varepsilon + \dfrac{\alpha}{2} < \beta$, together with two sufficiently small constants $\delta_\eta > 0$ and $\delta_\sigma > 0$. Then there exists a constant $K > 0$ so that for any $T > 1$, any $0 < \eta \leq \delta_\eta$ and any $0 \leq \sigma \leq \delta_\sigma T^{-1/2}$ we have the bound*

$$E \sup_{0 \leq t \leq \tau_{\varepsilon,\alpha}(T,\eta)} N_{\varepsilon,\alpha}\big(t\big) \;\; \leq \;\; K\|V(0)\|_{H^1}^2. \tag{2.9.10}$$

Exploiting the technique used in Stannat [105], these bounds can be turned into estimates concerning the probabilities

$$p_{\varepsilon,\alpha}(T,\eta) = P\Big(\sup_{0 \leq t \leq T}\big[N_{\varepsilon,\alpha}(t)\big] > \eta\Big). \tag{2.9.11}$$

This allows our main stability theorems to be established.

Corollary 2.9.3. *Consider the setting of Proposition 2.9.1. Then there exists a constant $K > 0$ so that for any $T > 1$, any $0 < \eta \le \delta_\eta$ and any $0 \le \sigma \le \delta_\sigma T^{-1/2}$, we have the bound*

$$p_{\varepsilon,0}(T,\eta) \le \eta^{-1} K \big[\|V(0)\|_{H^1}^2 + \sigma^2 T \big]. \tag{2.9.12}$$

Proof. Upon computing

$$\begin{aligned} \eta p_{\varepsilon,0}(T,\eta) &= \eta P\big(\tau_{\varepsilon,0}(T,\eta) < T\big) \\ &= E\Big[\mathbf{1}_{\tau_{\varepsilon,0}(T,\eta)<T} N_{\varepsilon,0}\big(\tau_{\varepsilon,0}(T,\eta)\big)\Big] \\ &\le E N_{\varepsilon,0}\big(\tau_{\varepsilon,0}(T,\eta)\big) \\ &\le E \sup_{0\le t\le \tau_{\varepsilon,0}(T,\eta)} N_{\varepsilon,0}(t), \end{aligned} \tag{2.9.13}$$

the result follows from (2.9.9). □

Corollary 2.9.4. *Consider the setting of Proposition 2.9.2. Then there exists a constant $K > 0$ so that for any $T > 1$, any $0 < \eta \le \delta_\eta$ and any $0 \le \sigma \le \delta_\sigma T^{-1/2}$ we have the bound*

$$p_{\varepsilon,\alpha}(T,\eta) \le \eta^{-1} K \|V(0)\|_{H^1}^2. \tag{2.9.14}$$

Proof. Upon computing

$$\begin{aligned} \eta p_{\varepsilon,\alpha}(T,\eta) &= \eta P\big(\tau_{\varepsilon,\alpha}(T,\eta) < T\big) \\ &= E\Big[\mathbf{1}_{\tau_{\varepsilon,\alpha}(T,\eta)<T} N_{\varepsilon,\alpha}\big(\tau_{\varepsilon,\alpha}(T,\eta)\big)\Big] \\ &\le E N_{\varepsilon,\alpha}\big(\tau_{\varepsilon,\alpha}(T,\eta)\big) \\ &\le E \sup_{0\le t\le \tau_{\varepsilon,\alpha}(T,\eta)} N_{\varepsilon,\alpha}(t), \end{aligned} \tag{2.9.15}$$

the result follows from (2.9.10). □

Proof of Theorems 2.2.4 and 2.2.5. On account of Propositions 2.2.3 and 2.6.3, the map $\overline{V}$ defined in (2.6.6) satisfies the conditions of (hSol) with $\big(\overline{\beta}_\tau, \overline{\mathcal{F}}_\tau\big)_{\tau\ge 0}$ as the relevant Brownian motion. In addition, Proposition 2.8.1 guarantees that (hFB) is satisfied. The desired estimates now follow from Corollaries 2.9.3 and 2.9.4, using Proposition 2.6.4 to reverse the time-transform. □

2.9.1 Setup

In order to establish Propositions 2.9.1-2.9.2 we need to estimate each of the terms featuring in the identity (2.9.6). The regularity structure of the semigroup $S(t)$ is crucial for our purposes here, so we discuss this in some detail using the terminology used in [55, §10].

In particular, for any $0 < \varphi < \pi$ we introduce the sector

$$\Sigma_\varphi = \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \varphi\}, \tag{2.9.16}$$

in which we take $\arg(z) \in (-\pi, \pi)$. We recall that a linear operator $\mathcal{L} : D(\mathcal{L}) \subset X \to X$ on a Banach space X is called sectorial if the spectrum of $\mathcal{L}$ is contained in $\overline{\Sigma}_\omega$ for some $0 < \omega(\mathcal{L}) < \frac{\pi}{2}$, while the resolvent operators $R(z, \mathcal{L}) = (z - \mathcal{L})^{-1}$ satisfy the bound

$$\sup_{z \in \mathbb{C} \setminus \overline{\Sigma}_{\omega(\mathcal{L})}} \| z R(z, \mathcal{L}) \|_{\mathscr{L}(X,X)} < \infty. \tag{2.9.17}$$

Our spectral assumptions (HS) combined with the fact that $\mathcal{L}_{\text{tw}}$ is a lower-order perturbation to the diffusion operator A_* guarantee that $-\mathcal{L}_{\text{tw}}$ is sectorial. This means that $\mathcal{L}_{\text{tw}}$ generates an analytic semigroup. In order to isolate the behavior caused by the neutral eigenmode, we introduce the map $Q : L^2 \to L^2$ that acts as

$$Qv = v - \langle v, \psi_{\text{tw}} \rangle_{L^2} \Phi_0'. \tag{2.9.18}$$

This projection allows us to formulate several important estimates.

Lemma 2.9.5 (see [80]). *Assume that (HTw) and (HS) hold and consider the analytic semigroup $S(t)$ generated by $\mathcal{L}_{\text{tw}}$. Then there is a constant $M \geq 1$ for which we have the bounds*

$$\begin{array}{lcll} \| S(t) Q \|_{\mathscr{L}(L^2, L^2)} & \leq & M e^{-\beta t}, & 0 < t < \infty, \\ \| S(t) Q \|_{\mathscr{L}(L^2, H^1)} & \leq & M t^{-\frac{1}{2}}, & 0 < t \leq 2, \\ \| S(t) Q \|_{\mathscr{L}(L^2, H^1)} & \leq & M e^{-\beta t}, & t \geq 1, \\ \| [\mathcal{L}_{\text{tw}} - A_*] S(t) Q \|_{\mathscr{L}(L^2, L^2)} & \leq & M t^{-\frac{1}{2}}, & 0 < t \leq 2, \\ \| [\mathcal{L}_{\text{tw}}^{\text{adj}} - A_*] S(t) Q \|_{\mathscr{L}(L^2, L^2)} & \leq & M t^{-\frac{1}{2}}, & 0 < t \leq 2. \end{array} \tag{2.9.19}$$

In order to understand the combination $S(t)Q$ as an independent semigroup, we introduce the spaces

$$L^2_Q = \{ v \in L^2 : (I - Q) v = 0 \}, \quad H^2_Q = \{ v \in H^2 : (I - Q) v = 0 \} \tag{2.9.20}$$

and consider the operator $\mathcal{L}^Q_{\text{tw}} : H^2_Q \to L^2_Q$ that arises upon restricting $\mathcal{L}_{\text{tw}}$ to act on H^2_Q. Note that this is well-defined since $\text{Range}\big(\mathcal{L}_{\text{tw}}\big) = L^2_Q$. For any $\theta \in \mathbb{R}$, we now introduce the linear operators

$$B_\theta = -\big[\mathcal{L}_{\text{tw}} + \theta \big], \quad B^Q_\theta = -\big[\mathcal{L}^Q_{\text{tw}} + \theta \big]. \tag{2.9.21}$$

Lemma 2.9.6. *Assume that (HTw) and (HS) hold and pick any $0 \leq \theta \leq \beta$. Then the operator B^Q_θ is sectorial on L^2_Q and the semigroup generated by $-B^Q_\theta$ corresponds with the restriction of $e^{\theta t} S(t)$ to L^2_Q.*

Proof. Note first that $\mathcal{L}^Q_{\text{tw}}$ is bijective since we have projected out the one-dimensional kernel. For any $v \in L^2_Q$ and λ in the resolvent set of $\mathcal{L}_{\text{tw}}$, we may compute

$$\begin{array}{rcl} 0 & = & (I - Q) \mathcal{L}_{\text{tw}} R(\lambda, \mathcal{L}_{\text{tw}}) v \\ & = & (I - Q) \big[- v + \lambda R(\lambda, \mathcal{L}_{\text{tw}}) v \big] \\ & = & \lambda (I - Q) R(\lambda, \mathcal{L}_{\text{tw}}) v. \end{array} \tag{2.9.22}$$

which implies that $R(\lambda, \mathcal{L}_{\rm tw})v \in L^2_Q$. In particular, the resolvent set of $\mathcal{L}_{\rm tw}$ is contained in the resolvent set of $\mathcal{L}^Q_{\rm tw}$. The stated properties now follow in a standard fashion; see, for example, [80, Prop. 3.1.5]. □

In order to define our final regularity concept, we need to introduce the Hardy spaces

$$\begin{aligned} H^1(\Sigma_\varphi) &= \{f : \Sigma_\varphi \to \mathbb{C} \text{ holomorphic for which} \\ &\qquad \|f\|_{H^1(\Sigma_\varphi)} := \sup_{|\nu|<\varphi} \int_0^\infty t^{-1} f(e^{i\nu}t)\, dt < \infty\}, \\ H^\infty(\Sigma_\varphi) &= \{f : \Sigma_\varphi \to \mathbb{C} \text{ holomorphic for which} \\ &\qquad \|f\|_{H^\infty(\Sigma_\varphi)} := \sup_{z\in\Sigma_\varphi} |f(z)| < \infty\}. \end{aligned} \tag{2.9.23}$$

If $\mathcal{L}$ is sectorial on a Banach space X, then for any $\omega(\mathcal{L}) < \varphi < \pi$ and any $h \in H^1(\Sigma_\varphi)$ one can define

$$h(\mathcal{L}) = \frac{1}{2\pi i} \int_{\partial\Sigma_\nu} R(z, \mathcal{L}) h(z)\, dz \in \mathcal{L}(X, X) \tag{2.9.24}$$

by picking an arbitrary $\nu \in (\omega(\mathcal{L}), \varphi)$ and traversing the boundary in a downward fashion, keeping the spectrum of $\mathcal{L}$ on the left. It is however unclear if this integral converges if we take $h \in H^\infty(\Sigma_\varphi)$. The following result states that this is indeed the case for the sectorial operators discussed in Lemma 2.9.6. Indeed, one can use a density argument to extend the conclusion to the whole space $H^\infty(\Sigma_\varphi)$. Operators with this property are said to admit a bounded H^∞-calculus, which is crucial for our stochastic regularity estimates.

Lemma 2.9.7. *Assume that (HTw) and (HS) hold and pick any* $0 \le \theta \le \beta$. *There exists* $\varphi \in (\omega(B^Q_\theta), \frac{\pi}{2})$ *together with a constant* $K > 0$ *so that for any* $h \in H^1(\Sigma_\varphi) \cap H^\infty(\Sigma_\varphi)$ *we have*

$$\|h(B^Q_\theta)\| \le K \|h\|_{H^\infty_\varphi}. \tag{2.9.25}$$

Proof. Since $\mathcal{L}_{\rm tw} - A_*$ is a first order differential operator with continuous coefficients, the perturbation theory described in [114, §8] can be applied to our setting. In particular, we can find constants $\Theta_0 \gg 1$ and $C_1 > 0$ together with an angle $\varphi_0 \in (\omega(B_{-\Theta_0}), \frac{\pi}{2})$ for which

$$\|h(B_{-\Theta_0})\|_{\mathcal{L}(L^2,L^2)} \le C_1 \|h\|_{H^\infty_{\varphi_0}} \tag{2.9.26}$$

holds for all $h \in H^1(\Sigma_{\varphi_0}) \cap H^\infty(\Sigma_{\varphi_0})$. By restriction, we hence also have

$$\|h(B^Q_{-\Theta_0})\|_{\mathcal{L}(L^2_Q,L^2_Q)} \le \|h(B_{-\Theta_0})\|_{\mathcal{L}(L^2,L^2)} C_1 \|h\|_{H^\infty_{\varphi_0}} \tag{2.9.27}$$

for all such h. Fix two constants

$$\max\{\omega(B^Q_\theta), \varphi_0\} < \nu < \varphi < \frac{\pi}{2} \tag{2.9.28}$$

and pick $h \in H^1(\Sigma_\varphi) \cap H^\infty(\Sigma_\varphi)$. Using the resolvent identity, we may compute

$$\begin{array}{rcl} h(B^Q_\theta) - h(B^Q_{-\Theta_0}) & = & \frac{1}{2\pi i}\int_{\partial\Sigma_\nu} h(z)\big[R(z, B^Q_\theta) - R(z, B^Q_{-\Theta_0})\big]\, dz \\ & = & \frac{1}{2\pi i}\int_{\partial\Sigma_\nu} h(z)\big[R(z, B^Q_\theta) - R(z-\theta-\Theta_0, B^Q_\theta)\big]\, dz \\ & = & (\theta+\Theta_0)\frac{1}{2\pi i}\int_{\partial\Sigma_\nu} h(z)R(z, B^Q_\theta)R(z-\theta-\Theta_0, B^Q_\theta)\, dz. \end{array} \tag{2.9.29}$$

Since zero is contained in the resolvent set of B^Q_θ, there exists $C_2 > 0$ for which the estimate

$$\|R(z, B^Q_\theta)R(z-\theta-\Theta_0, B^Q_\theta)\|_{\mathcal{L}(L^2_Q, L^2_Q)} \le \frac{C_2}{1+|z|^2} \tag{2.9.30}$$

holds for all $z \in \partial\Sigma_\nu$. This decays sufficently fast to ensure that

$$\|h(B^Q_\theta) - h(B^Q_{-\Theta_0})\|_{\mathcal{L}(L^2_Q, L^2_Q)} \le C_3\|h\|_{H^\infty_\varphi} \tag{2.9.31}$$

for some $C_3 > 0$ that does not depend on the choice of h. The desired bound now follows from the inequality

$$\|h\|_{H^\infty_{\varphi_0}} \le \|h\|_{H^\infty_\varphi}. \tag{2.9.32}$$

□

Now that the formal framework has been set up, we are ready to return to the quantity $N_{\varepsilon,\alpha}(t)$ defined in (2.9.7), which is the main object of our interest. For convenience, we use the shorthand notation $\tau = \tau_{\varepsilon,\alpha}(T, \eta)$ ubiquitously throughout the remainder of this section. Writing $\nu = \alpha + \varepsilon$, we introduce the splitting

$$\begin{array}{rcl} N_{\varepsilon,\alpha;I}(t) & = & e^{\alpha t}\|V(t)\|^2_{L^2}, \\ N_{\varepsilon,\alpha;II}(t) & = & \int_0^t e^{-\varepsilon(t-s)}e^{\alpha s}\|V(s)\|^2_{H^1}\, ds \\ & = & e^{-\varepsilon t}\int_0^t e^{\nu s}\|V(s)\|^2_{H^1}\, ds. \end{array} \tag{2.9.33}$$

In order to understand $N_{\varepsilon,\alpha;I}$, we introduce the expression

$$\mathcal{E}_0(t) = S(t)QV(0), \tag{2.9.34}$$

together with the long-term integrals

$$\begin{array}{rcl} \mathcal{E}^{\rm lt}_{F;\rm lin}(t) & = & \int_0^{t-1} S(t-s)QF_{\rm lin}\big(V(s)\big)\mathbf{1}_{s<\tau}\, ds, \\ \mathcal{E}^{\rm lt}_{F;\rm nl}(t) & = & \int_0^{t-1} S(t-s)QF_{\rm nl}\big(V(s)\big)\mathbf{1}_{s<\tau}\, ds, \\ \mathcal{E}^{\rm lt}_{B;\rm lin}(t) & = & \int_0^{t-1} S(t-s)QB_{\rm lin}\big(V(s)\big)\mathbf{1}_{s<\tau}\, d\beta_s, \\ \mathcal{E}^{\rm lt}_{B;\rm cn}(t) & = & \int_0^{t-1} S(t-s)QB_{\rm cn}\mathbf{1}_{s<\tau}\, d\beta_s \end{array} \tag{2.9.35}$$

and their short-term counterparts

$$\begin{array}{lcl}\mathcal{E}^{\rm sh}_{F;\rm lin}(t) & = & \int_{t-1}^{t} S(t-s)QF_{\rm lin}\big(V(s)\big)\mathbf{1}_{s<\tau}\,ds,\\ \mathcal{E}^{\rm sh}_{F;\rm nl}(t) & = & \int_{t-1}^{t} S(t-s)QF_{\rm nl}\big(V(s)\big)\mathbf{1}_{s<\tau}\,ds,\\ \mathcal{E}^{\rm sh}_{B;\rm lin}(t) & = & \int_{t-1}^{t} S(t-s)QB_{\rm lin}\big(V(s)\big)\mathbf{1}_{s<\tau}\,d\beta_s,\\ \mathcal{E}^{\rm sh}_{B;\rm cn}(t) & = & \int_{t-1}^{t} S(t-s)QB_{\rm cn}\mathbf{1}_{s<\tau}\,d\beta_s.\end{array} \tag{2.9.36}$$

Here we use the convention that integrands are set to zero for $s < 0$. For convenience, we also write

$$\mathcal{E}_{F;\#}(t) = \mathcal{E}^{\rm lt}_{F;\#}(t) + \mathcal{E}^{\rm sh}_{F;\#}(t) \tag{2.9.37}$$

for $\# \in \{\rm lin, nl\}$ and

$$\mathcal{E}_{B;\#}(t) = \mathcal{E}^{\rm lt}_{B;\#}(t) + \mathcal{E}^{\rm sh}_{B;\#}(t) \tag{2.9.38}$$

for $\# \in \{\rm lin, cn\}$.

Turning to the terms in (2.9.6) that are relevant for evaluating $N_{\varepsilon,\alpha;II}$, we introduce the expression

$$\mathcal{I}_{\nu,\delta;0}(t) \quad = \quad \int_0^t e^{\nu s}\|S(\delta)\mathcal{E}_0(s)\|^2_{H^1}\,ds, \tag{2.9.39}$$

together with

$$\begin{array}{lcl}\mathcal{I}^{\#}_{\nu,\delta;F;\rm lin}(t) & = & \int_0^t e^{\nu s}\|S(\delta)\mathcal{E}^{\#}_{F;\rm lin}(s)\|^2_{H^1}\,ds,\\ \mathcal{I}^{\#}_{\nu,\delta;F;\rm nl}(t) & = & \int_0^t e^{\nu s}\|S(\delta)\mathcal{E}^{\#}_{F;\rm nl}(s)\|^2_{H^1}\,ds,\\ \mathcal{I}^{\#}_{\nu,\delta;B;\rm lin}(t) & = & \int_0^t e^{\nu s}\|S(\delta)\mathcal{E}^{\#}_{B;\rm lin}(s)\|^2_{H^1}\,ds,\\ \mathcal{I}^{\#}_{\nu,\delta;B;\rm cn}(t) & = & \int_0^t e^{\nu s}\|S(\delta)\mathcal{E}^{\#}_{B;\rm cn}(s)\|^2_{H^1}\,ds\end{array} \tag{2.9.40}$$

for $\# \in \{\rm lt, sh\}$. The extra $S(\delta)$ factor will be used to ensure that all the integrals we encounter are well-defined. We emphasize that all our estimates are uniform in $0 < \delta < 1$, allowing us to take $\delta \downarrow 0$. The estimates concerning $\mathcal{I}^{\rm sh}_{\nu,\delta;F;\rm nl}$ and $\mathcal{I}^{\rm sh}_{\nu,\delta;B;\rm lin}$ in Lemmas 2.9.12 and 2.9.18 are particularly delicate in this respect, as a direct application of the bounds in Lemma 2.9.5 would result in expressions that diverge as $\delta \downarrow 0$.

2.9.2 Deterministic regularity estimates

In this part we set out to analyze the deterministic integrals in (2.9.6). The main complication is that we only have integrated control over the squared H^1-norm of V. This is particularly delicate for $\mathcal{I}^{\rm sh}_{\nu,\delta;F;\rm nl}$, where the nonlinearity itself is quadratic in V.

Lemma 2.9.8. *Fix $T > 0$ and assume that (HA), (HTw), (HS), (Hβ), (hSol) and (hFB) all hold. Pick two constants $\varepsilon > 0$, $\alpha \geq 0$ for which $\varepsilon + \frac{\alpha}{2} < \beta$ and write $\nu = \alpha + \varepsilon$. Then for any $0 \leq \delta < 1$ and any $0 \leq t \leq T$, we have the bound*

$$e^{\alpha t}\|\mathcal{E}_0(t)\|_{L^2}^2 \leq M^2 e^{-\varepsilon t}\|V(0)\|_{L^2}^2, \tag{2.9.41}$$

together with

$$e^{-\varepsilon t}\mathcal{I}_{\nu,\delta;0}(t) \leq \frac{M^2}{2\beta - \nu} e^{-\varepsilon t}\|V(0)\|_{H^1}^2, \tag{2.9.42}$$

Proof. We compute

$$\begin{aligned} e^{\alpha t}\|\mathcal{E}_0(t)\|_{L^2}^2 &\leq M^2 e^{\alpha t} e^{-2\beta t}\|V(0)\|_{L^2}^2 \\ &\leq M^2 e^{-\varepsilon t}\|V(0)\|_{L^2}^2, \end{aligned} \tag{2.9.43}$$

together with

$$\begin{aligned} e^{-\varepsilon t}\mathcal{I}_{\nu,\delta;0}(t) &\leq M^2 e^{-\varepsilon t}\int_0^t e^{\nu s} e^{-2\beta(s+\delta)}\|V(0)\|_{H^1}^2\, ds \\ &\leq \frac{M^2}{2\beta - \nu} e^{-\varepsilon t}\|V(0)\|_{H^1}^2. \end{aligned} \tag{2.9.44}$$

□

Lemma 2.9.9. *Fix $T > 0$ and assume that (HA), (HTw), (HS), (Hβ), (hSol) and (hFB) all hold. Pick two constants $\varepsilon > 0$, $\alpha \geq 0$ for which $\varepsilon + \frac{\alpha}{2} < \beta$ and write $\nu = \alpha + \varepsilon$. Then for any $0 \leq \delta < 1$ and any $0 \leq t \leq \tau$, we have the bound*

$$e^{\alpha t}\|\mathcal{E}_{F;\mathrm{lin}}(t)\|_{L^2}^2 \leq K_{F;\mathrm{lin}}^2 \frac{M^2}{2\beta - \nu} N_{\varepsilon,\alpha;II}(t), \tag{2.9.45}$$

together with

$$e^{-\varepsilon t}\mathcal{I}^{\mathrm{lt}}_{\nu,\delta;F;\mathrm{lin}}(t) \leq K_{F;\mathrm{lin}}^2 \frac{M^2}{2(\beta + \frac{\alpha}{2} - \nu)\varepsilon} N_{\varepsilon,\alpha;II}(t). \tag{2.9.46}$$

Proof. We first observe that

$$\begin{aligned} \|\mathcal{E}_{F;\mathrm{lin}}(t)\|_{L^2}^2 &\leq K_{F;\mathrm{lin}}^2 M^2 \left(\int_0^t e^{-\beta(t-s)}\|V(s)\|_{H^1}\, ds\right)^2, \\ \|S(\delta)\mathcal{E}^{\mathrm{lt}}_{F;\mathrm{lin}}(t)\|_{H^1}^2 &\leq K_{F;\mathrm{lin}}^2 M^2 \left(\int_0^t e^{-\beta(t-s)}\|V(s)\|_{H^1}\, ds\right)^2. \end{aligned} \tag{2.9.47}$$

This allows us to compute

$$\begin{aligned} e^{\alpha t}\|\mathcal{E}_{F;\mathrm{lin}}(t)\|_{L^2}^2 &\leq K_{F;\mathrm{lin}}^2 M^2 e^{\alpha t}\left(\int_0^t e^{-(\beta - \frac{\nu}{2})(t-s)} e^{-\frac{\nu}{2}(t-s)}\|V(s)\|_{H^1}\, ds\right)^2 \\ &\leq K_{F;\mathrm{lin}}^2 \frac{M^2}{2\beta - \nu} e^{\alpha t}\int_0^t e^{-\nu(t-s)}\|V(s)\|_{H^1}^2\, ds \\ &= K_{F;\mathrm{lin}}^2 \frac{M^2}{2\beta - \nu} N_{\varepsilon,\alpha;II}(t). \end{aligned} \tag{2.9.48}$$

Exploiting the inequality $2\beta - \nu > \varepsilon$, we write

$$\gamma_2 = \frac{\varepsilon + \nu}{2\beta} < 1 \tag{2.9.49}$$

and observe that

$$2\gamma_2\beta - \nu = \varepsilon. \tag{2.9.50}$$

Upon fixing $\gamma_1 = 1 - \gamma_2$, we readily see that

$$2\gamma_1\beta = 2\beta - \varepsilon - \nu = 2\big(\beta + \frac{\alpha}{2} - \nu\big). \tag{2.9.51}$$

This allows us to compute

$$\begin{aligned}
e^{-\varepsilon t}\mathcal{I}^{\mathrm{lt}}_{\nu,\delta;F;\mathrm{lin}}(t) &\leq K^2_{F;\mathrm{lin}} M^2 e^{-\varepsilon t}\int_0^t e^{\nu s}\Big(\int_0^s e^{-\beta(s-s')}\|V(s')\|_{H^1}\,ds'\Big)^2\,ds \\
&\leq K^2_{F;\mathrm{lin}} M^2 e^{-\varepsilon t}\int_0^t e^{\nu s}\Big(\int_0^s e^{-2\gamma_1\beta(s-s')}\,ds'\Big)\times \\
&\qquad \Big(\int_0^s e^{-2\gamma_2\beta(s-s')}\|V(s')\|^2_{H^1}\,ds'\Big)\,ds \\
&\leq K^2_{F;\mathrm{lin}}\frac{M^2}{2\gamma_1\beta}e^{-\varepsilon t}\int_0^t e^{\nu s}\int_0^s e^{-2\gamma_2\beta(s-s')}\|V(s')\|^2_{H^1}\,ds'\,ds \\
&= K^2_{F;\mathrm{lin}}\frac{M^2}{2\gamma_1\beta}e^{-\varepsilon t}\int_0^t\int_{s'}^t e^{\nu s}e^{-2\gamma_2\beta(s-s')}\|V(s')\|^2_{H^1}\,ds\,ds' \\
&= K^2_{F;\mathrm{lin}}\frac{M^2}{2\gamma_1\beta}e^{-\varepsilon t}\int_0^t\Big[\int_{s'}^t e^{-(2\gamma_2\beta-\nu)s}\,ds\Big]e^{2\gamma_2\beta s'}\|V(s')\|^2_{H^1}\,ds' \\
&= K^2_{F;\mathrm{lin}}\frac{M^2}{(2\gamma_1\beta)(2\gamma_2\beta-\nu)}e^{-\varepsilon t}\int_0^t e^{-(2\gamma_2\beta-\nu)s'}e^{2\gamma_2\beta s'}\|V(s')\|^2_{H^1}ds' \\
&= K^2_{F;\mathrm{lin}}\frac{M^2}{(2\gamma_1\beta)(2\gamma_2\beta-\nu)}e^{-\varepsilon t}\int_0^t e^{\nu s'}\|V(s')\|^2_{H^1}\,ds' \\
&= K^2_{F;\mathrm{lin}}\frac{M^2}{2(\beta+\frac{\alpha}{2}-\nu)\varepsilon}N_{\varepsilon,\alpha;II}(t).
\end{aligned} \tag{2.9.52}$$

□

Lemma 2.9.10. *Fix $T > 0$ and assume that (HA), (HTw), (HS), (Hβ), (hSol) and (hFB) all hold. Pick two constants $\varepsilon > 0$, $\alpha \geq 0$ and write $\nu = \alpha + \varepsilon$. Then for any $0 \leq \delta < 1$ and any $0 \leq t \leq \tau$, we have the bound*

$$e^{-\varepsilon t}\mathcal{I}^{\mathrm{sh}}_{\nu,\delta;F;\mathrm{lin}}(t) \quad \leq \quad 4e^{\nu}M^2K^2_{F;\mathrm{lin}}N_{\varepsilon,\alpha;II}(t). \tag{2.9.53}$$

Proof. Using Cauchy-Schwarz, we compute

$$
\begin{aligned}
e^{-\varepsilon t}\mathcal{I}^{\mathrm{sh}}_{\nu,\delta;F;\mathrm{lin}}(t) &\leq M^2 K^2_{F;\mathrm{lin}} e^{-\varepsilon t}\int_0^t e^{\nu s}\Big(\int_{s-1}^s \frac{1}{\sqrt{s+\delta-s'}}\|V(s')\|_{H^1}ds'\Big)^2\,ds \\
&\leq M^2 K^2_{F;\mathrm{lin}} e^{-\varepsilon t}\int_0^t e^{\nu s}\Big(\int_{s-1}^s \frac{1}{\sqrt{s+\delta-s'}}\,ds'\Big) \\
&\qquad \Big(\int_{s-1}^s \frac{1}{\sqrt{s+\delta-s'}}\|V(s')\|^2_{H^1}ds'\Big)\,ds \\
&\leq 2M^2 K^2_{F;\mathrm{lin}} e^{-\varepsilon t}\int_0^t e^{\nu s}\Big(\int_{s-1}^s \frac{1}{\sqrt{s+\delta-s'}}\|V(s')\|^2_{H^1}ds'\Big)\,ds \\
&= 2M^2 K^2_{F;\mathrm{lin}} e^{-\varepsilon t}\int_0^t \Big[\int_{s'}^{\min\{t,s'+1\}} \frac{e^{\nu s}}{\sqrt{s+\delta-s'}}\,ds\Big]\|V(s')\|^2_{H^1}\,ds' \\
&\leq 4e^{\nu} M^2 K^2_{F;\mathrm{lin}} e^{-\varepsilon t}\int_0^t e^{\nu s'}\|V(s')\|^2_{H^1}\,ds' \\
&= 4e^{\nu} M^2 K^2_{F;\mathrm{lin}} N_{\varepsilon,\alpha;II}(t).
\end{aligned}
\tag{2.9.54}
$$

□

Lemma 2.9.11. *Fix $T > 0$ and assume that (HA), (HTw), (HS), (Hβ), (hSol) and (hFB) all hold. Pick two constants $\varepsilon > 0$, $\alpha \geq 0$ for which $\varepsilon + \dfrac{\alpha}{2} < \beta$ and write $\nu = \alpha + \varepsilon$. Then for any $\eta > 0$, any $0 \leq \delta < 1$ and any $0 \leq t \leq \tau$, we have the bound*

$$
e^{\alpha t}\|\mathcal{E}_{F;\mathrm{nl}}(t)\|^2_{L^2} \leq \eta K^2_{F;\mathrm{nl}} M^2 (1+\eta^m)^2 N_{\varepsilon,\alpha;II}(t), \tag{2.9.55}
$$

together with

$$
e^{-\varepsilon t}\mathcal{I}^{\mathrm{lt}}_{\nu,\delta;F;\mathrm{nl}}(t) \leq \eta K^2_{F;\mathrm{nl}}(1+\eta^m)^2 \frac{M^2}{\beta + \frac{\alpha}{2} - \nu} N_{\varepsilon,\alpha;II}(t). \tag{2.9.56}
$$

Proof. We first notice that

$$
\begin{aligned}
\|\mathcal{E}_{F;\mathrm{nl}}(t)\|^2_{L^2} &\leq K^2_{F;\mathrm{nl}}(1+\eta^m)^2 M^2 \Big(\int_0^t e^{-\beta(t-s)}\|V(s)\|^2_{H^1}\,ds\Big)^2, \\
\|S(\delta)\mathcal{E}^{\mathrm{lt}}_{F;\mathrm{nl}}(t)\|^2_{H^1} &\leq K^2_{F;\mathrm{nl}}(1+\eta^m)^2 M^2 \Big(\int_0^t e^{-\beta(t-s)}\|V(s)\|^2_{H^1}\,ds\Big)^2.
\end{aligned}
\tag{2.9.57}
$$

Using $\beta > \nu - \frac{1}{2}\alpha = \frac{1}{2}\alpha + \varepsilon$, we compute

$$
\begin{aligned}
\int_0^t e^{-\beta(t-s)}\|V(s)\|^2_{H^1}\,ds &= e^{\frac{\alpha}{2}t}\int_0^t e^{-\beta(t-s)}e^{-\frac{\alpha}{2}t}\|V(s)\|^2_{H^1}\,ds \\
&\leq e^{\frac{\alpha}{2}t}\int_0^t e^{-\beta(t-s)}e^{-\frac{\alpha}{2}(t-s)}\|V(s)\|^2_{H^1}\,ds \\
&\leq e^{\frac{\alpha}{2}t}\int_0^t e^{-\nu(t-s)}\|V(s)\|^2_{H^1}\,ds.
\end{aligned}
\tag{2.9.58}
$$

This yields the desired bound

$$\begin{aligned} e^{\alpha t}\|\mathcal{E}_{F;\mathrm{nl}}(t)\|_{L^2}^2 &\le K_{F;\mathrm{nl}}^2(1+\eta^m)^2M^2e^{\alpha t}\left(\int_0^t e^{-\beta(t-s)}\|V(s)\|_{H^1}^2\,ds\right)^2 \\ &\le K_{F;\mathrm{nl}}^2(1+\eta^m)^2M^2e^{2\alpha t}\left(\int_0^t e^{-\nu(t-s)}\|V(s)\|_{H^1}^2\,ds\right)^2 \\ &\le K_{F;\mathrm{nl}}^2(1+\eta^m)^2M^2\eta N_{\varepsilon,\alpha;II}(t). \end{aligned} \tag{2.9.59}$$

In a similar spirit, we compute

$$\begin{aligned} e^{-\varepsilon t}\mathcal{I}^{\mathrm{lt}}_{\nu,\delta;F;\mathrm{nl}}(t) &\le K_{F;\mathrm{nl}}^2(1+\eta^m)^2M^2e^{-\varepsilon t}\int_0^t e^{\nu s}\left(\int_0^s e^{-\beta(s-s')}\|V(s')\|_{H^1}^2\,ds'\right)^2 ds \\ &\le K_{F;\mathrm{nl}}^2(1+\eta^m)^2M^2e^{-\varepsilon t}\int_0^t e^{\nu s}e^{\frac{\alpha}{2}s}\left(\int_0^s e^{-\nu(s-s')}\|V(s')\|_{H^1}^2ds'\right) \\ &\qquad\times\left(\int_0^s e^{-\beta(s-s')}\|V(s')\|_{H^1}^2\,ds'\right) ds \\ &\le \eta K_{F;\mathrm{nl}}^2(1+\eta^m)^2M^2e^{-\varepsilon t}\int_0^t e^{(\nu-\frac{\alpha}{2})s}\int_0^s e^{-\beta(s-s')}\|V(s')\|_{H^1}^2\,ds'ds \\ &= \eta K_{F;\mathrm{nl}}^2(1+\eta^m)^2M^2e^{-\varepsilon t}\int_0^t \left[\int_{s'}^t e^{-(\frac{\alpha}{2}-\nu+\beta)s}ds\right]e^{\beta s'}\|V(s')\|_{H^1}^2ds' \\ &\le \eta K_{F;\mathrm{nl}}^2(1+\eta^m)^2\frac{M^2}{\beta+\frac{\alpha}{2}-\nu}e^{-\varepsilon t}\int_0^t e^{-(\frac{\alpha}{2}-\nu+\beta)s'}e^{\beta s'}\|V(s')\|_{H^1}^2ds' \\ &= \eta K_{F;\mathrm{nl}}^2(1+\eta^m)^2\frac{M^2}{\beta+\frac{\alpha}{2}-\nu}e^{-\varepsilon t}\int_0^t e^{\nu s'}e^{-\frac{\alpha}{2}s'}\|V(s')\|_{H^1}^2ds' \\ &\le \eta K_{F;\mathrm{nl}}^2(1+\eta^m)^2\frac{M^2}{\beta+\frac{\alpha}{2}-\nu}N_{\varepsilon,\alpha;II}(t). \end{aligned} \tag{2.9.60}$$

□

Lemma 2.9.12. *Fix $T>0$ and assume that (HA), (HTw), (HS), (Hβ), (hSol) and (hFB) all hold. Pick two constants $\varepsilon>0$, $\alpha\ge 0$ and write $\nu=\alpha+\varepsilon$. Then for any $\eta>0$, any $0\le\delta<1$ and any $0\le t\le\tau$, we have the bound*

$$e^{-\varepsilon t}\mathcal{I}^{\mathrm{sh}}_{\nu,\delta;F,\mathrm{nl}}(t) \;\le\; \eta M^2K_{F;\mathrm{nl}}^2(1+\eta^m)^2(1+\rho^{-1})e^{3\nu}(4+\nu)N_{\varepsilon,\alpha;II}(t). \tag{2.9.61}$$

Proof. We start by observing that

$$\|v\|_{H^1}^2=\|v\|_{L^2}^2+\rho^{-1}\|A_*^{1/2}v\|_{L^2}^2. \tag{2.9.62}$$

In addition, for any $w \in L^2$, $\vartheta > 0$, $\vartheta_A \geq -\vartheta$ and $\vartheta_B \geq -\vartheta$ we have

$$\begin{array}{lcl} \frac{d}{d\vartheta}\langle S(\vartheta+\vartheta_A)w, S(\vartheta+\vartheta_B)w\rangle_{L^2} & = & \langle \mathcal{L}_{\mathrm{tw}} S(\vartheta+\vartheta_A)w, S(\vartheta+\vartheta_B)w\rangle_{L^2} \\ & & \quad +\langle S(\vartheta+\vartheta_A)w, \mathcal{L}_{\mathrm{tw}} S(\vartheta+\vartheta_B)w\rangle_{L^2} \\ & = & \langle S(\vartheta+\vartheta_A)w, \mathcal{L}_{\mathrm{tw}}^{\mathrm{adj}} S(\vartheta+\vartheta_B)w\rangle_{L^2} \\ & & \quad +\langle S(\vartheta+\vartheta_A)w, \mathcal{L}_{\mathrm{tw}} S(\vartheta+\vartheta_B)w\rangle_{L^2} \\ & = & \langle S(\vartheta+\vartheta_A)w, \big[\mathcal{L}_{\mathrm{tw}}^{\mathrm{adj}} - A_*\big] S(\vartheta+\vartheta_B)w\rangle_{L^2} \\ & & \quad +\langle S(\vartheta+\vartheta_A)w, \big[\mathcal{L}_{\mathrm{tw}} - A_*\big] S(\vartheta+\vartheta_B)w\rangle_{L^2} \\ & & \quad -2\langle A_*^{1/2} S(\vartheta+\vartheta_A)w, A_*^{1/2} S(\vartheta+\vartheta_B)w\rangle_{L^2}. \end{array} \tag{2.9.63}$$

Assume for the moment that $\delta > 0$. For convenience, we introduce the expression

$$\mathcal{E}_{s,s',s'';\mathcal{H}} = \langle S(s+\delta-s')QF_{\mathrm{nl}}\big(V(s')\big), S(s+\delta-s'')QF_{\mathrm{nl}}\big(V(s'')\big)\rangle_{\mathcal{H}}, \tag{2.9.64}$$

where we allow $\mathcal{H} \in \{L^2, H^1\}$. Exploiting (2.9.63) and the fact that $\delta > 0$, we obtain the bound

$$\begin{array}{lcl} \mathcal{E}_{s,s',s'';H^1} & \leq & M^2 K_{F;\mathrm{nl}}^2 (1+\eta^m)^2 \|V(s')\|_{H^1}^2 \|V(s'')\|_{H^1}^2 \\ & & \quad + M^2 K_{F;\mathrm{nl}}^2 (1+\eta^m)^2 \rho^{-1} \frac{1}{\sqrt{s+\delta-s''}} \|V(s')\|_{H^1}^2 \|V(s'')\|_{H^1}^2 \\ & & \quad -\rho^{-1}\frac{1}{2}\frac{d}{ds}\mathcal{E}_{s,s',s'';L^2} \end{array} \tag{2.9.65}$$

for the values of (s, s', s'') that are relevant below. Upon introducing the integrals

$$\begin{array}{lcl} \mathcal{I}_I & = & e^{-\varepsilon t}\int_0^t e^{\nu s}\int_{s-1}^s\int_{s-1}^s \big[1+\frac{1}{\sqrt{s+\delta-s''}}\big]\|V(s')\|_{H^1}^2\|V(s'')\|_{H^1}^2\,ds''\,ds'\,ds, \\ \mathcal{I}_{II} & = & e^{-\varepsilon t}\int_0^t e^{\nu s}\int_{s-1}^s\int_{s-1}^s \frac{d}{ds}\mathcal{E}_{s,s',s'';L^2}ds''\,ds'\,ds, \end{array} \tag{2.9.66}$$

we hence readily obtain the estimate

$$e^{-\varepsilon t}\mathcal{I}_{\nu,\delta;F;\mathrm{nl}}^{\mathrm{sh}}(t) \quad \leq \quad (1+\rho^{-1})M^2K_{F;\mathrm{nl}}^2(1+\eta^m)^2\mathcal{I}_I - \frac{1}{2}\rho^{-1}\mathcal{I}_{II}. \tag{2.9.67}$$

Changing the order of the integrals, we find

$$\begin{array}{rcl}\mathcal{I}_I & = & e^{-\varepsilon t}\int_0^t \int_{s'-1}^{\min\{t,s'+1\}} \\ & & \Big[\int_{\max\{s',s''\}}^{\min\{t,s'+1,s''+1\}} e^{\nu s}\big[1+\frac{1}{\sqrt{s+\delta-s''}}\big]\,ds\Big]\|V(s')\|^2_{H^1}\|V(s'')\|^2_{H^1}\,ds''ds' \\ & \leq & 3e^{-\varepsilon t}\int_0^t e^{\nu s'}e^{\nu}\|V(s')\|^2_{H^1}\int_{s'-1}^{\min\{t,s'+1\}}\|V(s'')\|^2_{H^1}\,ds''ds' \\ & \leq & 3e^{-\varepsilon t}\int_0^t e^{\nu s'}e^{3\nu}\|V(s')\|^2_{H^1}\int_{s'-1}^{\min\{t,s'+1\}} e^{-\nu(\min\{t,s'+1\}-s'')}\|V(s'')\|^2_{H^1}\,ds''ds' \\ & \leq & 3\eta e^{3\nu}e^{-\varepsilon t}\int_0^t e^{\nu s'}\|V(s')\|^2_{H^1}e^{-\alpha\min\{t,s'+1\}}\,ds' \\ & \leq & 3\eta e^{3\nu}N_{\varepsilon,\alpha;II}(t).\end{array} \tag{2.9.68}$$

In a similar fashion, we may use an integration by parts to write

$$\begin{array}{rcl}\mathcal{I}_{II} & = & e^{-\varepsilon t}\int_0^t\int_{s'-1}^{\min\{t,s'+1\}}\Big[\int_{\max\{s',s''\}}^{\min\{t,s'+1,s''+1\}} e^{\nu s}\frac{d}{ds}\mathcal{E}_{s,s',s'';L^2}\,ds\Big]\,ds''\,ds' \\ & = & \mathcal{I}_{II;A}+\mathcal{I}_{II;B}+\mathcal{I}_{II;C},\end{array} \tag{2.9.69}$$

in which we have introduced

$$\begin{array}{rcl}\mathcal{I}_{II;A} & = & e^{-\varepsilon t}\int_0^t\int_{s'-1}^{\min\{t,s'+1\}} e^{\nu\min\{t,s'+1,s''+1\}}\mathcal{E}_{\min\{t,s'+1,s''+1\},s',s'';L^2}\,ds''\,ds', \\ \mathcal{I}_{II;B} & = & -e^{-\varepsilon t}\int_0^t\int_{s'-1}^{\min\{t,s'+1\}} e^{\nu\max\{s',s''\}}\mathcal{E}_{\max\{s',s''\},s',s'';L^2}\,ds''\,ds', \\ \mathcal{I}_{II;C} & = & -e^{-\varepsilon t}\int_0^t\int_{s'-1}^{\min\{t,s'+1\}}\Big[\int_{\max\{s',s''\}}^{\min\{t,s'+1,s''+1\}}\nu e^{\nu s}\mathcal{E}_{s,s',s'';L^2}\,ds\Big]\,ds''\,ds'.\end{array} \tag{2.9.70}$$

Note here that $\mathcal{I}_{II;B}$ is well defined because $\delta > 0$. A direct inspection of these terms yields the bound

$$\begin{array}{rcl}|\mathcal{I}_{II}| & \leq & e^{\nu}(2+\nu)M^2K^2_{F;\mathrm{nl}}(1+\eta^m)^2e^{-\varepsilon t}\times \\ & & \int_0^t e^{\nu s'}\|V(s')\|^2_{H^1}\int_{s'-1}^{\min\{t,s'+1\}}\|V(s'')\|^2_{H^1}\,ds''\,ds' \\ & \leq & e^{\nu}(2+\nu)M^2K^2_{F;\mathrm{nl}}(1+\eta^m)^2e^{-\varepsilon t}\times \\ & & \int_0^t e^{\nu s'}\|V(s')\|^2_{H^1}e^{2\nu}\int_{s'-1}^{\min\{t,s'+1\}} e^{-\nu(\min\{t,s'+1\}-s'')}\|V(s'')\|^2_{H^1}\,ds''\,ds' \\ & \leq & \eta e^{3\nu}(2+\nu)M^2K^2_{F;\mathrm{nl}}(1+\eta^m)^2e^{-\varepsilon t}\int_0^t e^{\nu s'}\|V(s')\|^2_{H^1}e^{-\alpha\min\{t,s'+1\}}\,ds' \\ & \leq & \eta e^{3\nu}(2+\nu)M^2K^2_{F;\mathrm{nl}}(1+\eta^m)^2N_{\varepsilon,\alpha;II}(t).\end{array} \tag{2.9.71}$$

It hence remains to consider the case $\delta = 0$. We may apply Fatou's lemma to conclude

$$\begin{aligned} \mathcal{I}^{\mathrm{sh}}_{\nu,0;F;\mathrm{nl}}(t) &= \int_0^t e^{\nu s}(\lim_{\delta\to 0}\|S(\delta)\mathcal{E}^{\mathrm{sh}}_{B;\mathrm{lin}}(s)\|_{H^1})^2\mathbf{1}_{s<\tau}\,ds \\ &\le \liminf_{\delta\to 0}\mathcal{I}^{\mathrm{sh}}_{\nu,\delta;F;\mathrm{nl}}(t). \end{aligned} \tag{2.9.72}$$

The result now follows from the fact that the bounds obtained above do not depend on δ. □

2

2.9.3 Stochastic regularity estimates

We are now ready to discuss the stochastic integrals in (2.9.6). These require special care because they cannot be bounded in a pathwise fashion, unlike the deterministic integrals above. Expectations of suprema are particularly delicate in this respect. Indeed, the powerful Burkholder-Davis-Gundy inequalities cannot be directly applied to the stochastic convolutions that arise in our mild formulation. However, the H^∞-calculus obtained in Lemma 2.9.7 allows us to use the following mild version, which is the source of the extra T factors that appear in our estimates.

Lemma 2.9.13. *Fix $T > 0$ and assume that (HA), (HTw), (HS) and (Hβ) all hold. There exists a constant $K_{\mathrm{cnv}} > 0$ so that for any $W \in \mathcal{N}^2([0,T];(\mathcal{F})_t;L^2)$ and any $0 \le \alpha \le 2\beta$ we have*

$$E\sup_{0\le t\le T} e^{\alpha t}\|\int_0^t S(t-s)QW(s)\,d\beta_s\|^2_{L^2} \le K_{\mathrm{cnv}}E\int_0^T e^{\alpha s}\|W(s)\|^2_{L^2}\,ds. \tag{2.9.73}$$

Proof. Lemma 2.9.7 implies that the generator $B^Q_\alpha = \mathcal{L}_{\mathrm{tw}} + \frac{1}{2}\alpha$ of the semigroup $e^{\frac{1}{2}\alpha t}S(t)$ on L^2_Q satisfies assumption (H) in [111]. On account of the identity

$$e^{\alpha t}\|\int_0^t S(t-s)QW(s)\,d\beta_s\|^2_{L^2} = \|\int_0^t e^{\frac{1}{2}\alpha(t-s)}S(t-s)Qe^{\frac{1}{2}\alpha s}W(s)\,d\beta_s\|^2_{L^2}, \tag{2.9.74}$$

the desired estimate is now an immediate consequence of [111, Thm. 1.1]. □

Lemma 2.9.14. *Fix $T > 0$ and assume that (HA), (HTw), (HS), (Hβ), (hSol) and (hFB) all hold. Then for any pair of constants $\varepsilon > 0$ and $0 \le \alpha \le 2\beta$ we have the bound*

$$E\sup_{0\le t\le\tau} e^{\alpha t}\|\mathcal{E}_{B;\mathrm{lin}}(t)\|^2_{L^2} \le (T+1)K_{\mathrm{cnv}}K^2_{B;\mathrm{lin}}e^{\varepsilon}E\sup_{0\le t\le\tau} N_{\varepsilon,\alpha;II}(t). \tag{2.9.75}$$

2

Proof. Using Lemma 2.9.13 we compute

$$\begin{array}{rcl} E \sup_{0 \le t \le \tau} e^{\alpha t} \|\mathcal{E}_{B;\mathrm{lin}}(t)\|_{L^2}^2 & \le & E \sup_{0 \le t \le T} e^{\alpha t} \|\mathcal{E}_{B;\mathrm{lin}}(t)\|_{L^2}^2 \\ & = & E \sup_{0 \le t \le T} e^{\alpha t} \| \int_0^t S(t-s) Q B_{\mathrm{lin}}\big(V(s)\big) \mathbf{1}_{s<\tau} \, d\beta_s \|_{L^2}^2 \\ & \le & K_{\mathrm{cnv}} E \int_0^T e^{\alpha s} \|B_{\mathrm{lin}}\big(V(s)\big)\mathbf{1}_{s<\tau}\|_{L^2}^2 \, ds \\ & \le & K_{\mathrm{cnv}} K_{B;\mathrm{lin}}^2 E \int_0^\tau e^{\alpha s} \|V(s)\|_{H^1}^2 \, ds. \end{array} \tag{2.9.76}$$

By dividing up the integral, we obtain

$$\begin{array}{rcl} \int_0^\tau e^{\alpha s} \|V(s)\|_{H^1}^2 \, ds & \le & e^{\varepsilon} \int_0^1 e^{-\varepsilon(1-s)} e^{\alpha s} \|V(s)\|_{H^1}^2 \mathbf{1}_{s<\tau} \, ds \\ & & + e^{\varepsilon} \int_1^2 e^{-\varepsilon(2-s)} e^{\alpha s} \|V(s)\|_{H^1}^2 \mathbf{1}_{s<\tau} \, ds \\ & & + \cdots + e^{\varepsilon} \int_{\lfloor T \rfloor}^{\lfloor T \rfloor + 1} e^{-\varepsilon(\lfloor T \rfloor + 1 - s)} e^{\alpha s} \|V(s)\|_{H^1}^2 \mathbf{1}_{s<\tau} \, ds \\ & \le & (T+1) e^{\varepsilon} \sup_{0 \le t \le T+1} \int_0^t e^{-\varepsilon(t-s)} e^{\alpha s} \|V(s)\|_{H^1}^2 \mathbf{1}_{s<\tau} \, ds \\ & \le & (T+1) e^{\varepsilon} \sup_{0 \le t \le \tau} \int_0^t e^{-\varepsilon(t-s)} e^{\alpha s} \|V(s)\|_{H^1}^2 \, ds \\ & = & (T+1) e^{\varepsilon} \sup_{0 \le t \le \tau} N_{\varepsilon,\alpha;II}(t), \end{array} \tag{2.9.77}$$

which yields the desired bound upon taking expectations. □

Lemma 2.9.15. *Fix $T > 0$ and assume that (HA), (HTw), (HS), (Hβ), (hSol) and (hFB) all hold. Then we have the bound*

$$E \sup_{0 \le t \le \tau} \|\mathcal{E}_{B;\mathrm{cn}}(t)\|_{L^2}^2 \le T K_{\mathrm{cnv}} K_{B;\mathrm{cn}}^2. \tag{2.9.78}$$

Proof. This bound follows directly from (2.9.76) by picking $\alpha = 0$ and making the substitutions

$$K_{B;\mathrm{lin}} \mapsto K_{B;\mathrm{cn}}, \qquad \|V(s)\|_{H^1}^2 \mapsto 1. \tag{2.9.79}$$

□

We now set out to bound the expectation of the supremum of the remaining double integrals $\mathcal{I}^{\#}_{\nu,\delta;B;\mathrm{lin}}(t)$ and $\mathcal{I}^{\#}_{\nu,\delta;B;\mathrm{cn}}(t)$ with $\# \in \{\mathrm{lt}, \mathrm{sh}\}$. This is performed in Lemmas 2.9.20 and 2.9.21, but we first compute several time independent bounds for the expectation of the integrals themselves.

Lemma 2.9.16. *Fix $T > 0$ and assume that (HA), (HTw), (HS), (Hβ), (hSol) and (hFB) all hold. Pick constants $\varepsilon > 0$, $\alpha \ge 0$ and write $\nu = \alpha + \varepsilon$. Then for any*

$0 \le \delta < 1$ and $0 \le t \le T$, we have the identities

$$\begin{aligned} E\mathcal{I}^{\mathrm{lt}}_{\nu,\delta;B;\mathrm{lin}}(t) &= E\int_0^t e^{\nu s}\int_0^{s-1} \|S(s+\delta-s')QB_{\mathrm{lin}}\big(V(s')\big)\|^2_{H^1}\mathbf{1}_{s'<\tau}\,ds'\,ds, \\ E\mathcal{I}^{\mathrm{lt}}_{\nu,\delta;B;\mathrm{cn}}(t) &= E\int_0^t e^{\nu s}\int_0^{s-1} \|S(s+\delta-s')QB_{\mathrm{cn}}\|^2_{H^1}\mathbf{1}_{s'<\tau}\,ds'\,ds, \end{aligned} \tag{2.9.80}$$

together with their short-time counterparts

$$\begin{aligned} E\mathcal{I}^{\mathrm{sh}}_{\nu,\delta;B;\mathrm{lin}}(t) &= E\int_0^t e^{\nu s}\int_{s-1}^{s} \|S(s+\delta-s')QB_{\mathrm{lin}}\big(V(s')\big)\|^2_{H^1}\mathbf{1}_{s'<\tau}\,ds'\,ds, \\ E\mathcal{I}^{\mathrm{sh}}_{\nu,\delta;B;\mathrm{cn}}(t) &= E\int_0^t e^{\nu s}\int_{s-1}^{s} \|S(s+\delta-s')QB_{\mathrm{cn}}\|^2_{H^1}\mathbf{1}_{s'<\tau}\,ds'\,ds. \end{aligned} \tag{2.9.81}$$

Proof. The linearity of the expectation operator, the Itô Isometry (see e.g. [93, §2.3]) and the integrability of the integrands imply that

$$\begin{aligned} E\mathcal{I}^{\mathrm{lt}}_{B;\mathrm{lin}}(t) &= E\int_0^t e^{\nu s}\|\int_0^{s-1} S(s+\delta-s')QB_{\mathrm{lin}}\big(V(s')\big)\mathbf{1}_{s'<\tau}d\beta_{s'}\|^2_{H^1}\,ds \\ &= E\int_0^t e^{\nu s}\int_0^{s-1} \|S(s+\delta-s')QB_{\mathrm{lin}}\big(V(s')\big)\|^2_{H^1}\mathbf{1}_{s'<\tau}\,ds'\,ds. \end{aligned} \tag{2.9.82}$$

The remaining expressions follow in a similar fashion. □

Lemma 2.9.17. *Fix $T > 0$ and assume that (HA), (HTw), (HS), (Hβ), (hSol) and (hFB) all hold. Pick constants $\varepsilon > 0$, $\alpha \ge 0$ for which $\varepsilon + \alpha < 2\beta$ and write $\nu = \alpha + \varepsilon$. Then for any $0 \le \delta < 1$ and any $0 \le t \le T$, we have the bound*

$$Ee^{-\varepsilon t}\mathcal{I}^{\mathrm{lt}}_{\nu,\delta;B;\mathrm{lin}}(t) \le \frac{M^2}{2\beta-\nu}K^2_{B;\mathrm{lin}}EN_{\varepsilon;\alpha;II}(t\wedge\tau). \tag{2.9.83}$$

Proof. Using (2.9.80) and switching the integration order, we obtain

$$\begin{aligned} Ee^{-\varepsilon t}\mathcal{I}^{\mathrm{lt}}_{\nu,\delta;B;\mathrm{lin}}(t) &\le M^2K^2_{B;\mathrm{lin}}Ee^{-\varepsilon t}\int_0^t e^{\nu s}\int_0^{s\wedge\tau} e^{-2\beta(s-s')}\|V(s')\|^2_{H^1}\,ds'\,ds \\ &= M^2K^2_{B;\mathrm{lin}}Ee^{-\varepsilon t}\int_0^{t\wedge\tau}\Big[\int_{s'}^t e^{-(2\beta-\nu)s}\,ds\Big]e^{2\beta s'}\|V(s')\|^2_{H^1}\,ds' \\ &\le \frac{M^2}{2\beta-\nu}K^2_{B;\mathrm{lin}}Ee^{-\varepsilon t}\int_0^{t\wedge\tau} e^{-(2\beta-\nu)s'}e^{2\beta s'}\|V(s')\|^2_{H^1}\,ds' \\ &= \frac{M^2}{2\beta-\nu}K^2_{B;\mathrm{lin}}Ee^{-\varepsilon t}\int_0^{t\wedge\tau} e^{\nu s'}\|V(s')\|^2_{H^1}\,ds' \\ &\le \frac{M^2}{2\beta-\nu}K^2_{B;\mathrm{lin}}Ee^{-\varepsilon t\wedge\tau}\int_0^{t\wedge\tau} e^{\nu s'}\|V(s')\|^2_{H^1}\,ds' \\ &= \frac{M^2}{2\beta-\nu}K^2_{B;\mathrm{lin}}EN_{\varepsilon,\alpha;II}(t\wedge\tau). \end{aligned} \tag{2.9.84}$$

□

Lemma 2.9.18. *Fix $T > 0$ and assume that (HA), (HTw), (HS), (Hβ), (hSol) and (hFB) all hold. Pick two constants $\varepsilon > 0$, $\alpha \geq 0$ and write $\nu = \alpha + \varepsilon$. Then for any $0 \leq \delta < 1$ and any $0 \leq t \leq T$, we have the bound*

$$Ee^{-\varepsilon t}\mathcal{I}^{\mathrm{sh}}_{\nu,\delta;B;\mathrm{lin}}(t) \quad \leq \quad K^2_{B;\mathrm{lin}}M^2(1+\rho^{-1})e^{\nu}(4+\nu)EN_{\varepsilon,\alpha;II}(t\wedge\tau). \tag{2.9.85}$$

Proof. We only consider the case $\delta > 0$ here, noting that the limit $\delta \downarrow 0$ can be handled as in the proof of Lemma 2.9.12. Applying the identity (2.9.63) with $\vartheta_A = \vartheta_B$, we obtain the bound

$$\begin{array}{rcl} \|S(s+\delta-s')QB_{\mathrm{lin}}\big(V(s')\big)\|^2_{H^1} & \leq & M^2K^2_{B;\mathrm{lin}}\|V(s')\|^2_{H^1} \\ & & +M^2K^2_{B;\mathrm{lin}}\rho^{-1}\dfrac{1}{\sqrt{s+\delta-s'}}\|V(s')\|^2_{H^1} \\ & & -\rho^{-1}\dfrac{1}{2}\dfrac{d}{ds}\|S(s+\delta-s')QB_{\mathrm{lin}}\big(V(s')\big)\|^2_{L^2} \end{array} \tag{2.9.86}$$

for the values of (s,s') that are relevant below. Upon writing

$$\begin{array}{rcl} \mathcal{I}_I & = & Ee^{-\varepsilon t}\displaystyle\int_0^t e^{\nu s}\int_{s-1}^s \big[1+\frac{1}{\sqrt{s+\delta-s'}}\big]\|V(s')\|^2_{H^1}\mathbf{1}_{s'<\tau}\,ds'\,ds, \\ \mathcal{I}_{II} & = & Ee^{-\varepsilon t}\displaystyle\int_0^t e^{\nu s}\int_{s-1}^s \frac{d}{ds}\|S(s+\delta-s')QB_{\mathrm{lin}}\big(V(s')\big)\|^2_{L^2}\mathbf{1}_{s'<\tau}ds'\,ds, \end{array} \tag{2.9.87}$$

we obtain the estimate

$$Ee^{-\varepsilon t}\mathcal{I}^{\mathrm{sh}}_{\nu,\delta;B;\mathrm{lin}}(t) \quad \leq \quad (1+\rho^{-1})M^2K^2_{B;\mathrm{lin}}\mathcal{I}_I - \frac{1}{2}\rho^{-1}\mathcal{I}_{II}. \tag{2.9.88}$$

Changing the integration order, we obtain

$$\begin{array}{rcl} \mathcal{I}_I & = & Ee^{-\varepsilon t}\displaystyle\int_0^t \Big[\int_{s'}^{\min\{t,s'+1\}} e^{\nu s}\big[1+\frac{1}{\sqrt{s+\delta-s'}}\big]\,ds\Big]\|V(s')\|^2_{H^1}\mathbf{1}_{s'<\tau}\,ds' \\ & \leq & 3e^{\nu}Ee^{-\varepsilon t}\displaystyle\int_0^t e^{\nu s'}\|V(s')\|^2_{H^1}\mathbf{1}_{s'<\tau}\,ds' \\ & \leq & 3e^{\nu}Ee^{-\varepsilon t\wedge\tau}\displaystyle\int_0^{t\wedge\tau} e^{\nu s'}\|V(s')\|^2_{H^1}\,ds' \\ & = & 3e^{\nu}EN_{\varepsilon,\alpha;II}(t\wedge\tau). \end{array} \tag{2.9.89}$$

Integrating by parts, we arrive at the identity

$$\begin{array}{rcl} \mathcal{I}_{II} & = & Ee^{-\varepsilon t}\displaystyle\int_0^t \Big[\int_{s'}^{\min\{t,s'+1\}} e^{\nu s}\frac{d}{ds}\|S(s+\delta-s')QB_{\mathrm{lin}}\big(V(s')\big)\|^2_{L^2}\,ds\Big]\mathbf{1}_{s'<\tau}\,ds' \\ & = & \mathcal{I}_{II;A}+\mathcal{I}_{II;B}+\mathcal{I}_{II;C}, \end{array} \tag{2.9.90}$$

2

in which we have introduced the expressions

$$\begin{aligned}
\mathcal{I}_{II;A} &= Ee^{-\varepsilon t}\int_0^t e^{\nu\min\{t,s'+1\}}\|S(\min\{t,s'+1\}+\delta-s')QB_{\mathrm{lin}}\big(V(s')\big)\|_{L^2}^2\mathbf{1}_{s'<\tau}\,ds',\\
\mathcal{I}_{II;B} &= -Ee^{-\varepsilon t}\int_0^t e^{\nu s'}\|S(\delta)QB_{\mathrm{lin}}\big(V(s')\big)\|_{L^2}^2\mathbf{1}_{s'<\tau}\,ds',\\
\mathcal{I}_{II;C} &= -Ee^{-\varepsilon t}\int_0^t\Big[\int_{s'}^{\min\{t,s'+1\}}\nu e^{\nu s}\|S(s+\delta-s')QB_{\mathrm{lin}}\big(V(s')\big)\|_{L^2}^2\,ds\Big]\mathbf{1}_{s'<\tau}\,ds'.
\end{aligned} \tag{2.9.91}$$

Inspecting these expressions, we readily obtain the bound

$$\begin{aligned}
|\mathcal{I}_{II}| &\leq e^{\nu}(2+\nu)M^2K_{B;\mathrm{lin}}^2Ee^{-\varepsilon t}\int_0^t e^{\nu s'}\|V(s')\|_{H^1}^2\mathbf{1}_{s'<\tau}\,ds'\\
&\leq e^{\nu}(2+\nu)M^2K_{B;\mathrm{lin}}^2EN_{\varepsilon,\alpha;II}(t\wedge\tau).
\end{aligned} \tag{2.9.92}$$

□

Lemma 2.9.19. *Fix $T>0$ and assume that (HA), (HTw), (HS), (Hβ), (hSol) and (hFB) all hold. Pick a constant $0<\varepsilon<2\beta$. Then for any $0\leq\delta<1$, and any $0\leq t\leq T$, we have the bounds*

$$\begin{aligned}
Ee^{-\varepsilon t}\mathcal{I}^{\mathrm{lt}}_{\varepsilon,\delta;B;\mathrm{cn}}(t) &\leq \frac{M^2}{(2\beta-\varepsilon)\varepsilon}K_{B;\mathrm{cn}}^2,\\
Ee^{-\varepsilon t}\mathcal{I}^{\mathrm{sh}}_{\varepsilon,\delta;B;\mathrm{cn}}(t) &\leq K_{B;\mathrm{cn}}^2\frac{M^2}{\varepsilon}(1+\rho^{-1})e^{\varepsilon}(4+\varepsilon).
\end{aligned} \tag{2.9.93}$$

Proof. Using the fact that

$$e^{-\varepsilon t}\int_0^t e^{\varepsilon s}\,ds\leq\frac{1}{\varepsilon}, \tag{2.9.94}$$

these bounds can be obtained from Lemmas 2.9.17 and 2.9.18 by picking $\alpha=0$ and making the substitutions

$$K_{B;\mathrm{lin}}\mapsto K_{B;\mathrm{cn}},\qquad EN_{\varepsilon,0;II}(t\wedge\tau)\mapsto\frac{1}{\varepsilon}. \tag{2.9.95}$$

□

Lemma 2.9.20. *Fix $T>0$ and assume that (HA), (HTw), (HS), (Hβ), (hSol) and (hFB) all hold. Pick two constants $\varepsilon>0$, $\alpha\geq0$ for which $\varepsilon+\alpha<2\beta$ and write $\nu=\alpha+\varepsilon$. Then we have the bounds*

$$\begin{aligned}
E\sup_{0\leq t\leq\tau}e^{-\varepsilon t}\mathcal{I}^{\mathrm{lt}}_{\nu,0;B;\mathrm{lin}}(t) &\leq e^{\varepsilon}(T+1)\frac{M^2}{2\beta-\nu}K_{B;\mathrm{lin}}^2E\sup_{0\leq t\leq\tau}N_{\varepsilon;\alpha;II}(t),\\
E\sup_{0\leq t\leq\tau}e^{-\varepsilon t}\mathcal{I}^{\mathrm{sh}}_{\nu,0;B;\mathrm{lin}}(t) &\leq e^{\varepsilon}(T+1)K_{B;\mathrm{lin}}^2M^2(1+\rho^{-1})e^{\nu}(4+\nu)E\sup_{0\leq t\leq\tau}N_{\varepsilon;\alpha;II}(t).
\end{aligned} \tag{2.9.96}$$

Proof. By splitting the integration interval we obtain

$$
\begin{aligned}
\sup_{0\le t\le\tau} e^{-\varepsilon t}\mathcal{I}^{\mathrm{lt}}_{\nu,0;B;\mathrm{lin}}(t) &\le \sup_{0\le t\le T} e^{-\varepsilon t}\mathcal{I}^{\mathrm{lt}}_{\nu,0;B;\mathrm{lin}}(t) \\
&= \sup_{0\le t\le T} e^{-\varepsilon t}\int_0^t e^{\nu s}\|\mathcal{E}^{\mathrm{lt}}_{B;\mathrm{lin}}(s)\|^2_{H^1}\,ds \\
&\le e^{\varepsilon}e^{-\varepsilon}\int_0^1 e^{\nu s}\|\mathcal{E}^{\mathrm{lt}}_{B;\mathrm{lin}}(s)\|^2_{H^1}\,ds \\
&\quad +e^{\varepsilon}e^{-2\varepsilon}\int_1^2 e^{\nu s}\|\mathcal{E}^{\mathrm{lt}}_{B;\mathrm{lin}}(s)\|^2_{H^1}\,ds \\
&\quad +e^{\varepsilon}e^{-(\lfloor T\rfloor+1)\varepsilon}\int_{\lfloor T\rfloor}^{\lfloor T\rfloor+1} e^{\nu s}\|\mathcal{E}^{\mathrm{lt}}_{B;\mathrm{lin}}(s)\|^2_{H^1}\,ds \\
&= e^{\varepsilon}\big[e^{-\varepsilon}\mathcal{I}^{\mathrm{lt}}_{\nu,0;B;\mathrm{lin}}(1)+e^{-2\varepsilon}\mathcal{I}^{\mathrm{lt}}_{\nu,0;B;\mathrm{lin}}(2) \\
&\quad +\ldots+e^{-\varepsilon(\lfloor T\rfloor+1)}\mathcal{I}^{\mathrm{lt}}_{\nu,0;B;\mathrm{lin}}(\lfloor T\rfloor+1)\big].
\end{aligned}
\tag{2.9.97}
$$

Applying Lemma 2.9.17, we hence see

$$
\begin{aligned}
&E\sup_{0\le t\le\tau} e^{-\varepsilon t}\mathcal{I}^{\mathrm{lt}}_{\nu,0;B;\mathrm{lin}}(t) \\
&\quad\le e^{\varepsilon}\frac{M^2}{2\beta-\nu}K^2_{B;\mathrm{lin}}E\big[N_{\varepsilon;\alpha;II}(1\wedge\tau)+\ldots+N_{\varepsilon;\alpha;II}((\lfloor T\rfloor+1)\wedge\tau)\big] \\
&\quad\le (T+1)e^{\varepsilon}\frac{M^2}{2\beta-\nu}K^2_{B;\mathrm{lin}}\sup_{0\le t\le\tau} N_{\varepsilon;\alpha;II}(t).
\end{aligned}
\tag{2.9.98}
$$

The same procedure works for the second estimate. □

Lemma 2.9.21. *Fix $T>0$ and assume that (HA), (HTw), (HS), (Hβ), (hSol) and (hFB) all hold. Pick a constant $\varepsilon>0$, $\alpha\ge 0$ for which $\varepsilon<2\beta$. Then we have the bounds*

$$
\begin{aligned}
E\sup_{0\le t\le\tau} e^{-\varepsilon t}\mathcal{I}^{\mathrm{lt}}_{\nu,0;B;\mathrm{cn}}(t) &\le e^{\varepsilon}(T+1)\frac{M^2}{(2\beta-\varepsilon)\varepsilon}K^2_{B;\mathrm{cn}}, \\
E\sup_{0\le t\le\tau} e^{-\varepsilon t}\mathcal{I}^{\mathrm{sh}}_{\nu,0;B;\mathrm{cn}}(t) &\le e^{\varepsilon}(T+1)K^2_{B;\mathrm{cn}}\frac{M^2}{\varepsilon}(1+\rho^{-1})e^{\varepsilon}(4+\varepsilon).
\end{aligned}
\tag{2.9.99}
$$

Proof. Following the procedure in the proof of Lemma 2.9.20, these bounds can be obtained from the estimates in Lemma 2.9.19 . □

Proof of Proposition 2.9.1. Pick $T>0$ and $0<\eta<\eta_0$ and write $\tau=\tau_{\varepsilon,\alpha}(T,\eta)$. Since the identities (2.9.3) with $v=V(t\wedge\tau)$ hold for all $0\le t\le T$, we may compute

$$
\begin{aligned}
E\sup_{0\le t\le\tau} N_{\varepsilon,0;I}(t) &\le 5E\sup_{0\le t\le\tau}\Big[\|\mathcal{E}_0(t)\|^2_{L^2}+\sigma^4\|\mathcal{E}_{F;\mathrm{lin}}(t)\|^2_{L^2}+\|\mathcal{E}_{F;\mathrm{nl}}(t)\|^2_{L^2} \\
&\quad +\sigma^2\|\mathcal{E}_{B;\mathrm{lin}}(t)\|^2_{L^2}+\sigma^2\|\mathcal{E}_{B;\mathrm{cn}}(t)\|^2_{L^2}\Big]
\end{aligned}
\tag{2.9.100}
$$

by applying Young's inequality. The inequalities in Lemmas 2.9.8-2.9.21 now imply that

$$E \sup_{0 \le t \le \tau} N_{\varepsilon,0;I}(t) \quad \le \quad C_1 \big[\|V(0)\|^2_{H^1} + \sigma^2 T + \Big(\eta + \sigma^2 T + \sigma^4 \Big) E \sup_{0 \le t \le \tau} N_{\varepsilon,0;II}(t) \big]. \tag{2.9.101}$$

In addition, we note that

$$\begin{aligned} E \sup_{0 \le t \le \tau} N_{\varepsilon,0;II}(t) \quad \le \quad & 9E \sup_{0 \le t \le \tau} e^{-\varepsilon t} \Big[\mathcal{I}_{\nu,0;0}(t) + \sigma^4 \mathcal{I}^{\mathrm{lt}}_{\nu,0;F;\mathrm{lin}}(t) + \sigma^4 \mathcal{I}^{\mathrm{sh}}_{\nu,0;F;\mathrm{lin}}(t) \\ & + \mathcal{I}^{\mathrm{lt}}_{\nu,0;F;\mathrm{nl}}(t) + \mathcal{I}^{\mathrm{sh}}_{\nu,0;F;\mathrm{nl}}(t) \\ & + \sigma^2 \mathcal{I}^{\mathrm{lt}}_{\nu,0;B;\mathrm{lin}}(t) + \sigma^2 \mathcal{I}^{\mathrm{sh}}_{\nu,0;B;\mathrm{lin}}(t) \\ & + \sigma^2 \mathcal{I}^{\mathrm{lt}}_{\nu,0;B;\mathrm{cn}}(t) + \sigma^2 \mathcal{I}^{\mathrm{sh}}_{\nu,0;B;\mathrm{cn}}(t) \Big]. \end{aligned} \tag{2.9.102}$$

The inequalities in Lemmas 2.9.8-2.9.21 now imply that

$$E \sup_{0 \le t \le \tau} N_{\varepsilon,0;II}(t) \quad \le \quad C_2 \big[\|V(0)\|^2_{H^1} + \sigma^2 T + \Big(\eta + \sigma^2 T + \sigma^4 \Big) \sup_{0 \le t \le \tau} N_{\varepsilon,0;II}(t) \big]. \tag{2.9.103}$$

In particular, we see that

$$E \sup_{0 \le t \le \tau} N_{\varepsilon,0}(t) \le C_3 \big[\|V(0)\|^2_{H^1} + \sigma^2 T + (\eta + \sigma^2 T + \sigma^4) E \sup_{0 \le t \le \tau} N_{\varepsilon,0}(t) \big]. \tag{2.9.104}$$

The desired bound hence follows by appropriately restricting the size of $\eta + \sigma^2 T + \sigma^4$. □

Proof of Proposition 2.9.2. Ignoring the contributions arising from B_{cn}, we can follow the proof of Proposition 2.9.1 to obtain the bound

$$E \sup_{0 \le t \le \tau} N_{\varepsilon,\alpha}(t) \le C_4 \big[\|V(0)\|^2_{H^1} + (\eta + \sigma^2 T + \sigma^4) E \sup_{0 \le t \le \tau} N_{\varepsilon,\alpha}(t) \big]. \tag{2.9.105}$$

The desired estimate hence follows by appropriately restricting the size of $\eta + \sigma^2 T + \sigma^4$. □

3

Systems of Reaction-Diffusion Equations with Scalar Noise

We consider reaction-diffusion equations that are stochastically forced by a small multiplicative noise term. We show that spectrally stable traveling wave solutions to the deterministic system retain their orbital stability if the amplitude of the noise is sufficiently small. By applying a stochastic phase-shift together with a time-transform, we obtain a quasi-linear SPDE that describes the fluctuations from the primary wave. We subsequently follow the semigroup approach developed in Chapter 2 to handle the nonlinear stability question. The main novel feature is that we no longer require the diffusion coefficients to be equal.

3.1 Introduction

In this chapter,[1] we consider stochastically perturbed versions of a class of reaction-diffusion equations that includes the FitzHugh-Nagumo equation

$$\begin{aligned} u_t &= u_{xx} + f_{\mathrm{cub}}(u) - w \\ w_t &= \varrho w_{xx} + \varepsilon[u - \gamma w]. \end{aligned} \tag{3.1.1}$$

Here we take $\varepsilon, \varrho, \gamma > 0$ and consider the standard bistable nonlinearity

$$f_{\mathrm{cub}}(u) = u(1-u)(u-a). \tag{3.1.2}$$

It has been known for quite some time that this system admits spectrally (and nonlinearly) stable traveling pulse solutions when $(\varrho, \gamma, \varepsilon)$ are all small [1]. Recently, such results have also become available for the equal-diffusion setting $\varrho = 1$ by using variational techniques together with the Maslov index [22, 24, 25].

Our goal here is to show that these spectrally stable wave solutions survive in a suitable sense upon adding a small pointwise multiplicative noise term to the underlying

[1] The content of this chapter has been published as *C.H.S. Hamster, H.J. Hupkes; Stability of Traveling Waves for Systems of Reaction-Diffusion Equations with Multiplicative Noise* in SIMA, see [47]

PDE. In particular, we generalize previous results in Chapter 2 where we were only able to consider the special case $\varrho = 1$. For example, we are now able to cover the Stochastic Partial Differential Equation (SPDE)

$$\begin{aligned} dU &= \big[U_{xx} + f_{\mathrm{cub}}(U) - W\big]dt + \sigma\chi(U)U(1-U)d\beta_t \\ dW &= \big[\varrho W_{xx} + \varepsilon(U - \gamma W)\big]dt \end{aligned} \tag{3.1.3}$$

for small $|\sigma|$, in which (β_t) is a Brownian motion and $\chi(U)$ is a cut-off function with $\chi(U) = 1$ for $|U| \leq 2$. The presence of this cut-off is required to enforce the global Lipschitz-smoothness of the noise term. In this regime, one can think of (3.1.3) as a version of the FitzHugh-Nagumo PDE (3.1.1) where the parameter a is replaced by $a + \sigma\dot{\beta}_t$. Notice that the noise vanishes at the asymptotic state $U = 0$ of the pulse.

Phase tracking Although the ability to include noise in models is becoming an essential tool in many disciplines [15, 16, 30, 35, 116], our understanding of the impact that such distortions have on basic patterns such as stripes, spots and waves is still in a preliminary stage [12, 14, 37, 43, 71, 79, 103, 112]. As explained in detail in §2.1, several approaches are being developed [57, 72, 104, 105] to analyze stochastically forced waves that each require a different set of conditions on the noise and structure of the system. The first main issue that often limits the application range of the results is that the underlying linear flow is required to be immediately contractive, which is (probably) not true for multi-component systems such as (3.1.1). The second main issue is that an appropriate phase needs to be defined for the wave. Various ad hoc choices have been made for this purpose, which typically rely on geometric intuition of some kind.

Inspired by the agnostic viewpoint described in the expository paper [117], we initiated a program in Chapter 2 that aims to define the phase, shape and speed of a stochastic wave purely by the technical considerations that arise when mimicking a deterministic nonlinear stability argument. In particular, the phase is constantly updated in such a way that the neutral part of the linearized flow is not felt by the nonlinear terms. The shape and speed of the stochastic wave are defined by the requirement that the resulting 'frozen wave' feels only (instantaneous) stochastic forcing. This allows us to obtain stability results, but also provides expressions for the leading order limiting behavior of the average speed experienced by the full stochastic system. We remark that the formal approach recently developed in [19] also touches upon several of the ideas underlying our approach.

Obstructions Applying the procedure sketched above to the FitzHugh-Nagumo SPDE (3.1.3), one can show that the deviation $(\tilde{U}, \tilde{W})$ from the phase-shifted stochastic wave satisfies a SPDE of the general form

$$\begin{aligned} d\tilde{U} &= \Big[\big(1 + \tfrac{1}{2}\sigma^2 b(\tilde{U}, \tilde{W})^2\big)\tilde{U}_{xx} + \mathcal{R}_U(\tilde{U}, \tilde{W}, \tilde{U}_x, \tilde{W}_x)\Big]dt + \mathcal{S}_U(\tilde{U}, \tilde{W}, \tilde{U}_x, \tilde{W}_x)\, d\beta_t, \\ d\tilde{W} &= \Big[\big(\varrho + \tfrac{1}{2}\sigma^2 b(\tilde{U}, \tilde{W})^2\big)\tilde{W}_{xx} + \mathcal{R}_W(\tilde{U}, \tilde{W}, \tilde{U}_x, \tilde{W}_x)\Big]dt + \mathcal{S}_W(\tilde{U}, \tilde{W}, \tilde{U}_x, \tilde{W}_x)\, d\beta_t \end{aligned} \tag{3.1.4}$$

in which b is a bounded scalar function. For $\sigma \neq 0$ this is a quasi-linear system, but the coefficients in front of the second order derivatives are constant with respect to the spatial variable x. These extra second order terms are a direct consequence of Itô's formula, which shows that second derivatives need to be included when applying the chain rule in a stochastic setting. In particular, deterministic phase-shifts lead to extra convective terms, while stochastic phase-shifts lead to extra diffusive terms.

These extra nonlinear diffusive terms cause short-term regularity issues that prevent a direct analysis of (3.1.4) in a semigroup framework. However, in the special case $\varrho = 1$ they can be transformed away by introducing a new time variable τ that satisfies

$$\tau'(t) = 1 + \frac{1}{2}\sigma^2 b(\tilde{U}, \tilde{W})^2. \tag{3.1.5}$$

This approach was taken in Chapter 2, where we studied reaction-diffusion systems with equal diffusion strengths.

In this chapter, we concentrate on the case $\varrho \neq 1$ and develop a more subtle version of this argument. In fact, we use a similar procedure to scale out the first of the two nonlinear diffusion terms. The remaining nonlinear second order term is only present in the equation for $\tilde{W}$, which allows us to measure its effect on $\tilde{U}$ via the off-diagonal elements of the associated semigroup. The key point is that these off-diagonal elements have better regularity properties than their on-diagonal counterparts, which allows us to side-step the regularity issues outlined above. Indeed, by commuting ∂_x with the semigroup, one can obtain an integral expression for $\tilde{U}$ that only involves $(\tilde{U}, \tilde{W}, \partial_x \tilde{U}, \partial_x \tilde{W})$ and that converges in $L^2(\mathbb{R})$. A second time-transform can be used to obtain similar results for $\tilde{W}$.

A second major complication in our stochastic setting is that $(\partial_x \tilde{U}, \partial_x \tilde{W})$ cannot be directly estimated in $L^2(\mathbb{R})$. Indeed, in order to handle the stochastic integrals we need tools such as the Itô Isometry, which requires square integrability in time. However, squaring the natural $\mathcal{O}(t^{-1/2})$ short-term behavior of the semigroup as measured in $\mathcal{L}(L^2; H^1)$ leads to integrals involving t^{-1} which diverge.

This difficulty was addressed in Chapter 2 by controlling temporal integrals of the H^1-norm. By performing a delicate integration-by-parts procedure one can explicitly isolate the troublesome terms and show that the divergence is in fact 'integrated out'. A similar approach works for our setting here, but the interaction between the separate time-transforms used for $\tilde{U}$ and $\tilde{W}$ requires a careful analysis with some non-trivial modifications.

Outlook Although this chapter relaxes the severe equal-diffusion requirement in Chapter 2, we wish to emphasize that our technical phase-tracking approach is still in a proof-of-concept state. For example, we rely heavily on the diffusive smoothening of the deterministic flow to handle the extra diffusive effects introduced by the stochastic phase shifts. Taking $\varrho = 0$ removes the former but keeps the latter, which makes it unclear at present how to handle such a situation. This is particularly relevant for many neural field models where the diffusion is modeled by convolution kernels rather than the standard Laplacian.

It is also unclear at present if our framework can be generalized to deal with branches of essential spectrum that touch the imaginary axis. This occurs when analyzing planar waves in two or more dimensions [8, 53, 54, 64] or when studying viscous shocks in the context of conservation laws [6, 7, 82]. In the deterministic case these settings require the use of pointwise estimates on Green's functions, which give more refined control on the linear flow than standard semigroup bounds.

We are more confident about the possibility of including more general types of noise in our framework. For instance, we believe that there is no fundamental obstruction to including noise that is colored in space[2], which arises frequently in many applications [27, 72]. In addition, it should also be possible to remove our dependence on the variational framework developed by Liu and Röckner [77]. Indeed, our estimates on the mild solutions appear to be strong enough to allow short-term existence results to be obtained for the original SPDE in the vicinity of the wave.

Organization This chapter is reasonably self-contained and the main narrative can be read independently of Chapter 2. However, we do borrow some results from Chapter 2 that do not depend on the structure of the diffusion matrix. This allows us to focus our attention on the parts that are essentially different.

We formulate our phase-tracking mechanism and state our main results in §3.2. In addition, we illustrate these results in the same section by numerically analyzing an example system of FitzHugh-Nagumo type. In §3.3 we decompose the semigroup associated to the linearization of the deterministic wave into its diagonal and off-diagonal parts. We focus especially on the short-time behavior of the off-diagonal elements and show that the commutator of ∂_x and the semigroup extends to a bounded operator on L^2. In §3.4 we describe the stochastic phase-shifts and time-shifts that are required to eliminate the problematic terms from our equations. We apply the results from §3.3 to recast the resulting SPDE into a mild formulation and establish bounds for the final nonlinearities. This allows us to close a nonlinear stability argument in §3.5 by carefully estimating each of the mild integrals.

3.2 Main results

In this chapter, we are interested in the stability of traveling wave solutions to SPDEs of the form

$$dU = \left[\rho \partial_{xx} U + f(U)\right] dt + \sigma g(U) d\beta_t. \qquad (3.2.1)$$

Here we take $U = U(x,t) \in \mathbb{R}^n$ with $x \in \mathbb{R}$ and $t \geq 0$.

We start by formulating two structural conditions on the deterministic and stochastic parts of (3.2.1). Together these imply that our system has a variational structure with a nonlinearity f that grows at most cubically. In particular, it is covered by the variational framework developed in [77] with $\alpha = 2$. The crucial difference between assumption (HDt) below and assumption (HA) in Chapter 2 is that the diagonal elements of ρ no longer have to be equal.

(HDt) The matrix $\rho \in \mathbb{R}^{n \times n}$ is a diagonal matrix with strictly positive diagonal elements $\{\rho_i\}_{i=1}^n$. In addition, we have $f \in C^3(\mathbb{R}^n; \mathbb{R}^n)$ and there exist $u_\pm \in \mathbb{R}^n$ for which $f(u_-) = f(u_+) = 0$. Finally, $D^3 f$ is bounded and there exists a constant $K_{\text{var}} > 0$ so that the one-sided inequality

$$\langle f(u_A) - f(u_B), u_A - u_B \rangle_{\mathbb{R}^n} \leq K_{\text{var}} \left| u_A - u_B \right|^2 \qquad (3.2.2)$$

[2] See Chapter 4 for results in this direction.

holds for all pairs $(u_A, u_B) \in \mathbb{R}^n \times \mathbb{R}^n$.

(HSt) The function $g \in C^2(\mathbb{R}^n; \mathbb{R}^n)$ is globally Lipschitz with $g(u_-) = g(u_+) = 0$. In addition, Dg is bounded and globally Lipschitz. Finally, the process $(\beta_t)_{t\geq 0}$ is a Brownian motion with respect to the complete filtered probability space

$$\Big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\Big). \tag{3.2.3}$$

We write $\rho_{\min} = \min\{\rho_i\} > 0$, together with $\rho_{\max} = \max\{\rho_i\}$. In addition, we introduce the shorthands

$$L^2 = L^2(\mathbb{R}; \mathbb{R}^n), \quad H^1 = H^1(\mathbb{R}; \mathbb{R}^n), \quad H^2 = H^2(\mathbb{R}; \mathbb{R}^n). \tag{3.2.4}$$

Our final assumption states that the deterministic part of (3.2.1) has a spectrally stable traveling wave solution that connects the two equilibria $u_\pm$ (which are allowed to be equal). This traveling wave should approach these equilibria at an exponential rate.

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(HTw) There exists a wavespeed $c_0 \in \mathbb{R}$ and a waveprofile $\Phi_0 \in C^2(\mathbb{R}; \mathbb{R}^n)$ that satisfies the traveling wave ODE

$$\rho\Phi_0'' + c_0\Phi_0' + f(\Phi_0) = 0 \tag{3.2.5}$$

and approaches its limiting values $\Phi_0(\pm\infty) = u_\pm$ at an exponential rate. In addition, the associated linear operator $\mathcal{L}_{\mathrm{tw}} : H^2 \to L^2$ that acts as

$$[\mathcal{L}_{\mathrm{tw}} v](\xi) = \rho v''(\xi) + c_0 v'(\xi) + Df\big(\Phi_0(\xi)\big)v(\xi), \tag{3.2.6}$$

has a simple eigenvalue at $\lambda = 0$ and has no other spectrum in the half-plane $\{\Re\lambda \geq -2\beta\} \subset \mathbb{C}$ for some $\beta > 0$.

The formal adjoint

$$\mathcal{L}_{\mathrm{tw}}^* : H^2 \to L^2 \tag{3.2.7}$$

of the operator (3.2.6) acts as

$$[\mathcal{L}_{\mathrm{tw}}^* w](\xi) = \rho w''(\xi) - c_0 w'(\xi) + (Df\big(\Phi_0(\xi)\big))^* w(\xi). \tag{3.2.8}$$

Indeed, one easily verifies that

$$\langle \mathcal{L}_{\mathrm{tw}} v, w\rangle_{L^2} = \langle v, \mathcal{L}_{\mathrm{tw}}^* w\rangle_{L^2} \tag{3.2.9}$$

whenever $(v, w) \in H^2 \times H^2$. Here $\langle\cdot,\cdot\rangle_{L^2}$ denotes the standard inner product on L^2. The assumption that zero is a simple eigenvalue for $\mathcal{L}_{\mathrm{tw}}$ implies that $\mathcal{L}_{\mathrm{tw}}^*\psi_{\mathrm{tw}} = 0$ for some $\psi_{\mathrm{tw}} \in H^2$ that we normalize to get

$$\langle \Phi_0', \psi_{\mathrm{tw}}\rangle_{L^2} = 1. \tag{3.2.10}$$

We remark here that it is advantageous to view SPDEs as evolutions on Hilbert spaces, since powerful tools are available in this setting. However, in the case where $u_- \neq u_+$, the waveprofile Φ_0 does not lie in the natural statespace L^2. In order to circumvent this problem, we use Φ_0 as a reference function that connects u_- to u_+, allowing us to measure deviations from this function in the Hilbert spaces H^1 and L^2. In order to highlight this dual role and prevent any confusion, we introduce the duplicate notation

$$\Phi_{\mathrm{ref}} = \Phi_0. \tag{3.2.11}$$

This allows us to introduce the sets

$$\mathcal{U}_{L^2} = \Phi_{\mathrm{ref}} + L^2, \quad \mathcal{U}_{H^1} = \Phi_{\mathrm{ref}} + H^1, \quad \mathcal{U}_{H^2} = \Phi_{\mathrm{ref}} + H^2, \tag{3.2.12}$$

which we will use as the relevant state-spaces to capture the solutions U to (3.2.1).

We now set out to couple an extra phase-tracking[3] SDE to our SPDE (3.2.1). As a preparation, we pick a sufficiently large constant $K_{\mathrm{high}} > 0$ together with two C^∞-smooth non-decreasing cut-off functions

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$$\chi_{\mathrm{low}} : \mathbb{R} \to [\frac{1}{4}, \infty), \quad \chi_{\mathrm{high}} : \mathbb{R} \to [-K_{\mathrm{high}} - 1, K_{\mathrm{high}} + 1] \tag{3.2.13}$$

that satisfy the identities

$$\chi_{\mathrm{low}}(\vartheta) = \frac{1}{4} \text{ for } \vartheta \leq \frac{1}{4}, \quad \chi_{\mathrm{low}}(\vartheta) = \vartheta \text{ for } \vartheta \geq \frac{1}{2}, \tag{3.2.14}$$

together with

$$\chi_{\mathrm{high}}(\vartheta) = \vartheta \text{ for } |\vartheta| \leq K_{\mathrm{high}}, \quad \chi_{\mathrm{high}}(\vartheta) = \mathrm{sign}(\vartheta)\big[K_{\mathrm{high}} + 1\big] \text{ for } |\vartheta| \geq K_{\mathrm{high}} + 1. \tag{3.2.15}$$

For any $u \in \mathcal{U}_{H^1}$ and $\psi \in H^1$, this allows us to introduce the function

$$b(u, \psi) \quad = \quad -\Big[\chi_{\mathrm{low}}\big(\langle \partial_\xi u, \psi \rangle_{L^2}\big)\Big]^{-1} \chi_{\mathrm{high}}\big(\langle g(u), \psi \rangle_{L^2}\big), \tag{3.2.16}$$

together with the diagonal $n \times n$-matrix

$$\kappa_\sigma(u, \psi) \quad = \quad \mathrm{diag}\{\kappa_{\sigma;i}(u, \psi)\}_{i=1}^n := \mathrm{diag}\{1 + \frac{1}{2\rho_i}\sigma^2 b(u, \psi)^2\}_{i=1}^n. \tag{3.2.17}$$

In addition, for any $u \in \mathcal{U}_{H^1}$, $c \in \mathbb{R}$ and $\psi \in H^2$ we write

$$\begin{aligned} a_\sigma(u, c, \psi) \quad = \quad & -\Big[\chi_{\mathrm{low}}\big(\langle \partial_\xi u, \psi \rangle_{L^2}\big)\Big]^{-1} \langle \kappa_\sigma(u, \psi) u, \rho \partial_{\xi\xi} \psi \rangle_{L^2} \\ & -\Big[\chi_{\mathrm{low}}\big(\langle \partial_\xi u, \psi \rangle_{L^2}\big)\Big]^{-1} \langle f(u) + c\partial_\xi u + \sigma^2 b(u, \psi) \partial_\xi [g(u)], \psi \rangle_{L^2}. \end{aligned} \tag{3.2.18}$$

The essential difference with the definitions of κ_σ and a_σ in Chapter 2 is that κ_σ is now a matrix instead of a constant. However, this does not affect the ideas and results in §3-4 and §7 of Chapter 2, which can be transferred to the current setting almost verbatim. Indeed, one simply replaces ρ by $\rho_{\min}$ or $\rho_{\max}$ as necessary.

[3] See §2.4 for a more intuitive explanation of this phase.

The traveling wave ODE (3.2.5) implies that $a_0(\Phi_0, c_0, \psi_{\mathrm{tw}}) = 0$. Following Proposition 2.2.2, one can show that there exists a branch of profiles and speeds (Φ_σ, c_σ) in $\mathcal{U}_{H^2} \times \mathbb{R}$ that is $\mathcal{O}(\sigma^2)$ close to (Φ_0, c_0), for which

$$a_\sigma(\Phi_\sigma, c_\sigma, \psi_{\mathrm{tw}}) = 0. \tag{3.2.19}$$

Upon introducing the right-shift operators

$$[T_\gamma u](\xi) = u(\xi - \gamma) \tag{3.2.20}$$

we can now formally introduce the coupled SPDE

$$\begin{aligned} dU &= \big[\rho \partial_{xx} U + f(U)\big]dt + \sigma g(U) d\beta_t, \\ d\Gamma &= \big[c_\sigma + a_\sigma\big(U, c_\sigma, T_\Gamma \psi_{\mathrm{tw}}\big)\big]dt + \sigma b\big(U, T_\Gamma \psi_{\mathrm{tw}}\big)\, d\beta_t, \end{aligned} \tag{3.2.21}$$

which is the main focus in this chapter. Following the procedure used to establish Proposition 2.2.1, one can show that this SPDE coupled with an initial condition

$$(U, \Gamma)(0) = (u_0, \gamma_0) \in \mathcal{U}_{H^1} \times \mathbb{R} \tag{3.2.22}$$

has solutions[4] $\big(U(t), \Gamma(t)\big) \in \mathcal{U}_{H^1} \times \mathbb{R}$ that can be defined for all $t \geq 0$ and are almost surely continuous as maps into $\mathcal{U}_{L^2} \times \mathbb{R}$.

For any initial condition $u_0 \in \mathcal{U}_{H^1}$ that is sufficiently close to Φ_σ, Proposition 2.2.3 shows that it is possible to pick γ_0 in such a way that

$$\langle T_{-\gamma_0} u_0 - \Phi_\sigma, \psi_{\mathrm{tw}} \rangle_{L^2} = 0. \tag{3.2.23}$$

This allows us to define the process

$$V_{u_0}(t) = T_{-\Gamma(t)}\big[U(t)\big] - \Phi_\sigma, \tag{3.2.24}$$

which can be thought of as the deviation of the solution $U(t)$ of (3.2.21)-(3.2.22) from the stochastic wave Φ_σ shifted to the position $\Gamma(t)$.

In order to measure the size of this deviation we pick $\varepsilon > 0$ and introduce the scalar function

$$N_{\varepsilon;u_0}(t) = \|V_{u_0}(t)\|_{L^2}^2 + \int_0^t e^{-\varepsilon(t-s)} \|V_{u_0}(s)\|_{H^1}^2 \, ds. \tag{3.2.25}$$

For each $T > 0$ and $\eta > 0$ we now define the probability

$$p_\varepsilon(T, \eta, u_0) = P\Big(\sup_{0 \leq t \leq T} N_{\varepsilon;u_0}(t) > \eta \Big). \tag{3.2.26}$$

Our main result shows that the probability that $N_{\varepsilon;u_0}$ remains small on timescales of order σ^{-2} can be pushed arbitrarily close to one by restricting the strength of the noise and the size of the initial perturbation. This extends Theorem 2.2.4 to the current setting where the diffusion matrix ρ need not be proportional to the identity.

Theorem 3.2.1 (see §3.5)**.** *Suppose that (HDt), (HSt) and (HTw) are all satisfied and pick sufficiently small constants $\varepsilon > 0$, $\delta_0 > 0$, $\delta_\eta > 0$ and $\delta_\sigma > 0$. Then there exists a constant $K > 0$ so that for every $0 \leq \sigma \leq \delta_\sigma T^{-1/2}$, any $u_0 \in \mathcal{U}_{H^1}$ that satisfies $\|u_0 - \Phi_\sigma\|_{L^2} < \delta_0$, any $0 < \eta \leq \delta_\eta$ and any $T > 0$, we have the inequality*

$$p_\varepsilon(T, \eta, u_0) \leq \eta^{-1} K \Big[\|u_0 - \Phi_\sigma\|_{H^1}^2 + \sigma^2 T \Big]. \tag{3.2.27}$$

[4] We refer to Proposition 2.2.1 for the precise notion of a solution.

3.2.1 Orbital drift

On account of the theory developed in [75, §12] to describe the suprema of finite-dimensional Gaussian processes, we suspect that the $\sigma^2 T$ term appearing in the bound (3.2.27) can be replaced by $\sigma^2 \ln(T)$. This would allow us to consider timescales of order $\exp[\delta_\sigma/\sigma^2]$, which are exponential in the noise-strength instead of merely polynomial. The key limitation is that the theory of stochastic convolutions in Hilbert spaces is still in the early stages of development.

In order to track the evolution of the phase over such long timescales, we follow Chapter 2 and introduce the formal Ansatz

$$\Gamma(t) = c_\sigma t + \sigma \Gamma_{\sigma;1}(t) + \sigma^2 \Gamma_{\sigma;2}(t) + \mathcal{O}(\sigma^3). \tag{3.2.28}$$

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The first-order term is the scaled Brownian motion

$$\Gamma_{\sigma;1}(t) = b(\Phi_\sigma, \psi_{\mathrm{tw}})\beta_t, \tag{3.2.29}$$

which naturally has zero mean and hence does not contribute to any deviation of the average observed wavespeed.

In order to understand the second order term, we introduce the orbital drift coefficient

$$c_{\sigma;2}^{\mathrm{od}} = \frac{1}{2}\int_0^\infty D_1^2 a_\sigma\big(\Phi_\sigma, c_\sigma, \psi_{\mathrm{tw}}\big)\Big[S(s)\big(g(\Phi_\sigma) + b(\Phi_\sigma, \psi_{\mathrm{tw}})\Phi_\sigma'\big)\Big]^2 ds, \tag{3.2.30}$$

in which $\{S(s)\}_{s\geq 0}$ denotes the semigroup generated by $\mathcal{L}_{\mathrm{tw}}$. In §2.4 we gave an explicit expression for $\Gamma_{\sigma;2}$ and showed that

$$\lim_{t\to\infty} t^{-1} E\Gamma_{\sigma;2}(t) = c_{\sigma;2}^{\mathrm{od}}. \tag{3.2.31}$$

Note that we are keeping the σ-dependence in these definitions for notational convenience, but in §3.2.2 we show how the leading order contribution can be determined.

The discussion above suggests that it is natural to introduce the expression

$$c_{\sigma;\mathrm{lim}}^{(2)} = c_\sigma + \sigma^2 c_{\sigma;2}^{\mathrm{od}}, \tag{3.2.32}$$

which satisfies $c_{\sigma;\mathrm{lim}}^{(2)} - c_0 = \mathcal{O}(\sigma^2)$. Our conjecture is that the expected value of the wavespeed for large times behaves as $c_{\sigma;\mathrm{lim}}^{(2)} + \mathcal{O}(\sigma^3)$. In order to interpret this, we note that the profile Φ_σ travels at an instantaneous velocity c_σ, but also experiences stochastic forcing. As a consequence of this forcing, which is mean reverting toward Φ_σ, the profile fluctuates in the orbital vicinity of Φ_σ. At leading order, the underlying mechanism behind this behavior resembles an Ornstein-Uhlenbeck process, which means that the amplitude of these fluctuations can be expected to stabilize for large times. This leads to an extra contribution to the observed wavespeed, which we refer to as an orbital drift. The second term in (3.2.32) describes the leading order contribution to this orbital drift.

3.2.2 Example

In order to illustrate our results, let us consider the FitzHugh-Nagumo system

$$\begin{aligned} dU &= \big[U_{xx} + f_{\mathrm{cub}}(U) - W\big]dt + \sigma g^{(u)}(U)d\beta_t, \\ dW &= \big[\varrho W_{xx} + \varepsilon(U - \gamma W)\big]dt \end{aligned} \tag{3.2.33}$$

in a parameter regime where (HDt), (HSt) and (HTw) all hold. We write $\Phi_0 = (\Phi_0^{(u)}, \Phi_0^{(w)})$ for the deterministic wave defined in (HTw) and recall the associated linear operator $\mathcal{L}_{\mathrm{tw}} : H^2(\mathbb{R};\mathbb{R}^2) \to L^2(\mathbb{R};\mathbb{R}^2)$ that acts as

$$\mathcal{L}_{\mathrm{tw}} = \begin{pmatrix} \partial_{\xi\xi} + c_0\partial_\xi + f'_{\mathrm{cub}}(\Phi_0^{(u)}) & -1 \\ \varepsilon & \varrho\partial_{\xi\xi} + c_0\partial_\xi - \varepsilon\gamma \end{pmatrix}. \tag{3.2.34}$$

The adjoint operator acts as

$$\mathcal{L}^*_{\mathrm{tw}} = \begin{pmatrix} \partial_{\xi\xi} - c_0\partial_\xi + f'_{\mathrm{cub}}(\Phi_0^{(u)}) & \varepsilon \\ -1 & \varrho\partial_{\xi\xi} - c_0\partial_\xi - \varepsilon\gamma \end{pmatrix} \tag{3.2.35}$$

and admits the eigenfunction $\psi_{\mathrm{tw}} = (\psi_{\mathrm{tw}}^{(u)}, \psi_{\mathrm{tw}}^{(w)})$ that can be normalized in such a way that

$$\langle \partial_\xi \Phi_0, \psi_{\mathrm{tw}} \rangle_{L^2(\mathbb{R};\mathbb{R}^2)} = 1. \tag{3.2.36}$$

To summarize, we have

$$\mathcal{L}_{\mathrm{tw}}\partial_\xi(\Phi_0^{(u)}, \Phi_0^{(w)})^T = 0, \quad \mathcal{L}^*_{\mathrm{tw}}(\psi_{\mathrm{tw}}^{(u)}, \psi_{\mathrm{tw}}^{(w)})^T = 0. \tag{3.2.37}$$

Upon writing $\Phi_\sigma = (\Phi_\sigma^{(u)}, \Phi_\sigma^{(w)})$, the stochastic wave equation $a_\sigma(\Phi_\sigma, c_\sigma, \psi_{\mathrm{tw}}) = 0$ can be written as

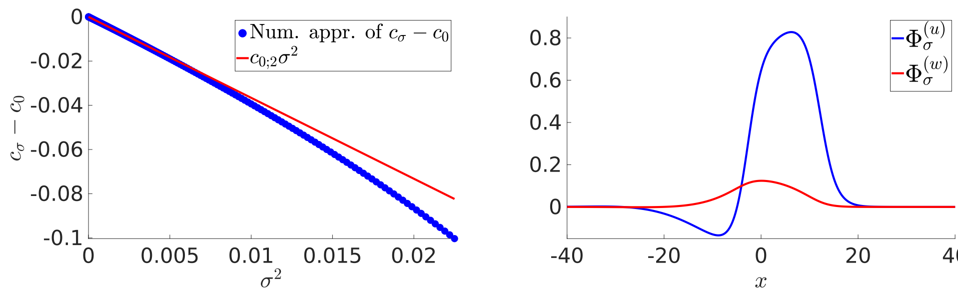
$$\begin{aligned} -c_\sigma\partial_\xi\Phi_\sigma^{(u)} &= \Big(1 + \frac{\sigma^2}{2}\tilde{b}(\Phi_\sigma)^2\Big)\partial_{\xi\xi}\Phi_\sigma^{(u)} + f_{\mathrm{cub}}(\Phi_\sigma^{(u)}) - \Phi_\sigma^{(w)} + \sigma^2\tilde{b}(\Phi_\sigma)\partial_\xi[g^{(u)}(\Phi_\sigma^{(u)})], \\ -c_\sigma\partial_\xi\Phi_\sigma^{(w)} &= \Big(\varrho + \frac{\sigma^2}{2}\tilde{b}(\Phi_\sigma)^2\Big)\partial_{\xi\xi}\Phi_\sigma^{(w)} + \varepsilon(\Phi_\sigma^{(u)} - \gamma\Phi_\sigma^{(w)}), \end{aligned} \tag{3.2.38}$$

where $\tilde{b}$ is given by

$$\tilde{b}(\Phi_\sigma) = -\frac{\langle g^{(u)}(\Phi_\sigma^{(u)}), \psi_{\mathrm{tw}}^{(u)} \rangle_{L^2(\mathbb{R};\mathbb{R})}}{\langle \partial_\xi\Phi_\sigma, \psi_{\mathrm{tw}} \rangle_{L^2(\mathbb{R};\mathbb{R}^2)}}. \tag{3.2.39}$$

We now introduce the expansions

$$\Phi_\sigma = \Phi_0 + \sigma^2\Phi_{0;2} + \mathcal{O}(\sigma^4), \quad c_\sigma = c_0 + \sigma^2 c_{0;2} + \mathcal{O}(\sigma^4) \tag{3.2.40}$$



(a) (b)

Figure 3.1: Numerical results for the solution (Φ_σ, c_σ) to equation (3.2.38). Figure (a) shows the numerical approximation of $c_\sigma - c_0$ and the first order approximation of this difference. We chose $g^{(u)}(u) = u$ with parameters $a = 0.1$, $\varrho = 0.01$, $\varepsilon = 0.01$, $\gamma = 5$. Using (3.2.43) we numerically computed $c_{0;2} = -3.66$. Figure (b) shows the two components of Φ_σ for $\sigma = 0.15$ for the same parameter values. On the scale of this figure they are almost identical to Φ_0.

with $\Phi_{0;2} = \big(\Phi_{0;2}^{(u)}, \Phi_{0;2}^{(w)}\big)$. Substituting these expressions into (3.2.38) and balancing the second order terms, we find

$$\begin{aligned} -c_{0;2}\partial_\xi \Phi_0^{(u)} - c_0 \partial_\xi \Phi_{0;2}^{(u)} &= \partial_{\xi\xi}\Phi_{0;2}^{(u)} + \frac{1}{2}\tilde{b}(\Phi_0)^2 \partial_{\xi\xi}\Phi_0^{(u)} + f'_{\mathrm{cub}}(\Phi_0^{(u)})\Phi_{0;2}^{(u)} - \Phi_{0;2}^{(w)} \\ &\quad + \tilde{b}(\Phi_0)\partial_\xi g^{(u)}(\Phi_0^{(u)}), \\ -c_{0;2}\partial_\xi \Phi_0^{(w)} - c_0 \partial_\xi \Phi_{0;2}^{(w)} &= \varrho \partial_{\xi\xi}\Phi_{0;2}^{(w)} + \frac{1}{2}\tilde{b}(\Phi_0)^2 \partial_{\xi\xi}\Phi_0^{(w)} + \varepsilon(\Phi_{0;2}^{(u)} - \gamma \Phi_{0;2}^{(w)}), \end{aligned} \tag{3.2.41}$$

which can be rephrased as

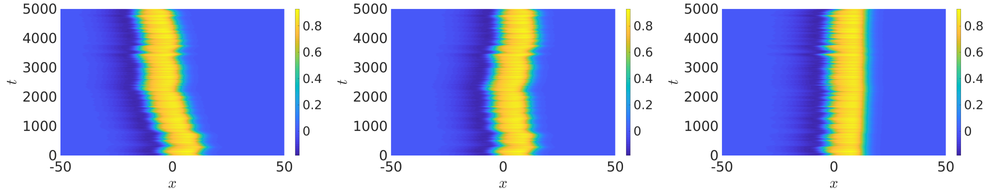
$$\mathcal{L}_{\mathrm{tw}}\Phi_{0;2} = -c_{0;2}\partial_\xi \Phi_0 - \frac{1}{2}\tilde{b}(\Phi_0)^2 \partial_{\xi\xi}\Phi_0 - \tilde{b}(\Phi_0)\big(\partial_\xi g^{(u)}(\Phi_0^{(u)}), 0\big)^T. \tag{3.2.42}$$

Using the normalization (3.2.36) together with the fact that $\langle \mathcal{L}_{\mathrm{tw}}\Phi_{0;2}, \psi_{\mathrm{tw}} \rangle_{L^2(\mathbb{R};\mathbb{R}^2)} = 0$, we find the explicit expression

$$c_{0;2} = -\frac{1}{2}\tilde{b}(\Phi_0)^2 \langle \partial_{\xi\xi}\Phi_0, \psi_{\mathrm{tw}} \rangle_{L^2(\mathbb{R};\mathbb{R}^2)} - \tilde{b}(\Phi_0)\langle \partial_\xi g^{(u)}(\Phi_0^{(u)}), \psi_{\mathrm{tw}}^{(u)} \rangle_{L^2(\mathbb{R};\mathbb{R})} \tag{3.2.43}$$

for the coefficient that governs the leading order behavior of $c_\sigma - c_0$. In Figure 3.1 we show numerically that $c_{0;2}\sigma^2$ indeed corresponds well with $c_\sigma - c_0$ for small values of σ^2.

In Figure 3.2 we illustrate the behavior of a representative sample solution to (3.2.33) by plotting it in three different moving frames. Figure 3.2a clearly shows that the deterministic speed c_0 overestimates the actual speed as the wave moves to the left. The situation is improved in Figure 3.2b, where we use a frame that travels with the stochastic speed c_σ. However, the position of the wave now fluctuates around a position that still moves slowly to the left as a consequence of the orbital drift. This is



(a) $U(\cdot + c_0 t, t)$ (b) $U(\cdot + c_\sigma t, t)$ (c) $U(\cdot + \Gamma(t), t)$

Figure 3.2: A single realization of the U-component of (3.2.33) with initial condition Φ_σ in 3 different reference frames. We chose $g^{(u)}(u) = u$ with parameters $a = 0.1$, $\sigma = 0.03$, $\varrho = 0.01$, $\varepsilon = 0.01$, $\gamma = 5$.

remedied in Figure 3.2c where we use the full stochastic phase $\Gamma(t)$. Indeed, the wave now appears to be at a fixed position, but naturally still experiences fluctuations in its shape. This shows that $\Gamma(t)$ is indeed a powerful tool to characterize the position of the wave.

In order to study the orbital drift mentioned above, we split the semigroup $S(t)$ generated by $\mathcal{L}_{\mathrm{tw}}$ into its components

$$S(t) = \begin{pmatrix} S^{(uu)}(t) & S^{(uw)}(t) \\ S^{(wu)}(t) & S^{(ww)}(t) \end{pmatrix} \tag{3.2.44}$$

and introduce the expression

$$\mathcal{I}(s) \quad = \quad S^{(uu)}(s) g^{(u)}(\Phi_0) + \tilde{b}(\Phi_0) S^{(uu)} \partial_\xi \Phi_0^{(u)} + \tilde{b}(\Phi_0) S^{(uw)} \partial_\xi \Phi_0^{(w)}, \tag{3.2.45}$$

together with

$$c_{0;2}^{\mathrm{od}} = -\frac{1}{2} \int_0^\infty \langle f''_{\mathrm{cub}}(\Phi_0^{(u)}) \mathcal{I}(s)^2, \psi_{\mathrm{tw}}^{(u)} \rangle_{L^2(\mathbb{R})} \, ds. \tag{3.2.46}$$

This last quantity is in fact the leading order term in the Taylor expansion of (3.2.30), which means that

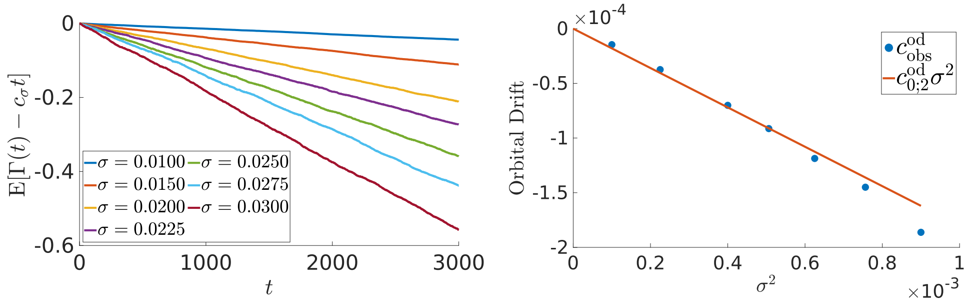
$$c_{\sigma;2}^{\mathrm{od}} = c_{0;2}^{\mathrm{od}} + \mathcal{O}(\sigma^2). \tag{3.2.47}$$

In particular, we see that

$$c_{\sigma;\lim}^{(2)} = c_0 + \sigma^2 \left[c_{0;2} + c_{0;2}^{\mathrm{od}} \right] + \mathcal{O}(\sigma^3), \tag{3.2.48}$$

which means that we have explicitly identified the leading order correction to the full limiting wavespeed.

To validate our prediction for the size of the orbital drift, we first approximated $E[\Gamma(t) - c_\sigma t]$ numerically by performing an average over a set of numerical simulations. In fact, to speed up the convergence rate, we first subtracted the term $\Gamma_{\sigma;1}(t)$ defined in (3.2.29) from each simulation, using the same realization of the Brownian motion that was used to generate the path for (U, W). The results can be found in Figure 3.3a.



(a) (b)

Figure 3.3: In (a) we computed the average $E[\Gamma(t) - c_\sigma t]$ over 1000 simulations of (3.2.33), using the procedure described in the main text for several values of σ. Notice that a clear trend is visible. In (b) we computed the corresponding orbital drift by evaluating the average (3.2.49) for the data in (a). Observe that there is a reasonable match with the predicted values $c^{\mathrm{od}}_{0;2}\sigma^2$. We chose $g^{(u)}(u) = u$ with parameters $a = 0.1$, $\varrho = 0.01$, $\varepsilon = 0.01$, $\gamma = 5$. We used the value $c^{\mathrm{od}}_{0;2} = -0.18$, which was found by evaluating (3.2.46) numerically.

In order to eliminate any transients from the data, we subsequently numerically computed the quantity

$$c^{\mathrm{od}}_{\mathrm{obs}} = \frac{2}{T}\int_{\frac{T}{2}}^{T} \frac{1}{t} E[\Gamma(t) - c_\sigma t]dt. \tag{3.2.49}$$

This corresponds with the average slope of the data in Figure 3.3a on the interval $[T/2, T]$, which is a useful proxy for the observed orbital drift. Figure 3.3b shows that these quantities are well-approximated by our leading order expression $\sigma^2 c^{\mathrm{od}}_{0;2}$.

3.3 Structure of the semigroup

In this section we analyze the analytic semigroup $S(t)$ generated by the linear operator $\mathcal{L}_{\mathrm{tw}}$, focusing specially on its off-diagonal elements. Assumption (HTw) implies that $\mathcal{L}_{\mathrm{tw}}$ has a spectral gap, which is essential for our computations. In order to exploit this, we introduce the maps $P : L^2 \to L^2$ and $Q : L^2 \to L^2$ that act as

$$Pv = \langle v, \psi_{\mathrm{tw}}\rangle_{L^2}\Phi_0', \quad Qv = v - Pv. \tag{3.3.1}$$

We also introduce the suggestive notation $P_\xi \in \mathcal{L}(L^2; L^2)$ to refer to the map

$$P_\xi v = -\langle v, \partial_\xi \psi_{\mathrm{tw}}\rangle_{L^2}\Phi_0', \tag{3.3.2}$$

noting that $P_\xi v = P\partial_\xi v$ whenever $v \in H^1$. These projections enable us to remove the simple eigenvalue at the origin and obtain the following bounds.

Lemma 3.3.1 (see [80]). *Assume that (HDt) and (HTw) hold. Then $\mathcal{L}_{\mathrm{tw}}$ generates an analytic semigroup $S(t)$ and there exists a constant $M \geq 1$ for which we have the bounds*

$$\begin{array}{llll} \|S(t)Q\|_{\mathcal{L}(L^2;L^2)} & \leq & Me^{-\beta t}, & 0 < t < \infty, \\ \|S(t)Q\|_{\mathcal{L}(L^2;H^1)} & \leq & Mt^{-\frac{1}{2}}, & 0 < t \leq 2, \\ \|S(t)P\|_{\mathcal{L}(L^2;H^2)} + \|S(t)P_\xi\|_{\mathcal{L}(L^2;H^2)} + \|S(t)\partial_\xi P\|_{\mathcal{L}(L^2;H^2)} & \leq & M, & 0 < t \leq 2, \\ \|S(t)Q\|_{\mathcal{L}(L^2;H^2)} & \leq & Me^{-\beta t}, & t \geq 1, \\ \|[\mathcal{L}_{\mathrm{tw}} - \rho\partial_{\xi\xi}]S(t)Q\|_{\mathcal{L}(L^2;L^2)} & \leq & Mt^{-\frac{1}{2}}, & 0 < t \leq 2, \\ \|[\mathcal{L}^*_{\mathrm{tw}} - \rho\partial_{\xi\xi}]S(t)Q\|_{\mathcal{L}(L^2;L^2)} & \leq & Mt^{-\frac{1}{2}}, & 0 < t \leq 2. \end{array} \tag{3.3.3}$$

Proof. Since $\rho\partial_{\xi\xi}$ generates n independent heat-semigroups, the analyticity of the semigroup $S(t)$ can be obtained from [80, Prop. 4.1.4]; see also Proposition 2.6.3.vi. The desired bounds follow from [80, Prop. 5.2.1] together with the fact that $\Phi_0' \in H^3$. □

In §3.4 we will show that the function $V(t)$ defined in (3.2.24) satisfies an SPDE that involves nonlinear terms containing second order derivatives. The short-term bounds above are too crude to handle such terms as they lead to divergences in the integrals governing short-time regularity. In addition, the variational framework in [77] only provides control on the H^1-norm of V.

In order to circumvent the first issue, we introduce the representation

$$S(t)v = \begin{pmatrix} S_{11}(t) & \dots & S_{1n}(t) \\ \vdots & \ddots & \vdots \\ S_{n1}(t) & \dots & S_{nn}(t) \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \tag{3.3.4}$$

with operators $S_{ij}(t) \in \mathcal{L}\big(L^2(\mathbb{R};\mathbb{R}); L^2(\mathbb{R};\mathbb{R})\big)$. Upon writing

$$S_{\mathrm{d}}(t) = \mathrm{diag}\big(S_{11}(t), \dots, S_{nn}(t)\big), \tag{3.3.5}$$

this allows us to make the splitting

$$S(t) = S_{\mathrm{d}}(t) + S_{\mathrm{od}}(t). \tag{3.3.6}$$

Our main result below shows that the off-diagonal terms $S_{\mathrm{od}}(t)$ have better short-term bounds than the original semigroup.

The second issue can be addressed by introducing the commutator

$$\Lambda(t) = [S(t)Q, \partial_\xi] = S(t)Q\partial_\xi - \partial_\xi S(t)Q \tag{3.3.7}$$

that initially acts on H^1. In fact, we show that this commutator can be extended to L^2 in a natural fashion and that it has better short-time bounds than $S(t)$. Upon writing

$$S(t)\partial_\xi v = S(t)Q\partial_\xi v + S(t)P_\xi v = \partial_\xi S(t)Qv + \Lambda(t)v + S(t)P_\xi v, \tag{3.3.8}$$

we hence see that the right-hand side of this identity is well-defined for $v \in L^2$. In §3.4 this observation will allow us to give a mild interpretation of the SPDE satisfied by $V(t)$ posed on the space H^1.

Proposition 3.3.2. *Suppose that (HDt) and (HTw) are satisfied. Then the operator $\Lambda(t)$ can be extended to L^2 for each $t \geq 0$. In addition, there is a constant $M > 0$ so that the short-term bound*

$$\|\Lambda(t)\|_{L^2 \to H^2} + \|S_{\mathrm{od}}(t)\|_{L^2 \to H^2} \leq M \tag{3.3.9}$$

holds for $0 < t \leq 1$, while the long-term bound

$$\|\Lambda(t)\|_{L^2 \to H^2} \leq M e^{-\beta t} \tag{3.3.10}$$

holds for $t \geq 1$.

3

3.3.1 Functional calculus

For any linear operator $\mathcal{L} : H^2 \to L^2$ we introduce the notation

$$R(\mathcal{L}, \lambda) = [\lambda - \mathcal{L}]^{-1} \tag{3.3.11}$$

for any λ in the resolvent set of $\mathcal{L}$. On account of (HTw) and the sectoriality of $\mathcal{L}_{\mathrm{tw}}$, we can find $\eta_+ \in (\frac{\pi}{2}, \pi)$ and $M > 0$ so that the sector

$$\Omega_{\mathrm{tw}} = \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg(\lambda)| < \eta_+\} \tag{3.3.12}$$

lies entirely in the resolvent set of $\mathcal{L}_{\mathrm{tw}}$, with

$$\|R(\mathcal{L}_{\mathrm{tw}}, \lambda)\|_{L^2 \to L^2} \leq \frac{M}{|\lambda|} \tag{3.3.13}$$

for all $\lambda \in \Omega_{\mathrm{tw}}$. Since $\lambda = 0$ is a simple eigenvalue for $\mathcal{L}_{\mathrm{tw}}$, we have the limit

$$\lambda R(\mathcal{L}_{\mathrm{tw}}, \lambda) \to P \tag{3.3.14}$$

as $\lambda \to 0$.

For any $r > 0$ and any $\eta \in (\frac{\pi}{2}, \eta_+)$, the curve given by

$$\gamma_{r,\eta} = \{\lambda \in \mathbb{C} : |\arg\lambda| = \eta, |\lambda| > r\} \cup \{\lambda \in \mathbb{C} : |\arg\lambda| \leq \eta, |\lambda| = r\} \tag{3.3.15}$$

lies entirely in Ω_{tw}. This curve can be used [80, (1.10)] to represent the semigroup S in the integral form

$$S(t) = \frac{1}{2\pi i} \int_{\gamma_{r,\eta}} e^{t\lambda} R(\mathcal{L}_{\mathrm{tw}}, \lambda) \, d\lambda \tag{3.3.16}$$

for any $t > 0$, where $\gamma_{r,\eta}$ is traversed in the upward direction.

We will analyze $\Lambda(t)$ and $S_{\mathrm{od}}(t)$ by manipulating this integral. As a preparation, we state two technical results concerning the convergence of contour integrals that are similar to (3.3.16). We note that our computations here are based rather directly on [80, §1.3].

Lemma 3.3.3. *Suppose that (HDt) and (HTw) are satisfied and pick $r > 0$ together with $\eta \in (\frac{\pi}{2}, \eta_+)$. Suppose furthermore that $\lambda \mapsto K(\lambda) \in \mathbb{C}$ is an analytic function on the resolvent set of $\mathcal{L}_{\mathrm{tw}}$ and that there exist constants $C > 0$ and $\vartheta \geq 1$ so that the estimate*

$$|K(\lambda)| \leq \frac{C}{|\lambda|^{\vartheta}} \tag{3.3.17}$$

holds for all $\lambda \in \Omega_{\mathrm{tw}}$. Then there exists $C_1 > 0$ so that

$$\left| \int_{\gamma_{r,\eta}} e^{\lambda t} K(\lambda)\, d\lambda \right| \leq C_1 t^{\vartheta - 1} \tag{3.3.18}$$

for all $t > 0$.

Proof. Writing

$$\mathcal{I}(t) = \int_{\gamma_{r,\eta}} e^{\lambda t} K(\lambda)\, d\lambda \tag{3.3.19}$$

and substituting $\lambda t = \xi$, the analyticity of K on Ω_{tw} implies

$$\mathcal{I}(t) = \int_{\gamma_{rt,\eta}} e^{\xi} K\left(\frac{\xi}{t}\right) \frac{1}{t}\, d\xi = \int_{\gamma_{r,\eta}} e^{\xi} K\left(\frac{\xi}{t}\right) \frac{1}{t}\, d\xi. \tag{3.3.20}$$

Using the obvious parametrization for $\gamma_{r,\eta}$, we find

$$\begin{aligned} \mathcal{I}(t) \quad = \quad & - \int_r^{\infty} e^{(\rho \cos(\eta) - i\rho \sin(\eta))} K\big(t^{-1} \rho e^{-i\eta}\big) e^{-i\eta} t^{-1} d\rho \\ & + \int_{\eta}^{\eta} e^{(r \cos(\alpha) - ir \sin(\alpha))} K\big(t^{-1} r e^{i\alpha}\big) i r e^{i\alpha} t^{-1} d\alpha \\ & + \int_r^{\infty} e^{(\rho \cos(\eta) - i\rho \sin(\eta))} K\big(t^{-1} \rho e^{i\eta}\big) e^{i\eta} t^{-1} d\rho. \end{aligned} \tag{3.3.21}$$

We hence obtain the desired estimate

$$\begin{aligned} |\mathcal{I}(t)| \quad & \leq \quad C t^{\vartheta - 1} \left(2 \int_r^{\infty} e^{\rho \cos(\eta)} \rho^{-\vartheta}\, d\rho + \int_{\eta}^{\eta} e^{r \cos(\alpha)} r^{1-\vartheta}\, d\alpha \right) \\ & := \quad C_1 t^{\vartheta - 1}. \end{aligned} \tag{3.3.22}$$

□

Lemma 3.3.4. *Suppose that (HDt) and (HTw) are satisfied and pick $r > 0$ together with $\eta \in (\frac{\pi}{2}, \eta_+)$. Suppose furthermore that $\lambda \mapsto K(\lambda)$ is an analytic function on the resolvent set of $\mathcal{L}_{\mathrm{tw}}$ and that there exists a constant $C > 0$ so that the estimate*

$$|K(\lambda)| \leq C \tag{3.3.23}$$

holds for all $\lambda \in \Omega_{\mathrm{tw}}$. Then there exists $C_2 > 0$ so that the bound

$$\left| \int_{\gamma_{r,\eta}} e^{\lambda t} K(\lambda)\, d\lambda \right| \leq C_2 e^{-\beta t} \tag{3.3.24}$$

holds for all $t \geq 1$.

Proof. Since K remains bounded for $\lambda \to 0$, this function can be analytically extended to a neighborhood of $\lambda = 0$. We can hence replace the curve $\gamma_{r,\eta}$ by the two half-lines

$$\tilde{\gamma}_{\eta'} = -\beta + \{\lambda \in \mathbb{C} : |\mathrm{arg}\lambda| = \eta'\} \tag{3.3.25}$$

for appropriate $\eta' \in (\frac{\pi}{2}, \eta_+)$. We can then compute

$$\begin{aligned} \left| \int_{\tilde{\gamma}_{\eta'}} e^{\lambda t} K(\lambda)\, d\lambda \right| &\leq 2Ce^{-\beta t} \int_0^\infty e^{\rho \cos(\eta')t}\, d\rho \\ &\leq 2Ce^{-\beta t} \int_0^\infty e^{\rho \cos(\eta')}\, d\rho \\ &:= C_2 e^{-\beta t}. \end{aligned} \tag{3.3.26}$$

□

3.3.2 The commutator $\Lambda(t)$

In this section we analyze $\Lambda(t)$ and establish the statements in Proposition 3.3.2 that concern this commutator. Based on the identity (3.3.16), we first set out to compute the commutator of $R(\mathcal{L}_{\mathrm{tw}}, \lambda)$ and ∂_ξ. As a preparation, we introduce the commutator

$$B = [\mathcal{L}_{\mathrm{tw}} Q, \partial_\xi] = [\mathcal{L}_{\mathrm{tw}}, \partial_\xi], \tag{3.3.27}$$

which can easily be seen to act as

$$Bv = -D^2 f(\Phi_0)\Phi_0' v \tag{3.3.28}$$

for any $v \in H^3$.

Lemma 3.3.5. *Suppose that (HDt) and (HTw) are satisfied and pick any λ in the resolvent set of $\mathcal{L}_{\mathrm{tw}}$. Then for any $g \in H^1$ we have the identity*

$$\begin{aligned} [R(\mathcal{L}_{\mathrm{tw}}, \lambda)Q, \partial_\xi]g &= R(\mathcal{L}_{\mathrm{tw}}, \lambda)Q\partial_\xi g - \partial_\xi R(\mathcal{L}_{\mathrm{tw}}, \lambda)Qg \\ &= R(\mathcal{L}_{\mathrm{tw}}, \lambda)\Big[BR(\mathcal{L}_{\mathrm{tw}}, \lambda)Qg - [P, \partial_\xi]g\Big]. \end{aligned} \tag{3.3.29}$$

Proof. Let us first write

$$v = [\lambda - \mathcal{L}_{\mathrm{tw}}]^{-1} Qg. \tag{3.3.30}$$

The definition (3.3.27) implies that

$$\begin{aligned} [\lambda - \mathcal{L}_{\mathrm{tw}}]Q\partial_\xi v &= \partial_\xi[\lambda - \mathcal{L}_{\mathrm{tw}}]Qv - Bv + \lambda[Q, \partial_\xi]v \\ &= \partial_\xi[\lambda - \mathcal{L}_{\mathrm{tw}}]v - \partial_\xi \lambda(I - Q)v - Bv + \lambda[Q, \partial_\xi]v \\ &= \partial_\xi[\lambda - \mathcal{L}_{\mathrm{tw}}]v - \lambda\partial_\xi Pv - Bv - \lambda[P, \partial_\xi]v \\ &= \partial_\xi Qg - Bv - \lambda P\partial_\xi v \\ &= Q\partial_\xi g - [P, \partial_\xi]g - Bv - \lambda P\partial_\xi v. \end{aligned} \tag{3.3.31}$$

Using $(\lambda - \mathcal{L}_{\mathrm{tw}})^{-1} P = \lambda^{-1} P$ we obtain

$$\begin{array}{rcl} [\lambda - \mathcal{L}_{\mathrm{tw}}]^{-1} Q \partial_\xi g & = & Q \partial_\xi v + [\lambda - \mathcal{L}_{\mathrm{tw}}]^{-1} B v + P \partial_\xi v + [\lambda - \mathcal{L}_{\mathrm{tw}}]^{-1} [P, \partial_\xi] g \\ & = & \partial_\xi [\lambda - \mathcal{L}_{\mathrm{tw}}]^{-1} Q g + [\lambda - \mathcal{L}_{\mathrm{tw}}]^{-1} B [\lambda - \mathcal{L}_{\mathrm{tw}}]^{-1} Q g \\ & & \quad + [\lambda - \mathcal{L}_{\mathrm{tw}}]^{-1} [P, \partial_\xi] g, \end{array} \tag{3.3.32}$$

which can be reordered to yield (3.3.29). □

On account of (3.3.29) we recall the definition (3.3.2) and introduce the operator $T_A \in \mathcal{L}(L^2; L^2)$ that acts as

$$T_A = \partial_\xi P - P_\xi. \tag{3.3.33}$$

In addition, we introduce the expression

$$T_B(\lambda) = B R(\mathcal{L}_{\mathrm{tw}}, \lambda) Q, \tag{3.3.34}$$

which is well-behaved in the following sense.

Lemma 3.3.6. *Suppose that (HDt) and (HTw) are satisfied. Then there exists a constant $C > 0$ so that for any λ in the resolvent set of $\mathcal{L}_{\mathrm{tw}}$ the operator $T_B(\lambda)$ satisfies the bound*

$$\| T_B(\lambda) \|_{L^2 \to L^2} \le \frac{C}{1 + |\lambda|}. \tag{3.3.35}$$

In additions, the maps

$$\lambda \mapsto T_B(\lambda) \in \mathcal{L}(L^2; L^2), \quad \lambda \mapsto \lambda^{-1} P \big[T_A + T_B(\lambda) \big] \in \mathcal{L}(L^2; L^2) \tag{3.3.36}$$

can be continued analytically into the origin $\lambda = 0$.

Proof. Since Φ_0 and Φ_0' are bounded functions, we have

$$\| B R(\mathcal{L}_{\mathrm{tw}}, \lambda) \|_{L^2 \to L^2} \le \frac{M}{|\lambda|} \| D^2 f(\Phi_0) \Phi_0' \|_\infty. \tag{3.3.37}$$

Using $P \mathcal{L}_{\mathrm{tw}} = 0$ and the resolvent identity

$$\mathcal{L}_{\mathrm{tw}} R(\mathcal{L}_{\mathrm{tw}}, \lambda) = -I + \lambda R(\mathcal{L}_{\mathrm{tw}}, \lambda), \tag{3.3.38}$$

we may compute

$$\begin{array}{rcl} P \big[T_A + T_B(\lambda) \big] & = & P_\xi P - P_\xi + P B R(\mathcal{L}_{\mathrm{tw}}, \lambda) Q \\ & = & P_\xi P - P_\xi + P \mathcal{L}_{\mathrm{tw}} \partial_\xi R(\mathcal{L}_{\mathrm{tw}}, \lambda) Q - P \partial_\xi \mathcal{L}_{\mathrm{tw}} R(\mathcal{L}_{\mathrm{tw}}, \lambda) Q \\ & = & P_\xi P - P_\xi + P_\xi Q - P \partial_\xi \lambda R(\mathcal{L}_{\mathrm{tw}}, \lambda) Q \\ & = & -P \partial_\xi \lambda R(\mathcal{L}_{\mathrm{tw}}, \lambda) Q. \end{array} \tag{3.3.39}$$

Since $\lambda \mapsto R(\mathcal{L}_{\mathrm{tw}}, \lambda) Q$ can be analytically continued to $\lambda = 0$ on account of (3.3.14), the same hence holds for the functions (3.3.36). □

Upon fixing $r > 0$ and $\eta \in (\frac{\pi}{2}, \eta_+)$, we now introduce the expressions

$$\begin{aligned}\Lambda_{\mathrm{ex};A}(t) &= \frac{1}{2\pi i}\int_{\gamma_{r,\eta}} e^{\lambda t} R(\mathcal{L}_{\mathrm{tw}}, \lambda) T_A \, d\lambda, \\ \Lambda_{\mathrm{ex};B}(t) &= \frac{1}{2\pi i}\int_{\gamma_{r,\eta}} e^{\lambda t} R(\mathcal{L}_{\mathrm{tw}}, \lambda) T_B(\lambda) \, d\lambda\end{aligned} \tag{3.3.40}$$

and write

$$\Lambda_{\mathrm{ex}}(t) = \Lambda_{\mathrm{ex};A}(t) + \Lambda_{\mathrm{ex};B}(t). \tag{3.3.41}$$

We note that

$$\Lambda_{\mathrm{ex};A}(t) = S(t)T_A = S(t)\partial_\xi P - S(t)P_\xi, \tag{3.3.42}$$

which for $0 < t \le 1$ is covered by the bounds in Lemma 3.3.1. The results below show that $\Lambda_{\mathrm{ex}}(t)$ is well-defined as an operator in $\mathcal{L}(L^2; H^2)$ and that it is indeed an extension of the commutator $\Lambda(t)$.

Lemma 3.3.7. *Suppose that (HDt) and (HTw) are satisfied. Then $\Lambda_{\mathrm{ex}}(t)$ is a well-defined operator in $\mathcal{L}(L^2; H^2)$ for all $t > 0$ that does not depend on $r > 0$ and $\eta \in (\frac{\pi}{2}, \eta_+)$. In addition, there exists a constant $C > 0$ so that the bound*

$$\|\Lambda_{\mathrm{ex}}(t)\|_{L^2 \to H^2} \le Ce^{-\beta t} \tag{3.3.43}$$

holds for all $t > 0$.

Proof. Note first that there exists a constant $C_1' > 0$ for which

$$\|v\|_{H^2} \le C_1' \big[\|\mathcal{L}_{\mathrm{tw}} v\|_{L^2} + \|v\|_{L^2} \big] \tag{3.3.44}$$

holds for all $v \in H^2$. On account of the identity

$$\mathcal{L}_{\mathrm{tw}} R(\mathcal{L}_{\mathrm{tw}}, \lambda)\big[T_A + T_B(\lambda)\big] = -\big[T_A + T_B(\lambda)\big] + \lambda R(\mathcal{L}_{\mathrm{tw}}, \lambda)\big[T_A + T_B(\lambda)\big] \tag{3.3.45}$$

and the analytic continuations (3.3.36), we see that there exist $C_2' > 0$ so that

$$\|\mathcal{L}_{\mathrm{tw}} R(\mathcal{L}_{\mathrm{tw}}, \lambda)\big[T_A + T_B(\lambda)\big]\|_{L^2 \to L^2} + \|R(\mathcal{L}_{\mathrm{tw}}, \lambda)\big[T_A + T_B(\lambda)\big]\|_{L^2 \to L^2} \le C_2' \tag{3.3.46}$$

for all $\lambda \in \Omega_{\mathrm{tw}}$. We can now apply Lemma 3.3.4 to obtain the desired bound for $t \ge 1$.

The bounds in Lemma 3.3.6 imply that there exists $C_3' > 0$ for which

$$\begin{aligned}\|\mathcal{L}_{\mathrm{tw}} R(\mathcal{L}_{\mathrm{tw}}, \lambda)\big[T_B(\lambda)\big]\|_{L^2 \to L^2} &\le \frac{C_3'}{1 + |\lambda|} \\ \|R(\mathcal{L}_{\mathrm{tw}}, \lambda)\big[T_B(\lambda)\big]\|_{L^2 \to L^2} &\le \frac{C_3'}{|\lambda|}\end{aligned} \tag{3.3.47}$$

holds for all $\lambda \in \Omega_{\mathrm{tw}}$. We can hence use Lemma 3.3.3 to find a constant $C_4' > 0$ for which we have the bound

$$\|\Lambda_{\mathrm{ex};B}(t)\|_{L^2 \to H^2} \le C_4' \tag{3.3.48}$$

for all $0 < t \le 1$. A direct application of Lemma 3.3.1 shows that also

$$\|\Lambda_{\mathrm{ex};A}(t)\|_{L^2 \to H^2} \le M \tag{3.3.49}$$

for all $0 < t \le 1$, which completes the proof. □

Corollary 3.3.8. *Suppose that (HDt) and (HTw) are satisfied. Then for any $g \in H^1$ we have*

$$\Lambda_{\rm ex}(t)g = \Lambda(t)g := [S(t)Q, \partial_\xi]g. \tag{3.3.50}$$

Proof. The result follows by integrating both sides of the identity (3.3.29) over the contour $\gamma_{r,\eta}$ and using (3.3.16) together with (3.3.40). □

3.3.3 Semigroup block structure

For the nonlinear stability proof in §3.5 we need to understand how the off-diagonal terms of $S(t)$ act on a second order nonlinearity. In order to do this, we first write $S_{\mathrm{d};I}(t)$ for the semigroup generated by

$$\mathcal{L}_{\rm tw;d} = \rho\partial_{\xi\xi}v + c_0 v_\xi, \tag{3.3.51}$$

which contains only diagonal terms. We also write

$$S_{\mathrm{od};I}(t) = S(t) - S_{\mathrm{d};I}(t) \tag{3.3.52}$$

for the rest of the semigroup. Note that $S_{\mathrm{od};I}(t)$ is not strictly off-diagonal, but it has the same off-diagonal elements as $S_{\mathrm{od}}(t)$.

Lemma 3.3.9. *Suppose that (HDt) and (HTw) are satisfied. Then there exists a constant $C > 0$ for which the short-term bound*

$$\|S_{\rm od;I}(t)\|_{L^2 \to H^2} \le C \tag{3.3.53}$$

holds for all $0 \le t \le 1$.

Proof. Possibly decreasing the size of η_+, we may assume that $\Omega_{\rm tw}$ is contained in the resolvent set of $\mathcal{L}_{\rm tw;d}$. We may also assume that the bound

$$\|R(\mathcal{L}_{\rm tw;d}, \lambda)\|_{L^2 \to L^2} \le \frac{M}{|\lambda|} \tag{3.3.54}$$

holds for $\lambda \in \Omega_{\rm tw}$ by increasing the size of $M > 0$ if necessary.

For any $r > 0$ and $\eta \in (\frac{\pi}{2}, \eta_+)$ we have

$$\begin{aligned} S_{\rm od;I}(t) &= \frac{1}{2\pi i}\int_{\gamma_{r,\eta}} e^{\lambda t}[R(\mathcal{L}_{\rm tw}, \lambda) - R(\mathcal{L}_{\rm tw;d}, \lambda)]\, d\lambda \\ &= \frac{1}{2\pi i}\int_{\gamma_{r,\eta}} e^{\lambda t} R(\mathcal{L}_{\rm tw}, \lambda)(\mathcal{L}_{\rm tw} - \mathcal{L}_{\rm tw;d})R(\mathcal{L}_{\rm tw;d}, \lambda)\, d\lambda \\ &= \frac{1}{2\pi i}\int_{\gamma_{r,\eta}} e^{\lambda t} R(\mathcal{L}_{\rm tw}, \lambda) Df(\Phi_0) R(\mathcal{L}_{\rm tw;d}, \lambda)\, d\lambda. \end{aligned} \tag{3.3.55}$$

On account of the identity

$$\mathcal{L}_{\rm tw} R(\mathcal{L}_{\rm tw}, \lambda) Df(\Phi_0) R(\mathcal{L}_{\rm tw;d}, \lambda) = \lambda R(\mathcal{L}_{\rm tw}, \lambda) Df(\Phi_0) R(\mathcal{L}_{\rm tw;d}, \lambda) - Df(\Phi_0) R(\mathcal{L}_{\rm tw;d}, \lambda) \tag{3.3.56}$$

we have the bounds

$$\begin{aligned} \|\mathcal{L}_{\mathrm{tw}} R(\mathcal{L}_{\mathrm{tw}}, \lambda) Df(\Phi_0) R(\mathcal{L}_{\mathrm{tw;d}}, \lambda)\|_{L^2 \to L^2} &\leq \|Df(\Phi_0)\|_\infty \frac{M(M+1)}{|\lambda|}, \\ \|R(\mathcal{L}_{\mathrm{tw}}, \lambda) Df(\Phi_0) R(\mathcal{L}_{\mathrm{tw;d}}, \lambda)\|_{L^2 \to L^2} &\leq \|Df(\Phi_0)\|_\infty \frac{M^2}{|\lambda|^2}. \end{aligned} \tag{3.3.57}$$

The desired estimate hence follows from Lemma 3.3.3. □

Proof of Proposition 3.3.2. The statements concerning $\Lambda(t)$ follow directly from Lemma 3.3.7 and Corollary 3.3.8. The bound for $S_{\mathrm{od}}(t)$ follows from Lemma 3.3.9 since $S_{\mathrm{od;I}}(t)$ contains all the non-trivial elements of $S_{\mathrm{od}}(t)$. □

3

3.4 Stochastic transformations

In this section we set out to derive a mild formulation for the SPDE satisfied by the process

$$V(t) = T_{-\Gamma(t)}[U(t)] - \Phi_\sigma, \tag{3.4.1}$$

which measures the deviation from the traveling wave Φ_σ in the coordinate $\xi = x - \Gamma(t)$. After recalling several results from Chapter 2 concerning the stochastic phaseshift, we focus on the new extra second order nonlinearity that appears in our setting. We use the results from §3.3 to rewrite this term in such a way that an effective mild integral equation can be formulated that does not involve second derivatives. We obtain estimates on all the nonlinear terms in §3.4.1 and rigorously verify that V indeed satisfies this mild equation in §3.4.2.

We start by introducing the nonlinearity

$$\begin{aligned} \mathcal{R}_\sigma(v) = \; & \kappa_\sigma(\Phi_\sigma + v, \psi_{\mathrm{tw}}) \rho \partial_{\xi\xi}[\Phi_\sigma + v] \\ & + f(\Phi_\sigma + v) + \sigma^2 b(\Phi_\sigma + v, \psi_{\mathrm{tw}}) \partial_\xi [g(\Phi_\sigma + v)] \\ & + \Big[c_\sigma + a_\sigma\big(\Phi_\sigma + v, c_\sigma, \psi_{\mathrm{tw}}\big)\Big][\Phi_\sigma' + v'], \end{aligned} \tag{3.4.2}$$

together with

$$\mathcal{S}_\sigma(v) = g(\Phi_\sigma + v) + b(\Phi_\sigma + v, \psi_{\mathrm{tw}})[\Phi_\sigma' + v']. \tag{3.4.3}$$

In §2.5 we established that the shifted process V can be interpreted as a weak solution to the SPDE

$$dV = \mathcal{R}_\sigma(V)\, dt + \sigma \mathcal{S}_\sigma(V) d\beta_t. \tag{3.4.4}$$

However, in our case here κ_σ is a matrix rather than a scalar. This means that we cannot transform (3.4.4) into a semilinear problem by a simple time transformation. But, we can improve individual components of the system by rescaling time with the diagonal elements $\kappa_{\sigma;i}$.

To this end, we follow Lemma 2.3.6 to find a constant $K_\kappa > 0$ for which

$$1 \leq \kappa_{\sigma;i}(\Phi_\sigma + v, \psi_{\mathrm{tw}}) \leq K_\kappa \tag{3.4.5}$$

holds for every $\sigma \in (-\delta_\sigma, \delta_\sigma)$, every $v \in H^1$ and every $1 \le i \le n$. Upon introducing the transformed time

$$\tau_i(t,\omega) = \int_0^t \kappa_{\sigma;i}\big(\Phi_\sigma + V(s,\omega), \psi_{\mathrm{tw}}\big)\, ds, \tag{3.4.6}$$

the bound (3.4.5) allows us to conclude that $t \mapsto \tau_i(t)$ is a continuous strictly increasing $(\mathcal{F}_t)$-adapted process that satisfies

$$t \le \tau_i(t) \le K_\kappa t \tag{3.4.7}$$

for $0 \le t \le T$. In particular, we can define a map

$$t_i : [0,T] \times \Omega \to [0,T] \tag{3.4.8}$$

for which

$$\tau_i(t_i(\tau,\omega),\omega) = \tau. \tag{3.4.9}$$

This in turn allows us to introduce the time-transformed map

$$\overline{V}_i : [0,T] \times \Omega \to L^2 \tag{3.4.10}$$

that acts as

$$\overline{V_i}(\tau,\omega) \quad = \quad V\big(t_i(\tau,\omega),\omega\big). \tag{3.4.11}$$

Upon introducing

$$\overline{\mathcal{R}}_{\sigma;i}(v) \quad = \quad \kappa_{\sigma;i}(\Phi_\sigma + v, \psi_{\mathrm{tw}})^{-1}\mathcal{R}_\sigma(v) - \mathcal{L}_{\mathrm{tw}} v \tag{3.4.12}$$

together with

$$\overline{\mathcal{S}}_{\sigma;i}(v) \quad = \quad \kappa_{\sigma;i}(\Phi_\sigma + v, \psi_{\mathrm{tw}})^{-1/2}\mathcal{S}_\sigma(v), \tag{3.4.13}$$

it is possible to follow Proposition 2.6.3 to show that $\overline{V}_i$ is a weak solution of

$$d\overline{V}_i = \big[\mathcal{L}_{\mathrm{tw}}\overline{V}_i + \overline{\mathcal{R}}_{\sigma;i}(\overline{V}_i)\big]\, d\tau + \sigma\overline{\mathcal{S}}_{\sigma;i}(\overline{V}_i) d\overline{\beta}_{\tau;i} \tag{3.4.14}$$

for every $1 \le i \le n$, in which $(\overline{\beta}_{\tau;i})_{\tau\ge 0}$ denotes the time-transformed Brownian motion that is now adapted to an appropriately transformed filtration $(\overline{\mathcal{F}}_{\tau;i})_{\tau \ge 0}$; see Lemma 2.6.2.

The nonlinearity $\overline{\mathcal{R}}_{\sigma;i}$ is less well-behaved than its counterpart from Proposition 2.6.3 since it still contains second order derivatives. In order to isolate these terms, we pick any $v \in H^1$ and introduce the diagonal matrix

$$\phi_{\sigma;i}(v) = \big[\kappa_{\sigma;i}(\Phi_\sigma + v, \psi_{\mathrm{tw}})\big]^{-1}\kappa_\sigma(\Phi_\sigma + v, \psi_{\mathrm{tw}}) - I \tag{3.4.15}$$

together with the function

$$\Upsilon_{\sigma;i}(v) = \rho\phi_{\sigma;i}(v)\partial_\xi v. \tag{3.4.16}$$

We note that $\partial_\xi \Upsilon_{\sigma;i}$ can be considered as the error caused by allowing unequal diffusion coefficients in our main structural assumption (HDt). Indeed, upon defining our final nonlinearity implicitly by imposing the splitting

$$\overline{\mathcal{R}}_{\sigma;i}(v) = \mathcal{W}_{\sigma;i}(v) + \partial_\xi \Upsilon_{\sigma;i}(v), \tag{3.4.17}$$

our first main result states that $\mathcal{W}_{\sigma;i}$ is well-behaved in the sense that it admits bounds that are similar to those derived for the full nonlinearity $\overline{\mathcal{R}}$ in Chapter 2. Indeed, it depends at most quadratically on $\|v\|_{H^1}$ but not on $\|v\|_{H^2}$. Note furthermore that Φ_σ was constructed in such a way that $\overline{\mathcal{R}}(0) = 0$.

Proposition 3.4.1. *Assume that (HDt), (HSt) and (HTw) all hold and fix $1 \leq i \leq n$. Then there exist constants $K > 0$ and $\delta_v > 0$ so that for any $0 \leq \sigma \leq \delta_\sigma$ and any $v \in H^1$, the following properties hold true.*

(i) We have the bound

$$\|\mathcal{W}_{\sigma;i}(v)\|_{L^2} \quad \leq \quad K\sigma^2 \|v\|_{H^1} + K\|v\|_{H^1}^2 \big[1 + \|v\|_{L^2}^2 + \sigma^2 \|v\|_{L^2}^3\big], \tag{3.4.18}$$

together with

$$\|\Upsilon_{\sigma;i}(v)\|_{L^2} \leq K\sigma^2 \|v\|_{H^1}. \tag{3.4.19}$$

(ii) We have the estimate

$$\|\overline{\mathcal{S}}_{\sigma;i}(v)\|_{L^2} \leq K\big[1 + \|v\|_{H^1}\big]. \tag{3.4.20}$$

(iii) If $\|v\|_{L^2} \leq \delta_v$, then we have the identities

$$\langle \overline{\mathcal{R}}_{\sigma;i}(v), \psi_{\mathrm{tw}} \rangle_{L^2} = \langle \overline{\mathcal{S}}_{\sigma;i}(v), \psi_{\mathrm{tw}} \rangle_{L^2} = 0. \tag{3.4.21}$$

The second main result of this section formulates a mild representation for solutions to (3.4.14). Items (i)-(iv) are included for completeness and are analogous to the results in Proposition 2.6.3. However, item (v) is specific to our situation because of the presence of the error term $\Upsilon_{\sigma;i}$. Indeed, we shall need to exploit the techniques developed in §3.3 to transfer the troublesome ∂_ξ present in (3.4.17) from the $\Upsilon_{\sigma;i}$ term to the semigroup. Nevertheless, the integral involving $\partial_\xi S$ is integrable in H^{-1} but not necessarily in L^2.

Proposition 3.4.2. *Assume that (HDt), (HSt), (HTw) are all satisfied. Then the map*

$$\overline{V}_i : [0,T] \times \Omega \to L^2 \tag{3.4.22}$$

defined by the transformations (3.4.1) and (3.4.11) satisfies the following properties.

(i) For almost all $\omega \in \Omega$, the map $\tau \mapsto \overline{V_i}(\tau;\omega)$ is of class $C\big([0,T];L^2\big)$.

(ii) For all $\tau \in [0,T]$, the map $\omega \mapsto \overline{V_i}(\tau,\omega)$ is $(\mathcal{F}_{\tau;i})$-measurable.

(iii) We have the inclusion

$$\overline{V}_i \in \mathcal{N}^2\big([0,T];(\overline{\mathcal{F}})_{\tau;i};H^1\big), \tag{3.4.23}$$

together with

$$\overline{\mathcal{S}}_{\sigma;i}(\overline{V}_i) \in \mathcal{N}^2\big([0,T];(\overline{\mathcal{F}})_{\tau;i};L^2\big). \tag{3.4.24}$$

(iv) For almost all $\omega \in \Omega$, we have the inclusion

$$\mathcal{W}_{\sigma;i}\big(\overline{V}_i(\cdot,\omega)\big) \in L^1([0,T];L^2) \tag{3.4.25}$$

together with

$$\Upsilon_{\sigma;i}\big(\overline{V}_i(\cdot,\omega)\big) \in L^1([0,T];L^2). \tag{3.4.26}$$

(v) For almost all $\omega \in \Omega$, the identity

$$\begin{aligned}\overline{V}_i(\tau) &= S(\tau)\overline{V}_i(0) \\ &\quad + \int_0^\tau S(\tau-\tau')\mathcal{W}_{\sigma;i}\big(\overline{V}_i(\tau')\big)d\tau' + \sigma\int_0^\tau S(\tau-\tau')\overline{\mathcal{S}}_{\sigma;i}\big(\overline{V}_i(\tau')\big)d\overline{\beta}_{\tau';i} \\ &\quad + \int_0^\tau \partial_\xi S(\tau-\tau')Q\Upsilon_{\sigma;i}\big(\overline{V}_i(\tau')\big)\,d\tau' + \int_0^\tau \Lambda(\tau-\tau')\Upsilon_{\sigma;i}\big(\overline{V}_i(\tau')\big)\,d\tau' \\ &\quad + \int_0^\tau S(\tau-\tau')P_\xi\Upsilon_{\sigma;i}\big(\overline{V}_i(\tau')\big)\,d\tau'\end{aligned} \tag{3.4.27}$$

holds for all $\tau \in [0,T]$.

3.4.1 Bounds on nonlinearities

In this section we set out to prove Proposition 3.4.1. In order to be able to write the nonlinearities in a compact fashion, we introduce the expression

$$\mathcal{J}_\sigma(u) \quad = \quad \kappa_\sigma(u,\psi_{\mathrm{tw}})^{-1}\Big[f(u) + c_\sigma\partial_\xi u + \sigma^2 b(u,\psi_{\mathrm{tw}})\partial_\xi[g(u)]\Big] \tag{3.4.28}$$

for any $u \in \mathcal{U}_{H^1}$. This allows us to define

$$\mathcal{Q}_\sigma(v) = \mathcal{J}_\sigma(\Phi_\sigma + v) - \mathcal{J}_\sigma(\Phi_\sigma) + [\rho\partial_{\xi\xi} - \mathcal{L}_{\mathrm{tw}}]v \tag{3.4.29}$$

for any $v \in H^1$, which is the residual upon linearizing $\mathcal{J}_\sigma(\Phi_\sigma + V)$ around Φ_σ, up to $O(\sigma^2)$ corrections. Indeed, we can borrow the following bound from Chapter 2.

Corollary 3.4.3. *Consider the setting of Proposition 3.4.1. There exists $K > 0$ so that for any $0 \le \sigma \le \delta_\sigma$ and any $v \in H^1$ we have the estimate*

$$\begin{aligned}\|\mathcal{Q}_\sigma(v)\|_{L^2} \quad &\le \quad K\big[\sigma^2 + \|v\|_{L^2}\big]\|v\|_{H^1} \\ &\qquad + K\big[1 + (1+\sigma^2)\|v\|_{L^2} + \sigma^2\|v\|_{L^2}^2\big]\|v\|_{H^1}^2,\end{aligned} \tag{3.4.30}$$

together with

$$\begin{array}{rcl} |\langle \mathcal{Q}_\sigma(v), \psi_{\mathrm{tw}} \rangle_{L^2}| & \leq & K\big[1 + \|v\|_{H^1}\big] \|v\|_{L^2} \|v\|_{L^2} \\ & & +K\big[\sigma^2 + \|v\|_{L^2}\big] \|v\|_{L^2} \\ & & +K\sigma^2 \|v\|_{H^1} \|v\|_{L^2}^2 \|v\|_{L^2} \\ & & +K\sigma^2 \|v\|_{L^2}^2 \|v\|_{H^1}. \end{array} \tag{3.4.31}$$

Proof. Recalling the function $\mathcal{M}$ that was defined in equation (2.7.2), we observe that

$$\mathcal{Q}_\sigma(v) = \mathcal{M}_{\sigma;\Phi_\sigma,c_\sigma}(v,0) - \mathcal{M}_{\sigma;\Phi_\sigma,c_\sigma}(0,0). \tag{3.4.32}$$

In particular, the desired bounds follow directly from Corollary 2.7.5. □

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We now introduce the function

$$\mathcal{W}_{\sigma;I,i}(v) = \mathcal{Q}_\sigma(v) + \phi_{\sigma;i}(v)\Big[\mathcal{J}_\sigma(\Phi_\sigma + v) - \mathcal{J}_\sigma(\Phi_\sigma)\Big] \tag{3.4.33}$$

together with the notation

$$\begin{array}{rcl} \mathcal{I}_{\sigma;I,i}(v) & = & \Big[\chi_{\mathrm{low}}\big(\langle \partial_\xi[\Phi_\sigma + v], \psi_{\mathrm{tw}}\rangle_{L^2}\big)\Big]^{-1} \langle \mathcal{W}_{\sigma;I,i}(v), \psi_{\mathrm{tw}}\rangle_{L^2} \\ & & -\Big[\chi_{\mathrm{low}}\big(\langle \partial_\xi[\Phi_\sigma + v], \psi_{\mathrm{tw}}\rangle_{L^2}\big)\Big]^{-1} \langle \Upsilon_{\sigma;i}(v), \partial_\xi \psi_{\mathrm{tw}}\rangle_{L^2}. \end{array} \tag{3.4.34}$$

The following result shows that these two expressions allow us to split off the a_σ-contribution to $\overline{\mathcal{R}}_{\sigma;i}$ that is visible in (3.4.2).

Lemma 3.4.4. *Consider the setting of Proposition 3.4.1. Then for any $0 \leq \sigma \leq \delta_\sigma$ and $v \in H^1$, we have the inclusion $\mathcal{W}_{\sigma;i}(v) \in L^2$ together with the identity*

$$\mathcal{W}_{\sigma;i}(v) = \mathcal{W}_{\sigma;I,i}(v) - \mathcal{I}_{\sigma;I,i}(v)[\Phi_\sigma' + v']. \tag{3.4.35}$$

Proof. For any $u \in \mathcal{U}_{H^2}$, the definition (3.2.18) implies that

$$a_\sigma(u, c_\sigma, \psi_{\mathrm{tw}}) = -\Big[\chi_{\mathrm{low}}\big(\langle \partial_\xi u, \psi_{\mathrm{tw}}\rangle_{L^2}\big)\Big]^{-1} \langle \kappa_\sigma(u, \psi_{\mathrm{tw}})\big[\rho \partial_{\xi\xi} u + \mathcal{J}_\sigma(u)\big], \psi_{\mathrm{tw}}\rangle_{L^2}. \tag{3.4.36}$$

The implicit definition $a_\sigma(\Phi_\sigma, c_\sigma, \psi_{\mathrm{tw}}) = 0$ hence yields

$$\mathcal{J}_\sigma(\Phi_\sigma) = -\rho \phi_\sigma''. \tag{3.4.37}$$

For any $v \in H^2$, this allows us to compute

$$\mathcal{Q}_\sigma(v) = \mathcal{J}_\sigma(\Phi_\sigma + v) + \rho[\Phi_\sigma'' + v''] - \mathcal{L}_{\mathrm{tw}} v, \tag{3.4.38}$$

which gives

$$\begin{array}{rcl} \mathcal{W}_{\sigma;I,i}(v) + \partial_\xi \Upsilon_{\sigma;i}(v) & = & [\kappa_{\sigma;i}(\Phi_\sigma + v, \psi_{\mathrm{tw}})]^{-1} \kappa_\sigma(\Phi_\sigma + v, \psi_{\mathrm{tw}})\big[\rho[\Phi_\sigma'' + v'']\big] \\ & & +\mathcal{J}_\sigma(\Phi_\sigma + v) - \mathcal{L}_{\mathrm{tw}} v. \end{array} \tag{3.4.39}$$

Using the fact that $\mathcal{L}_{\mathrm{tw}}^* \psi_{\mathrm{tw}} = 0$, we now readily verify that for $v \in H^2$ we have

$$\mathcal{I}_{\sigma;I,i}(v) = [\kappa_{\sigma;i}(\Phi_\sigma + v, \psi_{\mathrm{tw}})]^{-1} a_\sigma(\Phi_\sigma + v, \psi_{\mathrm{tw}}). \tag{3.4.40}$$

The result hence follows by rewriting the definition (3.4.2) in the form

$$\begin{aligned} \mathcal{R}_\sigma(v) &= \kappa_\sigma(\Phi_\sigma + v, \psi_{\mathrm{tw}})\Big[\rho\partial_{\xi\xi}[\Phi_\sigma + v] + \mathcal{J}_\sigma(\Phi_\sigma + v)\Big] \\ &\quad + a_\sigma(\Phi_\sigma + v, c_\sigma, \psi_{\mathrm{tw}})[\Phi_\sigma' + v'] \end{aligned} \tag{3.4.41}$$

and substituting this into the definition (3.4.12) of $\overline{\mathcal{R}}_{\sigma;i}$. □

In order to obtain the estimates in Proposition 3.4.1 it hence suffices to obtain bounds for $\phi_{\sigma;i}$, $\mathcal{W}_{\sigma;I,i}$ and $\mathcal{I}_{\sigma;I,i}$. This can be done in a direct fashion.

Lemma 3.4.5. *Assume that (HDt) and (HSt) are satisfied. Then there exists a constant $K_\phi > 0$ so that*

$$|\phi_{\sigma;i}(v)| \leq \sigma^2 K_\phi \tag{3.4.42}$$

holds for any $v \in L^2$ and $0 \leq \sigma \leq \delta_\sigma$.

Proof. For any $x, y \geq 0$ we have the inequality

$$\left| \frac{1 + \frac{1}{2\rho_j} x}{1 + \frac{1}{2\rho_i} x} - \frac{1 + \frac{1}{2\rho_i} y}{1 + \frac{1}{2\rho_i} y} \right| = \frac{1}{4\rho_i\rho_j} \frac{|x-y|}{(1 + \frac{1}{2\rho_i}x)(1 + \frac{1}{2\rho_i}y)} \leq \frac{1}{4\rho_i\rho_j}|x-y|. \tag{3.4.43}$$

Applying these bounds with $y = 0$, we obtain

$$|\phi^j_{\sigma;i}(v)| \leq \frac{\sigma^2}{4\rho_i\rho_j}|b(\Phi_\sigma + v)|^2 \leq \frac{\sigma^2}{4\rho_{\min}^2} K_b^2, \tag{3.4.44}$$

where the last bound on b follows from Lemma 2.3.6. The result now readily follows. □

Lemma 3.4.6. *Consider the setting of Proposition 3.4.1. Then there exists $K > 0$ so that for any $v \in H^1$ and $0 \leq \sigma \leq \delta_\sigma$ we have the bound*

$$\|\mathcal{W}_{\sigma;I,i}(v)\|_{L^2} \leq K\sigma^2\|v\|_{H^1} + K\|v\|_{H^1}^2\big[1 + \|v\|_{L^2} + \sigma^2\|v\|_{L^2}^2\big], \tag{3.4.45}$$

together with

$$|\mathcal{I}_{\sigma;I,i}(v)| \leq K\|v\|_{L^2}\big[\sigma^2 + \|v\|_{L^2}\big] + K\|v\|_{H^1}\big[\|v\|_{L^2}^2 + \sigma^2\|v\|_{L^2}^3\big]. \tag{3.4.46}$$

Proof. Note first that we can write $\mathcal{W}_{\sigma;I,i}(v)$ as

$$\mathcal{W}_{\sigma;I,i}(v) = \mathcal{Q}_\sigma(v) + \phi_{\sigma;i}(v)\Big[\mathcal{Q}_\sigma(v) + (\mathcal{L}_{\mathrm{tw}} - \rho\partial_{\xi\xi})v\Big] \tag{3.4.47}$$

and hence

$$\|\mathcal{W}_{\sigma;I,i}(v)\|_{L^2} \leq \|\mathcal{Q}_\sigma(v)\|_{L^2} + |\phi_{\sigma;i}(v)|\Big[\|\mathcal{Q}_\sigma(v)\|_{L^2} + \|(\mathcal{L}_{\mathrm{tw}} - \rho\partial_{\xi\xi})v\|_{L^2}\Big]. \tag{3.4.48}$$

The definition of $\mathcal{L}_{\text{tw}}$ implies that there exists $C_1 > 0$ for which

$$\|[\mathcal{L}_{\text{tw}} - \rho\partial_{\xi\xi}]v\|_{L^2} \leq C_1\|v\|_{H^1} \tag{3.4.49}$$

holds. The desired bound hence follows from Corollary 3.4.3 and Lemma 3.4.5.

Turning to the second estimate, we note that there is a positive constant C_2 for which we have

$$|\mathcal{I}_{\sigma;I,i}(v)| \leq C_2\big[\|\mathcal{W}_{\sigma;I,i}(v)\|_{L^2} + \|\Upsilon_{\sigma;i}(v)\|_{L^2}\big]. \tag{3.4.50}$$

We can hence again apply Corollary 3.4.3 and Lemma 3.4.5, which yields expressions that can all be absorbed into (3.4.46). □

Proof of Proposition 3.4.1. To obtain (3.4.18), we use (3.4.35) together with Lemma 3.4.6 to compute

$$\begin{array}{rcl} \|\mathcal{W}_{\sigma;i}(v)\|_{L^2} & \leq & \|\mathcal{W}_{\sigma;i}\|_{L^2} + C_1\,|\mathcal{I}_{\sigma;I,i}(v)|\,\big[1 + \|v\|_{H^1}\big] \\ & \leq & C_2\sigma^2\|v\|_{H^1} + C_2\|v\|_{H^1}^2\big[1 + \|v\|_{L^2} + \sigma^2\|v\|_{L^2}^2\big] \\ & & \quad + C_2\|v\|_{L^2}\big[\sigma^2 + \|v\|_{L^2}\big]\big[1 + \|v\|_{H^1}\big] \\ & & \quad + C_2\|v\|_{H^1}\big[\|v\|_{L^2}^2 + \sigma^2\|v\|_{L^2}^3\big]\big[1 + \|v\|_{H^1}\big] \end{array} \tag{3.4.51}$$

for some constants $C_1 > 0$ and $C_2 > 0$. These terms can all be absorbed into (3.4.18). The bound (3.4.19) follows from Lemma 3.4.5 and (HDt), while (ii) and (iii) follow directly from Proposition 2.8.1. □

3.4.2 Mild formulation

In this section we establish Proposition 3.4.2. We note that items (i)-(iv) follow directly from Propositions 2.5.1 and 2.6.3, so we focus here on the integral identity (3.4.27). We first obtain this identity in a weak sense, bypassing the need to interpret the term involving $\Upsilon_{\sigma;i}$ in a special fashion. We note that $S^*(t)$ is the adjoint operator of $S(t)$, which coincides with the semigroup generated by $\mathcal{L}_{\text{tw}}^*$.

Lemma 3.4.7. *Consider the setting of Proposition 3.4.2 and pick any $\eta \in H^3$. Then for almost all $\omega \in \Omega$ the identity*

$$\begin{aligned} \langle \overline{V}_i(\tau), \eta\rangle_{L^2} =& \langle S(\tau)\overline{V}_i(0) + \int_0^\tau S(\tau - \tau')\mathcal{W}_{\sigma;i}\big(\overline{V}_i(\tau')\big)\,d\tau' \\ & + \sigma\int_0^\tau S(\tau - \tau')\overline{\mathcal{S}}_{\sigma;i}\big(\overline{V}_i(\tau')\big)d\overline{\beta}_{\tau';i}, \eta\rangle_{L^2} \\ & + \int_0^\tau \langle\partial_\xi\Upsilon_{\sigma;i}\big(\overline{V}_i(\tau')\big), S^*(\tau - \tau')\eta\rangle_{H^{-1};H^1}\,d\tau' \end{aligned} \tag{3.4.52}$$

holds for any $\tau \in [0, T]$.

Proof. Pick any $\tau \in [0,T]$. Since $\overline{V_i} \in \mathcal{N}^2([0,T];(\mathcal{F}_t);H^1)$ is a weak solution to (3.4.14), the identity

$$\begin{aligned}\overline{V}_i(\tau) &= \overline{V}_i(0) + \int_0^\tau \big[\mathcal{L}_{\text{tw}}\overline{V}_i(\tau') + \overline{\mathcal{R}}_{\sigma;i}\big(\overline{V}_i(\tau')\big)\big]\, d\tau' \\ &\quad +\sigma \int_0^\tau \overline{S}_{\sigma;i}\big(\overline{V}_i(\tau')\big)\, d\overline{\beta}_{\tau';i}\end{aligned} \tag{3.4.53}$$

holds in H^{-1}; see Proposition 2.6.3. We note that these integrals are well defined by items (i)-(iv) of Proposition 3.4.2.

Following the proof of [67, Prop. 2.10], we pick $\eta \in H^3$ and define the function

$$\zeta(\tau') = S^*(\tau - \tau')\eta \tag{3.4.54}$$

on the interval $[0,\tau]$. Noting that $\zeta \in C^1([0,\tau], H^1)$, we may define the functional $\phi : [0,\tau] \times H^{-1} \to \mathbb{R}$ that acts as

$$\phi(\tau', v) = \langle v, \zeta(\tau')\rangle_{H^{-1};H^1}, \tag{3.4.55}$$

which is C^1-smooth in the first variable and linear in the second variable. Applying a standard Itô formula such as [27, Thm. 1] (with $S = I$) yields

$$\begin{aligned}\phi\big(\tau, \overline{V}_i(\tau)\big) &= \phi\big(0, \overline{V}_i(0)\big) \\ &\quad + \int_0^\tau \langle \overline{V}_i(\tau'), \zeta'(\tau')\rangle_{H^{-1};H^1}\, d\tau' \\ &\quad + \int_0^\tau \langle \mathcal{L}_{\text{tw}}\overline{V}_i(\tau'), \zeta(\tau')\rangle_{H^{-1};H^1}\, d\tau' \\ &\quad + \int_0^\tau \langle \overline{\mathcal{R}}_{\sigma;i}\big(\overline{V}_i(\tau')\big), \zeta(\tau')\rangle_{H^{-1};H^1}\, d\tau' \\ &\quad +\sigma \int_0^\tau \langle \overline{\mathcal{S}}_{\sigma;i}\big(\overline{V}_i(\tau')\big), \zeta(\tau')\rangle_{L^2}\, d\overline{\beta}_{\tau';i}.\end{aligned} \tag{3.4.56}$$

Since $\zeta'(t) = -\mathcal{L}_{\text{tw}}^*\zeta(t)$, the second line in the expression above disappears. Using the identities

$$\begin{aligned}\phi\big(\tau, \overline{V}_i(\tau)\big) &= \langle \overline{V}_i(\tau), \eta\rangle_{L^2}, \\ \phi\big(0, \overline{V}_i(0)\big) &= \langle \overline{V}_i(0), S^*(\tau)\eta\rangle_{L^2} \\ &= \langle S(\tau)\overline{V}_i(0), \eta\rangle_{L^2}\end{aligned} \tag{3.4.57}$$

we hence obtain

$$\begin{aligned}\langle \overline{V}_i(\tau), \eta\rangle_{L^2} &= \langle S(\tau)\overline{V}_i(0), \eta\rangle_{L^2} \\ &\quad + \int_0^\tau \langle S(\tau - \tau')\mathcal{W}_{\sigma;i}\big(\overline{V}_i(\tau')\big)\, d\tau', \eta\rangle_{L^2}\, d\tau' \\ &\quad + \int_0^\tau \langle \partial_\xi \Upsilon_{\sigma;i}\big(\overline{V}_i(\tau')\big), S^*(t-s)\eta\rangle_{H^{-1};H^1}\, d\tau' \\ &\quad +\sigma \int_0^\tau \langle S(\tau - \tau')\overline{S}_{\sigma;i}\big(\overline{V}_i(\tau')\big), \eta\rangle_{L^2}\, d\overline{\beta}_{\tau';i},\end{aligned} \tag{3.4.58}$$

as desired. □

Lemma 3.4.8. *Pick $v \in L^2$ together with $\eta \in H^1$ and $t > 0$. Then we have the identity*

$$\langle \partial_\xi v, S^*(t)\eta \rangle_{H^{-1};H^1} = \langle \partial_\xi S(t)Qv + \Lambda(t)v + S(t)P_\xi v, \eta \rangle_{L^2}. \tag{3.4.59}$$

Proof. For $v \in H^1$, this identity follows directly from (3.3.8). For fixed η and $t > 0$, both both sides of (3.4.59) can be interpreted as bounded linear functions on L^2 by Proposition 3.3.2. In particular, the result can be obtained by approximating $v \in L^2$ by H^1-functions. □

Proof of Proposition 3.4.2. As mentioned above, items (i)-(iv) follow directly from Propositions 2.5.1 and 2.6.3. Item (v) follows from Lemmas 3.4.7 and 3.4.8, using the density of H^3 in H^1 and the fact that H^{-1} is separable. □

3.5 Nonlinear stability of mild solutions

In this section we prove Theorem 3.2.1, which provides an orbital stability result for the stochastic wave (Φ_σ, c_σ). In particular, for any $\varepsilon > 0$, $T > 0$ and $\eta > 0$ we recall the notation

$$N_\varepsilon(t) = \|V(t)\|_{L^2}^2 + \int_0^t e^{-\varepsilon(t-s)} \|V(s)\|_{H^1}^2 ds \tag{3.5.1}$$

and introduce the $(\mathcal{F}_t)$-stopping time

$$t_{\mathrm{st}}(T, \varepsilon, \eta) = \inf \Big\{ 0 \le t < T : N_\varepsilon(t) > \eta \Big\}, \tag{3.5.2}$$

writing $t_{\mathrm{st}}(T, \varepsilon, \eta) = T$ if the set is empty. We derive a number of technical regularity estimates in §3.5.1 that allows us to exploit the integral identity (3.4.27) to bound the expectation of $\sup_{0 \le t \le t_{\mathrm{st}}(T,\varepsilon,\eta)} N_\varepsilon(t)$ in terms of itself, the noise-strength σ and the size of the initial condition $V(0)$. This leads to the following bound for this expectation.

Proposition 3.5.1. *Assume that (HDt), (HSt) and (HTw) are satisfied. Pick a constant $0 < \varepsilon < \beta$, together with two sufficiently small constants $\delta_\eta > 0$ and $\delta_\sigma > 0$. Then there exists a constant $K > 0$ so that for any $T > 0$, any $0 < \eta \le \delta_\eta$ and any $0 \le \sigma \le \delta_\sigma T^{-1/2}$ we have the bound*

$$E[\sup_{0 \le t \le t_{\mathrm{st}}(T,\varepsilon,\eta)} N_\varepsilon(t)] \quad \le \quad K\Big[\|V(0)\|_{H^1}^2 + \sigma^2 T\Big]. \tag{3.5.3}$$

Exploiting the technique used in Stannat [105], this bound can be turned into an estimate concerning the probability

$$p_\varepsilon(T, \eta) = P\Big(\sup_{0 \le t \le T} \big[N_\varepsilon(t)\big] > \eta \Big). \tag{3.5.4}$$

This allows our main stability result to be established in a straightforward fashion.

Proof of Theorem 3.2.1. Upon computing

$$\begin{aligned} \eta p_\varepsilon(T,\eta) &= \eta P\big(t_{\mathrm{st}}(T,\varepsilon,\eta) < T\big) \\ &= E\Big[\mathbf{1}_{t_{\mathrm{st}}(T,\varepsilon,\eta)<T} N_\varepsilon\big(t_{\mathrm{st}}(T,\varepsilon,\eta)\big)\Big] \\ &\leq E[N_\varepsilon\big(t_{\mathrm{st}}(T,\varepsilon,\eta)\big)] \\ &\leq E[\sup_{0\leq t\leq t_{\mathrm{st}}(T,\varepsilon,\eta)} N_\varepsilon(t)], \end{aligned} \tag{3.5.5}$$

the result follows from Proposition 3.5.3.

□

3

3.5.1 Setup

In this subsection we establish Proposition 3.5.1 by estimating each of the terms featuring in (3.4.27). In contrast to the situation in Chapter 2 we cannot estimate $N_\varepsilon(t)$ directly because the integral involving $\partial_\xi S(t-s)$ applied to $\Upsilon_{\sigma;i}\big(\overline{V}_i(s)\big)$ presents short-time regularity issues. Instead, we will obtain separate estimates for each of the components $N_\varepsilon^i(t)$, which are given by

$$N_\varepsilon^i(t) \;=\; \|V^i(t)\|_{L^2}^2 + \int_0^t e^{-\varepsilon(t-s)}\|V^i(s)\|_{H^1}^2\, ds. \tag{3.5.6}$$

Indeed, the definitions (3.4.15) and (3.4.16) imply that the i-th component of $\Upsilon_{\sigma;i}$ vanishes, which allows us to replace the problematic $\partial_\xi S(t-s)$ term with its off-diagonal components $\partial_\xi S_{\mathrm{od}}(t-s)$. More precisely, for $\tau' \geq \tau - 1$ when computing short time bounds, we will use

$$\begin{aligned} &\Big[\partial_\xi S(\tau-\tau')Q\Upsilon_{\sigma;i}\big(\overline{V}_i(\tau')\big)\Big]^i \\ &\quad=\Big[\partial_\xi S(\tau-\tau')(I-P)\Upsilon_{\sigma;i}\big(\overline{V}_i(\tau')\big)\Big]^i \\ &\quad=\Big[\partial_\xi S_{\mathrm{od}}(\tau-\tau')\Upsilon_{\sigma;i}\big(\overline{V}_i(\tau')\big) - \partial_\xi S(\tau-\tau')P\Upsilon_{\sigma;i}\big(\overline{V}_i(\tau')\big)\Big]^i. \end{aligned} \tag{3.5.7}$$

This will allow us to bound $N_\varepsilon^i(t)$ in terms of $N_\varepsilon(t)$.

In order to streamline our computations, we now introduce some notation that will help us to stay as close as possible to the framework developed in Chapter 2. First of all, we impose the splittings

$$\begin{aligned} N_{\varepsilon,I}(t) &= \|V(t)\|_{L^2}^2, \\ N_{\varepsilon,II}(t) &= \int_0^t e^{-\varepsilon(t-s)}\|V(s)\|_{H^1}^2\, ds, \end{aligned} \tag{3.5.8}$$

together with

$$\begin{array}{rcl} N^i_{\varepsilon;I}(t) & = & \|V^i(t)\|^2_{L^2} \\ & = & \|\overline{V}^i_i\big(\tau_i(t)\big)\|^2_{L^2}, \\ N^i_{\varepsilon;II}(t) & = & \int_0^t e^{-\varepsilon(t-s)}\|V^i(s)\|^2_{H^1}\, ds \\ & = & \int_0^t e^{-\varepsilon(t-s)}\|\overline{V}^i_i\big(\tau_i(s)\big)\|^2_{H^1}\, ds. \end{array} \tag{3.5.9}$$

In addition, we split $\mathcal{W}_{\sigma;i}$ into a linear and nonlinear part as

$$\mathcal{W}_{\sigma;i}(v) = \sigma^2 F_{\mathrm{lin}}(v) + F_{\mathrm{nl}}(v) \tag{3.5.10}$$

and we isolate the constant term in $\overline{\mathcal{S}}_{\sigma;i}$ by writing

$$\overline{\mathcal{S}}_{\sigma;i}(v) = B_{\mathrm{cn}} + B_{\mathrm{lin}}(v). \tag{3.5.11}$$

Proposition 3.4.1 implies that these functions satisfy the bounds

$$\begin{array}{rl} \|F_{\mathrm{lin}}(v)\|_{L^2} & \leq K_{\mathrm{F;lin}}\|v\|_{H^1}, \\ \|F_{\mathrm{nl}}(v)\|_{L^2} & \leq K_{\mathrm{F;nl}}\|v\|^2_{H^1}(1+\|v\|^3_{L^2}), \\ \|B_{\mathrm{cn}}\|_{L^2} & < \infty, \\ \|B_{\mathrm{lin}}(v)\|_{L^2} & \leq K_{\mathrm{B;lin}}\|v\|_{H^1} \end{array} \tag{3.5.12}$$

for appropriate constants $K_{\mathrm{F;lin}} > 0$, $K_{\mathrm{F;nl}} > 0$ and $K_{\mathrm{B;lin}} > 0$. In particular, they satisfy assumption (hFB) in Chapter 2, which gives us the opportunity to apply some of the ideas in §2.9.

For convenience we will write from now on t_{st} for $t_{\mathrm{st}}(T,\varepsilon,\eta)$. In order to understand $N^i_{\varepsilon;I}$, we introduce the expression

$$\mathcal{E}_0(t) \quad = \quad S\big(\tau_i(t)\big)QV(0), \tag{3.5.13}$$

together with the long-term integrals

$$\begin{array}{rcl} \mathcal{E}^{\mathrm{lt}}_{F;\mathrm{lin}}(t) & = & \int_0^{\tau_i(t)-1} S(\tau_i(t)-\tau)QF_{\mathrm{lin}}\big(\overline{V}_i(\tau)\big)\mathbf{1}_{\tau<\tau_i(t_{\mathrm{st}})}d\tau, \\ \mathcal{E}^{\mathrm{lt}}_{F;\mathrm{nl}}(t) & = & \int_0^{\tau_i(t)-1} S(\tau_i(t)-\tau)QF_{\mathrm{nl}}\big(\overline{V}_i(\tau)\big)\mathbf{1}_{\tau<\tau_i(t_{\mathrm{st}})}d\tau, \\ \mathcal{E}^{\mathrm{lt}}_{B;\mathrm{lin}}(t) & = & \int_0^{\tau_i(t)-1} S(\tau_i(t)-\tau)QB_{\mathrm{lin}}\big(\overline{V}_i(\tau)\big)\mathbf{1}_{\tau<\tau_i(t_{\mathrm{st}})}d\beta_\tau, \\ \mathcal{E}^{\mathrm{lt}}_{B;\mathrm{cn}}(t) & = & \int_0^{\tau_i(t)-1} S(\tau_i(t)-\tau)QB_{\mathrm{cn}}\mathbf{1}_{\tau<\tau_i(t_{\mathrm{st}})}d\beta_\tau, \\ \mathcal{E}^{\mathrm{lt}}_{\mathrm{so}}(t) & = & \int_0^{\tau_i(t)-1} \left[\partial_\xi S(\tau_i(t)-\tau)Q + \Lambda(\tau_i(t)-\tau)\right]\Upsilon_{\sigma;i}\big(\overline{V}_i(\tau)\big)\mathbf{1}_{\tau<\tau_i(t_{\mathrm{st}})}d\tau, \end{array} \tag{3.5.14}$$

the short-term integrals

$$\begin{aligned}
\mathcal{E}^{\rm sh}_{F;\rm lin}(t) &= \int_{\tau_i(t)-1}^{\tau_i(t)} S(\tau_i(t)-\tau)QF_{\rm lin}\big(\overline{V}_i(\tau)\big)\mathbf{1}_{\tau<\tau_i(t_{\rm st})}d\tau, \\
\mathcal{E}^{\rm sh}_{F;\rm nl}(t) &= \int_{\tau_i(t)-1}^{\tau_i(t)} S(\tau_i(t)-\tau)QF_{\rm nl}\big(\overline{V}_i(\tau)\big)\mathbf{1}_{\tau<\tau_i(t_{\rm st})}d\tau, \\
\mathcal{E}^{\rm sh}_{B;\rm lin}(t) &= \int_{\tau_i(t)-1}^{\tau_i(t)} S(\tau_i(t)-\tau)QB_{\rm lin}\big(\overline{V}_i(\tau)\big)\mathbf{1}_{\tau<\tau_i(t_{\rm st})}d\beta_\tau, \\
\mathcal{E}^{\rm sh}_{B;\rm cn}(t) &= \int_{\tau_i(t)-1}^{\tau_i(t)} S(\tau_i(t)-\tau)QB_{\rm cn}\mathbf{1}_{\tau<\tau_i(t_{\rm st})}d\beta_\tau,
\end{aligned} \tag{3.5.15}$$

and finally the split second order integrals

$$\begin{aligned}
\mathcal{E}^{\rm sh}_{{\rm so};A}(t) &= -\int_{\tau_i(t)-1}^{\tau_i(t)} \partial_\xi S(\tau_i(t)-\tau)P\Upsilon_{\sigma;i}\big(\overline{V}_i(\tau)\big)\mathbf{1}_{\tau<\tau_i(t_{\rm st})}d\tau, \\
\mathcal{E}^{\rm sh}_{{\rm so};B}(t) &= \int_{\tau_i(t)-1}^{\tau_i(t)} \Lambda(\tau_i(t)-\tau)\Upsilon_{\sigma;i}\big(\overline{V}_i(\tau)\big)\mathbf{1}_{\tau<\tau_i(t_{\rm st})}d\tau, \\
\mathcal{E}^{\rm sh}_{{\rm so};C}(t) &= \int_{\tau_i(t)-1}^{\tau_i(t)} \partial_\xi S_{\rm od}(\tau_i(t)-\tau)\Upsilon_{\sigma;i}\big(\overline{V}_i(\tau)\big)\,\mathbf{1}_{\tau<\tau_i(t_{\rm st})}d\tau.
\end{aligned} \tag{3.5.16}$$

Here we use the convention that integrands are set to zero for $\tau < 0$. Note that integration variables in the original time are represented by s, while integration variables in the rescaled time are denoted by τ. For $\eta > 0$ sufficiently small, our stopping time ensures that the identities (3.4.21) hold. This implies that we may assume

$$P_\xi\Upsilon_{\sigma;i}\big(\overline{V}_i(\tau)\big) + P\mathcal{W}_{\sigma;i}\big(\overline{V}_i(\tau)\big) = 0. \tag{3.5.17}$$

This explains why there is a Q in the first two lines of (3.5.14), as their P-counterparts are canceled against the $S(\tau_i(t)-\tau)P_\xi$ term that is present in (3.4.27) but absent from (3.5.14).

For convenience, we also write

$$\mathcal{E}_{F;\#}(t) = \mathcal{E}^{\rm lt}_{F;\#}(t) + \mathcal{E}^{\rm sh}_{F;\#}(t) \tag{3.5.18}$$

for $\# \in \{\rm lin, nl\}$, together with

$$\mathcal{E}_{B;\#}(t) = \mathcal{E}^{\rm lt}_{B;\#}(t) + \mathcal{E}^{\rm sh}_{B;\#}(t) \tag{3.5.19}$$

for $\# \in \{\rm lin, cn\}$ and finally

$$\mathcal{E}^{\rm sh}_{\rm so}(t) = \mathcal{E}^{\rm sh}_{{\rm so};A}(t) + \mathcal{E}^{\rm sh}_{{\rm so};B}(t) + \mathcal{E}^{\rm sh}_{{\rm so};C}(t) \tag{3.5.20}$$

for the short-term second order terms.

Turning to the terms that are relevant for evaluating $N^i_{\varepsilon;II}$, we introduce the expression

$$\mathcal{I}_{\varepsilon,\delta;0}(t) = \int_0^t e^{-\varepsilon(t-s)}\|S(\delta)\mathcal{E}_0(s)\|^2_{H^1}\,ds, \tag{3.5.21}$$

together with

$$
\begin{aligned}
\mathcal{I}^{\#}_{\varepsilon,\delta;F;\mathrm{lin}}(t) &= \int_0^t e^{-\varepsilon(t-s)} \|S(\delta)\mathcal{E}^{\#}_{F;\mathrm{lin}}(s)\|^2_{H^1}\, ds, \\
\mathcal{I}^{\#}_{\varepsilon,\delta;F;\mathrm{nl}}(t) &= \int_0^t e^{-\varepsilon(t-s)} \|S(\delta)\mathcal{E}^{\#}_{F;\mathrm{nl}}(s)\|^2_{H^1}\, ds, \\
\mathcal{I}^{\#}_{\varepsilon,\delta;B;\mathrm{lin}}(t) &= \int_0^t e^{-\varepsilon(t-s)} \|S(\delta)\mathcal{E}^{\#}_{B;\mathrm{lin}}(s)\|^2_{H^1}\, ds, \\
\mathcal{I}^{\#}_{\varepsilon,\delta;B;\mathrm{cn}}(t) &= \int_0^t e^{-\varepsilon(t-s)} \|S(\delta)\mathcal{E}^{\#}_{B;\mathrm{cn}}(s)\|^2_{H^1}\, ds, \\
\mathcal{I}^{\#}_{\varepsilon,\delta;\mathrm{so}}(t) &= \int_0^t e^{-\varepsilon(t-s)} \|S(\delta)\mathcal{E}^{\#}_{\mathrm{so}}(s)\|^2_{H^1}\, ds
\end{aligned}
\tag{3.5.22}
$$

for $\# \in \{\mathrm{lt}, \mathrm{sh}\}$. The extra $S(\delta)$ factor will be used to ensure that all the integrals we encounter are well-defined. We emphasize that all our estimates are uniform in $0 < \delta < 1$, allowing us to take $\delta \downarrow 0$. The estimates concerning $\mathcal{I}^{\mathrm{sh}}_{\varepsilon,\delta;F;\mathrm{nl}}$ and $\mathcal{I}^{\mathrm{sh}}_{\varepsilon,\delta;B;\mathrm{lin}}$ in Lemmas 3.5.5 and 3.5.11 are particularly delicate in this respect, as a direct application of the bounds in Lemma 3.3.1 would result in expressions that diverge as $\delta \downarrow 0$.

The main difference between the approach here and the computations in §2.9 is that we need to keep track of several time transforms simultaneously, which forces us to use the original time t in the definitions (3.5.8)-(3.5.9). The following result plays a key role in this respect, as it shows that decay rates in the τ-variable are stronger than decay rates in the original time.

Lemma 3.5.2. *Assume that (HDt), (HSt) and (HTw) are satisfied and pick $0 \le \sigma \le \delta_\sigma$. Then for any pair $t > s \ge 0$ we have the inequality*

$$\tau_i(t) - \tau_i(s) \ge t - s, \tag{3.5.23}$$

while for any $s \ge t_i(1)$ we have

$$t_i(\tau_i(s) - 1) \ge s - 1. \tag{3.5.24}$$

Proof. The first inequality can be verified by using (3.4.5) to compute

$$
\begin{aligned}
\tau_i(t) - \tau_i(s) &= \int_s^t \kappa_{\sigma;i}(\Phi_\sigma + V(s'), \psi_{\mathrm{tw}}) ds' \\
&\ge (t-s) \min_{s \le s' \le t} \kappa_{\sigma;i}(\Phi_\sigma + V(s'), \psi_{\mathrm{tw}}) \\
&\ge t - s.
\end{aligned}
\tag{3.5.25}
$$

To obtain the second inequality, we write $\tilde{s} = t_i(1) \le 1$ and compute

$$\tau_i(s) - 1 = \tau_i(s) - \tau_i(\tilde{s}) \ge s - \tilde{s} \ge s - 1. \tag{3.5.26}$$

□

We now set out to bound all the terms appearing in $N^i_\varepsilon(t)$. Following Chapter 2, we first study the deterministic integrals and afterwards use H^∞-calculus to bound the stochastic integrals.

3.5.2 Deterministic Regularity Estimates

First, we collect some results from §2.9.2 that are easily adapted to the present situation.

Lemma 3.5.3. *Fix $T > 0$, assume that (HDt), (HSt) and (HTw) all hold and pick a constant $0 < \varepsilon < \beta$. Then for any $\eta > 0$, any $0 \le \delta < 1$ and any $0 \le t \le t_{\mathrm{st}}$, we have the bounds*

$$\begin{array}{lcl} \|\mathcal{E}_0(t)\|_{L^2}^2 & \le & M^2 e^{-2\beta t}\|V(0)\|_{L^2}^2, \\ \|\mathcal{E}_{F;\mathrm{lin}}(t)\|_{L^2}^2 & \le & K_\kappa^2 K_{F;\mathrm{lin}}^2 \dfrac{M^2}{2\beta-\varepsilon} N_{\varepsilon;II}(t), \\ \|\mathcal{E}_{F;\mathrm{nl}}(t)\|_{L^2}^2 & \le & \eta K_\kappa^2 K_{F;\mathrm{nl}}^2 M^2 (1+\eta^3)^2 N_{\varepsilon;II}(t), \end{array} \tag{3.5.27}$$

together with

$$\begin{array}{lcl} \mathcal{I}_{\varepsilon,\delta;0}(t) & \le & \dfrac{M^2}{2\beta-\varepsilon} e^{-\varepsilon t}\|V(0)\|_{H^1}^2, \\ \mathcal{I}^{\mathrm{lt}}_{\varepsilon,\delta;F;\mathrm{lin}}(t) & \le & K_\kappa^2 K_{F;\mathrm{lin}}^2 \dfrac{M^2}{2(\beta-\varepsilon)\varepsilon} N_{\varepsilon;II}(t), \\ \mathcal{I}^{\mathrm{sh}}_{\varepsilon,\delta;F;\mathrm{lin}}(t) & \le & 4e^{\varepsilon} M^2 K_\kappa K_{F;\mathrm{lin}}^2 N_{\varepsilon;II}(t), \\ \mathcal{I}^{\mathrm{lt}}_{\varepsilon,\delta;F;\mathrm{nl}}(t) & \le & \eta K_\kappa^2 K_{F;\mathrm{nl}}^2 (1+\eta^3)^2 \dfrac{M^2}{\beta-\varepsilon} N_{\varepsilon;II}(t). \end{array} \tag{3.5.28}$$

Proof. Observe first that

$$\|\mathcal{E}_{F;\mathrm{lin}}(t)\|_{L^2}^2 \quad \le \quad K_{F;\mathrm{lin}}^2 M^2 \left(\int_0^{\tau_i(t)} e^{-\beta(\tau_i(t)-\tau)} \|\overline{V}_i(\tau)\|_{H^1}\, d\tau \right)^2. \tag{3.5.29}$$

Substituting $s = t_i(\tau)$ we find

$$\|\mathcal{E}_{F;\mathrm{lin}}(t)\|_{L^2}^2 \le K_{F;\mathrm{lin}}^2 M^2 \left(\int_0^{t} e^{-(\beta-\frac{\varepsilon}{2})(\tau_i(t)-\tau_i(s))} e^{-\frac{\varepsilon}{2}(\tau_i(t)-\tau_i(s))} \|V(s)\|_{H^1} \tau_i'(s)\, ds \right)^2. \tag{3.5.30}$$

Applying (3.5.23) and using (3.4.5) to bound the extra integration factor $\tau_i'(s)$ by K_κ, we obtain

$$\|\mathcal{E}_{F;\mathrm{lin}}(t)\|_{L^2}^2 \quad \le \quad K_\kappa^2 K_{F;\mathrm{lin}}^2 M^2 \left(\int_0^{t} e^{-(\beta-\frac{\varepsilon}{2})(t-s)} e^{-\frac{\varepsilon}{2}(t-s)} \|V(s)\|_{H^1}\, ds \right)^2. \tag{3.5.31}$$

Cauchy-Schwarz now yields the desired bound

$$\begin{array}{lcl} \|\mathcal{E}_{F;\mathrm{lin}}(t)\|_{L^2}^2 & \le & K_\kappa^2 K_{F;\mathrm{lin}}^2 \dfrac{M^2}{2\beta-\varepsilon} \displaystyle\int_0^t e^{-\varepsilon(t-s)} \|V(s)\|_{H^1}^2\, ds \\ & = & K_\kappa^2 K_{F;\mathrm{lin}}^2 \dfrac{M^2}{2\beta-\varepsilon} N_{\varepsilon;II}(t). \end{array} \tag{3.5.32}$$

The remaining estimates follow in an analogous fashion by making similar small adjustments to the proofs of Lemmas 2.9.9-2.9.11. □

Our next result discusses the novel second order terms. The crucial ingredient here is that we no longer have to consider the dangerous $\partial_\xi S(t_i(\tau) - \tau) Q \Upsilon_{\sigma;i}\big(\overline{V}(\tau)\big)$ term for $\tau \geq t_i(\tau) - 1$. Indeed, this term need not be integrable even in L^2 because of the divergent $(\tau_i(t) - \tau)^{-1/2}$ behavior of $\partial_\xi S$ and the fact that we only have square-integrable control of the H^1-norm of $\overline{V}_i(\tau)$.

Lemma 3.5.4. *Fix $T > 0$ and assume that (HDt), (HSt) and (HTw) all hold. Pick a constant $0 < \varepsilon < 2\beta$. Then for any $0 \leq \delta < 1$ and any $0 \leq t \leq t_{\mathrm{st}}$, we have the bounds*

$$\begin{array}{lcl} \|\mathcal{E}^{\mathrm{sh}}_{\mathrm{so}}(t)\|^2_{L^2} & \leq & 9\sigma^4 e^{2\beta} K^2 K_\kappa M^2 N_{\varepsilon;II}(t), \\ \|\mathcal{E}^{\mathrm{lt}}_{\mathrm{so}}(t)\|^2_{L^2} & \leq & 4\sigma^4 K^2 K_\kappa \dfrac{M^2}{2\beta - \varepsilon} N_{\varepsilon;II}(t), \end{array} \qquad (3.5.33)$$

together with

$$\begin{array}{lcl} \mathcal{I}^{\mathrm{sh}}_{\varepsilon,\delta;\mathrm{so}}(t) & \leq & 9\sigma^4 e^{2\beta} K^2 K_\kappa M^2 N_{\varepsilon;II}(t) \\ \mathcal{I}^{\mathrm{lt}}_{\varepsilon,\delta;\mathrm{so}}(t) & \leq & 4\sigma^4 K^2 K_\kappa \dfrac{M^2}{2(\beta - \varepsilon)\varepsilon} N_{\varepsilon;II}(t). \end{array} \qquad (3.5.34)$$

Proof. For $\tau \geq \tau_i(t) - 1$ we may use Lemma 3.3.1 together with Proposition 3.4.1 to obtain the estimate

$$\begin{array}{lcl} \|\partial_\xi S(\tau_i(t) - \tau) P \Upsilon_{\sigma;i}\big(\overline{V}_i(\tau)\big)\|_{H^1} & \leq & \sigma^2 K M \|\overline{V}_i(\tau)\|_{H^1} \\ & \leq & e^\beta \sigma^2 K M e^{-\beta(\tau_i(t) - \tau)} \|\overline{V}_i(\tau)\|_{H^1}. \end{array} \qquad (3.5.35)$$

In the same fashion we obtain

$$\begin{array}{lcl} \|\Lambda(\tau_i(t) - \tau) \Upsilon_{\sigma;i}\big(\overline{V}_i(\tau)\big)\|_{H^1} & \leq & e^\beta \sigma^2 K M e^{-\beta(\tau_i(t) - \tau)} \|\overline{V}_i(\tau)\|_{H^1}, \\ \|\partial_\xi S_{\mathrm{od}}(\tau_i(t) - \tau) \Upsilon_{\sigma;i}\big(\overline{V}_i(\tau)\big)\|_{H^1} & \leq & e^\beta \sigma^2 K M e^{-\beta(\tau_i(t) - \tau)} \|\overline{V}_i(\tau)\|_{H^1}. \end{array} \qquad (3.5.36)$$

In addition, for $\tau \leq \tau_i(t) - 1$ we obtain

$$\|\big[\partial_\xi S(\tau_i(t) - \tau) Q + \Lambda(\tau_i(t) - \tau)\big] \Upsilon_{\sigma;i}\big(\overline{V}_i(\tau)\big)\|_{H^1} \leq 2KM\sigma^2 \|\overline{V}_i(\tau)\|_{H^1} e^{-\beta(\tau_i(t) - \tau)}. \qquad (3.5.37)$$

The desired estimates can hence be obtained in the same fashion as the bounds for $\mathcal{E}_{F;\mathrm{lin}}(t)$ and $\mathcal{I}^{\mathrm{lt}}_{\varepsilon,\delta;F;\mathrm{lin}}(t)$ in Lemma 3.5.3 . □

The following results at times do require the computations in Chapter 2 to be modified in a subtle non-trivial fashion. We therefore provide full proofs here, noting however that the main ideas remain unchanged.

Lemma 3.5.5. *Fix $T > 0$ and assume that (HDt), (HSt) and (HTw) all hold. Pick a constant $\varepsilon > 0$. Then for any $\eta > 0$, any $0 \leq \delta < 1$ and any $0 \leq t \leq t_{\mathrm{st}}$, we have the bound*

$$\mathcal{I}^{\mathrm{sh}}_{\varepsilon,\delta;F,\mathrm{nl}}(t) \quad \leq \quad \eta M^2 e^{3\varepsilon} K_\kappa^2 K_{F;\mathrm{nl}}^2 (1 + \eta^3)^2 (1 + \rho_{\min}^{-1})(3K_\kappa + 2) N_{\varepsilon;II}(t). \qquad (3.5.38)$$

Proof. We first introduce the inner product

$$\langle v, w \rangle_{H^1_\rho} = \langle v, w \rangle_{L^2} + \langle \sqrt{\rho}\partial_\xi v, \sqrt{\rho}\partial_\xi w \rangle_{L^2} \tag{3.5.39}$$

and note that

$$\|v\|^2_{H^1} \leq \|v\|^2_{L^2} + \rho_{\min}^{-1}\|\sqrt{\rho}\partial_\xi v\|^2_{L^2} \leq \big(1 + \rho_{\min}^{-1}\big)\langle v, v\rangle_{H^1_\rho}. \tag{3.5.40}$$

For $\# \in \{L^2, H^1_\rho\}$ we introduce the expression

$$\mathcal{E}_{\tau,\tau',\tau'';\#} = \Big\langle S(\tau+\delta-\tau')QF_{\mathrm{nl}}\big(\overline{V}_i(\tau')\big), S(\tau+\delta-\tau'')QF_{\mathrm{nl}}\big(\overline{V}_i(\tau'')\big)\Big\rangle_{\#}, \tag{3.5.41}$$

which allows us to obtain the estimate

$$\begin{aligned} \mathcal{I}^{\mathrm{sh}}_{\varepsilon,\delta;F,\mathrm{nl}}(t) &\leq \big(1+\rho_{\min}^{-1}\big)\int_0^t e^{\varepsilon(t-s)}\int_{\tau_i(s)-1}^{\tau_i(s)}\int_{\tau_i(s)-1}^{\tau_i(s)} \mathcal{E}_{\tau_i(s),\tau',\tau'';H^1_\rho}\, d\tau'' d\tau' ds \\ &\leq \big(1+\rho_{\min}^{-1}\big)\int_0^t e^{\varepsilon(t-s)}\big[t_i'\big(\tau_i(s)\big)\big]^{-1}\int_{\tau_i(s)-1}^{\tau_i(s)}\int_{\tau_i(s)-1}^{\tau_i(s)} \mathcal{E}_{\tau_i(s),\tau',\tau'';H^1_\rho}\, d\tau'' d\tau' ds. \end{aligned} \tag{3.5.42}$$

The extra term involving the function t_i', which takes values in $[K_\kappa^{-1}, 1]$, was included for technical reasons that will become clear in what follows.

For any $v, w \in L^2$, $\vartheta > 0$, $\vartheta_A \geq -\vartheta$ and $\vartheta_B \geq \vartheta$, we have

$$\begin{aligned} \frac{d}{d\vartheta}\langle S(\vartheta+\vartheta_A)v, S(\vartheta+\vartheta_B)w\rangle_{L^2} &= \langle \mathcal{L}_{\mathrm{tw}} S(\vartheta+\vartheta_A)v, S(\vartheta+\vartheta_B)w\rangle_{L^2} \\ &\quad + \langle S(\vartheta+\vartheta_A)v, \mathcal{L}_{\mathrm{tw}} S(\vartheta+\vartheta_B)w\rangle_{L^2} \\ &= \langle S(\vartheta+\vartheta_A)v, \mathcal{L}^*_{\mathrm{tw}} S(\vartheta+\vartheta_B)w\rangle_{L^2} \\ &\quad + \langle S(\vartheta+\vartheta_A)v, \mathcal{L}_{\mathrm{tw}} S(\vartheta+\vartheta_B)w\rangle_{L^2} \\ &= \langle S(\vartheta+\vartheta_A)v, \big[\mathcal{L}^*_{\mathrm{tw}} - \rho\partial_{\xi\xi}\big] S(\vartheta+\vartheta_B)w\rangle_{L^2} \\ &\quad + \langle S(\vartheta+\vartheta_A)v,\ \big[\mathcal{L}_{\mathrm{tw}} - \rho\partial_{\xi\xi}\big] S(\vartheta+\vartheta_B)w\rangle_{L^2} \\ &\quad - 2\langle \sqrt{\rho}\partial_\xi S(\vartheta+\vartheta_A)v, \sqrt{\rho}\partial_\xi S(\vartheta+\vartheta_B)w\rangle_{L^2}. \end{aligned} \tag{3.5.43}$$

Upon taking $\delta > 0$ for the moment and choosing $v = QF_{\mathrm{nl}}\big(\overline{V}_i(\tau')\big)$, $w = QF_{\mathrm{nl}}(\overline{V}_i(\tau''))$, $\vartheta = \tau_i(s) + \delta$, $\vartheta_A = -\tau'$ and $\vartheta_B = -\tau''$, we may rearrange (3.5.43) to obtain the estimate

$$\begin{aligned} \mathcal{E}_{\tau_i(s),\tau',\tau'';H^1_\rho} &\leq M^2 K^2_{F;\mathrm{nl}}(1+\eta^3)^2\|\overline{V}_i(\tau')\|^2_{H^1_\rho}\|\overline{V}_i(\tau'')\|^2_{H^1} \\ &\quad + M^2 K^2_{F;\mathrm{nl}}(1+\eta^3)^2\frac{1}{\sqrt{\tau_i(s)+\delta-\tau''}}\|\overline{V}_i(\tau')\|^2_{H^1}\|\overline{V}_i(\tau'')\|^2_{H^1} \\ &\quad - \frac{1}{2}\partial_1\mathcal{E}_{\tau_i(s),\tau',\tau'';L^2} \end{aligned} \tag{3.5.44}$$

for the values of (s, τ', τ'') that are relevant.

3

Upon introducing the integrals

$$\begin{aligned}\mathcal{I}_I &= \int_0^t e^{-\varepsilon(t-s)}\big[t_i'\big(\tau_i(s)\big)\big]^{-1}\int_{\tau_i(s)-1}^{\tau_i(s)}\int_{\tau_i(s)-1}^{\tau_i(s)} \\ &\qquad [1+\frac{1}{\sqrt{\tau_i(s)+\delta-\tau''}}]\|\overline{V}_i(\tau')\|_{H^1}^2\|\overline{V}_i(\tau'')\|_{H^1}^2\,d\tau''\,d\tau'\,ds, \\ \mathcal{I}_{II} &= \int_0^t e^{-\varepsilon(t-s)}\big[t_i'\big(\tau_i(s)\big)\big]^{-1}\int_{\tau_i(s)-1}^{\tau_i(s)}\int_{\tau_i(s)-1}^{\tau_i(s)}\partial_1\mathcal{E}_{\tau_i(s),\tau',\tau'';L^2}\,d\tau''\,d\tau'\,ds,\end{aligned} \tag{3.5.45}$$

we hence readily obtain the estimate

$$\mathcal{I}^{\mathrm{sh}}_{\varepsilon,\delta;B;\mathrm{nl}}(t) \;\leq\; (1+\rho_{\min}^{-1})M^2K_{F;\mathrm{nl}}^2(1+\eta^3)^2\mathcal{I}_I-\frac{1}{2}(1+\rho_{\min}^{-1})\mathcal{I}_{II}. \tag{3.5.46}$$

Using Lemma 3.5.2 we see that

$$\mathcal{I}_I \;\leq\; K_\kappa^3\int_0^t e^{-\varepsilon(t-s)}\int_{s-1}^{s}\int_{s-1}^{s}[1+\frac{1}{\sqrt{s+\delta-s''}}]\|V(s')\|_{H^1}^2\|V(s'')\|_{H^1}^2\,ds''\,ds'\,ds, \tag{3.5.47}$$

which allows us to repeat the computation (2.9.68) and conclude

$$\mathcal{I}_I\leq 3\eta e^{3\varepsilon}K_\kappa^3N_{\varepsilon;II}(t). \tag{3.5.48}$$

To understand $\mathcal{I}_{II}$ it is essential to change the order of integration and integrate with respect to s before switching τ' and τ'' back to the original time. Rearranging the integrals in (3.5.45), we find

$$\mathcal{I}_{II}=\int_0^{\tau_i(t)}e^{-\varepsilon t}\int_{\max\{0,\tau'-1\}}^{\min\{\tau_i(t),\tau'+1\}}\int_{\tau^-(\tau',\tau'')}^{\tau^+(\tau',\tau'')}\frac{e^{\varepsilon s}}{t_i'\big(\tau_i(s)\big)}\partial_1\mathcal{E}_{\tau_i(s),\tau',\tau'';L^2}\,ds\,d\tau''\,d\tau', \tag{3.5.49}$$

where we introduced the notation

$$\tau^+(\tau',\tau'')=\min\{\tau_i(t),\tau'+1,\tau''+1\},\quad \tau^-(\tau',\tau'')=\max\{\tau',\tau''\}. \tag{3.5.50}$$

The substitution $\tau=\tau_i(s)$ now yields

$$\mathcal{I}_{II} \;=\; \int_0^{\tau_i(t)}e^{-\varepsilon t}\int_{\max\{0,\tau'-1\}}^{\min\{\tau_i(t),\tau'+1\}}\Big[\int_{\tau^-(\tau',\tau'')}^{\tau^+(\tau',\tau'')}e^{\varepsilon t_i(\tau)}\partial_1\mathcal{E}_{\tau,\tau',\tau'';L^2}\,d\tau\Big]\,d\tau''\,d\tau'. \tag{3.5.51}$$

We emphasize here that the integration factor associated to this substitution cancels out against the additional term introduced in (3.5.42). Integrating by parts, we find

$$\mathcal{I}_{II} \;=\; \mathcal{I}_{II;A}+\mathcal{I}_{II;B}+\mathcal{I}_{II;C} \tag{3.5.52}$$

in which we have introduced

$$\begin{aligned}
\mathcal{I}_{II;A} &= \int_0^{\tau_i(t)} e^{-\varepsilon t} \int_{\max\{0,\tau'-1\}}^{\min\{\tau_i(t),\tau'+1\}} e^{\varepsilon t_i(\tau)} \mathcal{E}_{\tau,\tau',\tau'';L^2}\big|_{\tau=\tau^+(\tau',\tau'')} \, d\tau'' d\tau', \\
\mathcal{I}_{II;B} &= -\int_0^{\tau_i(t)} e^{-\varepsilon t} \int_{\max\{0,\tau'-1\}}^{\min\{\tau_i(t),\tau'+1\}} e^{\varepsilon t_i(\tau)} \mathcal{E}_{\tau,\tau',\tau'';L^2}\big|_{\tau=\tau^-(\tau',\tau'')} d\tau'' d\tau', \\
\mathcal{I}_{II;C} &= -\int_0^{\tau_i(t)} e^{-\varepsilon t} \int_{\max\{0,\tau'-1\}}^{\min\{\tau_i(t),\tau'+1\}} \left[\int_{\tau^-(\tau',\tau'')}^{\tau^+(\tau',\tau'')} \left(\frac{d}{d\tau} e^{\varepsilon t_i(\tau)} \right) \mathcal{E}_{\tau,\tau',\tau'';L^2} d\tau \right] d\tau'' d\tau'.
\end{aligned} \tag{3.5.53}$$

Note here that $\mathcal{I}_{II;B}$ is well defined because $\delta > 0$.

Using the substitutions

$$s' = t_i(\tau'), \quad s'' = t_i(\tau'') \tag{3.5.54}$$

together with the bound

$$\begin{aligned}
t_i\big(\tau_-(\tau',\tau'')\big) &\leq t_i\big(\tau^+(\tau',\tau'')\big) \\
&\leq \min\{t, t_i(\tau'+1), t_i(\tau''+1)\} \\
&\leq \min\{t, t_i(\tau')+1, t_i(\tau'')+1\}, \\
&\leq \min\{s', s''\}+1 \\
&\leq s'+1,
\end{aligned} \tag{3.5.55}$$

we find

$$\int_{\tau^-(\tau',\tau'')}^{\tau^+(\tau',\tau'')} \left| \frac{d}{d\tau} e^{\varepsilon t_i(\tau)} \right| d\tau = \int_{\tau^-(\tau',\tau'')}^{\tau^+(\tau',\tau'')} \frac{d}{d\tau} e^{\varepsilon t_i(\tau)} d\tau = e^{\varepsilon t_i(\tau)}\big|_{\tau^-(\tau',\tau'')}^{\tau^+(\tau',\tau'')} \leq 2 e^{\varepsilon} e^{\varepsilon s'}. \tag{3.5.56}$$

Applying Cauchy-Schwarz to the inner product $\mathcal{E}$, we hence obtain

$$|\mathcal{I}_{II}| \leq 4 e^{\varepsilon} M^2 K_\kappa^2 K_{F;\mathrm{nl}}^2 (1+\eta^3)^2 \int_0^t e^{-\varepsilon(t-s')} \|V(s')\|_{H^1}^2 \mathcal{J}(s') \, ds', \tag{3.5.57}$$

in which we have introduced the function

$$\mathcal{J}(s') = \int_{\max\{0,t_i(\tau_i(s')-1)\}}^{\min\{t,t_i(\tau_i(s')+1)\}} \|V(s'')\|_{H^1}^2 \, ds''. \tag{3.5.58}$$

Exploiting Lemma 3.5.2 again, we can bound

$$\begin{aligned}
\mathcal{J}(s') &\leq \int_{\max\{0,s'-1\}}^{\min\{t,s'+1\}} \|V(s'')\|_{H^1}^2 \, ds'' \\
&\leq \int_{\max\{0,s'-1\}}^{\min\{t,s'+1\}} e^{2\varepsilon} e^{-\varepsilon(\min\{t,s'+1\}-s'')} \|V(s'')\|_{H^1}^2 \, ds'' \\
&\leq e^{2\varepsilon}\eta,
\end{aligned} \tag{3.5.59}$$

which hence gives

$$|\mathcal{I}_{II}| \quad \leq \quad 4\eta e^{3\varepsilon} M^2 K_\kappa^2 K_{F;\mathrm{nl}}^2 (1+\eta^3)^2 N_{\varepsilon;II}(t), \tag{3.5.60}$$

as desired. It hence remains to consider the case $\delta = 0$. We may apply Fatou's Lemma to conclude

$$\begin{array}{rcl} \mathcal{I}^{\mathrm{sh}}_{\varepsilon,0;F;\mathrm{nl}}(t) & = & \int_0^t e^{\varepsilon(t-s)} (\lim_{\delta\to 0} \|S(\delta)\mathcal{E}^{\mathrm{sh}}_{B;\mathrm{lin}}(s)\|_{H^1})^2 \mathbf{1}_{s<t_{\mathrm{st}}}\, ds \\ & \leq & \liminf_{\delta\to 0} \mathcal{I}^{\mathrm{sh}}_{\varepsilon,\delta;F;\mathrm{nl}}(t). \end{array} \tag{3.5.61}$$

The result now follows from the fact that the bounds obtained above do not depend on δ. □

3

3.5.3 Stochastic Regularity Estimates

We are now ready to discuss the stochastic integrals. These require special care because they cannot be bounded in a pathwise fashion, unlike the deterministic integrals above. Expectations of suprema are particularly delicate in this respect. Indeed, the powerful Burkholder-Davis-Gundy inequalities cannot be directly applied to the stochastic convolutions that arise in our mild formulation. However, as was shown in Lemma 2.9.7, we can obtain an H^∞-calculus for our linear operator $\mathcal{L}_{\mathrm{tw}}$ which allows us to use the following mild version, which is the source of the extra T factors that appear in our estimates.

Lemma 3.5.6. *Fix $T > 0$ and assume that (HDt), (HSt) and (HTw) all hold. There exists a constant $K_{\mathrm{cnv}} > 0$ so that for any $W \in \mathcal{N}^2([0,T]; (\mathcal{F})_t; L^2)$ we have*

$$E \sup_{0\leq t\leq T} \| \int_0^t S(t-s) Q W(s)\, d\beta_s\|^2_{L^2} \leq K_{\mathrm{cnv}} E \int_0^T \|W(s)\|^2_{L^2}\, ds. \tag{3.5.62}$$

Proof. This is a direct result of the computations in §2.9.1, which are based on the main theorem of [111]. □

Lemma 3.5.7. *Fix $T > 0$ and assume that (HDt), (HSt), and (HTw) all hold. Then for any $\varepsilon > 0$ we have the bound*

$$E \sup_{0\leq t\leq t_{\mathrm{st}}} \|\mathcal{E}_{B;\mathrm{lin}}(t)\|^2_{L^2} \quad \leq \quad (T+1) K_{\mathrm{cnv}} K^2_{B;\mathrm{lin}} e^{\varepsilon} E \sup_{0\leq t\leq t_{\mathrm{st}}} N^i_{\varepsilon;II}(t). \tag{3.5.63}$$

Proof. Using Lemma 3.5.6 we compute

$$\begin{array}{rcl} E \sup_{0\leq t\leq t_{\mathrm{st}}} \|\mathcal{E}_{B;\mathrm{lin}}(t)\|^2_{L^2} & \leq & E \sup_{0\leq t\leq T} \|\mathcal{E}_{B;\mathrm{lin}}(t)\|^2_{L^2} \\ & = & E \sup_{0\leq\tau\leq\tau_i(T)} \| \int_0^\tau S(\tau-\tau') Q B_{\mathrm{lin}}\big(\overline{V}_i(\tau')\big) \mathbf{1}_{\tau'<\tau_i(t_{\mathrm{st}})}\, d\beta_{\tau'} \|^2_{L^2} \\ & \leq & K_{\mathrm{cnv}} E \int_0^{\tau_i(T)} \|B_{\mathrm{lin}}\big(\overline{V}_i(\tau)\big)\mathbf{1}_{\tau<\tau_i(t_{\mathrm{st}})}\|^2_{L^2}\, d\tau \\ & \leq & K_\kappa K_{\mathrm{cnv}} K^2_{B;\mathrm{lin}} E \int_0^{t_{\mathrm{st}}} \|V(s)\|^2_{H^1}\, ds. \end{array} \tag{3.5.64}$$

By dividing up the integral, we obtain

$$\begin{aligned}
\int_0^{t_{\mathrm{st}}} \|V(s)\|_{H^1}^2 \, ds &\leq e^{\varepsilon} \int_0^1 e^{-\varepsilon(1-s)} \|V(s)\|_{H^1}^2 \mathbf{1}_{s<t_{\mathrm{st}}} \, ds \\
&\quad + e^{\varepsilon} \int_1^2 e^{-\varepsilon(2-s)} \|V(s)\|_{H^1}^2 \mathbf{1}_{s<t_{\mathrm{st}}} \, ds \\
&\quad + \ldots + e^{\varepsilon} \int_{\lfloor T \rfloor}^{\lfloor T \rfloor + 1} e^{-\varepsilon(\lfloor T \rfloor + 1 - s)} \|V(s)\|_{H^1}^2 \mathbf{1}_{s<t_{\mathrm{st}}} \, ds \\
&\leq (T+1) e^{\varepsilon} \sup_{0 \leq t \leq T+1} \int_0^t e^{-\varepsilon(t-s)} \|V(s)\|_{H^1}^2 \mathbf{1}_{s<t_{\mathrm{st}}} \, ds \\
&\leq (T+1) e^{\varepsilon} \sup_{0 \leq t \leq t_{\mathrm{st}}} \int_0^t e^{-\varepsilon(t-s)} \|V(s)\|_{H^1}^2 \, ds \\
&= (T+1) e^{\varepsilon} \sup_{0 \leq t \leq t_{\mathrm{st}}} N_{\varepsilon;II}(t),
\end{aligned} \tag{3.5.65}$$

which yields the desired bound upon taking expectations. □

Lemma 3.5.8. *Fix $T > 0$ and assume that (HDt), (HSt) and (HTw) all hold. Then we have the bound*

$$E \sup_{0 \leq t \leq t_{\mathrm{st}}} \|\mathcal{E}_{B;\mathrm{cn}}(t)\|_{L^2}^2 \leq T K_{\mathrm{cnv}} K_{B;\mathrm{cn}}^2. \tag{3.5.66}$$

Proof. This bound follows directly from (3.5.64) by making the substitutions

$$K_{B;\mathrm{lin}} \mapsto K_{B;\mathrm{cn}}, \qquad \|V(s)\|_{H^1}^2 \mapsto 1. \tag{3.5.67}$$

□

We now set out to bound the expectation of the suprema of the remaining double integrals $\mathcal{I}^{\#}_{\varepsilon,\delta;B;\mathrm{lin}}(t)$ and $\mathcal{I}^{\#}_{\varepsilon,\delta;B;\mathrm{cn}}(t)$ with $\# \in \{\mathrm{lt}, \mathrm{sh}\}$. This is performed in Lemma 3.5.13, but we first compute several time independent bounds for the expectation of the integrals themselves.

Lemma 3.5.9. *Fix $T > 0$ and assume that (HDt), (HSt) and (HTw) all hold. Pick a constant $\varepsilon > 0$. Then for any $0 \leq \delta < 1$ and $0 \leq t \leq T$, we have the identities*

$$\begin{aligned}
E\mathcal{I}^{\mathrm{lt}}_{\varepsilon,\delta;B;\mathrm{lin}}(t) &= E\int_0^t e^{-\varepsilon(t-s)} \int_0^{\tau_i(s)-1} \|S(\tau_i(s)+\delta-\tau')QB_{\mathrm{lin}}\big(\overline{V}_i(\tau')\big)\|_{H^1}^2 \mathbf{1}_{\tau'<\tau_i(t_{\mathrm{st}})} d\tau' ds, \\
E\mathcal{I}^{\mathrm{lt}}_{\varepsilon,\delta;B;\mathrm{cn}}(t) &= E\int_0^t e^{-\varepsilon(t-s)} \int_0^{\tau_i(s)-1} \|S(\tau_i(s)+\delta-\tau')QB_{\mathrm{cn}}\|_{H^1}^2 \mathbf{1}_{\tau'<\tau_i(t_{\mathrm{st}})} d\tau' ds
\end{aligned} \tag{3.5.68}$$

and their short-time counterparts

$$\begin{aligned}
E\mathcal{I}^{\mathrm{sh}}_{\varepsilon,\delta;B;\mathrm{lin}}(t) &= E\int_0^t e^{-\varepsilon(t-s)} \int_{\tau_i(s)-1}^{\tau_i(s)} \|S(\tau_i(s)+\delta-\tau')QB_{\mathrm{lin}}\big(\overline{V}_i(\tau')\big)\|_{H^1}^2 \mathbf{1}_{\tau'<\tau_i(t_{\mathrm{st}})} d\tau' ds, \\
E\mathcal{I}^{\mathrm{sh}}_{\varepsilon,\delta;B;\mathrm{cn}}(t) &= E\int_0^t e^{-\varepsilon(t-s)} \int_{\tau_i(s)-1}^{\tau_i(s)} \|S(\tau_i(s)+\delta-\tau')QB_{\mathrm{cn}}\|_{H^1}^2 \mathbf{1}_{\tau'<\tau_i(t_{\mathrm{st}})} d\tau' ds.
\end{aligned} \tag{3.5.69}$$

Proof. This follows directly from the Itô Isometry, see also Lemma 2.9.16. □

Lemma 3.5.10. *Fix $T > 0$, assume that (HDt), (HSt) and (HTw) all hold and pick a constant $0 < \varepsilon < 2\beta$. Then for any $0 \le \delta < 1$ and any $0 \le t \le T$, we have the bound*

$$E\mathcal{I}^{\mathrm{lt}}_{\varepsilon,\delta;B;\mathrm{lin}}(t) \quad \le \quad \frac{M^2}{2\beta - \varepsilon} K_\kappa K^2_{B;\mathrm{lin}} E N_{\varepsilon;II}(t \wedge t_{\mathrm{st}}). \tag{3.5.70}$$

Proof. Using (3.5.68) and switching the integration order, we obtain

$$\begin{aligned}
E\mathcal{I}^{\mathrm{lt}}_{\varepsilon,\delta;B;\mathrm{lin}}(t) &\le M^2 K^2_{B;\mathrm{lin}} E \int_0^t e^{-\varepsilon(t-s)} \int_0^{\tau_i(s)\wedge\tau_i(t_{\mathrm{st}})} e^{-2\beta(\tau_i(s)-\tau')} \|\overline{V}_i(\tau')\|^2_{H^1}\, d\tau'\, ds \\
&\le M^2 K_\kappa K^2_{B;\mathrm{lin}} E \int_0^t e^{-\varepsilon(t-s)} \int_0^{s\wedge t_{\mathrm{st}}} e^{-2\beta(s-s')} \|V(s')\|^2_{H^1}\, ds'\, ds \\
&= M^2 K_\kappa K^2_{B;\mathrm{lin}} E \int_0^{t\wedge t_{\mathrm{st}}} e^{-\varepsilon t} \Big[\int_{s'}^t e^{-(2\beta-\varepsilon)s}\, ds \Big] e^{2\beta s'} \|V(s')\|^2_{H^1}\, ds' \\
&\le \frac{M^2}{2\beta-\varepsilon} K_\kappa K^2_{B;\mathrm{lin}} E \int_0^{t\wedge t_{\mathrm{st}}} e^{-\varepsilon t} e^{-(2\beta-\varepsilon)s'} e^{2\beta s'} \|V(s')\|^2_{H^1}\, ds' \\
&\le \frac{M^2}{2\beta-\varepsilon} K_\kappa K^2_{B;\mathrm{lin}} E \int_0^{t\wedge t_{\mathrm{st}}} e^{-\varepsilon(t\wedge t_{\mathrm{st}} - s')} \|V(s')\|^2_{H^1}\, ds' \\
&= \frac{M^2}{2\beta-\varepsilon} K_\kappa K^2_{B;\mathrm{lin}} E N_{\varepsilon;II}(t\wedge t_{\mathrm{st}}).
\end{aligned} \tag{3.5.71}$$

□

Lemma 3.5.11. *Fix $T > 0$ and assume that (HDt), (HSt) and (HTw), all hold. Pick a constant $\varepsilon > 0$. Then for any $0 \le \delta < 1$, and any $0 \le t \le T$, we have the bound*

$$E\mathcal{I}^{\mathrm{sh}}_{\varepsilon,\delta;B;\mathrm{lin}}(t) \quad \le \quad K_\kappa K^2_{B;\mathrm{lin}} M^2 (1 + \rho^{-1}_{\min}) e^{\varepsilon} (3K_\kappa + 2) E N_{\varepsilon;II}(t \wedge t_{\mathrm{st}}). \tag{3.5.72}$$

Proof. We only consider the case $\delta > 0$ here, noting that the limit $\delta \downarrow 0$ can be handled as in the proof of Lemma 3.5.5. Applying the identity (3.5.43) with $w = v$ and $\vartheta_A = \vartheta_B$, we obtain

$$\begin{aligned}
\frac{d}{d\vartheta} \|S(\vartheta + \vartheta_A)v\|^2_{L^2} \quad = \quad & \langle S(\vartheta+\vartheta_A)v, \big[\mathcal{L}^*_{\mathrm{tw}} - \rho\partial_{\xi\xi}\big] S(\vartheta+\vartheta_A)v\rangle_{L^2} \\
& + \langle S(\vartheta+\vartheta_A)v,\ \big[\mathcal{L}_{\mathrm{tw}} - \rho\partial_{\xi\xi}\big] S(\vartheta+\vartheta_A)v\rangle_{L^2} \\
& - 2\|\sqrt{\rho}\,\partial_\xi S(\vartheta+\vartheta_A)v\|^2_{L^2}.
\end{aligned} \tag{3.5.73}$$

Recalling the inner product (3.5.39) and introducing the expression

$$\mathcal{E}_{\tau,\tau';\#} = \|S(\tau + \delta - \tau') Q B_{\mathrm{lin}}\big(\overline{V}_i(\tau')\big)\|^2_{\#} \tag{3.5.74}$$

for $\# \in \{L^2, H^1_\rho\}$, we obtain the bound

$$\begin{array}{rcl}\mathcal{E}_{\tau,\tau';H^1_\rho} & \leq & M^2 K^2_{B;\mathrm{lin}} \|\overline{V}_i(\tau')\|^2_{H^1} + M^2 K^2_{B;\mathrm{lin}} \frac{1}{\sqrt{\tau_i(s)+\delta-\tau'}} \|\overline{V}_i(\tau')\|^2_{H^1} \\ & & -\frac{1}{2}\partial_1 \mathcal{E}_{\tau,\tau';L^2}\end{array} \tag{3.5.75}$$

for the values of (s,τ') that are relevant below. Upon writing

$$\begin{aligned}\mathcal{I}_I &= E\int_0^t e^{-\varepsilon(t-s)}\big[t_i'\big(\tau_i(s)\big)\big]^{-1}\int_{\tau_i(s)-1}^{\tau_i(s)}\big[1+\frac{1}{\sqrt{\tau_i(s)+\delta-\tau'}}\big]\|\overline{V}_i(\tau')\|^2_{H^1}\mathbf{1}_{\tau'<\tau_i(t_{\mathrm{st}})}\,d\tau' ds,\\ \mathcal{I}_{II} &= E\int_0^t e^{-\varepsilon(t-s)}\big[t_i'\big(\tau_i(s)\big)\big]^{-1}\int_{\tau_i(s)-1}^{\tau_i(s)}\partial_1\mathcal{E}_{\tau_i(s),\tau';L^2}\mathbf{1}_{\tau'<\tau_i(t_{\mathrm{st}})}\,d\tau' ds,\end{aligned} \tag{3.5.76}$$

we obtain the estimate

$$E\,\mathcal{I}^{\mathrm{sh}}_{\nu,\delta;B;\mathrm{lin}}(t) \quad\leq\quad (1+\rho_{\min}^{-1})M^2K^2_{B;\mathrm{lin}}\mathcal{I}_I - \frac{1}{2}(1+\rho_{\min}^{-1})\mathcal{I}_{II}. \tag{3.5.77}$$

Changing the integration order, we obtain

$$\begin{aligned}\mathcal{I}_I =& E\int_0^{\tau_i(t\wedge t_{\mathrm{st}})}\Big[\int_{t_i(\tau')}^{\min\{t\wedge t_{\mathrm{st}},t_i(\tau'+1)\}}\frac{e^{-\varepsilon(t-s)}}{t_i'\big(\tau_i(s)\big)}\big[1+\frac{1}{\sqrt{\tau_i(s)+\delta-\tau'}}\big]ds\Big]\|\overline{V}_i(\tau')\|^2_{H^1}d\tau',\\ \mathcal{I}_{II} =& E\int_0^{\tau_i(t\wedge t_{\mathrm{st}})}\int_{t_i(\tau')}^{\min\{t\wedge t_{\mathrm{st}},t_i(\tau'+1)\}}\frac{e^{-\varepsilon(t-s)}}{t_i'\big(\tau_i(s)\big)}\partial_1\mathcal{E}_{\tau_i(s),\tau';L^2}\,ds\,d\tau'.\end{aligned} \tag{3.5.78}$$

The substitution $s' = t_i(\tau')$ together with Lemma 3.5.2 now yields

$$\begin{array}{rcl}\mathcal{I}_I & \leq & K_\kappa^2 E\int_0^{t\wedge t_{\mathrm{st}}} e^{-\varepsilon(t\wedge t_{\mathrm{st}})}\Big[\int_{s'}^{\min\{t\wedge t_{\mathrm{st}},t_i(\tau_i(s')+1)\}} e^{\varepsilon s}\times \\ & & \big[1+\frac{1}{\sqrt{\tau_i(s)+\delta-\tau(s')}}\big]\,ds\Big]\|V(s')\|^2_{H^1}\,ds' \\ & \leq & K_\kappa^2 E\int_0^{t\wedge t_{\mathrm{st}}} e^{-\varepsilon(t\wedge t_{\mathrm{st}})}\Big[\int_{s'}^{\min\{t\wedge t_{\mathrm{st}},s'+1\}} e^{\varepsilon s}\big[1+\frac{1}{\sqrt{s+\delta-s'}}\big]ds\Big]\|V(s')\|^2_{H^1}ds' \\ & \leq & 3e^{\varepsilon}K_\kappa^2 E\int_0^{t\wedge t_{\mathrm{st}}} e^{-\varepsilon(t\wedge t_{\mathrm{st}}-s')}\|V(s')\|^2_{H^1}\,ds' \\ & = & 3e^{\varepsilon}K_\kappa^2 E N_{\varepsilon;II}(t\wedge t_{\mathrm{st}}).\end{array} \tag{3.5.79}$$

For convenience, we introduce the notation

$$\tau^+(\tau') = \min\{\tau_i(t\wedge t_{\mathrm{st}}), \tau'+1\}. \tag{3.5.80}$$

3

Substituting $\tau = \tau_i(s)$ and integrating by parts, we may compute

$$\begin{array}{lcl} \mathcal{I}_{II} & = & E \int_0^{\tau_i(t \wedge t_{\mathrm{st}})} e^{-\varepsilon t} \int_{\tau'}^{\tau^+(\tau')} e^{\varepsilon t_i(\tau)} \partial_1 \mathcal{E}_{\tau,\tau';L^2} \, d\tau \, d\tau' \\ & = & \mathcal{I}_{II;A} + \mathcal{I}_{II;B} + \mathcal{I}_{II;C}, \end{array} \tag{3.5.81}$$

in which we have introduced the expressions

$$\begin{array}{lcl} \mathcal{I}_{II;A} & = & E \int_0^{\tau_i(t \wedge t_{\mathrm{st}})} e^{-\varepsilon t} e^{\varepsilon t_i(\tau^+(\tau'))} \mathcal{E}_{\tau^+(\tau'),\tau';L^2} \, d\tau', \\ \mathcal{I}_{II;B} & = & -E \int_0^{\tau_i(t \wedge t_{\mathrm{st}})} e^{-\varepsilon t} e^{\varepsilon t_i(\tau')} \mathcal{E}_{\tau',\tau';L^2} \, d\tau', \\ \mathcal{I}_{II;C} & = & -E \int_0^{\tau_i(t \wedge t_{\mathrm{st}})} e^{-\varepsilon t} \int_{\tau'}^{\tau^+(\tau')} \left(\frac{d}{d\tau} e^{\varepsilon t_i(\tau)} \right) \mathcal{E}_{\tau,\tau';L^2} \, d\tau \, d\tau'. \end{array} \tag{3.5.82}$$

Upon computing

$$\int_{\tau'}^{\tau^+(\tau')} \left| \frac{d}{d\tau} e^{\varepsilon t_i(\tau)} \right| d\tau = e^{\varepsilon t_i(\tau)} \Big|_{\tau'}^{\tau^+(\tau')} \leq 2 e^{\varepsilon} e^{\varepsilon t_i(\tau')}, \tag{3.5.83}$$

we can make the substitution $s' = t_i(\tau')$ and obtain the final estimate

$$\begin{array}{lcl} |\mathcal{I}_{II}| & \leq & 4 e^{\varepsilon} K_\kappa M^2 K_{B;\mathrm{lin}}^2 E \int_0^{t \wedge t_{\mathrm{st}}} e^{-\varepsilon(t \wedge t_{\mathrm{st}} - s')} \|V(s')\|_{H^1}^2 ds' \\ & \leq & 4 e^{\varepsilon} K_\kappa M^2 K_{B;\mathrm{lin}}^2 E N_{\varepsilon;II}(t \wedge t_{\mathrm{st}}). \end{array} \tag{3.5.84}$$

□

Lemma 3.5.12. *Fix $T > 0$ and assume that (HDt), (HSt) and (HTw) all hold. Pick a constant $0 < \varepsilon < \beta$. Then for any $0 \leq \delta < 1$, any $(\mathcal{F}_t)$-stopping time t_{st} and any $0 \leq t \leq T$, we have the bounds*

$$\begin{array}{lcl} E \mathcal{I}^{\mathrm{lt}}_{\varepsilon,\delta;B;\mathrm{cn}}(t) & \leq & \frac{M^2}{(2\beta - \varepsilon)\varepsilon} K_{B;\mathrm{cn}}^2, \\ E \mathcal{I}^{\mathrm{sh}}_{\varepsilon,\delta;B;\mathrm{cn}}(t) & \leq & \frac{1}{\varepsilon} K_\kappa K_{B;\mathrm{lin}}^2 M^2 (1 + \rho_{\min}^{-1}) e^{\varepsilon} (3 K_\kappa + 2). \end{array} \tag{3.5.85}$$

Proof. These results follows by repeating Lemmas 3.5.10 and 3.5.11. Since

$$\int_0^t e^{-\varepsilon(t-s)} \, ds \leq \frac{1}{\varepsilon}, \tag{3.5.86}$$

we can obtain the bounds by making the substitution

$$K_{B;\mathrm{lin}} \mapsto K_{B;\mathrm{cn}}, \qquad E N_{\varepsilon;II}(t \wedge t_{\mathrm{st}}) \mapsto \frac{1}{\varepsilon}. \tag{3.5.87}$$

□

Lemma 3.5.13. *Fix $T > 0$ and assume that (HDt), (HSt) and (HTw) all hold. Pick a constant $0 < \varepsilon < 2\beta$, then for any $0 \le \delta < 1$ we have the bounds*

$$\begin{array}{rcl} E \sup_{0 \le t \le t_{\mathrm{st}}} \mathcal{I}^{\mathrm{lt}}_{\varepsilon,\delta;B;\mathrm{lin}}(t) & \le & e^{\varepsilon}(T+1)\dfrac{M^2}{2\beta - \varepsilon} K_\kappa K^2_{B;\mathrm{lin}} E \sup_{0 \le t \le t_{\mathrm{st}}} N_{\varepsilon;II}(t), \\ E \sup_{0 \le t \le t_{\mathrm{st}}} \mathcal{I}^{\mathrm{sh}}_{\varepsilon,\delta;B;\mathrm{lin}}(t) & \le & e^{\varepsilon}(T+1) K^2_{B;\mathrm{lin}} M^2 (1 + \rho^{-1}) e^{\varepsilon}(3K_\kappa + 2) E \sup_{0 \le t \le t_{\mathrm{st}}} N_{\varepsilon;II}(t), \end{array} \tag{3.5.88}$$

and

$$\begin{array}{rcl} E \sup_{0 \le t \le t_{\mathrm{st}}} \mathcal{I}^{\mathrm{lt}}_{\varepsilon,\delta;B;\mathrm{cn}}(t) & \le & e^{\varepsilon}(T+1)\dfrac{M^2}{(2\beta - \varepsilon)\varepsilon} K_\kappa K^2_{B;\mathrm{cn}}, \\ E \sup_{0 \le t \le t_{\mathrm{st}}} \mathcal{I}^{\mathrm{sh}}_{\varepsilon,\delta;B;\mathrm{cn}}(t) & \le & e^{\varepsilon}(T+1) K_\kappa K^2_{B;\mathrm{cn}} \dfrac{M^2}{\varepsilon}(1 + \rho^{-1}) e^{\varepsilon}(3K_\kappa + 2). \end{array} \tag{3.5.89}$$

Proof. This follows directly from Lemmas 2.9.20 and 2.9.21. □

Proof of Proposition 3.5.1. Pick $T > 0$ and $0 < \eta < \eta_0$ and write $t_{\mathrm{st}} = t_{\mathrm{st}}(T, \varepsilon, \eta)$. Since the identities (3.4.21) with $v = V(t \wedge t_{\mathrm{st}})$ hold for all $0 \le t \le T$, we may compute

$$\begin{array}{rcl} E \sup_{0 \le t \le t_{\mathrm{st}}} [N^i_{\varepsilon;I}(t)] & \le & 7E \sup_{0 \le t \le t_{\mathrm{st}}} \Big[\|\mathcal{E}_0(t)\|^2_{L^2} + \sigma^4 \|\mathcal{E}_{F;\mathrm{lin}}(t)\|^2_{L^2} + \|\mathcal{E}_{F;\mathrm{nl}}(t)\|^2_{L^2} \\ & & + \sigma^2 \|\mathcal{E}_{B;\mathrm{lin}}(t)\|^2_{L^2} + \sigma^2 \|\mathcal{E}_{B;\mathrm{cn}}(t)\|^2_{L^2} \\ & & + \|\mathcal{E}^{\mathrm{lt}}_{\mathrm{so}}(t)\|^2_{L^2} + \|\mathcal{E}^{\mathrm{st}}_{\mathrm{so}}(t)\|^2_{L^2} \Big] \end{array} \tag{3.5.90}$$

by applying Young's inequality. The inequalities in Lemmas 3.5.3-3.5.13 now imply that

$$E \sup_{0 \le t \le t_{\mathrm{st}}} [N^i_{\varepsilon;I}(t)] \le C_1 \big[\|V(0)\|^2_{H^1} + (\eta + \sigma^2 T + \sigma^4) \sup_{0 \le t \le t_{\mathrm{st}}} N_{\varepsilon;II}(t) \big]. \tag{3.5.91}$$

In addition, we note that

$$\begin{array}{rcl} E \sup_{0 \le t \le t_{\mathrm{st}}} N^i_{\varepsilon,0;II}(t) & \le & 11E \sup_{0 \le t \le t_{\mathrm{st}}} \Big[\mathcal{I}_{\varepsilon,0;0}(t) + \sigma^4 \mathcal{I}^{\mathrm{lt}}_{\varepsilon,0;F;\mathrm{lin}}(t) + \sigma^4 \mathcal{I}^{\mathrm{sh}}_{\varepsilon,0;F;\mathrm{lin}}(t) \\ & & + \mathcal{I}^{\mathrm{lt}}_{\varepsilon,0;F;\mathrm{nl}}(t) + \mathcal{I}^{\mathrm{sh}}_{\varepsilon,0;F;\mathrm{nl}}(t) \\ & & + \sigma^2 \mathcal{I}^{\mathrm{lt}}_{\varepsilon,0;B;\mathrm{lin}}(t) + \sigma^2 \mathcal{I}^{\mathrm{sh}}_{\varepsilon,0;B;\mathrm{lin}}(t) \\ & & + \sigma^2 \mathcal{I}^{\mathrm{lt}}_{\varepsilon,0;B;\mathrm{cn}}(t) + \sigma^2 \mathcal{I}^{\mathrm{sh}}_{\varepsilon,0;B;\mathrm{cn}}(t) \\ & & + \mathcal{I}^{\mathrm{lt}}_{\varepsilon,0;\mathrm{so}}(t) + \mathcal{I}^{\mathrm{sh}}_{\varepsilon,0;\mathrm{so}}(t) \Big]. \end{array} \tag{3.5.92}$$

The inequalities in Lemmas 3.5.3-3.5.12 now imply that

$$E \sup_{0 \le t \le t_{\mathrm{st}}} N^i_{\varepsilon,0;II}(t) \le C_2 \big[\|V(0)\|^2_{H^1} + \sigma^2 T + (\eta + \sigma^2 T + \sigma^4) \sup_{0 \le t \le t_{\mathrm{st}}} N_{\varepsilon;II}(t) \big]. \tag{3.5.93}$$

In particular, we see that

$$E \sup_{0 \le t \le t_{\mathrm{st}}} N^i_{\varepsilon}(t) \le C_3 \big[\|V(0)\|^2_{H^1} + \sigma^2 T + (\eta + \sigma^2 T + \sigma^4) E \sup_{0 \le t \le t_{\mathrm{st}}} N_{\varepsilon}(t) \big]. \tag{3.5.94}$$

The desired bound hence follows by summing over i and appropriately restricting the size of $\eta + \sigma^2 T + \sigma^4$. □

4

Reaction-Diffusion Equations Forced by Translation Invariant Noise

Inspired by applications, we consider reaction-diffusion equations on $\mathbb{R}$ that are stochastically forced by a small multiplicative noise term that is white in time, coloured in space and invariant under translations. We show how these equations can be understood as a stochastic partial differential equation (SPDE) forced by a cylindrical Q-Wiener process and subsequently explain how to study stochastic travelling waves in this setting. In particular, we generalize the phase tracking framework that was developed in Chapters 2 and 3 for noise processes driven by a single Brownian motion. The main focus lies on explaining how this framework naturally leads to long term approximations for the stochastic wave profile and speed. We illustrate our approach by two fully worked-out examples, which highlight the predictive power of our expansions.

4.1 Introduction

In this chapter[1] we set out to study the propagation of wave solutions to stochastic equations of the form

$$u_t = \rho u_{xx} + f(u) + \sigma g(u)\xi(x,t), \tag{4.1.1}$$

in which ξ is a Gaussian process[2] that is white in time and coloured in space. In particular, we assume formally that

$$\begin{aligned} E[\xi(x,t)] &= 0, \\ E[\xi(x,t)\xi(x',t')] &= \delta(t-t')q(x-x'), \end{aligned} \tag{4.1.2}$$

[1] The content of this chapter has been published as *C.H.S. Hamster, H.J. Hupkes; Travelling Waves for Reaction-Diffusion Equations Forced by Translation Invariant Noise* in Physica D, see [50].

[2] Actually 'generalized Gaussian' would be a more accurate term, since we will see that ξ does not have the right properties to be a Gaussian random variable on $L^2(\mathbb{R})$.

for some smooth covariance function q that describes the correlation in space. Such equations have been used in a wide range of applications, for example to model the appearance of travelling waves in light-sensitive Belousov-Zhabotinsky chemical reactions [63] or to study the excitability of activator-inhibitor systems such as nerve fibres [38]. We refer to [39, §6.1] for an extended list of examples.

We will assume here that (4.1.1) with $\sigma = 0$ admits a spectrally stable deterministic travelling front or pulse and examine the impact of the multiplicative noise term for small σ. The nonlinearity g will be chosen to vanish at the endpoints of the deterministic wave. In particular, the full stochastic system is at rest whenever the deterministic portion is at rest. Such an assumption is typically used to examine the distortions on a system caused by *external* random effects, such as fluctuations in the intensity of the light driving a Belousov-Zhabotinsky reaction. In a controlled setting these effects can often be minimized or switched off completely, leading to the notion of the deterministic limit.

On the other hand, *internal* fluctuations arise from the microscopic properties of the system itself and cannot be readily eliminated. For example, the vibrations of the individual atoms are essential ingredients in the derivation of the ideal gas equations. It is more natural to use an additive noise term to model effects of this type, but we do not focus on this case here.

The main goal of this chapter is to uncover the corrections to the deterministic wave that are caused by the (small) multiplicative noise term. In particular, we develop a framework that allows the corrections to the speed and shape of the wave to be computed to any desired order in σ. We explicitly compute the second and third order correction terms and show numerically that these expansions are valid for long time scales. We also outline in which sense these predictions can be made rigorous, which involves casting the translationally invariant Stochastic Partial Differential Equation (SPDE) (4.1.1) into a mathematically precise form.

Example I: The Nagumo equation In order to set the stage, let us consider the stochastic Nagumo equation

$$u_t = \rho u_{xx} + f_{\text{cub}}(u; a) + \sigma u(1-u)\xi(x,t), \tag{4.1.3}$$

with the bistable cubic nonlinearity

$$f_{\text{cub}}(u; a) = u(1-u)(u-a), \qquad 0 < a < 1 \tag{4.1.4}$$

and the Gaussian covariance kernel

$$q(x) = \frac{1}{2} e^{\frac{-\pi x^2}{4}}. \tag{4.1.5}$$

For $0 < a < 1$, the deterministic system is known to have a spectrally stable wave solution $u(x,t) = \Phi_0(x - c_0 t)$ that connects the stable rest states zero and one [65]. In fact, the travelling wave ODE

$$\rho \Phi_0'' + c_0 \Phi_0' + f_{\text{cub}}(\Phi_0; a) = 0 \tag{4.1.6}$$

can be solved by using the explicit expressions

$$c_0 = \sqrt{2\rho_0}\left(\frac{1}{2} - a_0\right), \qquad \Phi_0(x) = \frac{1}{2}\left[1 - \tanh\left(\frac{1}{2\sqrt{2\rho_0}}x\right)\right] \tag{4.1.7}$$

with $(\rho_0, a_0) = (\rho, a)$.

Note that the form of the nonlinear terms in (4.1.3) allows us to recast the system as

$$u_t = \rho_0 u_{xx} + f_{\mathrm{cub}}\big(u; a_0 - \sigma\xi(x,t)\big), \tag{4.1.8}$$

showing that we are stochastically forcing the (external) parameter a. As a consequence, it is natural to ask whether effective σ-dependent parameters (ρ_σ, a_σ) can be derived that are able to capture the stochastic effects on the waves by replacing (4.1.7) with

$$c_\sigma = \sqrt{2\rho_\sigma}\left(\frac{1}{2} - a_\sigma\right), \qquad \Phi_\sigma(x) = \frac{1}{2}\left[1 - \tanh\left(\frac{1}{2\sqrt{2\rho_\sigma}}x\right)\right]. \tag{4.1.9}$$

In the case where $\xi(x,t)$ is replaced by the derivative of a single x-independent Brownian motion in time, this point of view can be made fully explicit and precise. Indeed, in this case the wave (4.1.9) with

$$(\rho_\sigma, a_\sigma) = \Big(\frac{\rho_0}{1+\sigma^2\rho_0}, a_0\Big) \tag{4.1.10}$$

is an exact solution to the underlying SPDE [19, 48], with a phase that follows a scaled Brownian motion.

The early results in [39] for general ξ can also be seen in this light. Indeed, the authors use a formal (partial) expansion to suggest the choices

$$\rho_\sigma = \frac{1}{1-\sigma^2 q(0)}, \qquad a_\sigma = \frac{2a_0 - \sigma^2 q(0)}{2 - 2\sigma^2 q(0)} \tag{4.1.11}$$

where $\rho_0 = 1$. However, the waves found in this way are only approximate solutions[3] to (4.1.3). We show in §4.3 how our techniques can be used to significantly improve the quality of this approximation.

In order to discuss the stability of these waves, we introduce the linearized operator

$$\mathcal{L}_{\mathrm{tw}} v = \rho v'' + c_0 v' + f'_{\mathrm{cub}}(\Phi_0; a)v, \tag{4.1.12}$$

together with its formal adjoint

$$\mathcal{L}^*_{\mathrm{tw}} w = \rho w'' - c_0 w' + f'_{\mathrm{cub}}(\Phi_0; a)w. \tag{4.1.13}$$

By direct substitution, it can be verified that $\psi_{\mathrm{tw}}(\xi) = \kappa e^{c_0\xi/\rho}\Phi_0'(\xi)$ is an element of the kernel of $\mathcal{L}^*_{\mathrm{tw}}$; see also [65, ex. 2.3.1 and 4.1.4] for more intuition. The constant κ can be chosen in such a way that

$$\mathcal{L}_{\mathrm{tw}}\Phi_0' = \mathcal{L}^*_{\mathrm{tw}}\psi_{\mathrm{tw}} = 0, \qquad \langle \Phi_0', \psi_{\mathrm{tw}}\rangle_{L^2(\mathbb{R})} = 1, \tag{4.1.14}$$

[3] Note that these scalings hold for the Stratonovich interpretation, while the results from [19, 48] hold for the Itô interpretation.

which allows us to write

$$Pv = \langle v, \psi_{\mathrm{tw}} \rangle_{L^2(\mathbb{R})} \Phi_0' \tag{4.1.15}$$

for the spectral projection onto the simple zero eigenspace of $\mathcal{L}_{\mathrm{tw}}$.

Since the remainder of the spectrum of $\mathcal{L}_{\mathrm{tw}}$ lies strictly to the left of the imaginary axis, general considerations [80] can be used to show that the associated semigroup satisfies the bound

$$\| e^{\mathcal{L}_{\mathrm{tw}} t} (I - P) v \|_{L^2(\mathbb{R})} \le M e^{-\beta t} \| (I - P) v \|_{L^2(\mathbb{R})} \tag{4.1.16}$$

for some $\beta > 0$ and $M \ge 1$. The approach in this chapter shows how this bound can be exploited to show that the stochastic waves discussed above are robust against small perturbations.

With additional ad hoc work [104] it is even possible to show that $M = 1$ holds in (4.1.16). Based on this latter property we say that the semigroup is immediately contractive. Indeed, perturbations are not able to grow even on short timescales, but always decay exponentially fast back to the wave. We do not use this property here, but it has played an essential role in many previous studies on stochastic waves [57, 68].

4

Example II: The FitzHugh-Nagumo system Our second main example is the two-component FitzHugh-Nagumo system

$$\begin{aligned} u_t &= u_{xx} + f_{\mathrm{cub}}(u; a) - w + \sigma u \xi(x, t), \\ w_t &= \varepsilon w_{xx} + \varrho(u - \gamma w), \end{aligned} \tag{4.1.17}$$

in which $\varepsilon > 0$ and $\varrho > 0$ are small parameters and $\gamma > 0$ is not too large. For convenience, we reuse the covariance kernel q given in (4.1.5). In the deterministic case $\sigma = 0$, this system can be used to describe signal propagation through nerve fibres. It is famous for its fast and slow travelling pulses that make an excursion from the stable 0 state. Indeed, the construction of these pulses sparked many developments in the area of singular perturbation theory [18, 51, 60–62]. Unlike the previous example, explicit expressions are not available for the profiles and wave speeds. Nevertheless, it is known that the fast pulses are spectrally stable [1]. This allows the framework developed in this chapter to be applied to (4.1.17), leading to the numerical and theoretical discussion in §4.4. Let us emphasize that the specific structure of the noise term in (4.1.17) is just for illustrative purposes. Indeed, our conditions in §4.2 are rather general and allow cross-talk between the noise on the u and w components.

For systems such as (4.1.17), we typically expect $M > 1$ to hold for the general bound (4.1.16). This means that larger excursions from the wave are possible before the exponential decay of the linear semigroup steps in. In particular, (4.1.17) does not fit into the framework of any previous results in this area. In fact, besides the results in Chapters 2 and 3, there do not seem to be many rigorous studies of travelling waves for multi-component SPDEs in the literature.

Translational invariance Notice that (4.1.1) is translationally invariant, in the sense that the deterministic terms are autonomous while the correlation function depends only on the distance between two points. This is a natural assumption, as any explicit dependence on x and y individually would imply some a priori knowledge about the noise that is often not available. Indeed, in many applications [3, 10, 38, 39] translationally invariant noise is considered to be the preferred modelling tool.

However, this choice does present certain mathematical issues that do not arise when replacing $q(x-y)$ by $\tilde{q}(x,y)$ in (4.1.1) and assuming that $\tilde{q}$ is square integrable with respect to (x,y). This breaks the translational symmetry, but does allow the noise-term to be expanded as a countable sum of Brownian motions. This approach is taken in several other works such as [19].

In the following paragraph we will explain how to set up a framework to study translationally invariant noise, but we emphasize that this is only relatively straightforward in the case of multiplicative noise [73, §2.2.1]. Indeed, additive noise of this type cannot be treated directly, but needs a far more abstract machinery that is still under development [45].

We recall that the goal of our approach is to understand the long-term behaviour of the travelling waves under consideration, which move freely throughout the entire spatial domain. We therefore believe that the elegance of the translationally invariant point of view in combination with the direct relevance for applications outweighs the additional mathematical complications.

Cylindrical Wiener process At present, (4.1.1) should be interpreted as a pre-equation rather than an actual SPDE. Our first task is to give a mathematical interpretation to the stochastic term involving the process ξ. To this end, we assume that the correlation function q is integrable, which allows us to define a bounded[4] linear convolution operator $Q : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ that acts as

$$[Qv](x) = [q * v](x) = \int_{\mathbb{R}} q(x-y)v(y)\,dy. \tag{4.1.18}$$

Assuming furthermore that the Fourier transform $\widehat{q}$ is a non-negative function, one can show that Q is a non-negative symmetric operator. More concretely, we have $\langle Qv, v\rangle_{L^2(\mathbb{R})} \geq 0$ for all $v \in L^2(\mathbb{R})$ and it is possible to define a square-root $Q^{1/2} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$.

However, we caution the reader that typically Q has infinite trace and is not even compact. In particular, it is not generally possible to construct a countable orthonormal basis of $L^2(\mathbb{R})$ that consists of the eigenfunctions of Q. This prevents us from using the Brownian-motion expansion discussed above. Stated in technical terms, we cannot interpret ξ as the derivative of a 'regular' Q-Wiener process.

These difficulties can be resolved through the use of cylindrical Wiener processes. Historically, such processes were developed to handle noise that is white (i.e. completely decorrelated) both in space and time. In our notation, this means that q is replaced by the delta-function to yield $Q = I$, which is clearly not of finite trace. This approach only requires that Q is bounded, self-adjoint and nonnegative and hence applies to the class of operators Q introduced above [92, §4.3].

To set the stage, we define the subspace

$$L^2_Q = Q^{1/2}\big(L^2(\mathbb{R})\big) \subset L^2(\mathbb{R}) \tag{4.1.19}$$

[4] The boundedness of Q follows from Young's convolution inequality: $\|q*v\|_{L^2(\mathbb{R})} \leq \|q\|_{L^1(\mathbb{R})}\|v\|_{L^2(\mathbb{R})}$.

equipped with the inner product

$$\langle v, w\rangle_{L^2_Q} = \langle Q^{-\frac{1}{2}}v, Q^{-\frac{1}{2}}w\rangle_{L^2(\mathbb{R})}. \tag{4.1.20}$$

In addition, we follow the construction in [93, §2.5] to define a Hilbert space L^2_{ext} that contains $L^2(\mathbb{R})$ and has the special property that the inclusion $L^2_Q \subset L^2_{\mathrm{ext}}$ is Hilbert-Schmidt.

One can now follow the procedure in [93, §2.5] or [66], to construct the so-called cylindrical Q-Wiener process W^Q_t. This process arises as a limit of processes on L^2_Q that converges as a process on L^2_{ext}, where it can be understood as a 'regular' $\bar{Q}$-Wiener process for some compact $\bar{Q} : L^2_{\mathrm{ext}} \to L^2_{\mathrm{ext}}$. This means that W^Q_t does not necessarily attain values in $L^2(\mathbb{R})$, but fortunately, the exact choice for L^2_{ext} is immaterial[5] for two important reasons.

First, it turns out [66, Prop. 2] that the expression $\langle W^Q_t, v\rangle_{L^2(\mathbb{R})}$ is well-defined for any $v \in L^2(\mathbb{R})$. In fact, it can be interpreted as a scaled Brownian motion that satisfies the correlations

$$E\Big[\langle W^Q_t, v\rangle_{L^2(\mathbb{R})}\langle W^Q_s, w\rangle_{L^2(\mathbb{R})}\Big] = (t \wedge s)\langle Qv, w\rangle_{L^2(\mathbb{R})}. \tag{4.1.21}$$

In particular, formally replacing v and w by delta functions $\delta_x(\cdot)$ and $\delta_y(\cdot)$ and taking the derivative with respect to t and s, we see that

$$E\Big[\langle dW^Q_t, \delta_x\rangle_{L^2(\mathbb{R})}\langle dW^Q_s, \delta_y\rangle_{L^2(\mathbb{R})}\Big] = \delta(t-s)q(x-y). \tag{4.1.22}$$

Comparing this with (4.1.2), we see that $\frac{d}{dt}W^Q_t(x)$ and $\xi(x,t)$ are natural counterparts.

The second reason is that all the essential stochastic estimates we will need only rely on the space L^2_Q. For example, the full noise term in (4.1.1) is well-defined if the pointwise multiplication

$$v(\xi) \mapsto g\big(U(\xi)\big)v(\xi) \tag{4.1.23}$$

can be interpreted as a Hilbert-Schmidt operator from L^2_Q into $L^2(\mathbb{R})$ for any relevant function U. We will show in Appendix 4.A that this can be achieved by imposing simple bounds on the scalar function $g : \mathbb{R} \to \mathbb{R}$ and its derivative.

Interpretation In many applications involving external noise, it is natural to interpret the stochastic terms in the Stratonovich sense [110]. Indeed, this interpretation yields the correct limit when approximating a Wiener process by regularized versions that can be fitted into the standard deterministic framework (a so-called Wong-Zakai Theorem). Upon using the process W^Q_t to recast (4.1.1) as a SPDE in Stratonovich form, we arrive at

$$dU = [\rho\partial_{xx}U + f(U)]dt + \sigma g(U) \circ dW^Q_t. \tag{4.1.24}$$

[5] For translation invariant processes it is possible to explicitly characterize a choice for L^2_{ext} in terms of the dual of a Schwartz space, see [90].

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The equivalent Itô formulation is given by

$$dU = \Big[\rho\partial_{xx}U + f(U) + \mu\frac{\sigma^2}{2}q(0)g'(U)g(U)\Big]dt + \sigma g(U)dW_t^Q \tag{4.1.25}$$

with $\mu = 1$. From a mathematical point of view it is more convenient to work in this formulation, since most of the technical machinery for SPDEs is based on Itô calculus. The choice $\mu = 0$ allows us to interpret the noise in (4.1.1) in the Itô sense directly. Our results in this chapter cover both cases, in order to ease the comparison with previous work and to illustrate how the two types of noise impact the deterministic waves in different ways.

Previous results Rigorous results concerning the well-posedness of SPDEs of type (4.1.25) are widely available by now, see e.g. [93]. However, the dynamics of this type of equation is less well studied in the math community. Several authors have considered the dynamics of stochastic waves driven by Q-Wiener processes, which means that the noise is necessarily localized in space. For example, the shape and speed of stochastic waves for Nagumo-type SPDEs were computed numerically in [79] and derived formally in [19] using a collective coordinate approach. In addition, short-time stability results for immediately contractive systems can be found in [57, 68]. The results in [86] do use cylindrical Q-Wiener processes for waves in the Fisher-KPP equation, but there the smooth covariance function q is replaced by a delta-function in order to model noise that is white in space and time. A more detailed overview of results on stochastic travelling waves can be found in the review by Kuehn [69].

Turning to non-rigorous results for (4.1.1) from other fields, we refer to [39] for an interesting overview of studies that have appeared in the physics and chemistry literature. For the Nagumo SPDE (4.1.3), the dynamics up to first order in σ of have been formally computed [39, eq. (6.11)]. At this order, the shape of the wave is equal to the deterministic shape and the phase of the wave follows a Brownian motion with a variance that can be expressed in closed form. We will see in §4.3 how these conclusions can be recovered as special cases from our expansions.

Phase tracking Our work here builds on the framework developed in Chapters 2 and 3 to study travelling waves in stochastic reaction-diffusion equations forced by a single Brownian motion. The main idea is to use a phase-tracking approach that is based purely on technical considerations rather than ad hoc geometric intuition. Inspired by the overview in [117], this allows us to adapt modern tools developed for deterministic stability results for use in a stochastic setting.

In order to explain the key concepts, we turn to the deterministic Nagumo PDE that arises by taking $\sigma = 0$ in (4.1.3). The translational invariance of the travelling wave $u(x,t) = \Phi_0(x - c_0 t)$ can be captured by introducing an Ansatz of the form

$$u\big(\cdot + \gamma(t), t\big) = \Phi_0(\cdot) + v(\cdot, t),$$

in which $\gamma(t)$ can be interpreted as the phase of u. We now demand that the evolution of the phase is governed by

$$\dot{\gamma}(t) = c_0 + a\big(v(\cdot, t)\big), \tag{4.1.26}$$

for some (nonlinear) functional $a : L^2(\mathbb{R}) \to \mathbb{R}$ that we are still free to choose. The resulting equation for v is then given by

$$\partial_t v(t) = \mathcal{L}_{\mathrm{tw}} v(t) + N\big(v(t)\big) + a\big(v(t)\big)\partial_\xi\big[\Phi_0 + v(t)\big], \tag{4.1.27}$$

in which $N(v) = f_{\mathrm{cub}}(\Phi_0 + v) - f_{\mathrm{cub}}(\Phi_0) - f'_{\mathrm{cub}}(\Phi_0)v$. This can be recast into the mild form

$$v(t) = e^{\mathcal{L}_{\mathrm{tw}} t} v_0 + \int_0^t e^{\mathcal{L}_{\mathrm{tw}}(t-s)}\Big[N\big(v(s)\big) + a\big(v(s)\big)\partial_\xi\big[\Phi_0 + v(s)\big]\Big] ds, \tag{4.1.28}$$

inviting us to apply the bound (4.1.16).

In order to apply exponential bounds such as (4.1.16) to the semigroup $e^{\mathcal{L}_{\mathrm{tw}} t}$, we must avoid the neutral non-decaying part of the semigroup. In order to force the integrand to be orthogonal to the zero eigenspace, we recall the spectral projection (4.1.15) and choose

$$a(v) = -\frac{\langle N(v), \psi_{\mathrm{tw}}\rangle_{L^2(\mathbb{R})}}{\langle \partial_\xi(\Phi_0 + v), \psi_{\mathrm{tw}}\rangle_{L^2(\mathbb{R})}}. \tag{4.1.29}$$

In fact, one arrives at the same choice if one directly imposes the orthogonality condition $\langle v(t), \psi_{\mathrm{tw}}\rangle_{L^2(\mathbb{R})} = 0$. By a standard bootstrapping procedure one can now establish the limits $\|v(t)\|_{L^2(\mathbb{R})} \to 0$ and $t^{-1}\gamma(t) \to c_0$ for $t \to \infty$, provided that v_0 is sufficiently small. This allows us to conclude that the travelling wave is orbitally stable.

In our stochastic setting, the pair (v, γ) is replaced by its stochastic counterpart (V, Γ), which we always write in capitals. The resulting equations for this pair are naturally much more complicated. They both consist of a deterministic and a stochastic part, resulting in *two* free functionals that can be tuned to ensure $\langle V, \psi_{\mathrm{tw}}\rangle_{L^2(\mathbb{R})} = 0$, see §4.2.2.

Besides the fact that we use the adjoint eigenfunction ψ_{tw} instead of Φ_0', the main difference with the phase tracking approaches developed in [57, 68] is that our perturbation V is taken relative to a novel pair (Φ_σ, c_σ) that we refer to as the instantaneous[6] stochastic wave. This pair is chosen in such a way that the deterministic part of the equation for V vanishes at $V = 0$. However, this does not hold for the stochastic part, leading to persistent fluctuations that must be controlled.

Stability Our first contribution is that we establish that the wave (Φ_σ, c_σ) is stable, in the sense that the perturbation $V(t)$ remains small over time scales of $\mathcal{O}(\sigma^{-2})$. In particular, we show that the semigroup techniques developed in our earlier chapters are general enough to remain applicable in the present more convoluted setting. The main effort is to verify that certain technical estimates remain valid, which is possible by the powerful theory that has been developed for cylindrical Q-Wiener processes.

The procedure in Chapters 2 and 3 is rather delicate in order to compensate for the lack of immediate contractivity. Indeed, the H^1-norm of $V(t)$ must be kept under control, resulting in apparent singularities in the stochastic integrals that must be handled with care. The time scale mentioned above arises as a consequence of the

[6] See §4.2.4 for a justification for this terminology.

mild Burkholder-Davis-Gundy inequality that we used to obtain supremum bounds on stochastic integrals. However, these bounds are known to be suboptimal. Indeed, we believe that our phase tracking approach can be maintained for time scales that are exponential in σ. This is confirmed by the numerical results at the end of §4.3.

Expansions in σ The second – and main – contribution in this chapter is that we explicitly show how to expand the fluctuations around the stochastic wave (Φ_σ, c_σ) in powers of the noise strength σ. In particular, we show that our framework yields a natural procedure to compute Taylor expansions for the pair $\big(V(t), \Gamma(t)\big)$. These results extend the pioneering work in [68, 72], where related multi-scale expansions were achieved in a variety of settings on short time-scales.

An important advantage of our semigroup approach is that the resulting terms have expectations that are well-defined in the limit $t \to \infty$. In particular, we are able to uncover the long-term stochastic corrections to the speed and shape of the travelling waves.

We provide explicit formula's for the first and second order corrections, which all crucially involve the semigroup $e^{\mathcal{L}_{\mathrm{tw}} t}$. In addition, we show how to compute the third order corrections in the phase Γ from these second order corrections. In principle, the expansions can be computed to any desired order in σ, but the process quickly becomes unwieldy.

At first order in σ, our predictions concur with many earlier results [16, 19, 39, 68], which show that the phase $\Gamma(t)$ of the wave behaves as a Brownian motion centred around the deterministic position $c_0 t$. In addition, the shape of the wave fluctuates at first order like an infinite dimensional Ornstein-Uhlenbeck process around the deterministic wave Φ_0.

At second order in σ, two distinct effects start to play a role. The first is that differences start to appear between the instantaneous stochastic wave (Φ_σ, c_σ) and the deterministic wave (Φ_0, c_0). This generalizes the wave steepening effect (4.1.10) discussed in Chapter 2 and [19, 79]. On top of this, there is an additional contribution to the average speed and shape that is caused by the feedback of the first order fluctuations of $V(t)$. These effects – which we refer to as orbital drift – become visible after an initial transient period. Besides a brief discussion in our earlier chapters, we do not believe that this long-term behaviour has been systematically explored before.

Taken together, we now have a quantitative procedure that is able to accurately describe the numerical results in [79] and the formal computations in [19] for the Nagumo SPDE (4.1.3), both for the Stratonovich and Itô formulation. This allows us to understand the differences in speed and shape between both interpretations analytically. These predictions are confirmed by our numerical results, which compare the solutions of the full SPDE with our explicit formula's and exhibit the rate of convergence with respect to σ.

Outlook In this chapter we will not treat space-time white noise, i.e. $q(x-y) = \delta(x-y)$, as our mathematical framework does not yet allow distributions to be used as kernels. This can already be seen from the fact that (4.1.24) depends explicitly on $q(0)$, which of course is not well-defined for distributions. In fact, it is still a subject of active research [45] to give a clear interpretation of (4.1.1) in this case. In the Itô interpretation however, many of our computations concerning the shape and speed of the stochastic wave have a well-defined limit if we let q converge to δ. In addition, many of our expressions still make sense for $g = 1$, which suggests that our expansions could also be used to make predictions for additive noise.

In addition, we expect that our methods can be extended to other types of equations. For example, stochastic neural field models have attracted a lot of attention in recent years, but they lack the smoothening effect of the diffusion operator. Finally, we are exploring techniques that would allow us to extend the $\mathcal{O}(\sigma^{-2})$ time scales in our results to the exponentially long scales observed in the numerical computations.

Organization In §4.2 we state our assumptions and give a step-by-step overview of the steps that lead to our expansions. In addition, we provide explicit formula's that describe the first and second order terms in the expansions for (V, Γ). In §4.3 and §4.4 we illustrate how our results can be applied to the Nagumo and FitzHugh-Nagumo SPDEs and verify the predictions with numerical computations.

The remaining sections contain the technical heart of this chapter and provide the link between our setting here and the bootstrapping procedure developed in Chapters 2 and 3. In particular, we show in §4.5 how the stochastic evolution equation for V can be computed using Itô calculus, which represents the core computation of this work. In §4.6 we explain how the technical machinery available for cylindrical Q-Wiener processes can be used to follow the steps of Chapters 2 and 3. Appendix 4.A provides the main link between these chapters, showing how the estimates in Chapter 2 can be generalized to the nonlinearities appearing here.

4.2 Main results

In this chapter we study the properties of travelling wave solutions to stochastic reaction-diffusion systems with translationally invariant noise. In §4.2.1 we introduce the class of systems that we are interested in. The main steps of our approach are outlined in §4.2.2, which allows us to expand the stochastic corrections to the shape and speed of the waves in powers of the noise strength. The precise form of these expansions is described in §4.2.3. Finally, we discuss several consequences of our results in §4.2.4 and compare them to earlier work in this area.

4.2.1 Setup

In this section we formulate the conditions that we need to impose on our stochastic reaction-diffusion system. Taken together, these conditions ensure that the noise-term is well-defined, that the SPDE is well-posed and that the deterministic part admits a spectrally stable travelling wave.

Noise Process We start by discussing the covariance function q that underpins the noise process, which we assume to have m components. In particular, we impose the following condition on the $m \times m$ components of the function q and its Fourier transform $\widehat{q}$.

(Hq) We have $q \in H^1(\mathbb{R}, \mathbb{R}^{m\times m}) \cap L^1(\mathbb{R}, \mathbb{R}^{m\times m})$, with $q(-\xi) = q(\xi)$ and $q^T(\xi) = q(\xi)$ for all $\xi \in \mathbb{R}$. In addition, for each $k \in \mathbb{R}$ the $m \times m$ matrix $\widehat{q}(k)$ is non-negative definite.

Since the Fourier transform maps Gaussians onto Gaussians, this condition can easily be verified for $q(x) = \exp(-x^2)$ in the scalar case $m = 1$. Other examples include the exponential $q(x) = \exp\big(-|x|\big)$ and the tent function $q(x) = 1 - |x|$ supported on $[-1, 1]$.

The integrability of q allows us to introduce a bounded linear operator Q on $L^2(\mathbb{R}, \mathbb{R}^m)$ that acts as

$$[Qv](x) = [q * v](x) = \int_{\mathbb{R}} q(x-y)v(y)\,dy. \tag{4.2.1}$$

The remaining conditions in (Hq) show that Q is symmetric and that $\langle Qv, v\rangle_{L^2(\mathbb{R},\mathbb{R}^m)} \geq 0$ holds for all $v \in L^2(\mathbb{R}, \mathbb{R}^m)$. As explained in §4.1 and §4.5.1, this allows us to follow [93, §2.5] and [66] to define a cylindrical Q-Wiener process W_t^Q over $L^2(\mathbb{R}, \mathbb{R}^m)$. In particular, for any $v, w \in L^2(\mathbb{R}, \mathbb{R}^m)$ we have

$$E[\langle W_t^Q, v\rangle_{L^2(\mathbb{R},\mathbb{R}^m)}\langle W_s^Q, w\rangle_{L^2(\mathbb{R},\mathbb{R}^m)}] = s \wedge t\,\langle q * v, w\rangle_{L^2(\mathbb{R},\mathbb{R}^m)}. \tag{4.2.2}$$

Upon writing $\{e_i\}$ for the standard unit vectors together with $v(x) = \delta(x - x_0)e_i$ and $w(x) = \delta(x - x_1)e_j$, this reduces formally to the familiar expression

$$E[dW_t^Q(x_0)dW_s^Q(x_1)] = \delta(t-s)q_{ij}(x_0 - x_1), \tag{4.2.3}$$

after taking the time derivative with respect to t and s. This highlights the role that the correlation function q plays in our setup.

Stochastic reaction-diffusion equation The main SPDE that we will study can now be formulated as

$$dU = \big[\rho\partial_{xx}U + f(U) + \sigma^2 h(U)\big]dt + \sigma g(U)dW_t^Q. \tag{4.2.4}$$

Here we take $U = U(x,t) \in \mathbb{R}^n$ with $x \in \mathbb{R}$ and $t \geq 0$. The nonlinearities $f : \mathbb{R}^n \to \mathbb{R}^n$, $h : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are considered to act in a pointwise fashion. In order to proceed, we need to assume that these nonlinearities have a common pair of equilibria.

(HEq) There exist $u_\pm \in \mathbb{R}^n$ so that

$$f(u_\pm) = g(u_\pm) = h(u_\pm) = 0. \tag{4.2.5}$$

If $u_- \neq u_+$, then the relevant solutions U to (4.2.4) cannot be captured in the Hilbert space $L^2(\mathbb{R},\mathbb{R}^n)$. In order to remedy this, we pick a smooth reference function Φ_{ref} that has the limits $\Phi_{\mathrm{ref}}(\pm\infty) = u_\pm$ and introduce the affine spaces

$$\mathcal{U}_{H^1} = \Phi_{\mathrm{ref}} + H^1(\mathbb{R},\mathbb{R}^n), \qquad \mathcal{U}_{H^2} = \Phi_{\mathrm{ref}} + H^2(\mathbb{R},\mathbb{R}^n). \tag{4.2.6}$$

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Naturally, we can simply take $\Phi_{\mathrm{ref}} = 0$ if $u_- = u_+$. We will see that (4.2.4) is well-posed as a stochastic evolution equation on $\mathcal{U}_{H^1}$. In fact, it is advantageous to study $X(t) = U(t) - \Phi_{\mathrm{ref}}$, which solves the SPDE[7]

$$dX = \big[\rho\partial_{xx}(X + \Phi_{\mathrm{ref}}) + f(X + \Phi_{\mathrm{ref}}) + \sigma^2 h(X + \Phi_{\mathrm{ref}})\big]dt + \sigma g(X + \Phi_{\mathrm{ref}})dW_t^Q \tag{4.2.7}$$

and hence attains values in $H^1(\mathbb{R},\mathbb{R}^n)$. This decomposition will be used in §4.5-§4.6.

Stochastic terms In order to ensure that the stochastic term in (4.2.4) is well-defined, we impose the following growth bound on g.

(HSt) We have $g \in C^2(\mathbb{R}^n,\mathbb{R}^{n\times m})$. In addition, the derivative Dg is bounded and globally Lipschitz continuous.

Indeed, in Appendix 4.A this assumption is used to establish that the pointwise map

$$[g(U)v](x) = g\big(U(x)\big)v(x) \tag{4.2.8}$$

is a Hilbert-Schmidt operator from

$$L_Q^2 = Q^{1/2}\big(L^2(\mathbb{R},\mathbb{R}^m)\big) \tag{4.2.9}$$

into $L^2(\mathbb{R},\mathbb{R}^n)$ for any $U \in \mathcal{U}_{H^1}$. We note that the existence of the square-root $Q^{1/2}$ follows from the fact that Q is a nonnegative operator. This square-root has a convolution kernel p that is also translationally invariant; see Appendix 4.A.1 for the details.

For clarity, we take the noise intensity σ to be a scalar factor in front of g. In principle however, each of the $n \times m$ components of g could have its own scaling. This can also be fitted into our framework, but it would unnecessarily complicate the expansions we are after.

[7] We emphasize that the reference function Φ_{ref} does not depend on time, which is why (4.2.7) contains no additional time derivatives.

Deterministic terms Turning to the deterministic part of (4.2.4), we first note that h is a function that can be used to represent the appropriate Itô-Stratonovich correction terms. For example, in the scalar case $n = m = 1$ we saw in §4.1 that the choice $h(U) = \frac{1}{2}q(0)g'(U)g(U)$ allows us to interpret (4.2.4) as the Stratonovich SPDE

$$dU = \big[\rho\partial_{xx}U + f(U)\big]dt + \sigma g(U) \circ dW_t^Q. \tag{4.2.10}$$

We refer to [33, 108] for further information concerning the construction of similar correction terms for multi-component systems.

We now impose the following conditions on the nonlinearities f and h.

(HDt) The matrix $\rho \in \mathbb{R}^{n\times n}$ is a diagonal matrix with strictly positive diagonal elements $\{\rho_i\}_{i=1}^n$. In addition, we have $f, h \in C^3(\mathbb{R}^n, \mathbb{R}^n)$. Finally, $D^3 f$ and $D^3 h$ are bounded and there exists a constant $K_{\mathrm{var}} > 0$ so that the one-sided inequality

$$\langle f(u_A) - f(u_B), u_A - u_B\rangle_{\mathbb{R}^n} + \sigma^2\langle h(u_A) - h(u_B), u_A - u_B\rangle_{\mathbb{R}^n} \le K_{\mathrm{var}}\,|u_A - u_B|^2 \tag{4.2.11}$$

holds for all pairs $(u_A, u_B) \in \mathbb{R}^n \times \mathbb{R}^n$ and all $0 \le \sigma \le 1$.

The precise form of these assumptions is strongly motivated by the setup in [77]. Indeed, the four conditions (Hq), (HEq), (HSt) and (HDt) together allow us to apply [77, Thm 1.1]. This implies that our system (4.2.4) has a unique solution in $\mathcal{U}_{H^1}$ that is defined for all $t \ge 0$. A precise statement on the properties of these solutions can be found in Proposition 4.5.2.

Travelling wave The following assumption states that the deterministic part of (4.2.4) has a spectrally stable travelling wave solution that connects the two equilibria $u_\pm$. We remark again that these two limiting values are allowed to be equal.

(HTw) There exists a wavespeed $c_0 \in \mathbb{R}$ and a waveprofile $\Phi_0 \in C^2(\mathbb{R}, \mathbb{R}^n)$ that satisfies the travelling wave ODE

$$\rho\Phi_0'' + c_0\Phi_0' + f(\Phi_0) = 0 \tag{4.2.12}$$

and approaches its limiting values $\Phi_0(\pm\infty) = u_\pm$ at an exponential rate. In addition, the associated linear operator $\mathcal{L}_{\mathrm{tw}} : H^2(\mathbb{R}, \mathbb{R}^n) \to L^2(\mathbb{R}, \mathbb{R}^n)$ that acts as

$$[\mathcal{L}_{\mathrm{tw}}v](\xi) = \rho v''(\xi) + c_0 v'(\xi) + Df\big(\Phi_0(\xi)\big)v(\xi) \tag{4.2.13}$$

has a simple eigenvalue at $\lambda = 0$ and has no other spectrum in the half-plane $\{\Re\lambda \ge -2\beta\} \subset \mathbb{C}$ for some $\beta > 0$.

The formal adjoint

$$\mathcal{L}_{\mathrm{tw}}^* : H^2(\mathbb{R}, \mathbb{R}^n) \to L^2(\mathbb{R}, \mathbb{R}^n) \tag{4.2.14}$$

of the operator (4.2.13) acts as

$$[\mathcal{L}_{\mathrm{tw}}^* w](\xi) = \rho w''(\xi) - c_0 w'(\xi) + Df\big(\Phi_0(\xi)\big)^T w(\xi). \tag{4.2.15}$$

Indeed, one easily verifies that

$$\langle \mathcal{L}_{\mathrm{tw}} v, w \rangle_{L^2(\mathbb{R},\mathbb{R}^n)} = \langle v, \mathcal{L}^*_{\mathrm{tw}} w \rangle_{L^2(\mathbb{R},\mathbb{R}^n)} \tag{4.2.16}$$

holds whenever $(v, w) \in H^2(\mathbb{R}, \mathbb{R}^n) \times H^2(\mathbb{R}, \mathbb{R}^n)$.

The assumption that zero is a simple eigenvalue for $\mathcal{L}_{\mathrm{tw}}$ implies that $\mathcal{L}^*_{\mathrm{tw}} \psi_{\mathrm{tw}} = 0$ for some $\psi_{\mathrm{tw}} \in H^2(\mathbb{R}, \mathbb{R}^n)$ that can be normalized to have

$$\langle \Phi_0', \psi_{\mathrm{tw}} \rangle_{L^2(\mathbb{R},\mathbb{R}^n)} = 1. \tag{4.2.17}$$

These assumptions imply [65, §4] that the family of travelling wave solutions

$$U(x, t) = \Phi_0(x + c_0 t + \vartheta), \qquad \vartheta \in \mathbb{R}, \tag{4.2.18}$$

is nonlinearly stable under the dynamics of (4.2.4) at $\sigma = 0$. In particular, any small perturbation from such a wave converges exponentially fast to a nearby translate.

4.2.2 Overview

Guided by the short sketch in §4.1 of the ideas behind the deterministic stability proof, we now give a step-by-step description of the stochastic framework that we use to generalize this result and compute our expansions. At this stage we only give an overview of the key concepts, leaving the details to §4.2.3 and later sections.

Step 1: Stochastic phase. We introduce a wavespeed $c_\sigma \in \mathbb{R}$, together with a nonlinear functional $\overline{a}_\sigma : \mathcal{U}_{H^1} \times \mathbb{R} \to \mathbb{R}$. In addition, for any $U \in \mathcal{U}_{H^1}$ and $\Gamma \in \mathbb{R}$ we define a Hilbert-Schmidt operator $\overline{b}(U, \Gamma)$ that maps L^2_Q into $\mathbb{R}$. We emphasize that all three objects are unknown at present; see §4.5.2 for their precise definitions. However, they do allow us to define a stochastic phase $\Gamma(t)$ by coupling the SDE

$$d\Gamma = \big[c_\sigma + \overline{a}_\sigma(U, \Gamma)\big]\, dt + \sigma \overline{b}(U, \Gamma) dW^Q_t \tag{4.2.19}$$

to the SPDE (4.2.4) that governs U. This generalizes the deterministic phase that was introduced in (4.1.26). For convenience, we also write the phase in the integrated form

$$\Gamma(t) = \Gamma_0 + c_\sigma t + \int_0^t \overline{a}_\sigma\big(U(s), \Gamma(s)\big)\, ds + \sigma \int_0^t \overline{b}\big(U(s), \Gamma(s)\big) dW^Q_s. \tag{4.2.20}$$

Step 2: Decomposition of U. We now introduce a, yet unknown, waveprofile $\Phi_\sigma \in \mathcal{U}_{H^2}$. We can use this together with the phase Γ defined in Step 1 to define the perturbation

$$V(t) = U(\cdot + \Gamma(t), t) - \Phi_\sigma, \tag{4.2.21}$$

which measures the deviation from Φ_σ of U after shifting it to the left by $\Gamma(t)$. We note that V takes values in $H^1(\mathbb{R}, \mathbb{R}^n)$ in a sense that is made precise in Proposition 4.5.4. In addition, we will from now on write

$$a_\sigma(V) = \overline{a}_\sigma(\Phi_\sigma + V, 0), \qquad b_\sigma(V) = \overline{b}(\Phi_\sigma + V, 0). \tag{4.2.22}$$

Using Itô calculus, we will show in §4.5 that $V(t)$ solves an equation of the form

$$dV = \big[F_\sigma(\Phi_\sigma, c_\sigma; b_\sigma) + \mathcal{L}_{\mathrm{tw}} V + N_\sigma(V; b_\sigma) + a_\sigma(V)\partial_\xi(\Phi_\sigma + V)\big]dt + \sigma \mathcal{S}_\sigma(V; b_\sigma) dW_t^Q. \tag{4.2.23}$$

Due to the second order terms in the Itô formula, the specific shapes of F_σ, N_σ and $\mathcal{S}_\sigma$ all depend on the functional b_σ. It is therefore helpful to make a choice for b_σ, which we will do in the next step.

However, at this point an important warning is in order. We are using the same symbol for the noise processes driving (4.2.4) and (4.2.23), because - by translation invariance - they are are indistinguishable from one another.

On the other hand, when one wants to compare $U(t)$ and $V(t)$ numerically for a specific *realisation* of W_t^Q, then one must take care to spatially translate this realisation by $\Gamma(t)$ when passing between (4.2.4) and (4.2.23). Indeed, these two equations are defined in separate coordinate systems. This distinction will be explained in detail in the proof of Proposition 4.5.4. For now, we remark that all the averages that we compute in this section are invariant under translations in the noise.

Step 3: Choice of b. For any $V \in H^1(\mathbb{R}, \mathbb{R}^n)$, the computations in §4.5 show that

$$\mathcal{S}_\sigma(V; b_\sigma)[v] = g(\Phi_\sigma + V)v + \partial_\xi(\Phi_\sigma + V) b_\sigma(V)[v] \tag{4.2.24}$$

for all $v \in L_Q^2$. As in the deterministic case, the goal is to achieve the identity

$$\big\langle \mathcal{S}_\sigma(V; b_\sigma)[v], \psi_{\mathrm{tw}} \big\rangle_{L^2(\mathbb{R},\mathbb{R}^n)} = 0 \tag{4.2.25}$$

for all $v \in L_Q^2$ in order to circumvent the neutral mode of the semigroup. Whenever $\|V\|_{L^2(\mathbb{R},\mathbb{R}^n)}$ is sufficiently small, this can be achieved by writing

$$b_\sigma(V)[v] = -\frac{\big\langle g(\Phi_\sigma + V)v, \psi_{\mathrm{tw}} \big\rangle_{L^2(\mathbb{R},\mathbb{R}^n)}}{\big\langle \partial_\xi(\Phi_\sigma + V), \psi_{\mathrm{tw}} \big\rangle_{L^2(\mathbb{R},\mathbb{R}^n)}}. \tag{4.2.26}$$

Having made this choice, we now drop the dependence on b_σ in $F_\sigma(V)$, $N_\sigma(V)$ and $\mathcal{S}_\sigma(V)$.

Step 4: Construction of (Φ_σ, c_σ). Ideally, we would like $V(t) = 0$ to be a solution to (4.2.23), since then $U(x,t) = \Phi_\sigma(x + \Gamma(t), t)$ would be an exact solution to (4.2.4). However, since the deterministic and stochastic terms both need to vanish simultaneously, this can only be achieved in very special situations[8]. In Prop. 4.5.1 we show that for small σ, it is possible to construct a pair (Φ_σ, c_σ) for which $F_\sigma(\Phi_\sigma, c_\sigma) = 0$. Since we will see below that $N_\sigma(0) = 0$ and $a_\sigma(0) = 0$, this ensures that the state $V = 0$ only experiences (instantaneous) stochastic forcing.

For any pair $(\Phi, c) \in \mathcal{U}_{H^2} \times \mathbb{R}$, the nonlinearity $F_\sigma(\Phi, c)$ can be decomposed as

$$F_\sigma(\Phi, c) = F_0(\Phi, c) + \sigma^2 F_{0;2}(\Phi). \tag{4.2.27}$$

[8] In the case of 1d Brownian motion, we explain in Chapter 2 how g can be chosen to make this possible.

The leading order term $F_0(\Phi, c)$ is related to the deterministic wave in the sense that

$$F_0(\Phi, c) = \rho\Phi'' + c\Phi' + f(\Phi), \tag{4.2.28}$$

while the correction term $F_{0;2}(\Phi)$ is found to be

$$F_{0;2}(\Phi) = \frac{1}{2}\frac{\langle g(\Phi)Qg^T(\Phi)\psi_{\mathrm{tw}}, \psi_{\mathrm{tw}}\rangle_{L^2(\mathbb{R},\mathbb{R}^n)}}{\langle \Phi', \psi_{\mathrm{tw}}\rangle^2_{L^2(\mathbb{R},\mathbb{R}^n)}}\Phi'' - \frac{(g(\Phi)Qg^T(\Phi)\psi_{\mathrm{tw}})'}{\langle \Phi', \psi_{\mathrm{tw}}\rangle_{L^2(\mathbb{R},\mathbb{R}^n)}} + h(\Phi) \tag{4.2.29}$$

whenever $\|\Phi - \Phi_0\|_{L^2(\mathbb{R},\mathbb{R}^n)}$ is sufficiently small. We emphasize here that the transpose is taken in a pointwise fashion.

Note here that the correction term $F_{0;2}(\Phi)$ depends on the operator $g(\Phi)Qg^T(\Phi)$, which is the covariance operator of the stochastic process

$$\int_0^t g(\Phi)dW_s^Q \tag{4.2.30}$$

for $t \to \infty$ [44]. Therefore, as we will see, the lowest order corrections of Φ_σ to Φ_0 can be understood in terms of the covariance of the stochastic process $\int_0^t g(\Phi_0)dW_s^Q$.

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Step 5: Choice of a_σ. As in the deterministic case, we now define a_σ in such a way that the deterministic part of (4.2.23) becomes orthogonal to ψ_{tw}. In particular, for V small we write

$$a_\sigma(V) = -\frac{\langle N_\sigma(V), \psi_{\mathrm{tw}}\rangle_{L^2(\mathbb{R},\mathbb{R}^n)}}{\langle \partial_\xi(\Phi_\sigma + V), \psi_{\mathrm{tw}}\rangle_{L^2(\mathbb{R},\mathbb{R}^n)}}. \tag{4.2.31}$$

For reference, we note that the non-linearity N_σ introduced in (4.2.23) is given by

$$N_\sigma(V) = F_\sigma(\Phi_\sigma + V, c_\sigma) - F_\sigma(\Phi_\sigma, c_\sigma) - \mathcal{L}_{\mathrm{tw}}V. \tag{4.2.32}$$

As we claimed in Step 4, we indeed see that $a_\sigma(0) = 0$ and $N_\sigma(0) = 0$. Upon introducing a nonlinearity $\mathcal{R}_\sigma$ that acts as

$$\mathcal{R}_\sigma\big(V\big) = F_\sigma(\Phi_\sigma + V, c_\sigma) + a_\sigma(V)\partial_\xi(\Phi_\sigma + V) \tag{4.2.33}$$

for small V, we conclude that V solves the equation

$$dV = \mathcal{R}_\sigma(V)dt + \sigma\mathcal{S}_\sigma(V)dW_t^Q. \tag{4.2.34}$$

The nonlinearities on the right hand side of this equation are now both orthogonal to ψ_{tw} for small V.

Step 6: Stability. We write $S(t)$ for the semigroup generated by the linear operator $\mathcal{L}_{\mathrm{tw}}$ and consider the mild formulation of (4.2.34), which is given by the integral equation

$$V(t) = S(t)V_0 + \int_0^t S(t-s)\widetilde{\mathcal{R}}_\sigma\big(V(s)\big)ds + \sigma\int_0^t S(t-s)\mathcal{S}_\sigma\big(V(s)\big)dW_s^Q, \tag{4.2.35}$$

in which we have

$$\widetilde{\mathcal{R}}_\sigma(V) = \mathcal{R}_\sigma(V) - \mathcal{L}_{\rm tw}V = N_\sigma(V) + a_\sigma(V)\partial_\xi(\Phi_\sigma + V). \tag{4.2.36}$$

By construction, we have achieved $\langle \widetilde{\mathcal{R}}_\sigma(V), \psi_{\rm tw}\rangle_{L^2(\mathbb{R},\mathbb{R}^n)} = \langle \mathcal{S}_\sigma(V), \psi_{\rm tw}\rangle_{L^2(\mathbb{R},\mathbb{R}^n)} = 0$ for small V. Whenever $U(0)$ is sufficiently close to Φ_0, we can also ensure $\langle V_0, \psi_{\rm tw}\rangle_{L^2(\mathbb{R},\mathbb{R}^n)} = 0$ by picking the initial phase $\Gamma(0)$ appropriately.

We caution the reader that it is hard to obtain estimates on V directly from (4.2.35), because the term $\widetilde{\mathcal{R}}_\sigma(V)$ still contains second order derivatives. Tackling this problem is the key part of Chapter 3, as we discuss in §4.6. Nevertheless, it is possible to show that the instantaneous wave (Φ_σ, c_σ) is stable in the sense that the size of $V(t)$ can be kept under control. An exact statement to this effect can be found in §4.6, but we here provide an informal summary.

Theorem 4.2.1 (see §4.6)**.** *Assume that (Hq), (HEq), (HDt), (HSt) and (HTw) all hold and that σ and V_0 are sufficiently small. Then for time scales up to $\mathcal{O}(\sigma^{-2})$, the perturbation $V(t)$ remains small and the phase $\Gamma(t)$ accurately represents the position of $U(t)$ relative to the wave (Φ_σ, c_σ).*

As we discussed in §4.1, we expect this result to remain valid up to exponentially long time scales. This is supported by the numerical evidence in §4.3.

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Step 7: Expansion in σ. In order to investigate the fluctuations around the instantaneous stochastic wave (Φ_σ, c_σ), we choose $(V_0, \Gamma_0) = (0,0)$ and expand our equations for (V, Γ) in powers of σ. In particular, we look for expansions of the form

$$V(t) = \sigma V_\sigma^{(1)}(t) + \sigma^2 V_\sigma^{(2)}(t) + V_{\rm res}(t) \tag{4.2.37}$$

and

$$\Gamma(t) = c_\sigma t + \sigma\Gamma_\sigma^{(1)}(t) + \sigma^2\Gamma_\sigma^{(2)}(t) + \sigma^3\Gamma_\sigma^{(3)}(t) + \mathcal{O}(\sigma^4). \tag{4.2.38}$$

For example, using (4.2.35) we may write

$$V_\sigma^{(1)}(t) = \int_0^t S(t-s)\mathcal{S}_\sigma(0)dW_s^Q, \tag{4.2.39}$$

which can be substituted back into (4.2.35) to find an expression for $V_\sigma^{(2)}(t)$ and so on. In addition, using (4.2.20) it is natural to write

$$\Gamma_\sigma^{(1)}(t) = \int_0^t b_\sigma(0)dW_s^Q. \tag{4.2.40}$$

Knowledge of $V_\sigma^{(1)}$ can subsequently be used to define $\Gamma_\sigma^{(2)}$, while $V_\sigma^{(2)}$ can be used to compute $\Gamma_\sigma^{(3)}$.

We provide explicit formula's for these expansion terms in §4.2.3 below. We mention here that we are including a σ-dependence in these terms as it often increases the readability to use (Φ_σ, c_σ) instead of (Φ_0, c_0). For example, $\mathcal{S}_\sigma(0)$ can be expanded in terms of σ to yield $V_\sigma^{(1)}(t) = V_0^{(1)}(t) + \mathcal{O}(\sigma^2)$, hence the difference between $\sigma V_\sigma^{(1)}(t)$ and $\sigma V_0^{(1)}(t)$ is only seen at third order.

Corollary 4.2.2 (see §4.6)**.** *Assume that (Hq), (HEq), (HSt), (HDt) and (HTw) all hold. Then $\sigma^{-2}V_{\rm res}$ remains small for time scales up to $\mathcal{O}(\sigma^{-2})$.*

Step 8: Formal limits. We are now in a position to address our main question concerning the average long-term behaviour of the speed and shape of $U(t)$. In particular, we are interested to see if - and in what sense - it is possible to define limiting quantities

$$\big(\Phi_{\sigma;\lim}, c_{\sigma;\lim}\big) = \text{`}\lim_{t\to\infty}\text{'}\ E\Big(U(\cdot + \Gamma(t), t), t^{-1}\Gamma(t)\Big). \tag{4.2.41}$$

Any rigorous definition of such a limit most likely requires the use of carefully constructed stopping times, since our wave-tracking mechanism almost surely fails at some finite time (see also §4.2.4). This delicate theoretical question is outside of the scope of this thesis unfortunately. From a practical point of view however, the numerical results in §4.3-4.4 displayed in the stochastic reference frame $\Gamma(t)$ clearly indicate that some type of fast convergence is taking place on long time scales. In fact, by evaluating the averages in (4.2.41) for sufficiently large values of t, we construct observed quantities $\big(\Phi^{\rm obs}_{\sigma;\lim}, c^{\rm obs}_{\sigma;\lim}\big)$ that we feel are useful proxies for the limits (4.2.41).

We emphasize that we expect these quantities to differ from the instantaneous stochastic wave (Φ_σ, c_σ). Indeed, the stochastic forcing leads to an effect that we refer to as 'orbital drift'. Upon (formally) writing

$$\big(V^{\rm od}_\sigma, c^{\rm od}_\sigma\big) = \big(\Phi_{\sigma;\lim} - \Phi_\sigma, c_{\sigma;\lim} - c_\sigma\big) \tag{4.2.42}$$

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to quantify this difference, we note that

$$\begin{array}{rcl} c^{\rm od}_\sigma & = & \text{`}\lim\limits_{t\to\infty}\text{'}\ E\,t^{-1}[\Gamma(t) - c_\sigma t], \\ V^{\rm od}_\sigma & = & \text{`}\lim\limits_{t\to\infty}\text{'}\ E\,V(t). \end{array} \tag{4.2.43}$$

Of course, the same theoretical issues discussed above apply to these limits.

The key point however, is that such limits *do* exist naturally for the individual terms in the expansions (4.2.37)-(4.2.38). In particular, it *is* possible to compute the expansions

$$\begin{array}{rcl} c^{\rm od}_{\sigma;i} & = & \lim\limits_{t\to\infty} E\,t^{-1}\Gamma^{(i)}_\sigma(t), \\ V^{\rm od}_{\sigma;i} & = & \lim\limits_{t\to\infty} E\,V^{(i)}_\sigma(t), \end{array} \tag{4.2.44}$$

which allows us to compute approximations for (4.2.41) that can be explicitly evaluated. In §4.3-4.4 we show that they agree remarkably well with the observed numerical proxies $\big(\Phi^{\rm obs}_{\sigma;\lim}, c^{\rm obs}_{\sigma;\lim}\big)$ for (4.2.41).

Giving an interpretation to the pair $(\Phi_{\sigma;\lim}, c_{\sigma;\lim})$ however is difficult. We do not have any ODE that it solves, but we think of $(\Phi_{\sigma;\lim}, c_{\sigma;\lim})$ as the ceasefire line between the stochastic term that pushes the solution away from (Φ_σ, c_σ) and the exponential decay of the deterministic part that pushes it back to (Φ_σ, c_σ). We remark that it might be possible to embed $(\Phi_{\sigma;\lim}, c_{\sigma;\lim})$ in some type of an invariant measure for the SPDE. There is a rich literature on the existence of invariant measures to stochastic Reaction-Diffusion equations, see e.g. [20] and we intend to study this in the future.

4.2.3 Explicit expansions

We now set out to explain in detail how the expansions discussed in §4.2.2 can be derived. We give general results here, but also show how they can be applied to two explicit examples in §4.3 and §4.4.

Expansions for (Φ_σ, c_σ) First, we examine the correction terms that are required to obtain the instantaneous stochastic wave from the deterministic wave (Φ_0, c_0). In particular, we recall the defining identity

$$F_0(\Phi_\sigma, c_\sigma) + \sigma^2 F_{0;2}(\Phi_\sigma, c_\sigma) = 0 \tag{4.2.45}$$

and write

$$\begin{aligned} \Phi_\sigma &= \Phi_0 + \sigma^2 \Phi_{0;2} + \mathcal{O}(\sigma^4), \\ c_\sigma &= c_0 + \sigma^2 c_{0;2} + \mathcal{O}(\sigma^4) \end{aligned} \tag{4.2.46}$$

for the solutions that are constructed in Proposition 4.5.1. We note that the $\mathcal{O}(1)$-terms in (4.2.45) indeed vanish because $F_0(\Phi_0, c_0) = 0$. Balancing the $\mathcal{O}(\sigma^2)$-terms, we find

$$\begin{aligned} \mathcal{L}_{\mathrm{tw}} \Phi_{0;2} =& -\frac{1}{2}\Phi_0'' \langle g(\Phi_0) Q g^T(\Phi_0)\psi_{\mathrm{tw}}, \psi_{\mathrm{tw}}\rangle^2_{L^2(\mathbb{R},\mathbb{R}^n)} - c_{0;2}\Phi_0' \\ &+ (g(\Phi_0)Qg^T(\Phi_0)\psi_{\mathrm{tw}})' - h(\Phi_0) \\ =& - F_{0;2}(\Phi_0, c_0) - c_{0;2}\Phi_0'. \end{aligned} \tag{4.2.47}$$

By the Fredholm alternative, we know that we can solve for $\Phi_{0;2}$ when the right hand side of this equation is orthogonal to ψ_{tw}. In view of the normalization $\langle \Phi_0', \psi_{\mathrm{tw}}\rangle_{L^2(\mathbb{R},\mathbb{R}^n)} = 1$, we hence find

$$c_{0;2} = -\langle F_{0;2}(\Phi_0, c_0), \psi_{\mathrm{tw}}\rangle_{L^2(\mathbb{R},\mathbb{R}^n)}. \tag{4.2.48}$$

The function $\Phi_{0;2}$ can now be computed by numerically (or analytically when possible) inverting $\mathcal{L}_{\mathrm{tw}}$ and solving (4.2.47).

First order: $(\Gamma_\sigma^{(1)}, V_\sigma^{(1)})$ We now turn our attention to the first order terms in the expansions (4.2.37)-(4.2.38). Expanding the expressions (4.2.39)-(4.2.40), we may write

$$\begin{aligned} V_\sigma^{(1)}(t) =& \int_0^t S(t-s) g(\Phi_\sigma) dW_s^Q \\ &- \int_0^t S(t-s)\Phi_\sigma' \frac{\langle \psi_{\mathrm{tw}}, g(\Phi_\sigma) dW_s^Q\rangle_{L^2(\mathbb{R},\mathbb{R}^n)}}{\langle \Phi_\sigma', \psi_{\mathrm{tw}}\rangle_{L^2(\mathbb{R},\mathbb{R}^n)}}, \end{aligned} \tag{4.2.49}$$

together with

$$\Gamma_\sigma^{(1)}(t) = -\int_0^t \frac{\langle \psi_{\mathrm{tw}}, g(\Phi_\sigma) dW_s^Q\rangle_{L^2(\mathbb{R},\mathbb{R}^n)}}{\langle \Phi_\sigma', \psi_{\mathrm{tw}}\rangle_{L^2(\mathbb{R},\mathbb{R}^n)}}. \tag{4.2.50}$$

We note that $E[V_\sigma^{(1)}(t)] = E[\Gamma_\sigma^{(1)}(t)] = 0$. On account of the decay of the semigroup, $V_\sigma^{(1)}$ can be regarded as a process of Ornstein-Uhlenbeck type. On the other hand, $\Gamma_\sigma^{(1)}$ behaves as a scaled Brownian motion with variance

$$\mathrm{Var}(\Gamma_\sigma^{(1)}(t)) = \langle g(\Phi_\sigma) Q g^T(\Phi_\sigma)\psi_{\mathrm{tw}}, \psi_{\mathrm{tw}}\rangle^2_{L^2(\mathbb{R},\mathbb{R}^n)} t. \tag{4.2.51}$$

Second order: $(\Gamma_\sigma^{(2)}, V_\sigma^{(2)})$ Substituting the first order term $V_\sigma^{(1)}$ into the right-hand-side of (4.2.35), we find that $V_\sigma^{(2)}(t)$ picks up a deterministic contribution coming from the quadratic terms in $\mathcal{R}_\sigma$, together with a stochastic contribution arising from the linear terms in $\mathcal{S}_\sigma$. In particular, we obtain

$$V_\sigma^{(2)}(t) = \int_0^t S(t-s)\mathcal{R}_\sigma^{(2)}[V_\sigma^{(1)}(s), V_\sigma^{(1)}(s)]\, ds + \int_0^t S(t-s)\mathcal{S}_\sigma^{(1)}\big(V_\sigma^{(1)}(s)\big)\, dW_s^Q, \tag{4.2.52}$$

in which we have

$$\mathcal{R}_\sigma^{(2)}[V,V] = \frac{1}{2}D^2 f(\Phi_\sigma)[V,V] - \frac{1}{2}\Phi_\sigma' \frac{\langle D^2 f(\Phi_\sigma)[V,V], \psi_{\mathrm{tw}}\rangle_{L^2(\mathbb{R},\mathbb{R}^n)}}{\langle \Phi_\sigma', \psi_{\mathrm{tw}}\rangle_{L^2(\mathbb{R},\mathbb{R}^n)}}, \tag{4.2.53}$$

together with

$$\begin{aligned} \mathcal{S}_\sigma^{(1)}(V)[w] =& Dg(\Phi_\sigma)[V]w - \partial_\xi V \frac{\langle \psi_{\mathrm{tw}}, g(\Phi_\sigma)w\rangle_{L^2(\mathbb{R},\mathbb{R}^n)}}{\langle \Phi_\sigma', \psi_{\mathrm{tw}}\rangle_{L^2(\mathbb{R},\mathbb{R}^n)}} \\ & - \Phi_\sigma' \frac{\langle \psi_{\mathrm{tw}}, Dg(\Phi_0)[V]w\rangle_{L^2(\mathbb{R},\mathbb{R}^n)}}{\langle \Phi_\sigma', \psi_{\mathrm{tw}}\rangle_{L^2(\mathbb{R},\mathbb{R}^n)}} \\ & + \Phi_\sigma' \langle \partial_\xi V, \psi_{\mathrm{tw}}\rangle_{L^2(\mathbb{R},\mathbb{R}^n)} \frac{\langle \psi_{\mathrm{tw}}, g(\Phi_\sigma)w\rangle_{L^2(\mathbb{R},\mathbb{R}^n)}}{\langle \Phi_\sigma', \psi_{\mathrm{tw}}\rangle^2_{L^2(\mathbb{R},\mathbb{R}^n)}} \end{aligned} \tag{4.2.54}$$

for any $w \in L_Q^2$. In a similar fashion, we find

$$\Gamma_\sigma^{(2)}(t) = \int_0^t a_\sigma^{(2)}(\Phi_\sigma)[V_\sigma^{(1)}(s), V_\sigma^{(1)}(s)]\, ds + \int_0^t b_\sigma^{(1)}(\Phi_\sigma)[V_\sigma^{(1)}(s)]\, dW_t^Q, \tag{4.2.55}$$

in which we have

$$a_\sigma^{(2)}[V,V] = -\frac{1}{2}\langle D^2 f(\Phi_\sigma)[V,V], \psi_{\mathrm{tw}}\rangle_{L^2(\mathbb{R},\mathbb{R}^n)}, \tag{4.2.56}$$

together with

$$\begin{aligned} b_\sigma^{(1)}(\Phi_\sigma)[V][w] =& -\frac{\langle \psi_{\mathrm{tw}}, Dg(\Phi_\sigma)[V]v\rangle_{L^2(\mathbb{R},\mathbb{R}^n)}}{\langle \Phi_\sigma', \psi_{\mathrm{tw}}\rangle_{L^2(\mathbb{R},\mathbb{R}^n)}} \\ & - \langle \partial_\xi V, \psi_{\mathrm{tw}}\rangle_{L^2(\mathbb{R},\mathbb{R}^n)} \frac{\langle \psi_{\mathrm{tw}}, g(\Phi_\sigma)w\rangle_{L^2(\mathbb{R},\mathbb{R}^n)}}{\langle \Phi_\sigma', \psi_{\mathrm{tw}}\rangle^2_{L^2(\mathbb{R},\mathbb{R}^n)}}. \end{aligned} \tag{4.2.57}$$

Note that the expressions for $V_\sigma^{(2)}(t)$ and $\Gamma_\sigma^{(2)}(t)$ depend on f, but not on h. This is due to the fact that the $\mathcal{O}(\sigma^2)$ part of the Itô-Stratonovich correction term is already absorbed in (Φ_σ, c_σ). If we would have started our computations around (Φ_0, c_0), the dependence of h would show up via $(\Phi_{0;2}, c_{0;2})$. However the extra second order terms together form (4.2.47) and therefore vanish.

We remark that both of these second order terms have a nonzero expectation, which can be explicitly computed using the Itô lemma. To this end, we introduce the notation

$$K_\sigma(s)[w_1, w_2] = \frac{1}{2}D^2 f(\Phi_\sigma)[S(s)\mathcal{S}_\sigma(0)w_1, S(s)\mathcal{S}_\sigma(0)w_2] \tag{4.2.58}$$

for any $v, w \in L^2_Q$. Upon choosing a basis (e_k) of $L^2(\mathbb{R}, \mathbb{R}^m)$ and applying Lemma 4.5.3, we find

$$E[\Gamma_\sigma^{(2)}(t)] = -\int_0^t \int_0^s \sum_{k=0}^{\infty} \langle K_\sigma(s')[\sqrt{Q}e_k, \sqrt{Q}e_k], \psi_{\mathrm{tw}}\rangle_{L^2(\mathbb{R},\mathbb{R}^n)}\, ds'ds, \tag{4.2.59}$$

together with

$$E[V_\sigma^{(2)}(t)] = \int_0^t S(t-s) \int_0^s \sum_{k=0}^{\infty} \Big[K_\sigma(s')[\sqrt{Q}e_k, \sqrt{Q}e_k] - \Phi'_\sigma \frac{\langle K_\sigma(s')[\sqrt{Q}e_k, \sqrt{Q}e_k], \psi_{\mathrm{tw}}\rangle_{L^2(\mathbb{R},\mathbb{R}^n)}}{\langle \Phi'_\sigma, \psi_{\mathrm{tw}}\rangle_{L^2(\mathbb{R},\mathbb{R}^n)}}\Big] ds'ds. \tag{4.2.60}$$

Sending $t \to \infty$, we can explicitly compute

$$c^{\mathrm{od}}_{\sigma;2} = \lim_{t\to\infty} t^{-1} E[\Gamma_\sigma^{(2)}(t)] = -\int_0^\infty \sum_{k=0}^{\infty} \langle K_\sigma(s')[\sqrt{Q}e_k, \sqrt{Q}e_k], \psi_{\mathrm{tw}}\rangle_{L^2(\mathbb{R},\mathbb{R}^n)}\, ds. \tag{4.2.61}$$

Note that this integral converges because $\mathcal{S}_\sigma(0)$ is orthogonal to ψ_{tw}, which circumvents the nondecaying mode of the semigroup.

In a similar fashion, we can obtain

$$V^{\mathrm{od}}_{\sigma;2} = \lim_{t\to\infty} E\left[V_\sigma^{(2)}(t)\right]. \tag{4.2.62}$$

Switching the integrals in (4.2.60) and applying the operator identity [80, Prop. 1.3.6]

$$\mathcal{L}_{\mathrm{tw}} \int_0^t S(s)ds = S(t) - I, \tag{4.2.63}$$

we arrive at

$$V^{\mathrm{od}}_{\sigma;2} = -\mathcal{L}_{\mathrm{tw}}^{-1} \int_0^\infty \sum_{k=0}^{\infty} \Big[K_\sigma(s)[\sqrt{Q}e_k, \sqrt{Q}e_k] - \Phi'_\sigma \frac{\langle K_\sigma(s)[\sqrt{Q}e_k, \sqrt{Q}e_k], \psi_{\mathrm{tw}}\rangle_{L^2(\mathbb{R},\mathbb{R}^n)}}{\langle \Phi'_\sigma, \psi_{\mathrm{tw}}\rangle_{L^2(\mathbb{R},\mathbb{R}^n)}}\Big] ds. \tag{4.2.64}$$

Third order: $\Gamma_\sigma^{(3)}$ Provided that the nonlinearities are sufficiently smooth, the methods in the previous paragraphs can in principle be extended to any desired order in σ, but the computations get more involved. However, it is important to note that in order to compute the n-th order approximation of $\Gamma(t)$, we only need information from $V(t)$ up to order $n-1$. In particular, upon inspecting equation (4.2.20) we find that

$$\sigma^3 \Gamma_\sigma^{(3)}(t) = \int_0^t a_\sigma\big(\sigma V_\sigma^{(1)}(s) + \sigma^2 V_\sigma^{(2)}(s)\big)\, ds + \sigma \int_0^t b_\sigma\big(\sigma V_\sigma^{(1)}(s) + \sigma^2 V_\sigma^{(2)}(s)\big)\, dW_s^Q - \sigma\Gamma_\sigma^{(1)}(t) - \sigma^2\Gamma_\sigma^{(2)}(t) + \mathcal{O}(\sigma^4). \tag{4.2.65}$$

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This gives us a convenient numerical procedure to compute $c_{0;3}^{\mathrm{od}}$ without having to explicitly compute $a_\sigma^{(3)}$ and $b_\sigma^{(2)}$. Indeed, we may write

$$c_{0;3}^{\mathrm{od}} = \lim_{\sigma\to 0,\, t\to\infty} \sigma^{-3} E\Big[\int_0^t a_\sigma\big(\sigma V_\sigma^{(1)}(s) + \sigma^2 V_\sigma^{(2)}(s)\big)\, ds - \sigma^2 \Gamma_\sigma^{(2)}(t)\Big]. \tag{4.2.66}$$

4.2.4 Predictions

Based upon the perturbation analysis in the previous section, we can make the following predictions on the behaviour of the wave.

Diffusive phase wandering At leading order in σ, we see that the phase wanders diffusively around the deterministic position $c_0 t$. Indeed, based upon (4.2.51) we predict that

$$\mathrm{Var}\big(\Gamma(t)\big) = \sigma^2 \langle g(\Phi_\sigma) Q g^T(\Phi_\sigma)\psi_{\mathrm{tw}}, \psi_{\mathrm{tw}}\rangle^2_{L^2(\mathbb{R},\mathbb{R}^n)} t + \mathcal{O}(\sigma^3). \tag{4.2.67}$$

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Note that this expression coincides with the mean square deviation from the deterministic phase $E[(\Gamma(t) - c_0 t)^2]$ up to $\mathcal{O}(\sigma^3)$. In the specific case of the stochastic Nagumo equation, this expression has been known for two decades already [39, eq. (6.25)]. Similar identities (with $g(u) = 1$) were found for almost translationally invariant additive noise [68, §3.4] and in the context of neural field equations [16, 72]. Remark that the difference between the Itô and Stratonovich interpretation cannot yet be observed at this level.

Short term behaviour Based on (4.2.59) we see that on short timescales we have

$$E[\Gamma_\sigma^{(2)}(t)] \sim t^2, \tag{4.2.68}$$

which does not contribute meaningfully to the speed for small t. Similarly, we have

$$\mathrm{Var}[V_\sigma^{(1)}(t)] \sim t, \qquad E[V_\sigma^{(2)}(t)] \sim t^2, \tag{4.2.69}$$

which shows that also the shape of the wave is relatively unaffected by these correction terms. In particular, we see that on short timescales the pair (Φ_σ, c_σ) indeed accurately describes the shape and speed of the wave. We feel that this justifies the use of our 'instantaneous stochastic wave' terminology.

Long term behaviour On longer timescales the Ornstein-Uhlenbeck-like process $V_\sigma^{(1)}$ starts to play an important role, causing fluctuations around (Φ_σ, c_σ) that lead to the orbital drift corrections. Using the Itô isometry, we predict that the size of the perturbations behaves as

$$\begin{aligned} E[\|V(t)\|^2_{L^2(\mathbb{R},\mathbb{R}^n)}] &= \sigma^2 E[\|V_0^{(1)}(t)\|^2_{L^2(\mathbb{R},\mathbb{R}^n)}] + \mathcal{O}(\sigma^3) \\ &= \sigma^2 \int_0^t \|S(s)\mathcal{S}_0(0)\|^2_{HS\left(L^2_Q, L^2(\mathbb{R},\mathbb{R}^n)\right)}\, ds + \mathcal{O}(\sigma^3), \end{aligned} \tag{4.2.70}$$

see [26, Ex. 4]. We point out that the integral actually converges (at an exponential rate) as $t \to \infty$. Naturally, the residual is predicted to behave as

$$E[\|V_{\rm res}(t)\|^2_{L^2(\mathbb{R},\mathbb{R}^n)}] = E[\|V(t) - V^{(1)}_\sigma(t) - V^{(2)}_\sigma(t)\|^2_{L^2(\mathbb{R},\mathbb{R}^n)}] \sim \mathcal{O}(\sigma^6). \tag{4.2.71}$$

Although the average of $V^{(1)}_0(t)$ can be kept under control, our phase tracking breaks down as soon as $\|V(t)\|_{L^2(\mathbb{R},\mathbb{R}^n)}$ exceeds a certain σ-independent threshold. Based on the hitting-time estimates for the scalar Ornstein-Uhlenbeck process that were obtained in [87, eq. (6a)], we conjecture that the expected break-down time increases exponentially with respect to σ^{-2}. In a similar vein, the quantity of interest for our stability analysis is the expectation of the *supremum* of $\|V(t)\|^2$ over the interval $[0,T]$. Based on detailed and very delicate computations for the standard scalar Ornstein-Uhlenbeck process [2, 91, 96], we conjecture that $E\sup_{0\le t\le T}\|V^{(1)}_0(t)\|^2_{L^2(\mathbb{R},\mathbb{R}^n)} \sim \ln(T)$, while the corresponding expression for $V^{(2)}_0$ grows as $\ln^2(T)$; see also Figure 4.8a.

Both these conjectures suggest that our framework remains valid for T up to $\mathcal{O}(e^{\eta\sigma^{-2}})$ for some small $\eta > 0$. However, it is not clear to us how the bounds above can be obtained in the infinite-dimensional semigroup setting. In our rigorous stability proof we are therefore forced to work with a weaker $\mathcal{O}(\sigma^2 T)$ bound for the supremum expectation, which understates the timescales over which we can keep track of the stochastic wave.

Once we have lost track of the wave, it could potentially be possible to restart the tracking mechanism by allowing for an instantaneous jump in the phase. In [57, §7] some first promising results in this direction were obtained by defining the phase as the - possibly discontinuous - solution to a global minimisation problem.

Turning to the limiting speed and shape of the wave, we arrive at the prediction

$$\begin{aligned} c_{\sigma;\rm lim} &= c_\sigma + \sigma^2 c^{\rm od}_{\sigma;2} + \sigma^3 c^{\rm od}_{\sigma;3} + \mathcal{O}(\sigma^4) \\ &= c_0 + \sigma^2[c_{0;2} + c^{\rm od}_{0;2}] + \sigma^3 c^{\rm od}_{0;3} + \mathcal{O}(\sigma^4), \end{aligned} \tag{4.2.72}$$

where we used $(\Phi_\sigma, c_\sigma) = (\Phi_0, c_0) + \mathcal{O}(\sigma^2)$ to conclude that the difference between $c^{\rm od}_{\sigma;2}$ and $c^{\rm od}_{0;2}$ is also of order σ^2. In a similar fashion, we obtain

$$\begin{aligned} \Phi_{\sigma;\rm lim} &= \Phi_\sigma + \sigma^2 V^{\rm od}_{0;2} + \mathcal{O}(\sigma^3) \\ &= \Phi_0 + \sigma^2\big[\Phi_{0;2} + V^{\rm od}_{0;2}\big] + \mathcal{O}(\sigma^3). \end{aligned} \tag{4.2.73}$$

The leading order terms in the expressions (4.2.72)-(4.2.73) can all be explicitly computed, which will allow us to test our predictions against numerical simulations in §4.3 and §4.4.

4.3 Example I: The Nagumo equation

In this section, we study the explicit example

$$dU = [\partial_{xx}U + f_{\rm cub}(U) + \frac{\mu\sigma^2}{2} q(0) g'(U) g(U)]dt + \sigma g(U) dW^Q_t, \tag{4.3.1}$$

in which μ is either zero (Itô) or one (Stratonovich), while the nonlinearities are given by

$$f_{\mathrm{cub}}(U) = U(1-U)(U-a), \qquad g(U) = U(1-U) \tag{4.3.2}$$

for some $a \in (0,1)$. We do remark that g does not have a bounded second derivative as demanded by our assumption (HSt). This technical problem can be remedied by applying a cut-off function to $g(U)$ to ensure that this value levels off for $U >> 1$.

Following [79], we use the normalized kernel

$$q(x) = \frac{1}{2\zeta} e^{\frac{-\pi x^2}{4\zeta^2}} \tag{4.3.3}$$

to generate the cylindrical Q-Wiener process W_t^Q over $L^2(\mathbb{R})$. Here the parameter $\zeta > 0$ is a measure for the spatial correlation length, which is defined [39] as the second moment of q, i.e. $\dfrac{2\zeta}{\pi}$. The kernel p of $\sqrt{Q}$ can be computed by taking the inverse Fourier transform of $\sqrt{\widehat{q}}$; see Appendix 4.A.1. This yields

$$p(x) = \sqrt[4]{\frac{\pi}{2}} \frac{e^{-\frac{\pi x^2}{2\zeta^2}}}{\zeta}. \tag{4.3.4}$$

4

Notice that the dimensions of the problem are $n = m = 1$, which means that $g = g^T$.

We now set out to carefully perform the computations in §4.2.3 and compare the results with our numerical simulations. These simulations are based on the algorithms from [78, Ch. 10]. In particular, we use a semi-implicit scheme in time and a straightforward central-difference discretization in space. In addition, we use circulant embedding [78, Alg. 6.8] to generate a stochastic Wiener process with the prescribed spatial correlation function.

Computing (Φ_σ, c_σ) As explained in §4.1, the wave (Φ_0, c_0) satisfies the ODE

$$\Phi_0'' + c_0 \Phi_0' + f_{\mathrm{cub}}(\Phi_0) = 0 \tag{4.3.5}$$

and is given by

$$\Phi_0 = \frac{1}{2}\left[1 - \tanh\left(\frac{1}{2\sqrt{2}}x\right)\right], \qquad c_0 = \sqrt{2}\left(\frac{1}{2} - a\right). \tag{4.3.6}$$

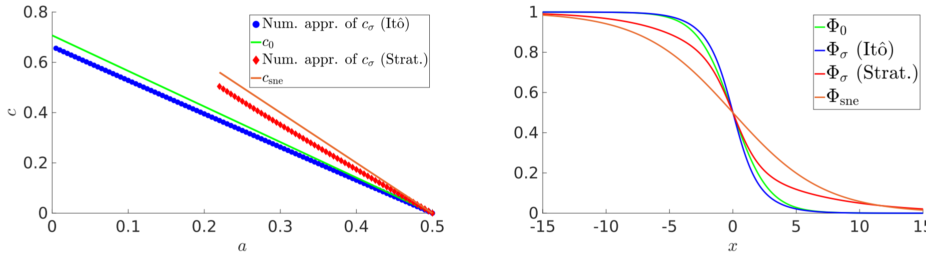
The linear operators $\mathcal{L}_{\mathrm{tw}}$ and $\mathcal{L}_{\mathrm{tw}}^*$ act as

$$\mathcal{L}_{\mathrm{tw}} v = v'' + c_0 v' + f_{\mathrm{cub}}'(\Phi_0) v, \qquad \mathcal{L}_{\mathrm{tw}}^* w = w'' - c_0 w' + f_{\mathrm{cub}}'(\Phi_0) w \tag{4.3.7}$$

and we write Φ_0' respectively $\psi_{\mathrm{tw}}(x) = e^{c_0 x}\Phi_0' / \langle \Phi_0', e^{c_0 \cdot}\Phi_0' \rangle_{L^2(\mathbb{R})}$ for their normalized simple eigenfunctions at zero.

In this scalar setting, the full equation $F_\sigma(\Phi_\sigma, c_\sigma) = 0$ can be written as

$$\begin{aligned}\Phi_\sigma'' + c_\sigma \Phi_\sigma' + f_{\mathrm{cub}}(\Phi_\sigma) = &- \frac{\sigma^2}{2} \frac{\langle q * (g(\Phi_\sigma)\psi_{\mathrm{tw}}), g(\Phi_\sigma)\psi_{\mathrm{tw}} \rangle_{L^2(\mathbb{R})}}{\langle \Phi_\sigma', \psi_{\mathrm{tw}} \rangle_{L^2(\mathbb{R})}^2} \Phi_\sigma'' \\ &+ \sigma^2 \frac{\big(g(\Phi_\sigma) q * (g(\Phi_\sigma)\psi_{\mathrm{tw}})\big)'}{\langle \Phi_\sigma', \psi_{\mathrm{tw}} \rangle_{L^2(\mathbb{R})}} - \frac{\mu\sigma^2}{2} q(0) g'(\Phi_\sigma) g(\Phi_\sigma).\end{aligned} \tag{4.3.8}$$



(a) (b)

Figure 4.1: Panel (a) compares the deterministic wave speed (4.3.6) (green), the instantaneous stochastic wave speed c_σ for the Ito (blue) and Stratonovich (red) interpretations and the speed c_{sne} derived in [39, eq. (6.33)] (orange), all for $\sigma = 1$ and $\zeta = 1$. The red and orange lines are only plotted for $a_{\mathrm{eff}} \in (0, 1/2)$. Panel (b) compares the associated wave profiles for $a = 0.45$ and $\sigma = 1.3$. Notice the steepening and flattening of the waves for the Itô respectively Stratonovich interpretations. The profiles are computed on the interval $[-40, 40]$, but here zoomed in to $[-15, 15]$ to highlight the differences.

It is interesting to compare this equation with the system

$$\Phi''_{\mathrm{sne}} + c_{\mathrm{sne}}\Phi'_{\mathrm{sne}} + f_{\mathrm{cub}}(\Phi_{\mathrm{sne}}) = -\frac{\mu\sigma^2}{2} q(0) g'(\Phi_{\mathrm{sne}}) g(\Phi_{\mathrm{sne}}) \tag{4.3.9}$$

used to construct the waves $(\Phi_{\mathrm{sne}}, c_{\mathrm{sne}})$ in [39] using their so-called small noise expansion technique. As the authors remark, this equation is not the result of a systematic perturbative expansion in σ, but rather a partial resummation of such an expansion. For example, the additional two $\mathcal{O}(\sigma^2)$ terms in (4.3.8) arise from the second order terms in the Itô formula which were neglected in [39].

In any case, for $\mu = 1$ we can rewrite (4.3.9) in the explicit form

$$\Phi''_{\mathrm{sne}} + c_{\mathrm{sne}}\Phi'_{\mathrm{sne}} + \big(1 - \sigma^2 q(0)\big) u(1-u)\Big(u - a_{\mathrm{eff}}\Big) = 0, \tag{4.3.10}$$

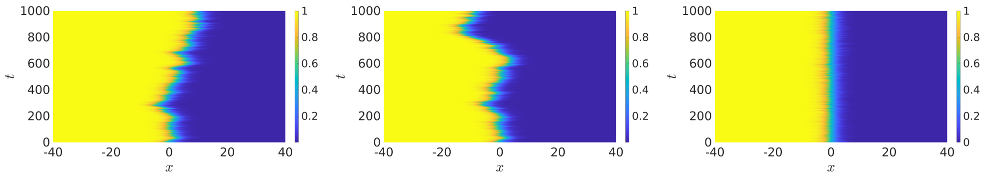
with a new effective detuning parameter

$$a_{\mathrm{eff}} = \frac{2a - \sigma^2 q(0)}{2 - 2\sigma^2 q(0)}. \tag{4.3.11}$$

This equation is just a scaled version of the original ODE, which can be solved by rescaling (4.3.6) as

$$\Phi_{\mathrm{sne}} = \frac{1}{2}\left[1 - \tanh\left(\frac{\sqrt{1-\sigma^2 q(0)}}{2\sqrt{2}} x\right)\right], \qquad c_{\mathrm{sne}} = \sqrt{2\big(1-\sigma^2 q(0)\big)}\big(\frac{1}{2} - a_{\mathrm{eff}}\big). \tag{4.3.12}$$

Our full system (4.3.8) cannot be solved explicitly, but in the bistable regime $a_{\mathrm{eff}} \in (0, 1)$ we were able to use a straightforward fixed point method to numerically



(a) $U(x + c_0 t, t)$ (b) $U(x + c_\sigma t, t)$ (c) $U(x + \Gamma(t), t)$

Figure 4.2: A single realisation of (4.3.1) in the Stratonovich interpretation with initial condition Φ_σ in 3 different reference frames, with parameters $a = 0.25$, $\sigma = 0.3$ and $\zeta = 1$. We can clearly see in (a) that the deterministic speed underestimates the stochastic speed. Replacing c_0 with c_σ in (b) captures the movement better, but the position is still fluctuating. Panel (c) shows that these fluctuations can be captured almost completely in the $\Gamma(t)$-frame.

approximate the solutions; see Figure 4.1a. These results show that c_{sne} is a reasonable approximation for c_σ, but in Figure 4.4b we shall see that c_{sne} compares less favourably with the full limiting wave speed. Note that our solutions are in agreement with the numerical observations from [79]: for the Stratonovich interpretation the wave moves faster and is less steep, but for the Itô interpretation the wave slows down and becomes steeper.

We now turn to expanding (Φ_σ, c_σ) in powers of σ. Following (4.2.48), the lowest order correction to c_σ becomes

$$\begin{aligned} c_{0;2} = &-\frac{1}{2}\langle \Phi_0'', \psi_{\text{tw}}\rangle_{L^2(\mathbb{R})}\langle q * (g(\Phi_0)\psi_{\text{tw}}), g(\Phi_0)\psi_{\text{tw}}\rangle_{L^2(\mathbb{R})} \\ &- \langle g(\Phi_0) q * (g(\Phi_0)\psi_{\text{tw}}), \psi_{\text{tw}}'\rangle_{L^2(\mathbb{R})} - \frac{\mu q(0)}{2}\langle g'(\Phi_0) g(\Phi_0), \psi_{\text{tw}}\rangle_{L^2(\mathbb{R})}. \end{aligned} \tag{4.3.13}$$

We can subsequently find $\Phi_{0;2}$ by numerically inverting the linear operator $\mathcal{L}_{\text{tw}}$ to solve

$$\begin{aligned} \mathcal{L}_{\text{tw}}\Phi_{0;2} = &-\frac{1}{2}\Phi_0''\langle q * (g(\Phi_0)\psi_{\text{tw}}), \psi_{\text{tw}}\rangle^2_{L^2(\mathbb{R})} - c_{0;2}\Phi_0' + \Big(g(\Phi_0) q * (g(\Phi_0)\psi_{\text{tw}})\Big)' \\ &- \frac{q(0)}{2} g'(\Phi_0) g(\Phi_0). \end{aligned} \tag{4.3.14}$$

We remark that these approximations can also be evaluated for additive noise ($g = 1$), or, in the Itô interpretation, for $q(x - y) = \delta(x - y)$. In Figure 4.3 we compare $(\Phi_\sigma - \Phi_0, c_\sigma - c_0)$ with our quadratic approximations for a range of different values of σ. There appears to be a good agreement, both for the Itô and Stratonovich interpretation.

Limiting wave speed In order to provide some insight on the effectiveness of our stochastic phase $\Gamma(t)$, Figure 4.2 describes the behaviour of $U(t)$ for a single realisation of (4.3.1) in three different reference frames. The first panel shows the wave in the deterministic co-moving frame, which clearly underestimates the wave speed. Replacing the speed c_0 by c_σ gives a better approximation, but the wave is still wandering. The right panel shows that these fluctuations can be largely eliminated by using $\Gamma(t)$, confirming that this an appropriate representation for the position of the wave.

At leading order, the fluctuations around $c_\sigma t$ are described by the scaled Brownian motion

$$\Gamma_0^{(1)}(t) = \int_0^t \langle \psi_{\mathrm{tw}}, g(\Phi_0) dW_s^Q \rangle_{L^2(\mathbb{R})}. \tag{4.3.15}$$

The corresponding variance is given by

$$\mathrm{Var}\big(\sigma \Gamma_0^{(1)}(t)\big) = \sigma^2 \langle q * (g(\Phi_0)\psi_{\mathrm{tw}}), g(\Phi_0)\psi_{\mathrm{tw}} \rangle_{L^2(\mathbb{R})} t, \tag{4.3.16}$$

which exactly matches [39, eq. (6.25)]. Since $E\,\Gamma_0^{(1)}(t) = 0$, the orbital drift corrections to the limiting wave speed are only visible at second order in σ. In particular, the lowest order contribution given in (4.2.61) reduces to

$$c_{0;2}^{\mathrm{od}} = -\frac{1}{2} \int_0^\infty \sum_{k=0}^\infty \langle f''_{\mathrm{cub}}(\Phi_0) \Big(S(s) \mathcal{S}_0(0)[p * e_k] \Big)^2, \psi_{\mathrm{tw}} \rangle_{L^2(\mathbb{R})}\, ds. \tag{4.3.17}$$

Here the square is taken in a pointwise fashion, with

$$\mathcal{S}_0(0)[p * e_k] = g(\Phi_0) p * e_k - \Phi_0' \langle p * e_k, g(\Phi_0)\psi_{\mathrm{tw}} \rangle_{L^2(\mathbb{R})}. \tag{4.3.18}$$

In order to evaluate this expression for $c_{0;2}^{\mathrm{od}}$, we need to choose an appropriate orthonormal basis for $L^2([-L, L]; \mathbb{R})$, where $[-L, L]$ is the domain that we use for the numerical simulations. Following [79, 103], we take

$$e_{k,c}^{(L)}(x) = \frac{1}{\sqrt{L}} \cos(\frac{\pi k x}{L}), \qquad e_{k,s}^{(L)}(x) = \frac{1}{\sqrt{L}} \sin(\frac{\pi k x}{L}) \tag{4.3.19}$$

for all integers $k \geq 0$ and introduce the quantities

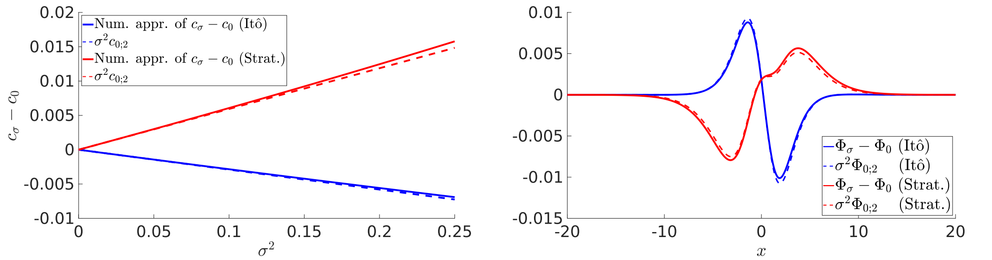
$$\lambda_{k;\mathrm{apx}} = \exp[-\pi k^2 \zeta^2 / L^2]. \tag{4.3.20}$$

A short computation shows that

$$Q e_{k,c}^{(L)} = q * e_{k,c}^{(L)} = \int_{-L}^{L} q(\cdot - y) e_{k,c}^{(L)}(y) dy \approx \int_{-\infty}^{\infty} q(\cdot - y) e_{k,c}^{(L)}(y) dy = \lambda_{k;\mathrm{apx}} e_{k,c}^{(L)} \tag{4.3.21}$$

and in the same fashion we find $Q e_{k,s}^{(L)} \approx \lambda_{k;\mathrm{apx}} e_{k,s}^{(L)}$. These observations can be used to approximate the expression (4.3.17) by writing

$$c_{0;2}^{\mathrm{od}} \approx -\frac{1}{2} \int_0^\infty \sum_{k=0}^{150} \sum_{\# \in \{c,s\}} \lambda_{k;\mathrm{apx}} \langle f''_{\mathrm{cub}}(\Phi_0) \big(S(s) \mathcal{I}_{k\#}^{(L)} \big)^2, \psi_{\mathrm{tw}} \rangle_{L^2([-L,L];\mathbb{R})}\, ds, \tag{4.3.22}$$



(a) (b)

Figure 4.3: These panels display the stochastic corrections $c_\sigma - c_0$ and $\Phi_\sigma - \Phi_0$ for the wave speed (a) and wave profile (b), together with their leading order approximations. We chose $a = 0.25$ and $\zeta = 1$, which results in $c_{0;2} = -0.0298$ (Itô) and $c_{0;2} = 0.0563$ (Stratonovich). The profiles in (b) were computed for $\sigma = 0.5$.

in which we have

$$\mathcal{I}^{(L)}_{k\#} = g(\Phi_0)e^{(L)}_{k\#} - \Phi_0'\langle e^{(L)}_{k\#}, g(\Phi_0)\psi_{\mathrm{tw}}\rangle_{L^2([-L,L];\mathbb{R})}. \tag{4.3.23}$$

We verified numerically that the resulting sum converges exponentially fast in both L and k.

In order to approximate the cubic coefficient $c^{\mathrm{od}}_{\sigma;3}$, we use the fact that $\Gamma^{(3)}_\sigma(t)$ depends only on $V^{(1)}_\sigma(t)$ and $V^{(2)}_\sigma(t)$. In particular, we made the approximation

$$\sigma^3 c^{\mathrm{od}}_{0;3} \approx c^{\mathrm{od}}_{\mathrm{cub}}(\sigma) \tag{4.3.24}$$

by numerically computing

$$c^{\mathrm{od}}_{\mathrm{cub}}(\sigma) = \frac{2}{T}\int_{\frac{T}{2}}^{T} \frac{1}{t} E\big[\Gamma_{\mathrm{apx}}(t) - c_\sigma t - \sigma\Gamma^{(1)}_\sigma(t) - \sigma^2\Gamma^{(2)}_\sigma(t)\big]\, dt, \tag{4.3.25}$$

in which

$$\Gamma_{\mathrm{apx}}(t) = c_\sigma t + \int_0^t a_\sigma\big(\sigma V^{(1)}_\sigma(s) + \sigma^2 V^{(2)}_\sigma(s)\big)\, ds + \int_0^t b_\sigma\big(\sigma V^{(1)}_\sigma(s) + \sigma^2 V^{(2)}_\sigma(s)\big)\, dW^Q_s \tag{4.3.26}$$

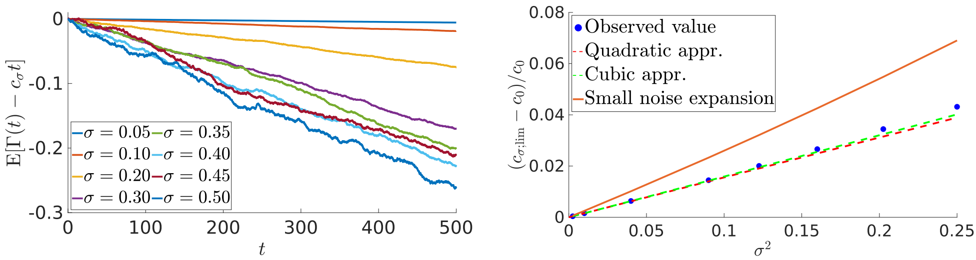
denotes the value for $\Gamma(t)$ that is obtained by integrating (4.2.20) using only the second order approximation of V.

Putting everything together, we obtain the prediction

$$c^{\mathrm{pred}}_{\sigma;\mathrm{lim}} = c_0 + \sigma^2[c_{0;2} + c^{\mathrm{od}}_{0;2}] + c^{\mathrm{od}}_{\mathrm{cub}}(\sigma) + \mathcal{O}(\sigma^4). \tag{4.3.27}$$

To get a feeling for the sizes of the perturbations in the Stratonovich interpretation, we remark that our computations for $a = 0.25$ and $\zeta = 1$ yield

$$c^{\mathrm{pred}}_{\sigma;\mathrm{lim}} = 0.3536 + \sigma^2[0.056 - 0.0043] + 0.0036\sigma^3 + \mathcal{O}(\sigma^4). \tag{4.3.28}$$



(a) (b)

Figure 4.4: In (a) we computed the average $E[\Gamma(t) - c_\sigma t]$ over 1000 simulations of (4.2.34) for the Stratonovich interpretation, using the procedure described in the main text for several values of σ. Notice that a clear trend is visible, which validates the orbital drift principle. In (b) we show the relative deviation of $c_{\sigma;\lim}$ from c_0. Here the observed limiting speed is computed by evaluating the average (4.3.29) for the data in (a), while the quadratic and cubic approximations were computed using the relevant terms in (4.3.27). The orange line is the prediction arising from the small noise expansion (4.3.12). Both plots use $a = 0.25$ and $\zeta = 1$.

Clearly, the contribution from the orbital drift is significantly smaller then the contribution from c_σ.

To test this prediction, we numerically computed a proxy for the limiting wave speed by evaluating the integral

$$c_{\sigma;\lim}^{\text{obs}} = c_\sigma + \frac{2}{T} \int_{\frac{T}{2}}^{T} \frac{1}{t} E[\Gamma(t) - c_\sigma t - \sigma \Gamma_\sigma^{(1)}(t)] \, dt, \tag{4.3.29}$$

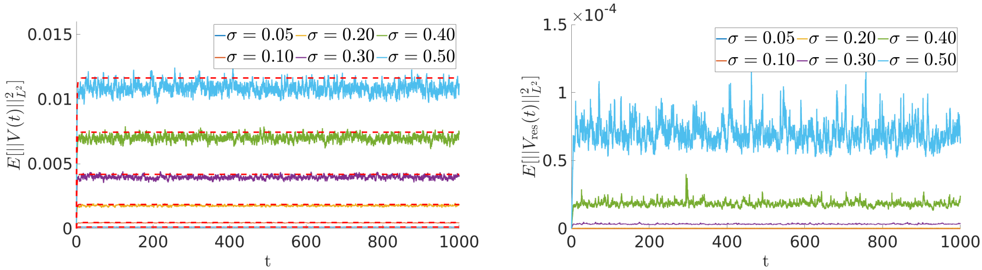
which computes the average speed over the interval $[T/2, T]$ in order to remove any transients from the data. Note that subtracting $\Gamma_\sigma^{(1)}(t)$ does not change the average but speeds up the convergence towards the average. This computation is motivated by the plots of $E[\Gamma(t) - c_\sigma t]$ contained in Figure 4.4a, which have a clear linear trend. This validates the concept of a limiting wavespeed, but also illustrates the need to include the orbital drift corrections to the instantaneous wavespeed c_σ.

In Figure 4.4b we show the relative deviation of $c_{\sigma;\lim}$ from c_0, i.e. $(c_{\sigma;\lim} - c_0)/c_0$. The blue dots represent the numerically observed values. The red dashed line shows the quadratic approximation $c_0 + \sigma^2[c_{0;2} + c_{0;2}^{\text{od}}]$ and there is indeed a good correspondence.

We also provide a cubic approximation to the wave speed by adding the term $c_{\text{cub}}^{\text{od}}(\sigma)$. This indeed improves the prediction, validating our computations. However, it also shows that the improvement is small and might not be worth the effort.

For completeness, we also included the predictions (4.3.9) arising from the small noise expansion technique. The results show that these predictions capture the overall behaviour of the limiting speed correctly, but the values deviate significantly.

Size of $V(t)$ Next, we turn our attention to the size of the perturbation $V(t)$ defined in (4.2.34). Although the leading order term $V_0^{(1)}(t)$ has zero mean, this does not hold



(a) (b)

Figure 4.5: In (a) we computed the average $E[\|V(t)\|^2_{L^2(\mathbb{R})}]$ over 1000 realisations of (4.2.34) in the Itô interpretation. The dashed line shows the numerical evaluation of the first order term (4.3.30). In (b) we computed the corresponding averages for the residual (4.3.32) by evaluating and subtracting $\sigma V_\sigma^{(1)}(t)$ and $\sigma^2 V_\sigma^{(2)}(t)$ for every realisation in (a). Note that both $V(t)$ and $V_{\rm res}(t)$ stabilize over time.

for its norm. Indeed, using (4.2.70) we find

$$\begin{aligned} E[\|V_0^{(1)}(t)\|^2_{L^2(\mathbb{R})}] &= \int_0^t \|S(s)\mathcal{S}_0(0)\|^2_{HS\left(L^2_Q, L^2(\mathbb{R})\right)} ds \\ &= \int_0^t \sum_{k=0}^{\infty} \|S(s)[g(\Phi_0)p * e_k - \Phi_0' \langle g(\Phi_0)\psi_{\rm tw}, p * e_k\rangle_{L^2(\mathbb{R})}]\|^2_{L^2(\mathbb{R})} \, ds. \end{aligned} \tag{4.3.30}$$

This expectation can be approximated using the same basis functions and eigenvalues that we used for the orbital drift. Stated more concretely, we recall (4.3.23) and write

$$E[\|V_0^{(1)}(t)\|^2_{L^2(\mathbb{R})}] \approx \int_0^t \sum_{k=0}^{150} \sum_{\# \in \{c,s\}} \lambda_{k;\rm apx} \|S(s)\mathcal{I}^{(L)}_{k\#}\|^2_{L^2([-L,L];\mathbb{R})} \, ds. \tag{4.3.31}$$

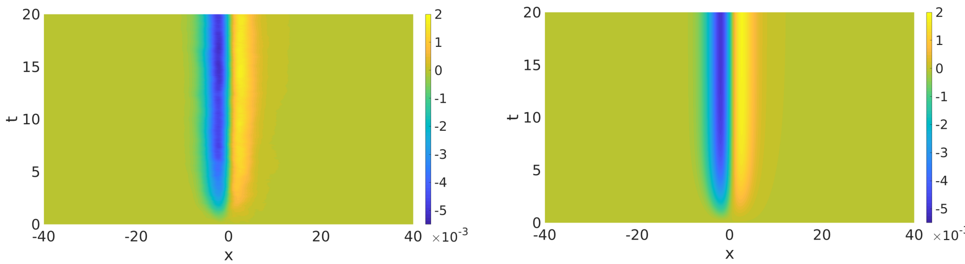
This function is represented by the red dashed line in Figure 4.5a. This agrees well with the numerical average of $E[\|V(t)\|^2_{L^2(\mathbb{R})}]$ that we computed directly from our simulations. The exponential behaviour for short time scales as well as the longer term stabilisation are nicely captured by these results. We note that we expect this limiting value to be of order $\mathcal{O}(\sigma^2)$. This is confirmed by Figure 4.7a, which shows how $E\|V(T)\|^2_{L^2(\mathbb{R})}$ scales with σ for $T = 1000$.

Similar behaviour was found during our simulations for the residual

$$V_{\rm res}(t) = V(t) - \sigma V_\sigma^{(1)}(t) - \sigma^2 V_\sigma^{(2)}(t). \tag{4.3.32}$$

Indeed, Figure 4.5b shows that this residual also stabilizes exponentially fast to a small value which we expect to be $\mathcal{O}(\sigma^6)$, as confirmed in Figure 4.7a.

We emphasize that we do not expect the running supremum of $\|V(t)\|_{L^2(\mathbb{R})}$ to stabilize in the same fashion. Indeed, we numerically computed $E[\sup_{0\le s\le t} \|V(s)\|^2_{L^2(\mathbb{R})}]$ for



(a) $E[V(t)]$ (b) $\sigma^2 E[V_0^{(2)}(t)]$

Figure 4.6: Panel (a) displays the average of $V(t)$ over 500 iterations of (4.3.1) in the Stratonovich interpretation for $a = 0.25$, $\zeta = 1$ and $\sigma = 0.5$. Panel (b) contains a numerical evaluation of (4.3.33) that includes the first 150 terms of the sum.

$0 \le t \le 1000$. The results strongly suggest that this supremum grows logarithmically in time (Figure 4.8a) and scales as σ^2 for large fixed t (Figure 4.8b). This is hence significantly better than the $\mathcal{O}(\sigma^2 t)$ bound that arises from the Burkholder-Davis-Gundy inequality and confirms our belief that our approach can be used to track waves over time scales that are exponential in σ.

Limiting wave profile Since $E[V_\sigma^{(1)}(t)] = 0$, we expect the leading order contribution to the average of $V(t)$ to be given by $\sigma^2 E[V_0^{(2)}(t)]$. Using (4.3.18) once more, we find that (4.2.60) can be written as

$$E[V_0^{(2)}(t)] = \frac{1}{2}\int_0^t S(t-s)\int_0^s \sum_{k=0}^{\infty}\Big[f''_{\rm cub}(\Phi_0)\left(S(s')\mathcal{S}_0(0)[p*e_k]\right)^2 - \Phi_0'\langle f''_{\rm cub}(\Phi_0)\left(S(s')\mathcal{S}_0(0)[p*e_k]\right)^2, \psi_{\rm tw}\rangle_{L^2(\mathbb{R})}\Big]ds'ds. \tag{4.3.33}$$

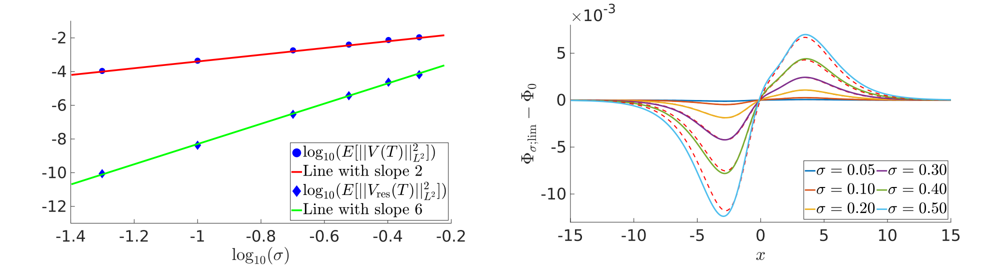
This can be evaluated using the same expressions for the eigenvalues and eigenfunctions that we used for the orbital drift. In order to compare this to our simulations, we numerically approximated $E[V(t)]$ by taking the average over 500 simulations of $V(t) - \sigma V_\sigma^{(1)}(t)$. Since $E[V_\sigma^{(1)}(t)] = 0$, this again speeds up the convergence to the mean. The results are contained in Figure 4.6, which shows that $\sigma^2 E[V_0^{(2)}(t)]$ is indeed very good approximation for $E[V(t)]$. These plots also show that the average shape indeed appears to converge to a limit, motivating us to write

$$\Phi_{\sigma;\rm lim}^{\rm obs} = \Phi_\sigma + E[V(20)]. \tag{4.3.34}$$

We recall our prediction

$$\Phi_{\sigma;\rm lim}^{\rm pred} = \Phi_0 + \sigma^2[\Phi_{0;2} + V_{0;2}^{\rm od}] + \mathcal{O}(\sigma^3) \tag{4.3.35}$$

for the limiting wave profile. We can now numerically approximate the expression



(a) (b)

Figure 4.7: The data points in (a) are computed from Figure 4.5 by evaluating the expectations at $T = 1000$ and plotting them as function of σ. We observe that $E[\|V(T)\|^2_{L^2(\mathbb{R})}]$ and $E[\|V_{\rm res}(T)\|^2_{L^2(\mathbb{R})}]$ scale as $\mathcal{O}(\sigma^2)$ and $\mathcal{O}(\sigma^6)$ respectively, as predicted. Panel (b) compares the observed (solid) and predicted (dashed) limiting deviations from Φ_0 for multiple values of σ in the Stratonovich interpretation, see (4.3.34) and (4.3.35).

(4.2.64) for $V^{\rm od}_{0;2}$ by computing

$$V^{\rm od}_{0;2} \approx -\frac{1}{2}\mathcal{L}^{-1}_{\rm tw} \int_0^T \sum_{k=0}^{150} \sum_{\# \in \{c,s\}} \lambda_{k;\rm apx}\Big[f''_{\rm cub}(\Phi_0)\big(S(s)\mathcal{I}^{(L)}_{k\#}\big)^2 - \Phi_0' \langle f''_{\rm cub}(\Phi_0)\big(S(s)\mathcal{I}^{(L)}_{k\#}\big)^2, \psi_{\rm tw}\rangle_{L^2([-L,L];\mathbb{R})}\Big] ds. \tag{4.3.36}$$

To test our prediction, we compare $\Phi^{\rm obs}_{\sigma;\rm lim} - \Phi_0$ against $\sigma^2[\Phi_{0;2} + V^{\rm od}_{0;2}]$ for multiple values of σ. The results are plotted in Figure 4.7b, which again confirms that there is a good match.

4.4 Example II: The FitzHugh-Nagumo system

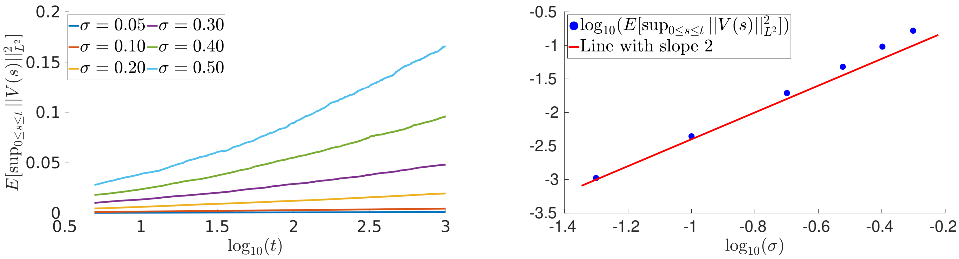
In this section we repeat the experiments from §4.3 for the two-component FitzHugh-Nagumo system

$$\begin{aligned} dU &= \big[U_{xx} + f_{\rm cub}(U) - W + \mu\sigma^2 h^{(u)}(U,W)\big]dt + \sigma g^{(u)}(U,W)dW^{Q_1}_t, \\ dW &= \big[\varrho V_{xx} + \varepsilon(U - \gamma W) + \mu\sigma^2 h^{(w)}(U,W)\big]dt + \sigma g^{(w)}(U,W)dW^{Q_2}_t, \end{aligned} \tag{4.4.1}$$

where $f_{\rm cub}$ is the same cubic polynomial as in §4.3 and $\varrho, \varepsilon, \gamma, \sigma > 0$. We assume that the two processes $W^{Q_1}_t$ and $W^{Q_2}_t$ are independent, allowing us to write

$$g(U,W) = \begin{pmatrix} g^{(u)}(U,W) & 0 \\ 0 & g^{(w)}(U,W) \end{pmatrix}, \quad Qv = \begin{pmatrix} Q_1 v_1 & 0 \\ 0 & Q_2 v_2 \end{pmatrix} = \begin{pmatrix} q_1 * v_1 & 0 \\ 0 & q_2 * v_2 \end{pmatrix} \tag{4.4.2}$$

for two convolution kernels q_1 and q_2. In particular, we have $n = m = 2$ and we assume that the combination $\mathrm{diag}(q_1, q_2)$ satisfies (Hq).



(a) (b)

Figure 4.8: Panel (a) shows the numerical evaluation of $E[\sup_{0\le s\le t}\|V(s)\|^2_{L^2(\mathbb{R})}]$ for different values of σ, where the average is computed over 500 iterations. The trend lines indicate that this supremum admits logarithmic growth. Panel (b) plots the supremum at $t = 1000$ against σ, illustrating the $\mathcal{O}(\sigma^2)$ behaviour. We used the Itô interpretation with $a = 0.25$ and $\zeta = 1$.

Upon combining the general computations in [33, p. 123] with the abstract infinite-dimensional framework developed in [108, §4.1], one can show that the Itô-Stratonovich correction term is given by

$$\begin{pmatrix} h^{(u)}(U,W) \\ h^{(w)}(U,W) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} q_1(0)D_1g^{(u)}(U,W)g^{(u)}(U,W) \\ q_2(0)D_2g^{(w)}(U,W)g^{(w)}(U,W) \end{pmatrix}. \tag{4.4.3}$$

As usual, we can switch between the Itô ($\mu = 0$) and Stratonovich ($\mu = 1$) interpretations for the noise term.

All the expressions that we derive in this section are valid for the general situation described in (4.4.2). However, in order to generate our plots we used the specific choices

$$g^{(u)}(U,W) = U, \qquad g^{(w)}(U,W) = 0, \qquad q_1(x) = q_2(x) = \frac{1}{2}e^{\frac{-\pi x^2}{4}}, \tag{4.4.4}$$

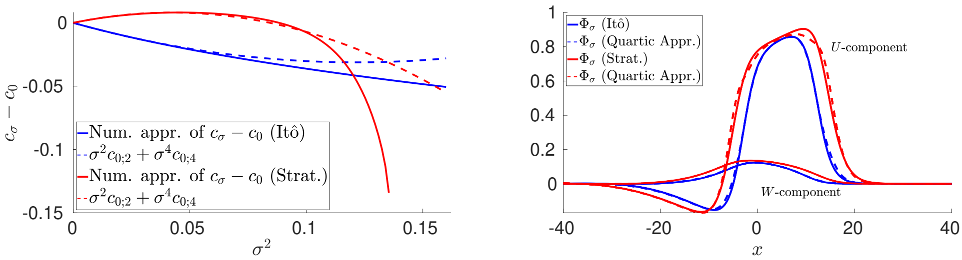
together with the parameter values $a = 0.1$, $\varrho = 0.01$, $\varepsilon = 0.01$ and $\gamma = 5$. Although we were unable to find prior work to which our results can be compared, we do point out that computations for the somewhat related Barkley model are discussed in [38].

Computing (Φ_σ, c_σ)**.** Assume for the moment that the deterministic travelling wave ODE

$$\begin{aligned} \partial_{\xi\xi}\Phi_0^{(u)} + c_0\partial_\xi\Phi_0^{(u)} + f_{\mathrm{cub}}(\Phi_0^{(u)}) - \Phi_0^{(w)} &= 0, \\ \varrho\partial_{\xi\xi}\Phi_0^{(w)} + c_0\partial_\xi\Phi_0^{(w)} + \varepsilon(\Phi_0^{(u)} - \gamma\Phi_0^{(w)}) &= 0 \end{aligned} \tag{4.4.5}$$

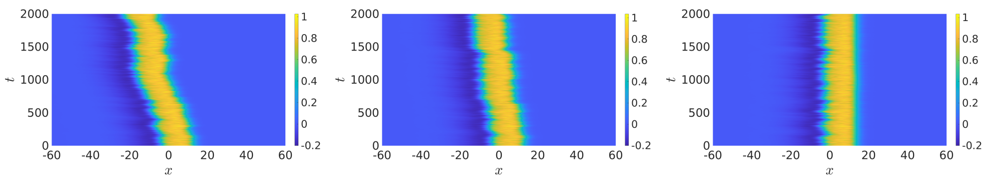
has a spectrally stable wave solution $\Phi_0 = (\Phi_0^{(u)}, \Phi_0^{(w)})$. We then recall the associated linear operator $\mathcal{L}_{\mathrm{tw}} : H^2(\mathbb{R},\mathbb{R}^2) \to L^2(\mathbb{R},\mathbb{R}^2)$ that acts as

$$\mathcal{L}_{\mathrm{tw}} = \begin{pmatrix} \partial_{\xi\xi} + c_0\partial_\xi + f'_{\mathrm{cub}}(\Phi_0^{(u)}) & -1 \\ \varepsilon & \varrho\partial_{\xi\xi} + c_0\partial_\xi - \varepsilon\gamma \end{pmatrix}, \tag{4.4.6}$$



(a) (b)

Figure 4.9: These panels display the stochastic corrections $c_\sigma - c_0$ for the wave speed (a) and the stochastic wave profiles Φ_σ for $\sigma = 0.3$ (b), together with their quartic approximations.



(a) $U(x + c_0 t, t)$ (b) $U(x + c_\sigma t, t)$ (c) $U(x + \Gamma(t), t)$

Figure 4.10: A single realisation of the U-component of (4.4.1) with $\sigma = 0.1$ in 3 different reference frames. The initial condition is given by $U(0) = \Phi_\sigma$.

together with the formal adjoint operator that is given by

$$\mathcal{L}_{\mathrm{tw}}^* = \begin{pmatrix} \partial_{\xi\xi} - c_0\partial_\xi + f'_{\mathrm{cub}}(\Phi_0^{(u)}) & \varepsilon \\ -1 & \varrho\partial_{\xi\xi} - c_0\partial_\xi - \varepsilon\gamma \end{pmatrix}. \tag{4.4.7}$$

The spectral stability implies that $\mathcal{L}_{\mathrm{tw}}^*$ admits an eigenfunction $\psi_{\mathrm{tw}} = (\psi_{\mathrm{tw}}^{(u)}, \psi_{\mathrm{tw}}^{(w)})$ that can be normalized in such a way that

$$\langle \partial_\xi \Phi_0, \psi_{\mathrm{tw}} \rangle_{L^2(\mathbb{R},\mathbb{R}^2)} = 1. \tag{4.4.8}$$

To summarise, we have

$$\mathcal{L}_{\mathrm{tw}} \partial_\xi (\Phi_0^{(u)}, \Phi_0^{(w)})^T = 0, \qquad \mathcal{L}_{\mathrm{tw}}^* (\psi_{\mathrm{tw}}^{(u)}, \psi_{\mathrm{tw}}^{(w)})^T = 0. \tag{4.4.9}$$

The existence of such spectrally stable waves has been obtained in various parameter regions [1, 24, 25, 62], but no explicit expressions are available for (Φ_0, c_0). However, they can readily be computed numerically.

Upon writing $\Phi_\sigma = (\Phi_\sigma^{(u)}, \Phi_\sigma^{(w)})$, the stochastic wave equation $F_\sigma(\Phi_\sigma, c_\sigma) = 0$

becomes

$$\begin{aligned}
\partial_{\xi\xi}\Phi_\sigma^{(u)} + c_\sigma\partial_\xi\Phi_\sigma^{(u)} + f_{\mathrm{cub}}(\Phi_\sigma^{(u)}) - \Phi_\sigma^{(w)} &= -\frac{\sigma^2}{2}\tilde{b}(\Phi_\sigma)\partial_{\xi\xi}\Phi_\sigma^{(u)} - \mu\sigma^2 h^{(u)}(\Phi_\sigma) \\
&\quad + \sigma^2\frac{\partial_\xi[g^{(u)}(\Phi_\sigma)q_1 * (g^{(u)}(\Phi_\sigma)\psi_{\mathrm{tw}}^{(u)})]}{\langle\partial_\xi\Phi_\sigma, \psi_{\mathrm{tw}}\rangle_{L^2(\mathbb{R},\mathbb{R}^2)}}, \\
\varrho\partial_{\xi\xi}\Phi_\sigma^{(w)} + c_\sigma\partial_\xi\Phi_\sigma^{(w)} + \varepsilon(\Phi_\sigma^{(u)} - \gamma\Phi_\sigma^{(w)}) &= -\frac{\sigma^2}{2}\tilde{b}(\Phi_\sigma)\partial_{\xi\xi}\Phi_\sigma^{(w)} - \mu\sigma^2 h^{(w)}(\Phi_\sigma) \\
&\quad + \sigma^2\frac{\partial_\xi[g^{(w)}(\Phi_\sigma)q_2 * (g^{(w)}(\Phi_\sigma)\psi_{\mathrm{tw}}^{(w)})]}{\langle\partial_\xi\Phi_\sigma, \psi_{\mathrm{tw}}\rangle_{L^2(\mathbb{R},\mathbb{R}^2)}},
\end{aligned} \tag{4.4.10}$$

where $\tilde{b}$ is given by

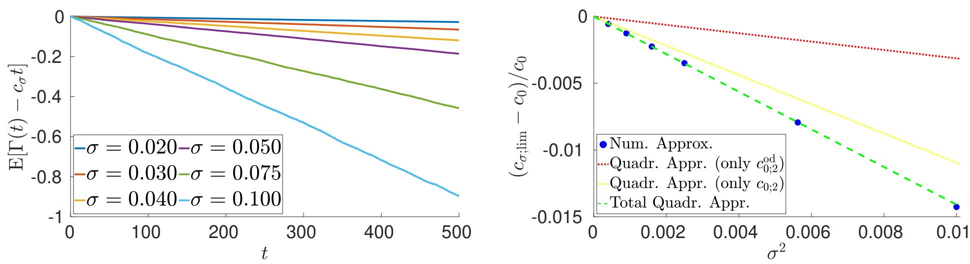
$$\tilde{b}(\Phi) = -\frac{\langle q_1 * (g^{(u)}(\Phi)\psi_{\mathrm{tw}}^{(u)}), g^{(u)}(\Phi)\psi_{\mathrm{tw}}^{(u)}\rangle_{L^2(\mathbb{R})} + \langle q_2 * (g^{(w)}(\Phi)\psi_{\mathrm{tw}}^{(w)}), g^{(w)}(\Phi)\psi_{\mathrm{tw}}^{(w)}\rangle_{L^2(\mathbb{R})}}{\langle\partial_\xi\Phi_\sigma, \psi_{\mathrm{tw}}\rangle^2_{L^2(\mathbb{R},\mathbb{R}^2)}}. \tag{4.4.11}$$

Using (4.4.10) to evaluate (4.2.48), we find that the lowest order correction to the speed c_σ reduces to

$$\begin{aligned}
c_{0;2} = &-\frac{1}{2}\tilde{b}(\Phi_0)\langle\partial_{\xi\xi}\Phi_0, \psi_{\mathrm{tw}}\rangle_{L^2(\mathbb{R},\mathbb{R}^2)} - \langle g^{(u)}(\Phi_0)q_1 * (g^{(u)}(\Phi_0)\psi_{\mathrm{tw}}^{(u)}), \partial_\xi\psi_{\mathrm{tw}}^{(u)}\rangle_{L^2(\mathbb{R})} \\
&- \langle g^{(w)}(\Phi_0)q_2 * (g^{(w)}(\Phi_0)\psi_{\mathrm{tw}}^{(w)}), \partial_\xi\psi_{\mathrm{tw}}^{(w)}\rangle_{L^2(\mathbb{R})} - \mu\langle h(\Phi_0), \psi_{\mathrm{tw}}\rangle_{L^2(\mathbb{R},\mathbb{R}^2)}.
\end{aligned} \tag{4.4.12}$$

In Figure 4.9a we numerically computed c_σ for the two interpretations. It turns out that the second order approximation above is only accurate for a range of σ that is much smaller than we saw for the Nagumo equation. By also including the quartic term $c_{0;4}$ in our expansion we are able to track Φ_σ reasonably well up to $\sigma = 0.3$. This is more than sufficient for practical purposes, as our simulations of the full system (4.4.1) revealed that the pulse is unstable for values of σ larger than approximately $\sigma = 0.15$. Figure 4.9b displays the shape of the instantaneous stochastic wave profile Φ_σ for the two different interpretations. It is striking that the wave becomes significantly wider for Stratonovich noise.

Limiting wave speed In Figure 4.10 we illustrate the behaviour of a representative sample solution to (4.4.1) by plotting it in three different moving frames. Figure 4.10a clearly shows that the deterministic speed c_0 overestimates the actual speed as the wave moves to the left. The situation is slightly improved in Figure 4.10b, where we use a frame that travels with the stochastic speed c_σ. However, the position of the wave now fluctuates around a position that still moves slowly to the left as a consequence of the orbital drift. This is remedied in Figure 4.10c, where we use the full stochastic phase $\Gamma(t)$. This again validates the idea of using $\Gamma(t)$ as position of the wave.



(a) (b)

Figure 4.11: In (a) we computed the average $E[\Gamma(t) - c_\sigma t]$ over 500 simulations of (4.2.34) with $\mu = 0$ for several values of σ, using the procedure described in the main text. Notice that a clear trend is visible. In (b) we show the relative deviation of $c_{\sigma;\mathrm{lim}}$ from c_0. Here the observed limiting speed is computed by evaluating the average (4.4.18) for the data in (a), while the various quadratic predictions are obtained from the relevant terms in (4.4.17).

As in the previous example, the variance of $\Gamma(t)$ is well-described by the variance of the leading order term $\Gamma_0^{(1)}$, which is given by

$$\begin{aligned}\mathrm{Var}\big(\Gamma_0^{(1)}(t)\big) &= \langle q_1 * (g^{(u)}(\Phi_0)\psi_{\mathrm{tw}}^{(u)}), g^{(u)}(\Phi_0)\psi_{\mathrm{tw}}^{(u)}\rangle_{L^2(\mathbb{R})} t \\ &\quad + \langle q_2 * (g^{(w)}(\Phi_0)\psi_{\mathrm{tw}}^{(w)}), g^{(w)}(\Phi_0)\psi_{\mathrm{tw}}^{(w)}\rangle_{L^2(\mathbb{R})} t.\end{aligned} \tag{4.4.13}$$

In order to explain the drift observed in Figure 4.10, we split the semigroup $S(t)$ into its two rows by writing $S(t) = \big(S^{(u)}(t), S^{(w)}(t)\big)^T$. The coefficient (4.2.61) can now be computed as

$$\begin{aligned}c_{0;2}^{\mathrm{od}} = \lim_{t\to\infty} t^{-1} E[\Gamma_0^{(2)}(t)] &= -\int_0^\infty \sum_{k=0}^\infty \langle K_0(s)[\sqrt{Q}e_k, \sqrt{Q}e_k], \psi_{\mathrm{tw}}\rangle_{L^2(\mathbb{R},\mathbb{R}^2)} ds \\ &= -\frac{1}{2}\int_0^\infty \sum_{k=0}^\infty \langle f''_{\mathrm{cub}}(\Phi_0^{(u)})\big(S^{(u)}(s)\mathcal{I}_k\big)^2, \psi_{\mathrm{tw}}^{(u)}\rangle_{L^2(\mathbb{R})}\, ds.\end{aligned} \tag{4.4.14}$$

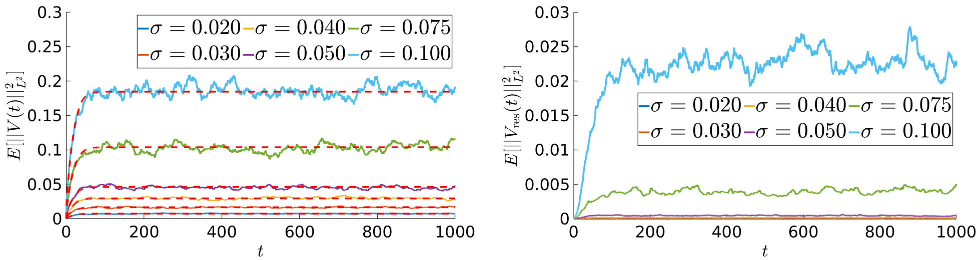
Here $\mathcal{I}_k$ is given by

$$\mathcal{I}_k = \begin{pmatrix} g^{(u)}(\Phi_0)p_1 * e_k^{(u)} \\ g^{(w)}(\Phi_0)p_2 * e_k^{(w)} \end{pmatrix} - \alpha_k \partial_\xi \begin{pmatrix} \Phi_0^{(u)} \\ \Phi_0^{(w)} \end{pmatrix}, \tag{4.4.15}$$

in which $(e_k) = (e_k^{(u)}, e_k^{(w)})$ is a basis of $L^2(\mathbb{R}, \mathbb{R}^2)$ and α_k is given by

$$\alpha_k = \frac{\langle p_1 * e_k^{(u)}, g^{(u)}(\Phi_0)\psi_{\mathrm{tw}}^{(u)}\rangle_{L^2(\mathbb{R})} + \langle p_2 * e_k^{(w)}, g^{(w)}(\Phi_0)\psi_{\mathrm{tw}}^{(w)}\rangle_{L^2(\mathbb{R})}}{\langle \partial_\xi \Phi_0, \psi_{\mathrm{tw}}\rangle_{L^2(\mathbb{R},\mathbb{R}^2)}}. \tag{4.4.16}$$

It is important to note here that the two components in the equation above mix even when $g^{(w)} = 0$ due to the presence of the semigroup.



(a) (b)

Figure 4.12: In (a) we computed the average $E[\|V(t)\|^2_{L^2(\mathbb{R},\mathbb{R}^2)}]$ over 500 realisations of (4.2.34) in the Itô interpretation. The dashed line shows the numerical evaluation of the first order term (4.4.20). In (b) we computed the corresponding averages for the residual (4.3.32) by evaluating and subtracting $\sigma V_\sigma^{(1)}(t)$ and $\sigma^2 V_\sigma^{(2)}(t)$ for every realisation in (a). Again, both $V(t)$ and $V_{\mathrm{res}}(t)$ stabilize over time.

In order to evaluate (4.4.14) numerically, we reuse the basis (4.3.19) for $L^2([-L,L];\mathbb{R})$ to construct a basis for $L^2([-L,L];\mathbb{R}) \times L^2([-L,L];\mathbb{R})$. Because Q is diagonal we can also recycle the approximate eigenvalues $\lambda_{k;\mathrm{apx}}$. For the Itô interpretation and the parameter values used in Figure 4.11a, we obtain

$$\begin{aligned} c^{\mathrm{pred}}_{\sigma;\mathrm{lim}} &= c_0 + \sigma^2[c_{0;2} + c^{\mathrm{od}}_{0;2}] + \mathcal{O}(\sigma^3) \\ &= 0.4693 - \sigma^2[0.5138 + 0.1470] + \mathcal{O}(\sigma^3). \end{aligned} \tag{4.4.17}$$

Clearly, for the FitzHugh-Nagumo equation the influence of the orbital drift is significant.

To validate our predictions, we again numerically compute

$$c^{\mathrm{obs}}_{\sigma;\mathrm{lim}} = c_\sigma + \frac{2}{T}\int_{\frac{T}{2}}^{T} \frac{1}{t} E[\Gamma(t) - c_\sigma t - \sigma\Gamma_\sigma^{(1)}(t)]\, dt \tag{4.4.18}$$

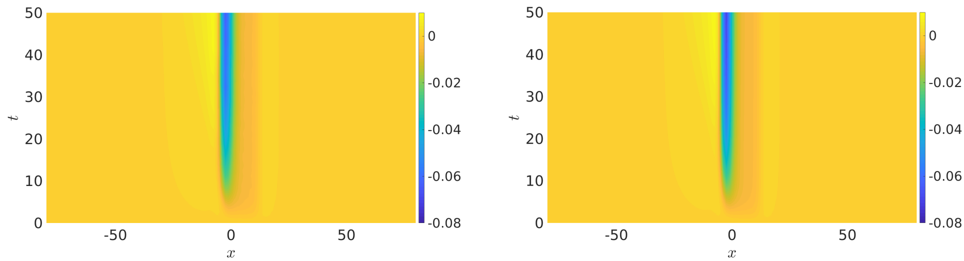
and compare the outcome with (4.4.17). Figure 4.11b shows that the total observed speed is indeed well approximated by the two leading order corrections, $\sigma^2 c_{0;2}$ and $\sigma^2 c^{\mathrm{od}}_{0;2}$.

Size of $V(t)$ We now turn our attention to the perturbation

$$V(t) = (V^{(u)}(t), V^{(w)}(t)) = \big(U(\cdot + \Gamma(t), t), W(\cdot + \Gamma(t), t)\big) - (\Phi_\sigma^{(u)}, \Phi_\sigma^{(w)}) \tag{4.4.19}$$

introduced in (4.2.34). As in §4.3, Figs. 4.12a and 4.14a show that $E\|V(t)\|^2_{L^2(\mathbb{R},\mathbb{R}^2)}$ stabilizes exponentially fast to a fixed value of size $\mathcal{O}(\sigma^2)$. These curves are nicely captured by the red dashed lines, which describe the integral

$$E[\|V_0^{(1)}(t)\|^2_{L^2(\mathbb{R},\mathbb{R}^2)}] = \int_0^t \sum_{k=0}^{\infty} \|S(s)\mathcal{I}_k\|^2_{L^2(\mathbb{R},\mathbb{R}^2)} ds \tag{4.4.20}$$



(a) $E[V^{(u)}(t)]$ (b) $\sigma^2 E[V_0^{(u,2)}(t)]$

Figure 4.13: Panel (a) shows the average of the first component of $V(t)$, computed over 500 iterations of (4.4.1) with $\sigma = 0.1$ and $\mu = 0$. Panel (b) shows the first component of the numerical evaluation of (4.4.22). As before, there is a good correspondence between the two figures.

that measure the size of the first order approximation

$$V_0^{(1)}(t) = \big(V_0^{(u,1)}(t), V_0^{(w,1)}(t)\big)^T. \tag{4.4.21}$$

Figure 4.12b shows that $E\|V_{\mathrm{res}}(t)\|^2_{L^2(\mathbb{R},\mathbb{R}^2)}$ also stabilizes over time, but Figure 4.14a indicates that the expected $\mathcal{O}(\sigma^6)$ scaling is not achieved (although the behaviour is significantly better than $\mathcal{O}(\sigma^4)$). We expect that this can be improved by utilising more advanced numerical schemes, but do not pursue this further here.

Limiting Wave Profile Turning our attention to the average shape of $V(t)$, we recall (4.4.15) and note that (4.2.60) can be computed as

$$\begin{aligned} E\,[V_0^{(2)}(t)] =& \frac{1}{2}\int_0^t S(t-s)\int_0^s \sum_{k=0}^{\infty}\Big[\begin{pmatrix} f''_{\mathrm{cub}}(\Phi_0)\big(S^{(u)}(s')\mathcal{I}_k\big)^2 \\ 0 \end{pmatrix} \\ & - \Phi_0'\langle f''_{\mathrm{cub}}(\Phi_0)\big(S^{(u)}(s')\mathcal{I}_k\big)^2, \psi_{\mathrm{tw}}^{(u)}\rangle_{L^2(\mathbb{R})}\Big]\, ds'ds. \end{aligned} \tag{4.4.22}$$

In Figure 4.13 we compare this second order expression with the numerical average of $E[V(t)]$ over 500 simulations of (4.4.1). To speed up the convergence of the average, we subtract both $\sigma V_\sigma^{(1)}(t)$ and the stochastic integral of $\sigma^2 V_\sigma^{(2)}(t)$ from $V(t)$. This does not change the outcome as both terms have zero expectation.

Notice that these two processes are almost indistinguishable from each other. To illustrate this, we provide snapshots of both processes at $t = 50$ in Figure 4.14b for various values of σ. Notice that the second order approximants follow the intricate shape of $E[V(t)]$ very closely.

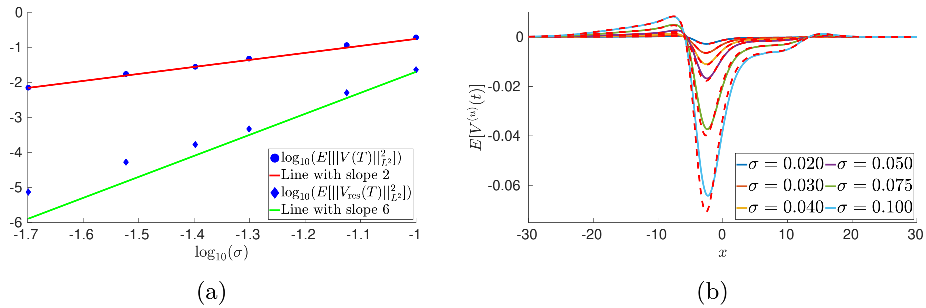


Figure 4.14: Panel (a) is computed from Figure 4.12 by evaluating the expectations at the last time step $T = 1000$ and plotting them as function of σ. We observe that $E[\|V(T)\|^2_{L^2(\mathbb{R},\mathbb{R}^2)}]$ scales as $\mathcal{O}(\sigma^2)$ as predicted and $E[\|V_{\mathrm{res}}(T)\|^2_{L^2(\mathbb{R},\mathbb{R}^2)}]$ scales significantly faster then $\mathcal{O}(\sigma^4)$, but not as the predicted $\mathcal{O}(\sigma^6)$. Panel (b) is computed from Figure 4.13, by evaluating $E[V^{(u)}(50)]$. The dashed lines correspond to the second order predictions $\sigma^2 E[V_0^{(u,2)}(50)]$.

4.5 The stochastic phase-shift

In this section we derive the SPDE (4.2.34) that we used to describe the behaviour of the phase-shifted perturbation

$$V(t) = T_{-\Gamma(t)}[X(t) + \Phi_{\mathrm{ref}}] - \Phi_\sigma \tag{4.5.1}$$

introduced in (4.2.21). Here T_γ stands for the right-shift operator[9] $T_\gamma U = U(\cdot - \gamma)$. We recall from §4.2 that the process X is a solution to the SPDE

$$dX = \big[\rho\partial_{xx}(X + \Phi_{\mathrm{ref}}) + f(X + \Phi_{\mathrm{ref}}) + \sigma^2 h(X + \Phi_{\mathrm{ref}})\big]dt + \sigma g(X + \Phi_{\mathrm{ref}})dW_t^Q \tag{4.5.2}$$

posed on the Hilbert space $L^2(\mathbb{R}, \mathbb{R}^n)$. In addition, the phase $\Gamma(t)$ was assumed to satisfy the SDE

$$d\Gamma = \big[c_\sigma + \overline{a}_\sigma(U, \Gamma)\big]\, dt + \sigma\overline{b}(U, \Gamma)dW_t^Q, \tag{4.5.3}$$

with nonlinearities $\overline{a}_\sigma$ and $\overline{b}$ that were only defined locally.

In §4.5.1 we sketch how the noise process dW_t^Q can be rigorously constructed. We subsequently introduce several cut-off functions in §4.5.2 that allow us to define $\overline{a}_\sigma$ and $\overline{b}$ in such a way that (4.5.3) remains well-posed globally. This allows us to formulate an appropriate Itô lemma in §4.5.3, which we use in §4.5.4 to perform the computations that lead to (4.2.34).

[9] These operators will always carry a subscript and should not be confused with the time T introduced in §4.2.

4.5.1 Background

In this section we briefly recall some of the functional analysis needed to set up the rigorous framework to study SPDEs. In order to ease the comparison with the earlier work in Chapter 2, it turns out to be convenient to work in an abstract setting for the moment. In particular, we consider noise that lives in an arbitrary separable Hilbert space $\mathcal{W}$ and pick a non-negative symmetric operator $Q \in \mathcal{L}(\mathcal{W}, \mathcal{W})$. We then write[10]

$$\mathcal{W}_Q = Q^{1/2}(\mathcal{W}), \tag{4.5.4}$$

which is again a separable Hilbert space with inner product

$$\langle v, w\rangle_{\mathcal{W}_Q} = \langle Q^{-1/2}v, Q^{-1/2}v\rangle_{\mathcal{W}}. \tag{4.5.5}$$

We now fix an orthonormal basis (e_k) for $\mathcal{W}$, which means that $(\sqrt{Q}e_k)$ is a basis for $\mathcal{W}_Q$. For any Hilbert space $\mathcal{H}$, we recall that a linear map $\Lambda : \mathcal{W}_Q \to \mathcal{H}$ is contained in the set of Hilbert-Schmidt operators $HS(\mathcal{W}_Q, \mathcal{H})$ if it satisfies $\langle \Lambda, \Lambda\rangle_{HS(\mathcal{W}_Q,\mathcal{H})} < \infty$. Here the inner product is given by

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$$\langle \Lambda_1, \Lambda_2\rangle_{HS(\mathcal{W}_Q,\mathcal{H})} := \sum_{k=0}^{\infty}\langle \Lambda_1\sqrt{Q}e_k, \Lambda_2\sqrt{Q}e_k\rangle_{\mathcal{H}}. \tag{4.5.6}$$

The construction in [93, §2.5] allow us to define a Hilbert space $\mathcal{W}_{\text{ext}} \supset \mathcal{W}$ so that the inclusion $\mathcal{W}_Q \subset \mathcal{W}_{\text{ext}}$ is such a Hilbert-Schmidt operator. This (non-unique) extension space is the key ingredient that allows our noise process to be rigorously constructed.

Turning to this task, we introduce a complete probability space $\Big(\Omega, \mathcal{F}, \mathbb{P}\Big)$, together with a normal filtration $(\mathcal{F}_t)_{t\geq 0}$ and a set of independent $(\mathcal{F}_t)$-Brownian motions (β_k). Following [66, eq. (2)], we introduce the formal sum

$$W_t^Q = \sum_{k=0}^{\infty} \sqrt{Q}e_k\beta_k(t), \tag{4.5.7}$$

which converges in $L^2(\Omega, \mathcal{F}, P; \mathcal{W}_{\text{ext}})$ for every $t \geq 0$. We will refer to this limiting process W_t^Q as a $(\mathcal{F}_t, Q)$-cylindrical Wiener process. The computations in [66, Prop. 2] show that the formal sums

$$\langle W_t^Q, w\rangle_{\mathcal{W}} = \sum_{k=0}^{\infty}\langle \sqrt{Q}e_k, w\rangle_{\mathcal{W}}\, \beta_k(t), \qquad w \in \mathcal{W} \tag{4.5.8}$$

define scalar Wiener processes that satisfy

$$E\Big[\langle W_t^Q, w_1\rangle_{\mathcal{W}}\langle W_s^Q, w_2\rangle_{\mathcal{W}}\Big] = (t \wedge s)\langle Qw_1, w_2\rangle_{\mathcal{W}}. \tag{4.5.9}$$

[10] In the literature, the pair $(\mathcal{W}_Q, \mathcal{W})$ is often denoted as (U_0, U), but in our setting this might be confusing with the solution $U(t)$.

For any Hilbert space $\mathcal{H}$ and any $T > 0$, we follow the convention in [93, 95] and introduce the space

$$\mathcal{N}^2\big([0,T];(\mathcal{F}_t);\mathcal{H}\big) = \{X \in L^2\big([0,T]\times\Omega; dt\otimes\mathbb{P};\mathcal{H}\big) : \\ X \text{ has a } (\mathcal{F}_t)\text{-progressively measurable version}\}, \tag{4.5.10}$$

For any process $B \in \mathcal{N}^2\big([0,T];(\mathcal{F}_t);HS(\mathcal{W}_Q,\mathcal{H})\big)$, we now use [66, eq. (7)] to define the stochastic integral

$$\int_0^t B(s)\,dW_s^Q = \lim_{m\to\infty}\sum_{k=0}^{m}\int_0^t B(s)[\sqrt{Q}e_k]\,d\beta_k(s) \tag{4.5.11}$$

for all $0 \le t \le T$. This limit can be taken directly in $L^2(\Omega,\mathcal{F},P;\mathcal{H})$ and hence avoids the use of the external space. In this setting, the Itô isometry can be stated as

$$E\langle\int_0^t B_1(s)\,dW_s^Q, \int_0^t B_2(s)\,dW_s^Q\rangle_{\mathcal{H}} = E\int_0^t \langle B_1(s),B_2(s)\rangle_{HS(\mathcal{W}_Q,\mathcal{H})}\,ds. \tag{4.5.12}$$

Returning to our main SPDE (4.2.4), we assume for the moment that $g(U)$ is a Hilbert-Schmidt operator from $\mathcal{W}_Q$ into $L^2(\mathbb{R},\mathbb{R}^n)$ for every $U \in \mathcal{U}_{H^1}$. The formal adjoint

$$g^{\mathrm{adj}}(U) : L^2(\mathbb{R},\mathbb{R}^n) \to \mathcal{W}_Q \tag{4.5.13}$$

is then defined in such a way that

$$\langle g(U)[w],\psi\rangle_{L^2(\mathbb{R},\mathbb{R}^n)} = \langle w, g^{\mathrm{adj}}(U)\psi\rangle_{\mathcal{W}_Q} = \langle Q^{-1/2}w, Q^{-1/2}g^{\mathrm{adj}}(U)[\psi]\rangle_{\mathcal{W}} \tag{4.5.14}$$

holds for any $w \in \mathcal{W}$ and $\psi \in L^2(\mathbb{R},\mathbb{R}^n)$. This point of view allows us to unify the framework of this chapter with the setup used in Chapters 2 and 3 where scalar noise is considered.

Indeed, for the setting described in §4.1-4.4 we can take $\mathcal{W} = L^2(\mathbb{R},\mathbb{R}^m)$ and $\mathcal{W}_Q = L^2_Q$. A simple computation shows that

$$g^{\mathrm{adj}}(U)[\psi] = Qg(U)^T\psi, \tag{4.5.15}$$

in which the matrix transpose is taken in a pointwise fashion. However, for $\mathcal{W} = \mathbb{R}^m$ we must take

$$g^{\mathrm{adj}}(U)[\psi] = Q\int_{\mathbb{R}} g\big(U(x)\big)^T\psi(x)\,dx, \tag{4.5.16}$$

which for $m = 1$ reduces further to

$$g^{\mathrm{adj}}(U)[\psi] = Q\langle g(U),\psi\rangle_{L^2(\mathbb{R})}. \tag{4.5.17}$$

We shall see that (4.5.17) can be used to recover the results in Chapters 2 and 3 from the expressions that we derive in this section.

4.5.2 Construction of $\overline{a}_\sigma$, $\overline{b}$, Φ_σ and c_σ

In order to ensure that the SDE for the phase $\Gamma(t)$ is well-defined and admits global solutions, we need to define the functions $\overline{a}_\sigma$ and $\overline{b}$ appearing in (4.2.19) in such a way that $\overline{b}$ is globally bounded, while the singularities in (4.2.26) and (4.2.31) are avoided.

To achieve this, we pick a C^∞-smooth non-decreasing cut-off function

$$\chi_{\text{low}} : \mathbb{R} \to [\frac{1}{4}, \infty), \tag{4.5.18}$$

that satisfies the identities

$$\chi_{\text{low}}(\vartheta) = \frac{1}{4} \text{ for } \vartheta \leq \frac{1}{4}, \qquad \chi_{\text{low}}(\vartheta) = \vartheta \text{ for } \vartheta \geq \frac{1}{2}. \tag{4.5.19}$$

In addition, we choose a C^∞-smooth non-increasing cut-off function

$$\chi_{\text{high}} : \mathbb{R}^+ \to [0,1], \tag{4.5.20}$$

for which we have

$$\chi_{\text{high}}(\vartheta) = 1 \text{ for } \vartheta \leq K_{\text{up}}, \qquad \chi_{\text{high}}(\vartheta) = 0 \text{ for } \vartheta \geq K_{\text{up}} + 1 \tag{4.5.21}$$

for some sufficiently large $K_{\text{up}} >> 1$.

For convenience, we now introduce the notation

$$\overline{\chi}_l(U,\Gamma) = \big[\chi_{\text{low}}\big(\langle \partial_\xi U, T_\Gamma \psi_{\text{tw}}\rangle_{L^2(\mathbb{R},\mathbb{R}^n)}\big)\big]^{-1}, \quad \overline{\chi}_h(U,\Gamma) = \chi_{\text{high}}(\|U - T_\Gamma \Phi_{\text{ref}}\|_{L^2(\mathbb{R},\mathbb{R}^n)}). \tag{4.5.22}$$

We remark that $\overline{\chi}_l$ and $\overline{\chi}_h$ are both uniformly bounded. Whenever $\|U - T_\Gamma \Phi_0\|_{L^2(\mathbb{R},\mathbb{R}^n)}$ is sufficiently small, we have

$$\overline{\chi}_l(U,\Gamma) = \big[\langle \partial_\xi U, T_\Gamma \psi_{\text{tw}}\rangle_{L^2(\mathbb{R},\mathbb{R}^n)}\big]^{-1}, \qquad \overline{\chi}_h(U,\Gamma) = 1. \tag{4.5.23}$$

We now define

$$\overline{b}(U,\Gamma)[v] = -\overline{\chi}_h(U,\Gamma)^2 \overline{\chi}_l(U,\Gamma)\big\langle g(U)v, T_\Gamma \psi_{\text{tw}}\big\rangle_{L^2(\mathbb{R},\mathbb{R}^n)}, \tag{4.5.24}$$

noting that the square on the high cut-off is simply for administrative reasons that will become clear in the sequel. A short computation shows that

$$\begin{aligned}
\|\overline{b}(U,\Gamma)\|^2_{HS(L^2_Q,\mathbb{R})} &= \overline{\chi}_h(U,\Gamma)^4 \overline{\chi}_l(U,\Gamma)^2 \sum_{k=0}^{\infty} \langle g(U)\sqrt{Q}e_k, T_\Gamma \psi_{\text{tw}}\rangle^2_{L^2(\mathbb{R},\mathbb{R}^n)} \\
&= \overline{\chi}_h(U,\Gamma)^4 \overline{\chi}_l(U,\Gamma)^2 \sum_{k=0}^{\infty} \langle \sqrt{Q}e_k, g^{\text{adj}}(U) T_\Gamma \psi_{\text{tw}}\rangle^2_{L^2_Q} \\
&= \overline{\chi}_h(U,\Gamma)^4 \overline{\chi}_l(U,\Gamma)^2 \langle g^{\text{adj}}(U) T_\Gamma \psi_{\text{tw}}, g^{\text{adj}}(U) T_\Gamma \psi_{\text{tw}}\rangle_{L^2_Q} \\
&= \overline{\chi}_h(U,\Gamma)^4 \overline{\chi}_l(U,\Gamma)^2 \langle g(U) g^{\text{adj}}(U) T_\Gamma \psi_{\text{tw}}, T_\Gamma \psi_{\text{tw}}\rangle_{L^2(\mathbb{R},\mathbb{R}^n)}.
\end{aligned} \tag{4.5.25}$$

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At this point, it is convenient to introduce the notation

$$\begin{aligned} \mathcal{K}_{\sigma;A}(U) &= \rho U'' + f(U) + \sigma^2 h(U), \\ \mathcal{K}_B(U,\Gamma) &= \frac{1}{2}\|\bar{b}(U,\Gamma)\|^2_{HS(L^2_Q,\mathbb{R})} U'', \\ \mathcal{K}_C(U,\Gamma) &= -\overline{\chi}_h^2(U,\Gamma)\overline{\chi}_l(U,\Gamma) g(U) g^{\mathrm{adj}}(U) T_\Gamma \psi_{\mathrm{tw}}, \end{aligned} \tag{4.5.26}$$

together with

$$\mathcal{K}_\sigma(U,\Gamma,c) = cU' + \mathcal{K}_{\sigma;A}(U) + \sigma^2 \mathcal{K}_B(U,\Gamma) + \sigma^2 \Big[\mathcal{K}_C(U,\Gamma)\Big]'. \tag{4.5.27}$$

In order to relate this back to §4.2, we write

$$F_\sigma(U,c) = \mathcal{K}_\sigma(U,0,c) \tag{4.5.28}$$

and note that this expression reduces to (4.2.27) whenever $\|U - \Phi_0\|_{L^2(\mathbb{R},\mathbb{R}^n)}$ is sufficiently small on account of (4.5.25) and (4.5.15). We are now in a position to construct the instantaneous stochastic waves (Φ_σ, c_σ) by looking for zeroes of F_σ.

Proposition 4.5.1. *Suppose that (Hq), (HEq), (HDt), (HSt) and (HTw) are all satisfied and pick a sufficiently large constant $K > 0$. Then there exists $\delta_\sigma > 0$ so that for every $0 \le \sigma \le \delta_\sigma$, there is a unique pair*

$$(\Phi_\sigma, c_\sigma) \in \mathcal{U}_{H^2} \times \mathbb{R} \tag{4.5.29}$$

that satisfies the system

$$\mathcal{K}_\sigma(\Phi_\sigma, 0, c_\sigma) = 0 \tag{4.5.30}$$

and admits the bound

$$\|\Phi_\sigma - \Phi_0\|_{H^2(\mathbb{R},\mathbb{R}^n)} + |c_\sigma - c_0| \le K\sigma^2. \tag{4.5.31}$$

Proof. On account of the estimates in Appendix 4.A, the bounds in 2.7 can be transferred to the current context. The result can hence be established by following the proof of 2.2.2 . □

Having defined $\bar{b}, \Phi_\sigma$ and c_σ, $\bar{a}_\sigma$ can now be written as

$$\bar{a}_\sigma(U,\Gamma) = -\overline{\chi}_l(U,\Gamma)\big\langle \mathcal{K}_\sigma(U,\Gamma,c_\sigma), T_\Gamma \psi_{\mathrm{tw}}\big\rangle_{L^2(\mathbb{R},\mathbb{R}^n)}. \tag{4.5.32}$$

The commutation relations

$$T_\Gamma f(U) = f(T_\Gamma U), \quad T_\Gamma g(U)[w] = g(T_\Gamma U)[T_\Gamma w], \quad T_\Gamma g^{\mathrm{adj}}(U)[\psi] = g^{\mathrm{adj}}(T_\Gamma U)[T_\Gamma \psi], \tag{4.5.33}$$

the latter of which exploits the translation invariance of Q, allow us to conclude the crucial identities

$$\bar{a}_\sigma(U,\Gamma) = \bar{a}_\sigma(T_{-\Gamma}U, 0), \qquad \bar{b}(U,\Gamma)[w] = \bar{b}(T_{-\Gamma}U,0)[T_{-\Gamma}w]. \tag{4.5.34}$$

This motivates the definitions

$$a_\sigma(V) = \bar{a}_\sigma(\Phi_\sigma + V, 0), \qquad b_\sigma(V) = \bar{b}(\Phi_\sigma + V, 0) \tag{4.5.35}$$

that were introduced in §4.2.1. In order to see that these expressions reduce to (4.2.26) and (4.2.31) when $\|V\|_{L^2(\mathbb{R},\mathbb{R}^n)}$ is small, we note that $F_\sigma(\Phi_\sigma, c_\sigma) = 0$ and $\langle \mathcal{L}_{\mathrm{tw}} V, \psi_{\mathrm{tw}}\rangle_{L^2(\mathbb{R},\mathbb{R}^n)} = 0$.

4.5.3 Itô lemma

Our goal here is to apply an appropriate version of the Itô lemma to the combined stochastic process $\mathbf{Z}(t) = \big(X(t), \Gamma(t)\big)$, which takes values in the Hilbert spaces

$$\mathcal{H}^1_{\mathbf{Z}} = H^1(\mathbb{R}, \mathbb{R}^n) \times \mathbb{R}, \qquad \mathcal{H}_{\mathbf{Z}} = L^2(\mathbb{R}, \mathbb{R}^n) \times \mathbb{R}, \qquad \mathcal{H}^{-1}_{\mathbf{Z}} = H^{-1}(\mathbb{R}, \mathbb{R}^n) \times \mathbb{R}. \tag{4.5.36}$$

Indeed, upon defining nonlinearities

$$\mathbf{A}_\sigma : \mathcal{H}^1_{\mathbf{Z}} \to \mathcal{H}^{-1}_{\mathbf{Z}}, \qquad \mathbf{B} : \mathcal{H}^1_{\mathbf{Z}} \to HS\big(L^2_Q, \mathcal{H}_{\mathbf{Z}}\big) \tag{4.5.37}$$

that act as

$$\mathbf{A}_\sigma(X, \Gamma) = \Big(\mathcal{K}_{\sigma;A}(X + \Phi_{\mathrm{ref}}), c_\sigma + \bar{a}_\sigma(X + \Phi_{\mathrm{ref}}, \Gamma)\Big), \tag{4.5.38}$$

together with

$$\mathbf{B}(X, \Gamma) = \Big(g(X + \Phi_{\mathrm{ref}}), \bar{b}(X + \Phi_{\mathrm{ref}}, \Gamma)\Big), \tag{4.5.39}$$

the coupled system for $\mathbf{Z}$ can formally be written as

$$d\mathbf{Z} = \mathbf{A}_\sigma(\mathbf{Z})\, dt + \sigma \mathbf{B}(\mathbf{Z})\, dW^Q_t. \tag{4.5.40}$$

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Our first result here clarifies how solutions to this system should be interpreted. We emphasize that our phase Γ is almost surely continuous, unlike its counterpart in [57] which admits jumps. This is a direct consequence of the fact that Γ is defined to be the solution of an SDE rather than the minimizer of a distance functional. The cut-off functions introduced in §4.5.2 ensure that the phase Γ remains well-defined even if the orthogonality condition $\langle V, \psi_{\mathrm{tw}}\rangle_{L^2(\mathbb{R},\mathbb{R}^n)} = 0$ can no longer be maintained.

Proposition 4.5.2. *Suppose that (Hq), (HEq), (HDt), (HSt) and (HTw) are all satisfied and fix* $T > 0$*,* $0 \le \sigma \le \delta_\sigma$ *and* $c_\sigma \in \mathbb{R}$*. In addition, pick an initial condition* $\mathbf{Z}_0 \in \mathcal{H}_{\mathbf{Z}}$*. Then there is a unique map* $\mathbf{Z} : [0, T] \times \Omega \to \mathcal{H}_{\mathbf{Z}}$ *that is of class*[11] $\mathcal{N}^2\big([0, T]; (\mathcal{F}_t); \mathcal{H}^1_{\mathbf{Z}}\big)$ *and satisfies the following properties.*

(i) For almost all $\omega \in \Omega$*, the map* $t \mapsto \mathbf{Z}(t, \omega)$ *is of class* $C([0, T]; \mathcal{H}_{\mathbf{Z}})$*.*

(ii) For all $t \in [0, T]$*, the map* $\omega \mapsto \mathbf{Z}(t, \omega) \in \mathcal{H}_{\mathbf{Z}}$ *is* $(\mathcal{F}_t)$*-measurable.*

(iii) We have the inclusion $\mathbf{B}(\mathbf{Z}) \in \mathcal{N}^2\big([0, T]; (\mathcal{F}_t); HS(L^2_Q, \mathcal{H}_{\mathbf{Z}})\big)$*.*

(iv) For almost all $\omega \in \Omega$*, the identity*

$$\mathbf{Z}(t) = \mathbf{Z}_0 + \int_0^t \mathbf{A}_\sigma\big(\mathbf{Z}(s)\big)\, ds + \sigma \int_0^t \mathbf{B}\big(\mathbf{Z}(s)\big)\, dW^Q_s \tag{4.5.41}$$

holds for all $0 \le t \le T$*.*

Proof. In light of the estimates obtained in Appendix 4.A, we can closely follow the proof of 2.2.1. Indeed, the existence of the $dt \otimes \mathbb{P}$ version of X that is $(\mathcal{F}_t)$-progressively measurable as a map into $H^1(\mathbb{R}, \mathbb{R}^n)$ follows from [93, Ex. 4.2.3]. The main result from

[11] Recall definition (4.5.10) for $\mathcal{N}^2$.

[77] with $\alpha = 2$ and $\beta = 4$ can be used to verify the remaining statements concerning X.

As in Chapter 2, the techniques developed in [93, Ch. 3] can be used to treat the second component of (4.5.40) as an SDE for Γ with random coefficients. The key ingredient is [93, Thm. 3.1.1], which however is stated only for finite dimensional noise. We claim here that the conclusions also extend to the current setting where a cylindrical Q-Wiener process drives the stochastic terms. To see this, we note that the Itô formula used in line 3.1.14 of the proof and the Burkholder-Davis-Gundy inequality used on page 56 both extend naturally to our infinite-dimensional setting. Most importantly, the local martingale defined in 3.1.14 remains a local martingale. The remaining details can now easily be filled in by the interested reader. □

The main ingredient to compute the equation for V is the Itô lemma. There are many versions available in the literature, but we choose to apply the formulation in [27] to our framework. Note here that $D\phi$ and $D^2\phi$ are Fréchet derivatives.

Lemma 4.5.3. *Consider the setting of Proposition 4.5.2 and pick a functional $\phi \in C^2(\mathcal{H}_{\mathbf{Z}}^{-1}, \mathbb{R})$. Then for almost all $\omega \in \Omega$, the identity*

$$\begin{aligned}\phi\big(\mathbf{Z}(t)\big) =& \phi\big(\mathbf{Z}(0)\big) + \int_0^t D\phi\big(\mathbf{Z}(s)\big)[\mathbf{A}_\sigma\big(\mathbf{Z}(s)\big)]\, ds + \sigma \int_0^t D\phi\big(\mathbf{Z}(s)\big)[\mathbf{B}\big(\mathbf{Z}(s)\big)]\, dW_s^Q \\ &+ \frac{1}{2}\sigma^2 \sum_{k=0}^{\infty} \int_0^t D^2\phi\big(\mathbf{Z}(s)\big)[\mathbf{B}\big(\mathbf{Z}(s)\big)\sqrt{Q}e_k, \mathbf{B}\big(\mathbf{Z}(s)\big)\sqrt{Q}e_k]\, ds\end{aligned} \tag{4.5.42}$$

holds for all $t > 0$.

Proof. Item (iii) of Proposition 4.5.2 and the identity (4.5.41) allow us to interpret $\mathbf{Z}(t)$ as a (standard) Itô process on $\mathcal{H}_{\mathbf{Z}}^{-1}$ in the sense of [27, Def. 1], with $S_{s,t} = I$. In particular, we can apply [27, Thm. 1] to obtain the result. □

4.5.4 SPDE for V

The defining identity (4.2.33) for $\mathcal{R}_\sigma$ can be formulated as

$$\begin{aligned}\mathcal{R}_\sigma(V) &= F_\sigma(\Phi_\sigma + V, c_\sigma) + a_\sigma(V)[\Phi_\sigma' + V'] \\ &= \mathcal{K}_\sigma(\Phi_\sigma + V, 0, c_\sigma) + a_\sigma(V)[\Phi_\sigma' + V'],\end{aligned} \tag{4.5.43}$$

which is now well-defined as an element of $H^{-1}(\mathbb{R}, \mathbb{R}^n)$ for all $V \in H^1(\mathbb{R}, \mathbb{R}^n)$. Recalling the definition

$$\mathcal{S}_\sigma(V)[w] = g(\Phi_\sigma + V)[w] + \partial_\xi(\Phi_\sigma + V)b_\sigma(V)[w], \tag{4.5.44}$$

we now set out to establish the following result.

Proposition 4.5.4. *Suppose that (Hq), (HEq), (HDt), (HSt) and (HTw) all hold. Then the map*

$$V : [0, T] \times \Omega \to L^2(\mathbb{R}, \mathbb{R}^n) \tag{4.5.45}$$

defined by (4.5.1) is of class $\mathcal{N}^2\big([0, T]; (\mathcal{F}_t); H^1(\mathbb{R}, \mathbb{R}^n)\big)$ and satisfies the following properties.

(i) *For almost all* $\omega \in \Omega$*, the map* $t \mapsto V(t, \omega)$ *is of class* $C\big([0,T]; L^2(\mathbb{R}, \mathbb{R}^n)\big)$.

(ii) *For all* $t \in [0,T]$*, the map* $\omega \mapsto V(t, \omega) \in L^2(\mathbb{R}, \mathbb{R}^n)$ *is* $(\mathcal{F}_t)$*-measurable.*

(iii) *We have the inclusion*
$\mathcal{S}_\sigma(V) \in \mathcal{N}^2\big([0,T]; (\mathcal{F}_t); HS\big(L^2_Q, L^2(\mathbb{R}, \mathbb{R}^n)\big)\big)$.

(iv) *For almost all* $\omega \in \Omega$*, we have the inclusion*

$$\mathcal{R}_\sigma\big(V(\cdot, \omega)\big) \in L^1\big([0,T]; H^{-1}(\mathbb{R}, \mathbb{R}^n)\big) \tag{4.5.46}$$

and the identity

$$V(t) \quad = \quad V(0) + \int_0^t \mathcal{R}_\sigma\big(V(s)\big)\, ds + \sigma \int_0^t \mathcal{S}_\sigma\big(V(s)\big) dW^Q_s \tag{4.5.47}$$

holds for all $0 \le t \le T$.

Our main task here is to establish (4.5.47). Taking derivatives of translation operators typically requires extra regularity of the underlying function, which prevents us from applying an Itô formula directly to (4.5.1). In order to circumvent this technical issue, we pick a test function $\zeta \in C^\infty_c(\mathbb{R}, \mathbb{R}^n)$ and consider the map

$$\phi_\zeta : H^{-1}(\mathbb{R}, \mathbb{R}^n) \times \mathbb{R} \to \mathbb{R} \tag{4.5.48}$$

that acts as

$$\phi_\zeta\big(X, \Gamma\big) \quad = \quad \langle X + \Phi_{\mathrm{ref}} - T_\Gamma \Phi_\sigma, T_\Gamma \zeta\rangle_{H^{-1};H^1}. \tag{4.5.49}$$

Here $\langle \cdot, \cdot \rangle_{H^{-1};H^1}$ denotes the duality pairing between $H^{-1}(\mathbb{R}, \mathbb{R}^n)$ and $H^1(\mathbb{R}, \mathbb{R}^n)$, which coincides with the inner product on $L^2(\mathbb{R}, \mathbb{R}^n)$ when both factors are from this space; see §2.2. This map does have sufficient smoothness for our purposes here and allows us to write

$$\langle V(t), \zeta \rangle = \phi_\zeta\big(X(t), \Gamma(t)\big). \tag{4.5.50}$$

We now introduce the notation

$$\overline{\mathcal{R}}_{\sigma;\zeta}(U, \Gamma) \quad = \quad \big\langle \mathcal{K}_\sigma(U, \Gamma, c_\sigma) + \overline{a}_\sigma(U, \Gamma) U', T_\Gamma \zeta \big\rangle_{H^{-1};H^1}, \tag{4.5.51}$$

together with

$$\overline{\mathcal{S}}_{\sigma;\zeta}(U, \Gamma)[w] \quad = \quad \langle g(U)[w], T_\Gamma \zeta \rangle_{L^2(\mathbb{R},\mathbb{R}^n)} + \langle U', T_\Gamma \zeta \rangle_{L^2(\mathbb{R},\mathbb{R}^n)} \overline{b}(U, \Gamma)[w]. \tag{4.5.52}$$

As usual, we have

$$\overline{\mathcal{R}}_{\sigma;\zeta}(U, \Gamma) = \overline{\mathcal{R}}_{\sigma;\zeta}(T_{-\Gamma} U, 0), \qquad \overline{\mathcal{S}}_{\sigma;\zeta}(U, \Gamma)[w] = \overline{\mathcal{S}}_{\sigma;\zeta}(T_{-\Gamma} U, 0)[T_{-\Gamma} w]. \tag{4.5.53}$$

In addition, we note that

$$\langle \mathcal{R}_\sigma\big(V\big), \zeta \rangle_{H^{-1};H^1} = \overline{\mathcal{R}}_{\sigma;\zeta}(\Phi_\sigma + V, 0), \qquad \langle \mathcal{S}_\sigma\big(V\big)[w], \zeta \rangle_{L^2(\mathbb{R},\mathbb{R}^n)} = \overline{\mathcal{S}}_{\sigma;\zeta}(\Phi_\sigma + V, 0)[w]. \tag{4.5.54}$$

These auxiliary functions can be used to formulate the equation that arises when applying Lemma 4.5.3 to the functional ϕ_ζ.

Lemma 4.5.5. *Suppose that (Hq), (HEq), (HDt), (HSt) and (HTw) all hold. Then for almost all $\omega \in \Omega$, the identity*

$$\phi_\zeta\big(X(t),\Gamma(t)\big) = \phi_\zeta\big(X(0),\Gamma(0)\big) + \int_0^t \overline{\mathcal{R}}_{\sigma;\zeta}\big(U(s),\Gamma(s)\big)\,ds + \int_0^t \overline{\mathcal{S}}_{\sigma;\zeta}\big(U(s),\Gamma(s)\big)\,dW_s^Q \tag{4.5.55}$$

holds for all $0 \le t \le T$, in which we have used $U(s) = X(s) + \Phi_{\mathrm{ref}}$.

Proof. For convenience, we introduce the splitting

$$\phi_\zeta(X,\Gamma) = \phi_{1;\zeta}(X,\Gamma) + \phi_{2;\zeta}(\Gamma) \tag{4.5.56}$$

with

$$\begin{aligned} \phi_{1;\zeta}\big(X,\Gamma\big) &= \langle X, T_\Gamma \zeta\rangle_{H^{-1};H^1}, \\ \phi_{2;\zeta}\big(\Gamma\big) &= \langle \Phi_{\mathrm{ref}} - T_\Gamma\Phi_\sigma, T_\Gamma\zeta\rangle_{H^{-1};H^1} \\ &= \langle T_{-\Gamma}\Phi_{\mathrm{ref}} - \Phi_\sigma, \zeta\rangle_{L^2(\mathbb{R},\mathbb{R}^n)}. \end{aligned} \tag{4.5.57}$$

We note that $\phi_{1;\zeta}$ and $\phi_{2;\zeta}$ are both C^2-smooth, with derivatives given by

$$\begin{aligned} D\phi_{1;\zeta}(X,\Gamma)[\widetilde{X},\widetilde{\Gamma}] &= D_1\phi_{1;\zeta}(X,\Gamma)[\widetilde{X}] + D_2\phi_{1;\zeta}(X,\Gamma)[\widetilde{\Gamma}] \\ &= \langle \widetilde{X}, T_\Gamma\zeta\rangle_{H^{-1};H^1} - \widetilde{\Gamma}\langle X, T_\Gamma\zeta'\rangle_{H^{-1};H^1}, \\ D\phi_{2;\zeta}(\Gamma)[\widetilde{\Gamma}] &= -\widetilde{\Gamma}\langle\Phi_{\mathrm{ref}}, T_\Gamma\zeta'\rangle_{L^2(\mathbb{R},\mathbb{R}^n)}, \end{aligned} \tag{4.5.58}$$

together with

$$\begin{aligned} D^2\phi_{1;\zeta}(X,\Gamma)[\widetilde{X},\widetilde{\Gamma}][\widetilde{X},\widetilde{\Gamma}] =& D_1^2\phi_{1;\zeta}(X,\Gamma)[\widetilde{X},\widetilde{X}] + 2D_{1,2}\phi_{1;\zeta}(X,\Gamma)[\widetilde{X},\widetilde{\Gamma}] \\ &+ D_2^2\phi_{1;\zeta}(X,\Gamma)[\widetilde{\Gamma},\widetilde{\Gamma}] \\ =& -2\widetilde{\Gamma}\langle\widetilde{X}, T_\Gamma\zeta'\rangle_{H^{-1};H^1} + \beta^2\langle X, T_\Gamma\zeta''\rangle_{H^{-1};H^1}, \\ D^2\phi_{2;\zeta}(\Gamma)[\widetilde{\Gamma},\widetilde{\Gamma}] =& \widetilde{\Gamma}^2\langle\Phi_{\mathrm{ref}}, T_\Gamma\zeta''\rangle_{L^2(\mathbb{R},\mathbb{R}^n)}. \end{aligned} \tag{4.5.59}$$

We hence see that

$$\begin{aligned} D\phi_\zeta\big(\mathbf{Z}(s)\big)\big[\mathbf{A}_\sigma\big(\mathbf{Z}(s)\big)\big] &= \langle \mathcal{K}_{A;\sigma}\big(U(s)\big), T_{\Gamma(s)}\zeta\rangle_{H^{-1};H^1} \\ &\quad -\big[c_\sigma + \overline{a}_\sigma\big(U(s),\Gamma(s)\big)\big]\langle U(s), T_{\Gamma(s)}\zeta'\rangle_{L^2(\mathbb{R},\mathbb{R}^n)} \\ D\phi_\zeta\big(\mathbf{Z}(s)\big)\big[\mathbf{B}\big(\mathbf{Z}(s)\big)w\big] &= \langle g\big(U(s)\big)[w], T_{\Gamma(s)}\zeta\rangle_{H^{-1};H^1} \\ &\quad -\overline{b}\big(U(s),\Gamma(s)\big)[w]\langle U(s), T_{\Gamma(s)}\zeta'\rangle_{L^2(\mathbb{R},\mathbb{R}^n)}. \end{aligned} \tag{4.5.60}$$

Upon writing

$$\begin{aligned} \mathcal{I}_k(U,\Gamma) &= -2\overline{b}(U,\Gamma)[\sqrt{Q}e_k]\langle g(U)[\sqrt{Q}e_k], T_\Gamma\zeta'\rangle_{L^2(\mathbb{R},\mathbb{R}^n)} \\ &\quad + \Big(\overline{b}(U,\Gamma)[\sqrt{Q}e_k]\Big)^2\langle U, T_\Gamma\zeta''\rangle_{L^2(\mathbb{R},\mathbb{R}^n)}, \end{aligned} \tag{4.5.61}$$

we also observe that

$$D^2\phi_\zeta\big(\mathbf{Z}(s)\big)\big[\mathbf{B}\big(\mathbf{Z}(s)\big)\sqrt{Q}e_k, \mathbf{B}\big(\mathbf{Z}(s)\big)\sqrt{Q}e_k\big] = \mathcal{I}_k\big(U(s), \Gamma(s)\big). \tag{4.5.62}$$

A short computation yields

$$\begin{aligned} \mathcal{I}_k(U,\Gamma) &= 2\overline{\chi}_h(U,\Gamma)^2\overline{\chi}_l(U,\Gamma)\big\langle g(U)[\sqrt{Q}e_k], T_\Gamma\psi_{\mathrm{tw}}\big\rangle_{L^2(\mathbb{R},\mathbb{R}^n)} \\ &\qquad \times\langle g\big(U\big)[\sqrt{Q}e_k], T_\Gamma\zeta'\rangle_{L^2(\mathbb{R},\mathbb{R}^n)} \\ &\qquad +\Big(\bar{b}(U,\Gamma)[\sqrt{Q}e_k]\Big)^2\langle U, T_\Gamma\zeta''\rangle_{L^2(\mathbb{R},\mathbb{R}^n)} \\ &= 2\overline{\chi}_h(U,\Gamma)^2\overline{\chi}_l(U,\Gamma)\big\langle\sqrt{Q}e_k, g^{\mathrm{adj}}(U)T_\Gamma\psi_{\mathrm{tw}}\big\rangle_{L^2_Q} \\ &\qquad \times\langle\sqrt{Q}e_k, g^{\mathrm{adj}}(U)T_\Gamma\zeta'\rangle_{L^2_Q} \\ &\qquad +\Big(\bar{b}(U,\Gamma)[\sqrt{Q}e_k]\Big)^2\langle U, T_\Gamma\zeta''\rangle_{L^2(\mathbb{R},\mathbb{R}^n)}. \end{aligned} \tag{4.5.63}$$

In particular, we see that

$$\begin{aligned} \sum_{k=0}^{\infty}\mathcal{I}_k(U,\Gamma) &= 2\overline{\chi}_h(U,\Gamma)^2\overline{\chi}_l(U,\Gamma)\big\langle g^{\mathrm{adj}}(U)T_\Gamma\psi_{\mathrm{tw}}, g^{\mathrm{adj}}(U)T_\Gamma\zeta'\big\rangle_{L^2_Q} \\ &\qquad +\|\bar{b}(U,\Gamma)\|^2_{HS(L^2_Q,\mathbb{R})}\langle U, T_\Gamma\zeta''\rangle_{L^2(\mathbb{R},\mathbb{R}^n)}, \end{aligned} \tag{4.5.64}$$

which yields

$$\begin{aligned} \sum_{k=0}^{\infty}\mathcal{I}_k\big(U(s),\Gamma(s)\big) &= -2\big\langle\mathcal{K}_C\big(U(s),\Gamma(s)\big), T_{\Gamma(s)}\zeta'\big\rangle_{L^2(\mathbb{R},\mathbb{R}^n)} \\ &\qquad +\|\bar{b}\big(U(s),\Gamma(s)\big)\|^2_{HS(L^2_Q,\mathbb{R})}\langle U(s), T_{\Gamma(s)}\zeta''\rangle_{L^2(\mathbb{R},\mathbb{R}^n)}. \end{aligned} \tag{4.5.65}$$

The derivatives can now be transferred from ζ to yield the desired expression. □

Corollary 4.5.6. *Suppose that (Hq), (HEq), (HDt), (HSt) and (HTw) all hold and pick a test-function $\zeta \in C_c^\infty(\mathbb{R},\mathbb{R}^n)$. Then for almost all $\omega \in \Omega$, the map V defined by (4.5.1) satisfies the identity*

$$\begin{aligned} \langle V(t),\zeta\rangle_{L^2(\mathbb{R},\mathbb{R}^n)} &= \langle V(0),\zeta\rangle_{L^2(\mathbb{R},\mathbb{R}^n)} + \int_0^t\langle\mathcal{R}_\sigma\big(V(s)\big),\zeta\rangle_{H^{-1};H^1}\,ds \\ &\qquad +\sigma\int_0^t\langle\mathcal{S}_\sigma\big(V(s)\big)T_{-\Gamma(s)}dW^Q_s,\zeta\rangle_{L^2(\mathbb{R},\mathbb{R}^n)} \end{aligned} \tag{4.5.66}$$

for all $0 \le t \le T$.

Proof. In view of Lemma 4.5.5, the result follows from (4.5.50) together with

$$\overline{\mathcal{R}}_{\sigma;\zeta}\big(U(s);\Gamma(s)\big) = \overline{\mathcal{R}}_{\sigma;\zeta}(T_{-\Gamma(s)}U(s),0) = \overline{\mathcal{R}}_{\sigma;\zeta}\big(\Phi_\sigma + V(s),0\big) = \langle\mathcal{R}_\sigma\big(V(s)\big),\zeta\rangle_{H^{-1};H^1} \tag{4.5.67}$$

and a similar identity involving $\mathcal{S}_\sigma$. □

Proof of Proposition 4.5.4. As a preparation, we modify the definition (4.5.7) and define a new process $\widetilde{W}_t^Q$ via the formal sum

$$\widetilde{W}_t^Q = \sum_{k=0}^{\infty} \int_0^t T_{-\Gamma(s)} \sqrt{Q} e_k d\beta_k(s). \tag{4.5.68}$$

The estimates in [66, §2] all remain valid since $T_{-\Gamma(s)}$ is an isometry. In particular, we can replace the $T_{-\Gamma(s)} dW_s^Q$ term in (4.5.66) by $d\widetilde{W}_s^Q$. The proof of Proposition 2.5.1 can then be readily applied to the current setting, yielding all the desired properties after replacing dW_s^Q by $d\widetilde{W}_s^Q$ in (4.5.47).

The key issue here is that - by design - stochastic integrals with respect to dW_s^Q and $d\widetilde{W}_s^Q$ are indistinguishable from each other in the sense that they generate the same statistical properties. To see this, we pick a Hilbert space $\mathcal{H}$ together with two processes

$$B_1, B_2 \in \mathcal{N}^2\big([0,T]; (\mathcal{F}_t); HS(L_Q^2, \mathcal{H})\big) \tag{4.5.69}$$

and consider the shifted inner product

$$\begin{aligned} \mathcal{I}_{1,2} &= E\langle \int_0^t B_1(s)\, d\widetilde{W}_s^Q, \int_0^t B_2(s)\, d\widetilde{W}_s^Q \rangle_{\mathcal{H}} \\ &= E\langle \int_0^t B_1(s)\, T_{-\Gamma(s)} dW_s^Q, \int_0^t B_2(s)\, T_{-\Gamma(s)} dW_s^Q \rangle_{\mathcal{H}}. \end{aligned} \tag{4.5.70}$$

The translational invariance of $\sqrt{Q}$ allows us to write

$$T_\gamma \sqrt{Q} e_k = \sqrt{Q} T_\gamma e_k \tag{4.5.71}$$

for any $\gamma \in \mathbb{R}$. In view of the fact that $(T_\gamma e_k)$ is also an orthonormal basis for $L^2(\mathbb{R}, \mathbb{R}^m)$, we have

$$\begin{aligned} \langle B_1(s) T_{-\Gamma(s)}, B_2(s) T_{-\Gamma(s)} \rangle_{HS(L_Q^2, \mathcal{H})} &= \sum_{k=0}^{\infty} \langle B_1(s) T_{-\Gamma(s)} \sqrt{Q} e_k, B_2(s) T_{-\Gamma(s)} \sqrt{Q} e_k \rangle_{\mathcal{H}} \\ &= \sum_{k=0}^{\infty} \langle B_1(s) \sqrt{Q} T_{-\Gamma(s)} e_k, B_2(s) \sqrt{Q} T_{-\Gamma(s)} e_k \rangle_{\mathcal{H}} \\ &= \langle B_1(s), B_2(s) \rangle_{HS(L_Q^2, \mathcal{H})} \end{aligned} \tag{4.5.72}$$

for all $0 \le s \le t$. The Itô isometry (4.5.12) hence allows us to compute

$$\begin{aligned} \mathcal{I}_{1,2} &= E \int_0^t \langle B_1(s) T_{-\Gamma(s)}, B_2(s) T_{-\Gamma(s)} \rangle_{HS(L_Q^2, \mathcal{H})}\, ds \\ &= E \int_0^t \langle B_1(s), B_2(s) \rangle_{HS(L_Q^2, \mathcal{H})}\, ds \\ &= E\langle \int_0^t B_1(s)\, dW_s^Q, \int_0^t B_2(s)\, dW_s^Q \rangle_{\mathcal{H}}. \end{aligned} \tag{4.5.73}$$

Applying this with $\mathcal{H} = \mathbb{R}$, $t = t_1 \wedge t_2$ and

$$B_1(s) = \mathbf{1}_{s<t_1} \langle \cdot, w_1 \rangle_{L^2(\mathbb{R},\mathbb{R}^m)}, \qquad B_2(s) = \mathbf{1}_{s<t_2} \langle \cdot, w_2 \rangle_{L^2(\mathbb{R},\mathbb{R}^m)}, \tag{4.5.74}$$

we recover the familiar correlations

$$E\Big[\langle \widetilde{W}^Q_{t_1}, w_1 \rangle_{L^2(\mathbb{R},\mathbb{R}^m)} \langle \widetilde{W}^Q_{t_2}, w_2 \rangle_{L^2(\mathbb{R},\mathbb{R}^m)}\Big] = (t_1 \wedge t_2) \langle Q w_1, w_2 \rangle_{L^2(\mathbb{R},\mathbb{R}^m)}. \tag{4.5.75}$$

In view of [29, Defn. 2.1], this means that $\widetilde{W}^Q_t$ is also a $(\mathcal{F}_t, Q)$-cylindrical Wiener process. We therefore follow the convention in [79, §2.2.2] and drop the distinction between W^Q_t and $\widetilde{W}^Q_t$. □

4.6 Stability

4

Our goal here is to provide a rigorous formulation of the two stability results provided in §4.2.1 and give a brief outline of their proofs. Given our preparatory work in §4.5 and Appendix 4.A, we can appeal to Chapter 3 for many of the details. However, we will need to generalize a stochastic time transformation result to our setting of cylindrical Q-Wiener processes.

Given an initial condition $U_0 \in \mathcal{U}_{H^1}$ that is sufficiently close to Φ_σ, it is possible to find a corresponding (Γ_0, V_0) so that $U_0 = T_{\Gamma_0}[V_0 + \Phi_\sigma]$ with $\langle V_0, \psi_{\mathrm{tw}} \rangle_{L^2(\mathbb{R},\mathbb{R}^n)} = 0$; see Proposition 2.2.3. Recalling the function V defined by

$$V(t) \quad = \quad V(0) + \int_0^t \mathcal{R}_\sigma\big(V(s)\big)\, ds + \sigma \int_0^t \mathcal{S}_\sigma\big(V(s)\big) dW^Q_s, \tag{4.6.1}$$

we fix a sufficiently small $\varepsilon > 0$ and introduce the scalar function

$$N_{U_0}(t) = \|V(t)\|^2_{L^2(\mathbb{R},\mathbb{R}^n)} + \int_0^t e^{-\varepsilon(t-s)} \|V(s)\|^2_{H^1(\mathbb{R},\mathbb{R}^n)}\, ds. \tag{4.6.2}$$

In addition, for any $\eta > 0$ we introduce the $(\mathcal{F}_t)$-stopping time

$$t_{\mathrm{st}}(U_0, T, \eta) = \inf \Big\{ 0 \le t < T : N_{U_0}(t) > \eta \Big\}, \tag{4.6.3}$$

writing $t_{\mathrm{st}}(U_0, T, \eta) = T$ if the set is empty.

The small (but fixed) parameter $\eta > 0$ allows us to keep the nonlinearities in the problem under control. Our main technical result provides a bound for N_{U_0} in terms of the initial perturbation and the noise strength.

Proposition 4.6.1. *Assume that (Hq), (HEq), (HDt), (HSt) and (HTw) are satisfied and pick two sufficiently small constants $\delta_\eta > 0$ and $\delta_\sigma > 0$. Then there exists a constant $K > 0$ so that for any $T > 0$, any $0 < \eta \le \delta_\eta$ and any $0 \le \sigma \le \delta_\sigma T^{-1/2}$ we have the bound*

$$E\Big[\sup_{0 \le t \le t_{\mathrm{st}}(U_0,T,\eta)} N_{U_0}(t) \Big] \quad \le \quad K\Big[\|V(0)\|^2_{H^1(\mathbb{R},\mathbb{R}^n)} + \sigma^2 T \Big]. \tag{4.6.4}$$

In a standard fashion, this bound can be used to show that the probability of hitting η can be made arbitrarily small by reducing the noise strength and the size of the initial perturbation. Indeed, upon writing

$$p(U_0, T, \eta) = P\Big(\sup_{0 \le t \le T} \big[N_{U_0}(t) \big] > \eta \Big), \tag{4.6.5}$$

we can compute

$$\begin{array}{rcl} \eta p(U_0, T, \eta) & = & \eta P\big(t_{\mathrm{st}} < T\big) \\ & = & E\,\big[\mathbf{1}_{t_{\mathrm{st}} < T} N_{U_0}\big(t_{\mathrm{st}}\big)\big] \\ & \le & E\,\big[N_{U_0}\big(t_{\mathrm{st}}\big)\big] \\ & \le & E\,\big[\sup_{0 \le t \le t_{\mathrm{st}}} N_{U_0}(t) \big] \\ & \le & K\big[\|V(0)\|^2_{H^1(\mathbb{R},\mathbb{R}^n)} + \sigma^2 T\big]. \end{array} \tag{4.6.6}$$

This is the rigorous interpretation of the informal statement contained in Theorem 4.2.1.

We now set out to quantify the residual resulting from the expansion process outlined in §4.2. To this end, we take $U_0 = \Phi_\sigma$ (i.e. $V(0) = 0$) and make the decomposition $V(t) = V_{\mathrm{apx}}(t) + V_{\mathrm{res}}(t)$. Here

$$V_{\mathrm{apx}}(t) = \sigma V^1_\sigma(t) + \sigma^2 V^2_\sigma(t) \tag{4.6.7}$$

denotes the second order approximation obtained formally in §4.2.1. We subsequently introduce the scalar quantity

$$\begin{array}{rcl} N_{\mathrm{res}}(t) & = & \sigma^4 \|V_{\mathrm{apx}}(t)\|^2_{L^2(\mathbb{R},\mathbb{R}^n)} + \|V_{\mathrm{res}}(t)\|^2_{L^2(\mathbb{R},\mathbb{R}^n)} \\ & & \quad + \displaystyle\int_0^t e^{-\varepsilon(t-s)} \Big[\sigma^4 \|V_{\mathrm{apx}}(s)\|^2_{H^1(\mathbb{R},\mathbb{R}^n)} + \|V_{\mathrm{res}}(s)\|^2_{H^1(\mathbb{R},\mathbb{R}^n)}\Big]\, ds, \end{array} \tag{4.6.8}$$

together with the $(\mathcal{F}_t)$-stopping time

$$t_{\mathrm{st}}(T, \sigma, \eta; \mathrm{res}) = \inf \Big\{ 0 \le t < T : N_{\mathrm{res}}(t) > \sigma^4 \eta \Big\}, \tag{4.6.9}$$

writing $t_{\mathrm{st}}(T, \sigma, \eta; \mathrm{res}) = T$ if the set is empty. Note that the scalings imply that V_{apx} remains bounded by η as long as the stopping time is not hit, which allows the nonlinear terms to be controlled in the same fashion as in the proof of Proposition 4.6.1. Since all the quadratic terms have now been accounted for, we arrive at the following estimate.

Corollary 4.6.2. *Assume that (Hq), (HEq), (HDt), (HSt) and (HTw) are satisfied and pick two sufficiently small constants $\delta_\eta > 0$ and $\delta_\sigma > 0$. Then there exists a constant $K > 0$ so that for any $T > 0$, any $0 < \eta \le \delta_\eta$ and any $0 \le \sigma \le \delta_\sigma T^{-1/2}$, we have the bound*

$$E\,\big[\sup_{0 \le t \le t_{\mathrm{st}}(T,\sigma,\eta;\mathrm{res})} N_{\mathrm{res}}(t) \big] \quad \le \quad K\sigma^6 T. \tag{4.6.10}$$

In order to turn this into a probability estimate, we write

$$p_{\mathrm{res}}(T, \sigma, \eta) = P\Big(\sup_{0 \le t \le T} N_{\mathrm{res}}(t) > \sigma^4 \eta \Big) \tag{4.6.11}$$

and compute

$$\begin{aligned} \sigma^4 \eta p_{\mathrm{res}}(T, \sigma, \eta) &= \sigma^4 \eta P\big(t_{\mathrm{st}}(T, \sigma, \eta; \mathrm{res}) < T \big) \\ &= E\left[\mathbf{1}_{t_{\mathrm{st}}(T,\sigma,\eta;\mathrm{res})<T} N_{\mathrm{res}}\big(t_{\mathrm{st}}(T, \sigma, \eta; \mathrm{res}) \big) \right] \\ &\le K \sigma^6 T. \end{aligned} \tag{4.6.12}$$

In particular, in the setting of Corollary 4.6.2 we find

$$p_{\mathrm{res}}(T, \sigma, \eta) \le \eta^{-1} K \sigma^2 T, \tag{4.6.13}$$

which is the quantitative version of Corollary 4.2.2.

4.6.1 Stochastic time transform

We now set out to outline how the techniques developed in Chapter 3 can be used to establish Proposition 4.6.1. The key issue is that we cannot study (4.6.1) or its mild counterpart in a direction fashion because it is a quasi-linear system. The offending component is $\mathcal{K}_B$, which represents an extra nonlinear - but spatially homogeneous - diffusive term that arises as a consequence of the Itô lemma.

Our strategy is to partially eliminate these terms by appropriate time transforms. In particular, for each component $1 \le i \le n$ we define the function

$$\kappa_{\sigma;i}(V) = 1 + \frac{\sigma^2}{2\rho_i} \| b_\sigma(V) \|^2_{HS\left(L^2_Q, L^2(\mathbb{R},\mathbb{R}^n)\right)} \tag{4.6.14}$$

and observe that $\rho_i \kappa_{\sigma;i}(V)$ corresponds precisely with the coefficient in front of V_i'' that appears in $\mathcal{R}_\sigma(V)$. In order to reset this single coefficient to the value ρ_i, we introduce the (faster) transformed time

$$\tau_i(t) = \int_0^t \kappa_{\sigma;i}\big(V(s) \big) \, ds \ge t. \tag{4.6.15}$$

The map $t \mapsto \tau_i(t)$ is a continuous strictly increasing $(\mathcal{F}_t)$-adapted process that hence admits an inverse $t_i(\tau)$, i.e.,

$$\tau_i\big(t_i(\tau) \big) = \tau, \qquad t_i\big(\tau_i(t) \big) = t. \tag{4.6.16}$$

This allows us to define the time-transformed function

$$\overline{V}_{(i)}(\tau) = V\big(t_i(\tau) \big), \tag{4.6.17}$$

for which an appropriate SPDE can be derived.

Lemma 4.6.3. *Consider the setting of Proposition 4.5.4 and pick* $1 \le i \le n$. *Then there exists a filtration* $(\overline{\mathcal{F}}_\tau)_{\tau \ge 0}$ *together with a cylindrical* $(\overline{\mathcal{F}}_\tau, Q)$*-Wiener process* $\overline{W}^Q_\tau$ *so that the following properties hold.*

(i) For almost all $\omega \in \Omega$, *the map* $\tau \mapsto \overline{V}_{(i)}(\tau; \omega)$ *is of class* $C\big([0,T]; L^2\big)$.

(ii) For all $\tau \in [0,T]$, *the map* $\omega \mapsto \overline{V}_{(i)}(\tau, \omega)$ *is* $(\overline{\mathcal{F}}_\tau)$*-measurable.*

(iii) The map $\tau \mapsto \kappa_{\sigma;i}^{-1/2}\big(\overline{V}_{(i)}(\tau)\big)\mathcal{S}_\sigma\big(\overline{V}_{(i)}(\tau)\big)$ *is of class* $\mathcal{N}^2\big([0,T]; (\overline{\mathcal{F}})_\tau; HS\big(L^2_Q, L^2(\mathbb{R}, \mathbb{R}^n)\big)\big)$.

(iv) For almost all $\omega \in \Omega$, *the identity*

$$\begin{aligned} \overline{V}_{(i)}(\tau) \;&=\; \overline{V}_{(i)}(0) + \int_0^\tau \kappa_{\sigma;i}^{-1}\big(\overline{V}_{(i)}(\tau')\big)\mathcal{R}_\sigma\big(\overline{V}_{(i)}(\tau')\big)\, d\tau' \\ &\quad + \sigma \int_0^\tau \kappa_{\sigma;i}^{-1/2}\big(\overline{V}_{(i)}(\tau')\big)\mathcal{S}_\sigma\big(\overline{V}_{(i)}(\tau')\big)\, d\overline{W}^Q_{\tau'} \end{aligned} \tag{4.6.18}$$

holds for all $0 \le \tau \le T$.

Proof. Recall the set of independent $(\mathcal{F}_t)$-Brownian motions β_k used to define W^Q_t in §4.5.1. Following the proof of Lemma 2.6.2, we now construct the processes

$$\overline{\beta}_k(\tau) = \int_0^\tau \frac{1}{\sqrt{\partial_\tau t_i(\tau')}}\, d\beta_k\big(t_i(\tau')\big). \tag{4.6.19}$$

These are independent Brownian motions with respect to the filtration $(\overline{\mathcal{F}}_t)$ defined in (2.6.14). As explained in §4.5.1, the sum

$$\overline{W}^Q_\tau = \sum_k \sqrt{Q} e_k \overline{\beta}_k(\tau) \tag{4.6.20}$$

hence defines a cylindrical $(\overline{\mathcal{F}}_\tau, Q)$-Wiener process. We can now apply the transformation rule from Lemma 2.6.2 for individual Brownian motions to compute the desired transformation

$$\begin{aligned} \int_0^{t_i(\tau)} \mathcal{S}_\sigma\big(V(s)\big)\, dW^Q_s \;&=\; \lim_{m\to\infty} \sum_{k=1}^m \int_0^{t_i(\tau)} \mathcal{S}_\sigma\big(V(s)\big)[\sqrt{Q} e_k]\, d\beta_k(s) \\ &=\; \lim_{m\to\infty} \sum_{k=1}^m \int_0^\tau \kappa_{\sigma;i}^{-1/2}\big(\overline{V}_{(i)}(\tau')\big)\mathcal{S}_\sigma\big(\overline{V}_{(i)}(s)\big)[\sqrt{Q} e_k]\, d\overline{\beta}_k(\tau') \\ &=\; \int_0^\tau \kappa_{\sigma;i}^{-1/2}\big(\overline{V}_{(i)}(\tau')\big)\mathcal{S}_\sigma\big(\overline{V}_{(i)}(s)\big)\, d\overline{W}^Q_{\tau'}. \end{aligned} \tag{4.6.21}$$

The remaining statements can now be established as in the proof of Proposition 2.6.3. □

We remark that the diffusion coefficient for the i-th component of $\overline{V}_{(i)}$ is now again equal to ρ_i. This allows this component to be appropriately estimated by analysing the mild formulation of (4.6.18). The key here is that the off-diagonal elements of the semigroup $S(t)$ have better smoothening properties than the diagonal elements. Since all the relevant estimates carry over on account of §4.A, the computations in Chapter 3 can be used to establish Proposition 4.6.1. and Corollary 4.6.2.

Appendix

4.A Estimates

In this section we set out to derive certain key estimates that will build a bridge between our setting here and the extensive computations in Chapters 2 and 3. The main issues are that the functions g and b now need to be bounded in an appropriate Hilbert-Schmidt norm and that the term $\mathcal{K}_C$ has a more delicate structure than its counterpart in Chapter 2.

Throughout this section, we will often use a general pair (Φ, c) for our estimates, since a priori the wave (Φ_σ, c_σ) has not been constructed yet. This pair is assumed to satisfy the following conditions.

(hPar) The condition (HTw) holds and the pair $(\Phi, c) \in \mathcal{U}_{H^1} \times \mathbb{R}$ satisfies the bounds

$$\|\Phi - \Phi_0\|_{H^1(\mathbb{R},\mathbb{R}^n)} \le \min\{1, [4\|\psi_{\mathrm{tw}}\|_{L^2(\mathbb{R},\mathbb{R}^n)}]^{-1}\}, \qquad |c - c_0| \le 1. \tag{4.A.1}$$

We start in §4.A.1 by deriving some preliminary estimates. This will help us in §4.A.2 to formulate the 'bridge' estimates on the three functions discussed above, which concern both their size and their Lipschitz properties.

4.A.1 Preliminaries

On account of (Hq), the function $k \mapsto \sqrt{\widehat{q}(k)}$ is well-defined. It is hence tempting to construct a convolution kernel p for $\sqrt{Q}$ by taking the inverse Fourier transform of this map, since then one formally has $q * v = p * p * v$. Our first result shows that this is indeed possible.

Lemma 4.A.1. *Suppose that (Hq) is satisfied. Then the map $k \mapsto \sqrt{\widehat{q}(k)}$ is contained in $L^2(\mathbb{R}, \mathbb{R}^{m\times m})$.*

Proof. It suffices to show that $\widehat{q} \in L^1(\mathbb{R}, \mathbb{R}^{m\times m})$, which follows from the bound

$$\|\widehat{q}\|_{L^1(\mathbb{R},\mathbb{R}^{m\times m})} = \int_{\mathbb{R}} \frac{1}{(1+|k|^2)^{\frac{1}{2}}} (1+|k|^2)^{\frac{1}{2}} |\hat{q}(k)| dk \le K \|q\|_{H^1(\mathbb{R}^{m\times m})}. \tag{4.A.2}$$

□

Using this L^2-bound on p, one can now show that any $z \in L^2(\mathbb{R}^{n\times m})$ can be interpreted as a Hilbert-Schmidt operator from L^2_Q into $L^2(\mathbb{R}, \mathbb{R}^n)$. As usual, this proceeds via the pointwise multiplication $z[w](x) = z(x)w(x)$.

Lemma 4.A.2. *Suppose that (Hq) is satisfied. There exists $K > 0$ so that for any $z \in L^2(\mathbb{R}, \mathbb{R}^{n\times m})$, we have $z \in HS\big(L^2_Q, L^2(\mathbb{R}, \mathbb{R}^n)\big)$ with*

$$\|z\|_{HS\left(L^2_Q, L^2(\mathbb{R},\mathbb{R}^n)\right)} \le K \|z\|_{L^2(\mathbb{R},\mathbb{R}^{n\times m})}. \tag{4.A.3}$$

Proof. Writing out the various matrix multiplications in a component-wise fashion, we obtain

$$\begin{aligned}
&\|z\|^2_{HS\left(L^2_Q, L^2(\mathbb{R},\mathbb{R}^n)\right)} \\
&= \sum_{k=0}^{\infty} \|z[\sqrt{Q}e_k]\|^2_{L^2(\mathbb{R},\mathbb{R}^n)} \\
&= \sum_{k=0}^{\infty} \sum_{i=1}^{n} \sum_{j,j'=1}^{m} \int z_{ij}(x) \langle p_{j\cdot}(x-\cdot), e_k \rangle_{L^2(\mathbb{R},\mathbb{R}^m)} z_{ij'} \langle p_{j'\cdot}(x-\cdot), e_k \rangle_{L^2(\mathbb{R},\mathbb{R}^m)}\, dx \\
&= \sum_{i=1}^{n} \sum_{j,j'=1}^{m} \int z_{ij}(x) z_{ij'}(x) \langle p_{j\cdot}(x-\cdot), p_{j'\cdot}(x-\cdot) \rangle_{L^2(\mathbb{R},\mathbb{R}^m)}\, dx \\
&= \sum_{i=1}^{n} \sum_{j,j',l=1}^{m} \langle p_{jl}, p_{j'l} \rangle_{L^2(\mathbb{R})} \int z_{ij}(x) z_{ij'}(x)\, dx
\end{aligned} \tag{4.A.4}$$

The result now follows by appealing to Cauchy-Schwarz. □

Our final two results concern a bound on the cut-off functions (4.5.22) and a bound on the L^2-norm of g that we borrow from Chapter 2. This is especially useful when combined with the bound in Lemma 4.A.2.

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Lemma 4.A.3. *Suppose that (HEq), (Hg) and (hPar) are satisfied. Then there exists a constant $K > 0$, which does not depend on the pair (Φ, c), so that the following holds true. For any $v \in H^1(\mathbb{R}, \mathbb{R}^n)$ and $\gamma \in \mathbb{R}$ we have the bound*

$$|\overline{\chi}_l(\Phi + v, \gamma)| + |\overline{\chi}_h(\Phi + v, \gamma)| \quad \leq \quad K, \tag{4.A.5}$$

while for any pair $(v_A, v_B) \in H^1(\mathbb{R}, \mathbb{R}^n) \times H^1(\mathbb{R}, \mathbb{R}^n)$ and $(\gamma_A, \gamma_B) \in \mathbb{R}^2$ we have the estimates

$$\begin{aligned}
|\overline{\chi}_l(\Phi + v_A, \gamma_A) - \overline{\chi}_l(\Phi + v_B, \gamma_B)| &\leq K\big[\|v_A - v_B\|_{L^2(\mathbb{R},\mathbb{R}^n)} \\
&\qquad + (1 + \|v_A\|_{L^2(\mathbb{R},\mathbb{R}^n)})\, |\gamma_1 - \gamma_2|\big], \\
|\overline{\chi}_h(\Phi + v_A, \gamma_A) - \overline{\chi}_h(\Phi + v_B, \gamma_B)| &\leq K\big[\|v_A - v_B\|_{L^2(\mathbb{R},\mathbb{R}^n)} + |\gamma_A - \gamma_B|\big].
\end{aligned} \tag{4.A.6}$$

Proof. The bound (4.A.5) follows directly from the definition of the cut-off functions. The first Lipschitz bound in (4.A.6) can be found in Lemma 2.3.3, while the second bound follows from the observation

$$\begin{aligned}
\big|\|\Phi + v_A - T_{\gamma_A}\Phi_{\mathrm{ref}}\|_{L^2(\mathbb{R},\mathbb{R}^n)} - \|\Phi + v_B - T_{\gamma_B}\Phi_{\mathrm{ref}}\|_{L^2(\mathbb{R},\mathbb{R}^n)}\big| \\
\leq K\big[\|v_A - v_B\|_{L^2(\mathbb{R},\mathbb{R}^n)} + |\gamma_A - \gamma_B|\big].
\end{aligned} \tag{4.A.7}$$

□

Lemma 4.A.4. *Suppose that (HEq), (Hg) and (hPar) are satisfied. Then there exists a constant $K > 0$, which does not depend on the pair (Φ, c), so that the following holds true. For any $v \in H^1(\mathbb{R}, \mathbb{R}^n)$ we have the bounds*

$$\begin{aligned}
\|g(\Phi + v)\|_{L^2(\mathbb{R}^{n\times m})} &\leq K[1 + \|v\|_{L^2(\mathbb{R},\mathbb{R}^n)}], \\
\|\partial_\xi g(\Phi + v)\|_{L^2(\mathbb{R}^{n\times m})} &\leq K[1 + \|v\|_{H^1(\mathbb{R},\mathbb{R}^n)}],
\end{aligned} \tag{4.A.8}$$

while for any pair $(v_A, v_B) \in H^1(\mathbb{R},\mathbb{R}^n) \times H^1(\mathbb{R},\mathbb{R}^n)$ *we have the estimates*

$$\begin{array}{rcl} \|g(\Phi + v_A) - g(\Phi + v_B)\|_{L^2(\mathbb{R}^{n\times m})} & \leq & K\|v_A - v_B\|_{L^2(\mathbb{R},\mathbb{R}^n)}, \\ \|\partial_\xi[g(\Phi + v_A) - g(\Phi + v_B)]\|_{L^2(\mathbb{R}^{n\times m})} & \leq & K\big[1 + \|v_A\|_{H^1(\mathbb{R},\mathbb{R}^n)}\big]\|v_A - v_B\|_{H^1(\mathbb{R},\mathbb{R}^n)}. \end{array} \tag{4.A.9}$$

Proof. This follows using the same techniques as in Lemma 2.3.2. □

4.A.2 Estimates for g, b_σ and $\mathcal{K}_C$

By combining the estimates in Lemmas 4.A.2 and 4.A.4 above, we immediately obtain bounds on $g(U)$ viewed as a pointwise multiplication operator from L^2_Q into $L^2(\mathbb{R},\mathbb{R}^n)$. These correspond precisely with the L^2-bounds for the function $g(U)$ itself, allowing the follow-up estimates to be readily transferred from Chapter 2 to the current setting.

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Corollary 4.A.5. *Suppose that (Hq), (HEq), (HSt) and (hPar) are satisfied. Then there exists a constant* $K > 0$*, which does not depend on* (Φ, c) *so that the following holds true. For any* $v \in H^1(\mathbb{R},\mathbb{R}^n)$ *we have the bounds*

$$\|g(\Phi + v)\|_{HS\left(L^2_Q, L^2(\mathbb{R},\mathbb{R}^n)\right)} \leq K[1 + \|v\|_{L^2(\mathbb{R},\mathbb{R}^n)}], \tag{4.A.10}$$

while for any pair $(v_A, v_B) \in H^1(\mathbb{R},\mathbb{R}^n) \times H^1(\mathbb{R},\mathbb{R}^n)$ *we have the estimates*

$$\|g(\Phi + v_A) - g(\Phi + v_B)\|_{HS\left(L^2_Q, L^2(\mathbb{R},\mathbb{R}^n)\right)} \leq K\|v_A - v_B\|_{L^2(\mathbb{R},\mathbb{R}^n)}. \tag{4.A.11}$$

Turning to the nonlinearity $\mathcal{K}_C$ defined in (4.5.26), our goal here is to derive estimates for $\partial_\xi \mathcal{K}_C(U, \gamma)$ that are comparable to those obtained for the product $b(U,\gamma)\partial_\xi g(U)$ in the context of Chapter 2, where b evaluates to a scalar. To this end, we introduce the auxiliary function

$$\widetilde{\mathcal{K}}_C(U,\Gamma) = \overline{\chi}_l(U,\Gamma)\overline{\chi}_h(U,\Gamma)Qg^T(U)T_\Gamma\psi_{\mathrm{tw}}, \tag{4.A.12}$$

which in view of the identification (4.5.15) allows us to write

$$\mathcal{K}_C(U,\gamma) = -\overline{\chi}_h(U,\Gamma)g(U)\widetilde{\mathcal{K}}_C(U,\Gamma). \tag{4.A.13}$$

The strategy is to use the splitting

$$\begin{array}{rcl} \|\partial_\xi \mathcal{K}_C(U,\Gamma)\|_{L^2(\mathbb{R},\mathbb{R}^n)} & \leq & \|\overline{\chi}_h(U,\Gamma)\partial_\xi g(U)\|_{HS\left(L^2_Q, L^2(\mathbb{R},\mathbb{R}^n)\right)}\|\widetilde{\mathcal{K}}_C(U,\Gamma)\|_{L^2_Q} \\ & & +\|\overline{\chi}_h(U,\Gamma)g(U)\|_{HS\left(L^2_Q, L^2(\mathbb{R},\mathbb{R}^n)\right)}\|\partial_\xi\widetilde{\mathcal{K}}_C(U,\Gamma)\|_{L^2_Q} \end{array} \tag{4.A.14}$$

together with its natural analogue for $\partial_\xi[\mathcal{K}_C(U_A,\Gamma_A) - \mathcal{K}_C(U_B,\Gamma_B)]$. The following two results provide bounds for the factors in (4.A.14) that show that both products on the right hand side lead to similar expressions as those obtained in Chapter 2. In fact, we obtain slightly better estimates as a consequence of a more refined use of the cutoff functions.

Corollary 4.A.6. *Suppose that (Hg), (HEq) and (hPar) are satisfied. Then there exists a constant $K > 0$, which does not depend on the pair (Φ, c), so that the following holds true. For any $v \in H^1(\mathbb{R}, \mathbb{R}^n)$ and $\gamma \in \mathbb{R}$ we have the bounds*

$$\begin{array}{lcl} \|\overline{\chi}_h(\Phi + v, \gamma) g(\Phi + v)\|_{L^2(\mathbb{R}^{n\times m})} & \leq & K, \\ \|\overline{\chi}_h(\Phi + v, \gamma)\partial_\xi g(\Phi + v)\|_{L^2(\mathbb{R}^{n\times m})} & \leq & K\big[1 + \|v\|_{H^1(\mathbb{R},\mathbb{R}^n)}\big]. \end{array} \tag{4.A.15}$$

In addition, for any pair $(v_A, v_B) \in H^1(\mathbb{R}, \mathbb{R}^n) \times H^1(\mathbb{R}, \mathbb{R}^n)$ and $(\gamma_A, \gamma_B) \in \mathbb{R}^2$, the expression

$$\Delta_{AB}\overline{\chi}_h g = \overline{\chi}_h(\Phi + v_A, \gamma_A) g(\Phi + v_A) - \overline{\chi}_h(\Phi + v_B, \gamma_B) g(\Phi + v_B) \tag{4.A.16}$$

satisfies the estimates

$$\begin{array}{lcl} \|\Delta_{AB}\overline{\chi}_h g\|_{L^2(\mathbb{R}^{n\times m})} & \leq & K\big[\|v_A - v_B\|_{L^2(\mathbb{R},\mathbb{R}^n)} + |\gamma_1 - \gamma_2|\big], \\ \|\partial_\xi \Delta_{AB}\|_{L^2(\mathbb{R}^{n\times m})} & \leq & K\big[\|v_A - v_B\|_{H^1(\mathbb{R},\mathbb{R}^n)} + |\gamma_A - \gamma_B|\big]\big[1 + \|v_A\|_{H^1(\mathbb{R},\mathbb{R}^n)}\big]. \end{array} \tag{4.A.17}$$

Proof. The estimates (4.A.15) follow directly from Lemma 4.A.4, using the fact that the cut-off allows us to assume an a priori bound for $\|v\|_{L^2(\mathbb{R},\mathbb{R}^n)}$. The Lipschitz bounds (4.A.17) can be obtained by writing

$$\begin{array}{rcl} \Delta_{AB}\overline{\chi}_h g & = & \big[\overline{\chi}_h(\Phi + v_A, \gamma_A) - \overline{\chi}_h(\Phi + v_B, \gamma_B)\big] g(\Phi + v_A) \\ & & + \overline{\chi}_h(\Phi + v_B, \gamma_B)\big[g(\Phi + v_A) - g(\Phi + v_B)\big] \end{array} \tag{4.A.18}$$

and applying the results from Lemmas 4.A.3 and 4.A.4. □

Lemma 4.A.7. *Suppose that (Hq), (HEq), (HSt) and (hPar) are satisfied. Then there exists a constant $K > 0$, which does not depend on the pair (Φ, c), so that the following holds true. For any $v \in H^1(\mathbb{R}, \mathbb{R}^n)$ and $\gamma \in \mathbb{R}$ we have the bounds*

$$\begin{array}{lcl} \|\widetilde{\mathcal{K}}_C(\Phi + v, \gamma)\|_{L^2_Q} & \leq & K, \\ \|\partial_\xi \widetilde{\mathcal{K}}_C(\Phi + v, \gamma)\|_{L^2_Q} & \leq & K[1 + \|v\|_{H^1(\mathbb{R},\mathbb{R}^n)}]. \end{array} \tag{4.A.19}$$

In addition, for any pair $(v_A, v_B) \in H^1(\mathbb{R}, \mathbb{R}^n) \times H^1(\mathbb{R}, \mathbb{R}^n)$ and $(\gamma_A, \gamma_B) \in \mathbb{R}^2$, the expression

$$\Delta_{AB}\widetilde{T}_C = \widetilde{\mathcal{K}}_C(\Phi + v_A, \gamma_A) - \widetilde{\mathcal{K}}_C(\Phi + v_B, \gamma_B) \tag{4.A.20}$$

satisfies the estimates

$$\begin{array}{lcl} \|\Delta_{AB}\widetilde{T}_C\|_{L^2_Q} & \leq & K\big[\|v_A - v_B\|_{L^2(\mathbb{R},\mathbb{R}^n)} + |\gamma_A - \gamma_B|\big], \\ \|\partial_\xi \Delta_{AB}\widetilde{T}_C\|_{L^2_Q} & \leq & K\big[1 + \|v_A\|_{H^1(\mathbb{R},\mathbb{R}^n)}\big]\big[\|v_A - v_B\|_{H^1(\mathbb{R},\mathbb{R}^n)} + |\gamma_A - \gamma_B|\big]. \end{array} \tag{4.A.21}$$

Proof. Note first that for any $z \in H^1(\mathbb{R}, \mathbb{R}^{m\times n})$ and any $\psi \in W^{1,\infty}(\mathbb{R}, \mathbb{R}^n)$, we have

$$\|Qz\psi\|^2_{L^2_Q} = \langle Qz\psi, z\psi\rangle_{L^2(\mathbb{R},\mathbb{R}^m)} \leq \|q\|_{L^1(\mathbb{R},\mathbb{R}^{m\times m})}\|z\|^2_{L^2(\mathbb{R},\mathbb{R}^{n\times m})}\|\psi\|^2_\infty \tag{4.A.22}$$

together with

$$\begin{array}{rcl}\|\partial_\xi Q z\psi\|_{L^2_Q}^2 = \|Q\partial_\xi[z\psi]\|_{L^2_Q}^2 & \leq & \|q\|_{L^1(\mathbb{R},\mathbb{R}^{m\times m})}\|\partial_\xi[z\psi]\|_{L^2(\mathbb{R},\mathbb{R}^m)}^2 \\ & \leq & \|q\|_{L^1(\mathbb{R},\mathbb{R}^{m\times m})}\|z\|_{H^1(\mathbb{R},\mathbb{R}^{n\times m})}^2[\|\psi\|_\infty + \|\psi'\|_\infty]^2.\end{array} \tag{4.A.23}$$

The bounds (4.A.19) hence follow directly from Lemma 4.A.4, using the cut-off function again to eliminate the dependence on $\|v\|_{L^2(\mathbb{R},\mathbb{R}^n)}$.

Turning to the Lipschitz estimates (4.A.21), we first compute

$$\begin{array}{rcl}\Delta_{AB}\widetilde{\mathcal{K}}_C & = & \big[\overline{\chi}_l(\Phi+v_A,\gamma_A) - \overline{\chi}_l(\Phi+v_B,\gamma_B)\big]Q\overline{\chi}_h(\Phi+v_A,\gamma_A)g^T(\Phi+v_A)T_{\gamma_A}\psi_{\mathrm{tw}} \\ & & +\overline{\chi}_l(\Phi+v_B,\gamma_B)Q\overline{\chi}_h(\Phi+v_A,\gamma_A)g^T(\Phi+v_A)\big[T_{\gamma_A}\psi_{\mathrm{tw}} - T_{\gamma_B}\psi_{\mathrm{tw}}\big] \\ & & +\overline{\chi}_l(\Phi+v_B,\gamma_B)Q\big[\Delta_{AB}\overline{\chi}_h g\big]^T T_{\gamma_B}\psi_{\mathrm{tw}}.\end{array} \tag{4.A.24}$$

If $\overline{\chi}_h(\Phi+v_A,\gamma_A)\neq 0$, then we can use an a priori bound on $\|v_A\|_{L^2(\mathbb{R},\mathbb{R}^n)}$ to obtain the result directly from Lemma 4.A.3 and Corollary 4.A.6. On the other hand, if we have an a priori bound on $\|v_B\|_{L^2(\mathbb{R},\mathbb{R}^n)}$, we can exploit symmetry to replace the $\|v_A\|_{L^2(\mathbb{R},\mathbb{R}^n)}$ term in (4.A.9) by $\|v_B\|_{L^2(\mathbb{R},\mathbb{R}^n)}$ and obtain the same result. □

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We are now ready to consider the final nonlinearity $\overline{b}$ that was defined in (4.5.24). Fortunately, our estimates for $\widetilde{\mathcal{K}_C}$ can also be used to establish the following bounds, which correspond precisely to those obtained in Chapter 2.

Lemma 4.A.8. *Suppose that (Hq), (HEq), (Hg), (HSt) and (hPar) are satisfied. Then there exist constants $K_b > 0$ and $K > 0$, which do not depend on the pair (Φ, c) so that the following holds true. For any $v \in H^1(\mathbb{R},\mathbb{R}^n)$ and $\gamma\in\mathbb{R}$ we have the bound*

$$\|\overline{b}(\Phi+\gamma,\psi)\|_{HS(L^2_Q,\mathbb{R})} \leq K_b, \tag{4.A.25}$$

while for any set of pairs $(v_A, v_B)\in H^1(\mathbb{R},\mathbb{R}^n)\times H^1(\mathbb{R},\mathbb{R}^n)$ and $(\gamma_A,\gamma_B)\in\mathbb{R}^2$ we have the estimate

$$\begin{array}{rcl}\|\overline{b}(\Phi+v_A,\gamma_A) - \overline{b}(\Phi+v_B,\gamma_B)\|_{HS(L^2_Q,\mathbb{R})} & \leq & K\|v_A - v_B\|_{L^2(\mathbb{R},\mathbb{R}^n)} \\ & & +K\big[1+\|v_B\|_{L^2(\mathbb{R},\mathbb{R}^n)}\big]\,|\gamma_A-\gamma_B|\,.\end{array} \tag{4.A.26}$$

Proof. The computation (4.5.25) shows that

$$\|\overline{b}(\Phi+v,\gamma)\|_{HS(L^2_Q,\mathbb{R})}^2 = \overline{\chi}_h(\Phi+V,\gamma)^2\|\widetilde{\mathcal{K}}_C(\Phi+v_A,\gamma_A)\|_{L^2_Q}^2, \tag{4.A.27}$$

which on account of (4.A.19) immediately implies (4.A.25).

Turning to the Lipschitz bound (4.A.26), we introduce the notation

$$\mathcal{I}_k = \overline{b}(v_A+\Phi,\gamma_A)[\sqrt{Q}e_k] - \overline{b}(v_B+\Phi,\gamma_B)[\sqrt{Q}e_k] \tag{4.A.28}$$

and note that

$$\|\overline{b}(\Phi+v_A,\gamma_A) - \overline{b}(\Phi+v_B,\gamma_B)\|_{HS(L^2_Q,\mathbb{R})}^2 = \sum_{k=0}^{\infty}\mathcal{I}_k^2. \tag{4.A.29}$$

We now compute

$$\begin{aligned}\mathcal{I}_k &= \overline{\chi}_h(v_A+\Phi,\gamma_A)^2\overline{\chi}_l(v_A+\Phi,\gamma_A)\langle g(\Phi+v_A)\sqrt{Q}e_k, T_{\gamma_A}\psi_{\mathrm{tw}}\rangle \\ &\qquad - \overline{\chi}_h(V_B+\Phi,\gamma_B)^2\overline{\chi}_l(V_B+\Phi,\gamma_B)\langle g(\Phi+V_A)\sqrt{Q}e_k, T_{\gamma_B}\psi_{\mathrm{tw}}\rangle \\ &= \langle\sqrt{Q}e_k, \overline{\chi}_h(v_A+\Phi,\gamma_A)\widetilde{\mathcal{K}}_C(\Phi+v_A,\gamma_A) - \overline{\chi}_h(v_B+\Phi,\gamma_B)\widetilde{\mathcal{K}}_C(\Phi+v_B,\gamma_B)\rangle_{L^2_Q}.\end{aligned} \tag{4.A.30}$$

In particular, we see that

$$\sum_{k=0}^{\infty}\mathcal{I}_k^2 = \|\overline{\chi}_h(v_A+\Phi,\gamma_A)\widetilde{\mathcal{K}}_C(\Phi+v_A,\gamma_A)-\overline{\chi}_h(v_B+\Phi,\gamma_B)\widetilde{\mathcal{K}}_C(\Phi+v_B,\gamma_B)\|^2_{L^2_Q}. \tag{4.A.31}$$

The desired bound now follows by combining Lemmas 4.A.3 and 4.A.7. □

5

Long Time Stability of Stochastic Travelling Waves

In this chapter we establish the meta-stability of travelling waves for a class of reaction-diffusion equations forced by a multiplicative noise term. In particular, we show that the phase-tracking technique developed in Chapters 2 and 4 can be maintained over timescales that are exponentially long with respect to the noise intensity. This is achieved by combining the generic chaining principle with a mild version of the Burkholder-Davis-Gundy inequality to establish logarithmic supremum bounds for stochastic convolutions in the critical regularity regime.

5.1 Introduction

In this chapter[1] we focus on the stochastic Nagumo equation

$$dU = [\rho \partial_{xx} U + f(U)]\, dt + \sigma g(U) dW_t^Q, \tag{5.1.1}$$

in which we take $U = U(x,t)$ with $x \in \mathbb{R}$ and $t \geq 0$. The nonlinearities are given by

$$f(u) = u(1-u)(u-a), \qquad g(u) = u(1-u)\chi(u) \tag{5.1.2}$$

for a parameter $a \in (0,1)$ and a smooth cut-off function $\chi(u)$ that forces g to be bounded and globally Lipschitz continuous on $\mathbb{R}$. The stochastic forcing is generated by the cylindrical Q-Wiener process W_t^Q characterized by the convolution operator

$$Q : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \qquad [Qv](x) = \int_{-\infty}^{\infty} e^{-(x-y)^2} v(y)\, dy. \tag{5.1.3}$$

In particular, our noise satisfies the formal relation

$$E\Big[dW_s^Q(x_0) dW_t^Q(x_1)\Big] = \delta(t-s) e^{-(x_0-x_1)^2} \tag{5.1.4}$$

[1] The content of this chapter has been accepted by SIADS as *C.H.S. Hamster, H.J. Hupkes; Stability of Travelling Waves on Exponentially Long Timescales in Stochastic Reaction-Diffusion Equations*, see [49].

and hence is white in time but coloured and translationally invariant in space. The well-posedness of such equations has been studied extensively [77, 93] and one can following Proposition 4.5.2 construct globally defined solutions in (for example) the affine space

$$\mathcal{U}_{H^1} = H^1(\mathbb{R}) + \frac{1}{2}\big(1 - \tanh(\cdot)\big). \tag{5.1.5}$$

The choice for this space is motivated by the fact that it contains the well-known deterministic travelling wave solution

$$U(x,t) = \Phi_0(x - c_0 t), \qquad \Phi_0(-\infty) = 1, \quad \Phi_0(+\infty) = 0 \tag{5.1.6}$$

for (5.1.1) with $\sigma = 0$. In Chapters 2 and 4 we showed that this pair (Φ_0, c_0) can be generalized to a branch of so-called *instantaneous stochastic waves* (Φ_σ, c_σ) for (5.1.1) that - at onset - travel with velocity velocity c_σ and feel only stochastic forcing. These waves can be shown to satisfy

$$\|\Phi_\sigma - \Phi_0\|_{H^2} + |c_\sigma - c_0| = \mathcal{O}(\sigma^2). \tag{5.1.7}$$

The key question is if one can understand the perturbations

$$V(t) = U(\cdot + \Gamma(t), t) - \Phi_\sigma \tag{5.1.8}$$

from these profiles, using an appropriate phase shift Γ to stochastically 'freeze' the solution U. In particular, we are interested in the behaviour of the stopping time

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$$t_{\mathrm{st}}(\eta) = \inf\{t \geq 0 : \|V(t)\|_{L^2}^2 + \int_0^t e^{-\varepsilon(t-s)}\|V(s)\|_{H^1}^2\, ds > \eta\} \tag{5.1.9}$$

for some small $\varepsilon > 0$, which measures when U exits an appropriate orbital η-neighbourhood of the profile Φ_σ. Our main result states that this exit-time is (with high probability) exponentially long with respect to the parameter $1/\sigma$. As such, it establishes the meta-stability of the deterministic travelling wave (5.1.6) under small stochastic forcing, significantly extending our earlier results in Chapters 2 and 4.

Theorem 5.1.1. *Pick a sufficiently large constant $K > 0$ and sufficiently small constants $\varepsilon > 0$, $\eta_0 > 0$, $\delta_0 > 0$ and $\delta_\sigma > 0$. Then for any $U(0) \in \mathcal{U}_{H^1}$ that satisfies $\|U(0) - \Phi_\sigma\|_{H^1} < \delta_0$ and any $0 \leq \sigma \leq \delta_\sigma$, there exists a scalar stochastic process Γ so that*

$$P(t_{\mathrm{st}}(\eta_0) < T) \leq K\left[\|U(0) - \Phi_\sigma\|_{H^1}^2 + \sigma\sqrt{\ln(T)}\right]. \tag{5.1.10}$$

holds for all $2 \leq T \leq \exp[\delta_\sigma^2/\sigma^2]$,

We remark here that general 'exit-problems' have been well-studied in finite-dimensional contexts [36], but much less is known in infinite dimensions [9, 41]. The recent paper by Salins and Spiliopoulous [98] discusses some of the main developments in this area, which chiefly focus on SPDEs with gradient-independent noise posed on finite domains. In this case, the associated semigroups are compact, allowing tightness results to be established that lead naturally to large deviation principles [21]. Such compactness properties do not apply in the current setting and we take a completely different approach.

Stochastic waves The impact of noise on pattern formation is an important topic that has attracted significant interest from the applied community [3, 13, 39, 102, 103, 112], but for which little rigorous mathematical theory is available [12, 52, 71, 86]. The Nagumo equation is a natural starting point for such investigations since it has served in the past as a prototypical system to analyse the interaction between two competing stable states in spatially extended domains [4, 5]. The deterministic travelling waves (5.1.6) represent a primary invasion mechanism by which the favourable state can spread throughout the entire domain. They are robust under perturbations, which allows them to be used as building blocks to understand the global behaviour of (5.1.1) in one [34, 65, 113] but also higher spatial dimensions [8, 64, 83].

The behaviour of these invasion waves under several types of stochastic forcing has been studied by various authors using a range of different techniques. The consensus emerging from a number of formal computations for (5.1.1) is that - to leading order in σ - the phase-shift of the wave follows a Brownian motion with a variance that can be expressed in closed form [16, 19, 39]. Various rigorous approaches have been pursued over the past five years that can successfully explain this diffusive behaviour on short time scales [57, 68, 104, 105]; see e.g [69] for a very recent overview and the introductions of Chapters 2 and 4 for a detailed technical discussion. Several of these techniques have been extended to stochastic neural field equations [15, 74, 81] and (very recently) to the FitzHugh-Nagumo system [31].

In the previous chapters, we pioneered a novel 'stochastic freezing' approach to rigorously analyse the behaviour of travelling fronts and pulses to a large class of reaction-diffusion equations (RDEs) - which includes (5.1.1) and the (fully diffusive) FitzHugh-Nagumo system. In essence, we developed a stochastic version of the freezing approach introduced by Beyn [11], which allows us to adopt the spirit behind the modern machinery for deterministic stability issues initiated by Howard and Zumbrun [118]. The power of this approach is that it leads naturally to *long-term* predictions concerning both the speed and the shape of the stochastic wave that can be computed to arbitrary order in σ. We demonstrated the accuracy of these novel predictions in Chapter 4 by performing a series of numerical experiments. As a consequence, we now have a quantitative explanation for the wave-steepening and speed-reduction phenomena that were illustrated numerically in [79] and - in a special case - derived formally in [19] using a collective coordinate approach.

Regularity issues The key novel feature of the approach in the previous chapters is that the perturbation V in the decomposition (5.1.8) is measured in the same reference frame as the frozen profile Φ_σ. This allows the delicate interaction between the speed and shape of the wave to be untangled, but also presents several fundamental complications that need to be carefully addressed. The most important of these is that the stochastic phase shift causes extra diffusive correction terms for V that are not seen in the deterministic context, together with a multiplicative noise term that involves the derivative of V. Unlike any of the previous approaches in this area, we hence need to keep the H^1-norm of $V(t)$ under control.

To be more specific, an essential step in our stability proofs is to obtain bounds for the expression

$$E \sup_{0 \le t \le T} \| \int_0^t S(t-s) B\big(V(s), \partial_x V(s)\big) dW_s^Q \|_{L^2}^2, \tag{5.1.11}$$

together with its integrated H^1-counterpart

$$E \sup_{0 \le t \le T} \int_0^t e^{-\varepsilon(t-s)} \| \int_0^s S(s-s') B\big(V(s'), \partial_x V(s')\big) dW_{s'}^Q \|_{H^1}^2 \, ds. \tag{5.1.12}$$

Here S denotes the semigroup associated to the linearization of (5.1.1) around the deterministic travelling wave (5.1.6). In the previous chapters we used the mild Burkholder-Davis-Gundy (BDG) inequality obtained by Veraar [111] to control (5.1.11), but the resulting bounds are unfortunately not optimal. As such, they restricted the validity range of our rigorous results to timescales of order $T \sim \sigma^{-2}$.

This shortfall is repaired by the bound in Theorem 5.1.1, which confirms that our phase-tracking can be maintained over the exponentially long timescales observed in the numerical results from Chapter 4. We emphasise that our improved bound also covers regimes where the stochastic phase Γ is expected to be very far away from its deterministic counterpart. This provides a solid theoretical underpinning to the formal predictions that we made in Chapter 4 concerning the stochastic corrections to (5.1.6).

To understand the issues that are involved, it is highly instructive to consider the scalar Ornstein-Uhlenbeck process

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$$X(t) = \int_0^t e^{-(t-s)} d\beta_s, \tag{5.1.13}$$

which here starts at $X(0) = 0$ and is driven by a standard Brownian motion β_t. Since $B(0,0) \neq 0$, the behaviour of X is highly comparable to that of V at lowest order in σ. Indeed, the deterministic dynamics pulls X towards zero at an exponential rate, but the stochastic forcing does not vanish there. Applying the mild Burkholder-Davis-Gundy inequality to (5.1.13) results in the bound

$$E \sup_{0 \le t \le T} |X(t)|^2 \le K \int_0^T 1 \, ds = KT. \tag{5.1.14}$$

This hence fails to reproduce the well-known fact that this expectation behaves as $\mathcal{O}\big(\ln(T)\big)$ for large T, which was originally established by examining crossing numbers [91] or analysing explicit probability distributions [2]. Fortunately, a more general abstract approach has been developed in recent years.

Chaining A powerful modern tool to derive supremum bounds for stochastic processes is commonly referred to as 'generic chaining'; see [106] for an accessible introduction.[2] Based on contributions from a range of authors, including Kolmogorov, Dudley, Fernique and Talagrand, it uses information on the increments of a stochastic process to establish

[2] This unpublished chapter by Pollard could also be useful: `http://www.stat.yale.edu/~pollard/Books/Mini/Chaining.pdf`

long-term supremum bounds. For instance, exploiting the fact that the Ornstein-Uhlenbeck process (5.1.13) is centred and Gaussian, one can obtain the tail bound

$$P(|X(t) - X(s)| > \vartheta) \leq 2e^{-\frac{\vartheta^2}{2d(t,s)^2}} \tag{5.1.15}$$

characterized by the metric $d(t,s) = \sqrt{E\big(X(t) - X(s)\big)^2}$. An explicit computation yields the bound

$$\begin{aligned} d(t,s)^2 &= \frac{1}{2}\big(2 - e^{-2t} - e^{-2s} - 2(e^{-|t-s|} - e^{-(t+s)})\big) \\ &\leq 1 - e^{-|t-s|} \\ &\leq \min\{|t-s|\,, 1\}. \end{aligned} \tag{5.1.16}$$

This shows that the covering number $N(T,d,\nu)$ - which measures the minimum number of intervals of length ν or less in the metric d required to cover $[0,T]$ - can be bounded by T/ν^2 for $\nu \in (0,1]$ and by 1 for $\nu \geq 1$. The main result in [106] - see Theorem 5.2.7 below - now provides the Dudley bound

$$\sup_{t\in[0,T]} X_t \sim \int_0^\infty \sqrt{\ln(N(T,d,\nu))}d\nu \leq \int_0^1 \sqrt{\ln\left(T/\nu^2\right)}d\nu \sim \sqrt{\ln(T)}, \tag{5.1.17}$$

which captures the desired logarithmic behaviour in a relatively straightforward manner.

Our main contribution in this chapter is that we extend this technique to provide similar sharp bounds for the stochastic integrals (5.1.11) and (5.1.12). On account of the regularity issues that are involved, this is a surprisingly delicate task. In fact, we are not aware of any related results in this direction besides the factorization method developed by Da Prato, Kwapień and Zabczyk [28], which typically only provides polynomial bounds in T. Let us remark that it was not immediately clear to us how this factorization technique should be applied in the present setting, because it introduces extra singularities into integrals that cannot be readily accommodated in our critical regularity regime.

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Obstructions In order to illustrate the key complications, let us consider the L^2-valued process

$$Y(t) = \int_0^t S(t-s)BdW_s^Q, \tag{5.1.18}$$

which can be seen as an infinite-dimensional version of (5.1.13). Here B is an appropriate constant Hilbert-Schmidt operator, which can be used to define the covariance operator

$$Q_\infty = \lim_{t\to\infty} \int_0^t S(t-s)BQB^*S^*(t-s)ds. \tag{5.1.19}$$

The analogue of the bound (5.1.16) is now given by[3]

$$d(t,s)^2 = E\|Y(t) - Y(s)\|^2_{L^2} \leq 2\,\mathrm{tr}\,\big((I - S(t-s))Q_\infty\big), \tag{5.1.20}$$

[3] This computation can be made rigorous using [44, §5].

but this time there is no $\alpha > 0$ for which one can extract a term of the form $|t-s|^{\alpha}$ from the difference $S(t-s) - I$. In principle this can be repaired by 'borrowing' some smoothness from B, but in our case this would again lead to unintegrable singularities.

In order to resolve this, it is crucial to combine the strong points of both the chaining technique and the mild Burkholder-Davis-Gundy inequality. Indeed, the former works well in the regime where $|t-s| \geq 1$ in (5.1.11), since here the decay and smoothening properties of the semigroup can both be put to excellent use. On the other hand, for $|t-s| \leq 1$, the H^{∞}-calculus underlying the mild Burkholder-Davis-Gundy inequality can resolve the critical regularity issues associated to supremum bounds without causing too much growth. The main issue is to set up an appropriate framework that allows this splitting to be achieved.

The second fundamental complication is that the integrands in (5.1.11) and (5.1.12) are time-dependent, which means that - in contrast to (5.1.18) - the stochastic integrals are not Gaussian. In this case, one must construct a metric such that a corresponding tail bound like (5.1.15) can be derived from scratch. Effectively, this requires us to control *all* the moments of the increments of (5.1.11). This is made possible by an effective use of stopping times in combination with a mild Itô formula.

Scope and outlook In order to make the arguments in this chapter as clear and concise as possible, we chose to restrict our attention to the single specific problem (5.1.1). However, we emphasise that our arguments transfer immediately to the general class of (multi-component) problems considered in Chapter 2 and Chapter 4, with the single restriction that all diffusion coefficients must be equal (condition (hA) in Chapter 2). This latter restriction can be removed by applying the spirit of Chapter 3, but this requires more complicated machinery that we will describe in an extensive forthcoming companion paper. There we will also address the long-term validity of the perturbation results from Chapter 4.

Organization We start in §5.2 by introducing some basic probabilistic and deterministic concepts. The heart of this chapter is contained in §5.3, where we provide logarithmic bounds for the stochastic integrals (5.1.11) and (5.1.12). Several supremum bounds for deterministic integrals are provided in §5.4, which allow for a streamlined proof of our main theorem in §5.5.

5.2 Preliminaries

In this section we collect several useful preliminary results that will streamline our arguments. We start in §5.2.1 by recalling well-known facts concerning the linearization of the Nagumo PDE around its travelling wave. We subsequently consider the relation between tail bounds and moment estimates for scalar stochastic processes in §5.2.2. Finally, in §5.2.3 we formulate the key technical tools that will allow us to apply the chaining principle to stochastic convolutions in the critical regularity regime.

5.2.1 Semigroup bounds

It is well-known that the Nagumo PDE (5.1.1) with $\sigma = 0$ admits a travelling front solution $U(x,t) = \Phi_0(x - c_0 t)$ that necessarily satisfies the travelling wave ODE

$$\rho\Phi_0'' + c_0\Phi_0' + f(\Phi_0) = 0, \qquad \Phi_0(-\infty) = 1, \quad \Phi_0(+\infty) = 0. \tag{5.2.1}$$

The associated linear operators

$$\mathcal{L}_{\mathrm{tw}} v = \rho v'' + c_0 v' + Df(\Phi_0)\, v, \qquad \mathcal{L}_{\mathrm{tw}}^* w = \rho w'' - c_0 w' + Df(\Phi_0)\, w, \tag{5.2.2}$$

which we view as maps from $H^2(\mathbb{R})$ into $L^2(\mathbb{R})$, both admit a simple eigenvalue at $\lambda = 0$ and have no other spectrum in the half-plane $\{\Re\lambda \geq -2\beta\} \subset \mathbb{C}$ for some $\beta > 0$. Writing P_{tw} for the spectral projection onto this neutral eigenvalue for $\mathcal{L}_{\mathrm{tw}}$, we can obtain the identifications

$$\mathrm{Ker}(\mathcal{L}_{\mathrm{tw}}) = \mathrm{span}\{\Phi_0'\}, \qquad \mathrm{Ker}(\mathcal{L}_{\mathrm{tw}}^*) = \mathrm{span}\{\psi_{\mathrm{tw}}\}, \qquad P_{\mathrm{tw}} v = \langle v, \psi_{\mathrm{tw}}\rangle_{L^2}\Phi_0' \tag{5.2.3}$$

by writing $\psi_{\mathrm{tw}}(\xi) = \kappa e^{-\frac{c_0\xi}{\rho}}\Phi_0'(\xi)$ for some κ that we fix by the requirement $\langle \Phi_0', \psi_{\mathrm{tw}}\rangle_{L^2} = 1$.

In fact, the operator $\mathcal{L}_{\mathrm{tw}}$ is sectorial and hence generates an analytic semigroup $S(t) = e^{\mathcal{L}_{\mathrm{tw}} t}$; see [80, Prop. 4.1.4] and Proposition 2.6.3. Upon introducing the notation

$$\begin{aligned} \mathcal{J}_{\mathrm{tw}}(t)[v,w] = {} & \langle S(t)v, S(t)w\rangle_{L^2} + \frac{1}{2\rho}\langle S(t)v, (\mathcal{L}_{\mathrm{tw}} - \rho\partial_{xx})S(t)w\rangle_{L^2} \\ & + \frac{1}{2\rho}\langle S(t)v, (\mathcal{L}_{\mathrm{tw}}^* - \rho\partial_{xx})S(t)w\rangle_{L^2}, \end{aligned} \tag{5.2.4}$$

a short computation (see §2.9.2) shows that

$$\langle S(t)v, S(t)w\rangle_{H^1} = \mathcal{J}_{\mathrm{tw}}(t)[v,w] - \frac{1}{2\rho}\frac{d}{dt}\langle S(t)v, S(t)w\rangle_{L^2} \tag{5.2.5}$$

holds for all $t > 0$ and $v, w \in L^2$. This identity allows the regularity issues that arise in §5.3 and §5.4 to be resolved.

Lemma 5.2.1. *Pick a sufficiently large $M \geq 1$ and write $\Pi = I - P_{\mathrm{tw}}$. Then for every $t > 0$ we have the bounds*

$$\begin{aligned} \|S(t)\Pi\|_{\mathcal{L}(L^2,L^2)} &\leq Me^{-\beta t}, \\ \|S(t)\Pi\|_{\mathcal{L}(L^2,H^1)} &\leq Mt^{-\frac{1}{2}}e^{-\beta t}, \\ \|[\mathcal{L}_{\mathrm{tw}} - \rho\partial_{\xi\xi}]S(t)\Pi\|_{\mathcal{L}(L^2,L^2)} &\leq Mt^{-\frac{1}{2}}e^{-\beta t}, \\ \|[\mathcal{L}_{\mathrm{tw}}^* - \rho\partial_{\xi\xi}]S(t)\Pi\|_{\mathcal{L}(L^2,L^2)} &\leq Mt^{-\frac{1}{2}}e^{-\beta t}, \\ \|(S(t) - I)S(1)\|_{\mathcal{L}(L^2,L^2)} &\leq M\,|t|\,. \end{aligned} \tag{5.2.6}$$

In particular, for any $t > 0$ and $v, w \in L^2$ we obtain the estimate

$$|\mathcal{J}_{\mathrm{tw}}(t)[\Pi v, \Pi w]| \leq M^2 e^{-2\beta t}\left(1 + \rho^{-1}t^{-1/2}\right)\|v\|_{L^2}\|w\|_{L^2}. \tag{5.2.7}$$

Proof. The bounds (5.2.6) can be deduced from [80, Prop. 5.2.1], while (5.2.7) follows readily by inspecting (5.2.4). □

5.2.2 Moment estimates and tail bounds

We briefly review here the technique that we use to pass back and forth between moment estimates and tail probabilities. The former are easier to estimate, but the latter are better suited for handling maxima. Our computations are based heavily on [106, Lem. 2.2.3] and [111].

Lemma 5.2.2. *Consider a random variable $Z \geq 0$ and suppose that there exists a $B > 0$ so that the bound*

$$E[Z^{2p}] \leq p^p B^{2p} \tag{5.2.8}$$

holds for all integers $p \geq 1$. Then for every $\vartheta > 0$ we have the estimate

$$P(Z > \vartheta) \leq 2 \exp\left[-\frac{\vartheta^2}{2eB^2}\right]. \tag{5.2.9}$$

Proof. For any $\nu > 0$ a formal computation shows that

$$\begin{aligned} P(Z > \vartheta) &= P(e^{\nu Z^2} > e^{\nu\vartheta^2}) \\ &\leq e^{-\nu\vartheta^2} E\left[e^{\nu Z^2}\right] \\ &\leq e^{-\nu\vartheta^2} E\left[\sum_{p=0}^{\infty} \frac{\nu^p}{p!} Z^{2p}\right] \\ &\leq e^{-\nu\vartheta^2} \sum_{p=0}^{\infty} \frac{\nu^p}{p!} p^p B^{2p}. \end{aligned} \tag{5.2.10}$$

Using $p! \geq p^p e^{-p}$ we obtain

$$P(Z > \vartheta) \leq e^{-\nu\vartheta^2} \sum_{p=0}^{\infty} \nu^p e^p B^{2p}, \tag{5.2.11}$$

which leads to (5.2.9) by choosing $\nu = (2eB^2)^{-1}$. □

Lemma 5.2.3. *Fix two constants $A \geq 2$ and $B > 0$ and consider a random variable $Z \geq 0$ that satisfies the estimate*

$$P(Z > \vartheta) \leq 2A \exp\left[-\frac{\vartheta^2}{2eB^2}\right] \tag{5.2.12}$$

for all $\vartheta > 0$. Then we have the moment bounds

$$E[Z] \leq 2\sqrt{2e}B\sqrt{\ln(A)}, \qquad E[Z^2] \leq 8eB^2 \ln(A). \tag{5.2.13}$$

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Proof. Starting with the second moment, we pick an arbitrary $u_0 > 0$ and compute

$$\begin{aligned} E[Z^2] &= \int_0^\infty P(Z^2 > u)\,du \\ &= \int_0^{u_0} P(Z > \sqrt{u})\,du + \int_{u_0}^\infty P(Z > \sqrt{u})\,du \\ &\le u_0 + \int_{u_0}^\infty 2Ae^{-u/(2eB^2)}\,du \\ &= u_0 + 4eAB^2 \exp\left(\frac{-u_0}{2eB^2}\right). \end{aligned} \tag{5.2.14}$$

Fixing $u_0 = 2eB^2 \ln(2A)$ and using $\ln(2A) > 1$, we obtain the desired estimate

$$E[Z^2] \le 2eB^2 \ln(2A) + 2eB^2 \le 8eB^2 \ln(A). \tag{5.2.15}$$

The bound for $E[Z]$ can now be deduced by taking square roots and applying Jensen's inequality. ☐

By applying a crude bound for tail-probabilities, Lemmas 5.2.2 and 5.2.3 can be combined to control maximum expectations. This results in the following useful logarithmic growth estimate.

Corollary 5.2.4. *Consider $N \ge 2$ non-negative random variables $Y_1, Y_2, ..., Y_N$ and suppose that there exists $B > 0$ so that the bound*

$$E\left[Y_i^{2p}\right] \le p^p B^{2p} \tag{5.2.16}$$

holds for all integers $p \ge 1$ and each $i \in \{1, .., N\}$. Then we have the bounds

$$E \max_{i\in\{1,\ldots,N\}} Y_i \le 2\sqrt{2e}B\sqrt{\ln(N)}, \qquad E \max_{i\in\{1,\ldots,N\}} Y_i^2 \le 8eB^2 \ln(N). \tag{5.2.17}$$

Proof. For any $\vartheta > 0$ we may use Lemma 5.2.2 to estimate

$$P\big(\max_{i\in\{1,\ldots,N\}} Y_i^2 > \vartheta\big) \le \sum_{i=1}^N P(Y_i^2 > \vartheta) \le 2N \exp\left(-\frac{\vartheta^2}{2eB^2}\right), \tag{5.2.18}$$

so we can directly apply Lemma 5.2.3. The proof for $E \max_{i\in\{1,\ldots,N\}} Y_i$ is identical. ☐

5.2.3 Supremum bounds

In this subsection we collect several key results that we will use to understand stochastic convolutions such as (5.1.11). In order to setup such integrals in a precise fashion, we follow the extensive discussion in §4.5 and introduce the Hilbert space

$$L^2_Q = L^2_Q(\mathbb{R}) = Q^{1/2}\left(L^2(\mathbb{R})\right), \tag{5.2.19}$$

together with the set of Hilbert-Schmidt operators

$$HS = HS(L^2_Q, L^2) = HS\big(L^2_Q(\mathbb{R}), L^2(\mathbb{R})\big) \tag{5.2.20}$$

that map $L^2_Q(\mathbb{R})$ into $L^2(\mathbb{R})$. Choosing an orthonormal basis (e_k) for $L^2(\mathbb{R})$, we recall that the Hilbert-Schmidt norm of the operator B is given by

$$\|B\|^2_{HS} = \sum_{k=0}^{\infty} \|B\sqrt{Q}e_k\|^2_{L^2}. \tag{5.2.21}$$

Fixing a complete filtered probability space $\Big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\Big)$, it turns out [66, 93, 95] that stochastic integrals against dW^Q_t are well-defined if the integrand is taken from the class

$$\begin{aligned}\mathcal{N}^2([0,T];(\mathcal{F}_t),HS) \quad := \quad & \{B \in L^2\big([0,T]\times\Omega; dt\otimes\mathbb{P}; HS\big) : \\ & B \text{ has a progressively } (\mathcal{F}_t)\text{-measurable version}\}.\end{aligned} \tag{5.2.22}$$

Our results in the previous chapters relied heavily on various versions of the Burkholder-Davis-Gundy inequality, but we only used the special case $p = 1$. The general form is stated below, where we highlight the p-dependence of the prefactors on the right-hand sides.

Lemma 5.2.5. *Pick*[4] *a sufficiently large $K_{\mathrm{cnv}} \geq 1$. Then for any $T > 0$, any integer $p \geq 1$ and any integrand $B \in \mathcal{N}^2\big([0,T];(\mathcal{F}_t);HS(L^2_Q, L^2)\big)$ we have the bound*

$$E \sup_{0\leq t\leq T} \|\int_0^t B(s)\, dW^Q_s\|^{2p}_{L^2} \leq K^{2p}_{\mathrm{cnv}} p^p E\left[\int_0^T \|B(s)\|^2_{HS}\, ds\right]^p, \tag{5.2.23}$$

together with its mild counterpart

$$E \sup_{0\leq t\leq T} \|\int_0^t S(t-s)B(s)\, dW^Q_s\|^{2p}_{L^2} \leq K^{2p}_{\mathrm{cnv}} p^p E\left[\int_0^T \|B(s)\|^2_{HS}\, ds\right]^p. \tag{5.2.24}$$

Proof. We note first that $L^2(\mathbb{R})$ is a Banach space of type 2. In particular, (5.2.23) follows from [111, Prop. 2.1 and Rem. 2.2]. In addition, the linear operator $\mathcal{L}_{\mathrm{tw}}$ admits a bounded H^∞-calculus (see Lemma 2.9.7), which allows us to apply [111, Thm. 1.1] and obtain (5.2.24). □

We remark that the inequalities (5.2.23)-(5.2.24) are very strong and useful on short time intervals, but on longer timescales it is no longer possible to exploit the decay properties of the semigroup. Indeed, the right-hand side of (5.2.24) grows linearly in time for integrands that are constant - as for the Ornstein-Uhlenbeck process. This changes if one drops the supremum.

Corollary 5.2.6. *Consider the setting of Lemma 5.2.5. Then for any $0 \leq t \leq T$ and any integer $p \geq 1$ we have the bound*

$$E\|\int_0^t S(t-s)B(s)\, dW^Q_s\|^{2p}_{L^2} \leq K^{2p}_{\mathrm{cnv}} p^p E\left[\int_0^t \|S(t-s)B(s)\|^2_{HS}\, ds\right]^p. \tag{5.2.25}$$

[4] Let us emphasise that all constants that appear in this chapter do not depend on T.

Proof. Note that

$$E\|\int_0^t S(t-s)B(s)\,dW_s^Q\|_{L^2}^{2p} \le E \sup_{0\le\tilde{t}\le t} \|\int_0^{\tilde{t}} S(t-s)B(s)\,dW_s^Q\|_{L^2}^{2p}, \tag{5.2.26}$$

so the result follows directly from (5.2.23). □

The following general result due to Talagrand [106, eq. (2.49)] is the key ingredient that will allow us to significantly improve the bound (5.2.24). It is based on the chaining principle, which requires us to understand the tail behaviour of the probability distribution for the temporal increments of stochastic processes.

Theorem 5.2.7 ([106])**.** *Pick a sufficiently large $C_{\mathrm{ch}} > 0$, choose an arbitrary $T > 0$ and consider a stochastic process $X : [0,T] \to L^2$ with paths that are almost-surely continuous. Suppose furthermore that there exists a metric $d = d(\cdot,\cdot)$ on $[0,T]$ so that the increments of X satisfy the estimate*

$$P\left(\|X(t_1) - X(t_2)\|_{L^2} > \vartheta\right) \le 2\exp\left(-\frac{\vartheta^2}{2d(t_1,t_2)^2}\right), \tag{5.2.27}$$

for every $t_1, t_2 \in [0,T]$ and $\vartheta > 0$. Then we have the bound

$$E \sup_{0\le t\le T} \|X(t)\|_{L^2}^2 \le C_{\mathrm{ch}} \left(\int_0^\infty \sqrt{\ln\left(N(T,d,\nu)\right)}\,d\nu\right)^2 \tag{5.2.28}$$

where $N(T,d,\nu)$ is the smallest number of intervals of length at most ν in the metric d required to cover $[0,T]$.

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5.3 Supremum bounds for stochastic integrals

In this section we develop the machinery needed to obtain bounds for two types of stochastic integrals. In particular, we introduce the L^2-valued integral

$$\mathcal{E}_B(t) = \int_0^t S(t-s)B(s)dW_s^Q \tag{5.3.1}$$

together with the scalar integral

$$\mathcal{I}_B^{\mathrm{s}}(t) = \int_0^t e^{-\varepsilon(t-s)} \int_0^s \langle S(s-s')V(s'), S(s-s')B(s')\,dW_{s'}^Q\rangle_{H^1}\,ds \tag{5.3.2}$$

and set out to obtain bounds for the quantities

$$E \sup_{0\le t\le T} \|\mathcal{E}_B(t)\|_{L^2}^2, \qquad E \max_{i\in\{1,\dots,T\}} |\mathcal{I}_B^{\mathrm{s}}(i)|\,. \tag{5.3.3}$$

We will use the first of these expressions in §5.5.1 to control the L^2-norm of $V(t)$, while the second term plays a crucial role in §5.5.2 where we bound the H^1-norm of $V(t)$ in an integrated sense. In both cases B will be replaced by a (complicated) function of V, but we make use of a generic placeholder here in order to emphasise the broad applicability of our techniques. Indeed, we only need to impose the following two general conditions on our integrands.

(hB) The process $B \in \mathcal{N}^2\left([0,T];(\mathcal{F}_t);HS(L^2_Q,L^2)\right)$ satisfies

$$\langle B(t)v, \psi_{\mathrm{tw}}\rangle_{L^2} = 0 \tag{5.3.4}$$

for all $t \in [0,T]$ and $v \in L^2_Q$. Furthermore, there exist $\varepsilon \in (0,\beta)$ and $\Theta_* > 0$ so that the following pathwise bounds hold for all $0 \le t \le T$:

$$\int_0^t e^{-\varepsilon(t-s)}\|B(s)\|^2_{HS}ds \le \Theta_*^2, \qquad \|S(1)B(t)\|^2_{HS} \le \Theta_*^2. \tag{5.3.5}$$

(hV) The process $V \in \mathcal{N}^2\left([0,T];(\mathcal{F}_t);H^1\right)$ satisfies

$$\langle V(t), \psi_{\mathrm{tw}}\rangle_{L^2} = 0 \tag{5.3.6}$$

for all $t \in [0,T]$. Furthermore, there exists a $\Lambda_* > 0$ so that the pathwise bound

$$\|V(t)\|_{L^2} \le \Lambda_* \tag{5.3.7}$$

holds for all $0 \le t \le T$.

We remark that (5.3.4) and (5.3.6) imply that B and V do not feel the neutral mode of the semigroup. This allows us to use the decay rates from Lemma 5.2.1 and establish our main result below. In particular, we obtain (5.3.8) in §5.3.1 and (5.3.9) in §5.3.2. For convenience, we will consider T to be an integer from now on. This will make the splitting of the integrals easier in the following sections and any results for non-integer T can be shown by rounding T up to the nearest integer.

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Proposition 5.3.1. *Fix a sufficiently large constant $K > 0$. Then for any integer $T \ge 2$ and any pair of processes (B,V) that satisfies (hB) and (hV), we have the supremum bound*

$$E \sup_{0\le t\le T} \|\mathcal{E}_B(t)\|^2_{L^2} \le K\Theta_*^2 \ln(T) \tag{5.3.8}$$

together with its counterpart

$$E \max_{i\in\{1,\ldots,T\}} |\mathcal{I}^{\mathrm{s}}_B(i)| \le K\Lambda_*\Theta_*\sqrt{\ln(T)}. \tag{5.3.9}$$

5.3.1 Estimates for $\mathcal{E}_B$

Motivated by the considerations in the introduction, we make the splitting

$$\mathcal{E}_B(t) = \mathcal{E}^{\mathrm{lt}}_B(t) + \mathcal{E}^{\mathrm{sh}}_B(t), \tag{5.3.10}$$

in which the short time (sh) and long time (lt) contributions are respectively given by

$$\mathcal{E}^{\mathrm{lt}}_B(t) = \int_0^{t-1} S(t-s)B(s)\,dW^Q_s, \qquad \mathcal{E}^{\mathrm{sh}}_B(t) = \int_{t-1}^{t} S(t-s)B(s)\,dW^Q_s, \tag{5.3.11}$$

where we interpret the boundary $t-1$ as $\max\{t-1,0\}$ if necessary. Both these terms need to be handled using separate techniques.

Short time bounds Remembering that T is an integer, we introduce the function

$$\Upsilon_B^{(i)} = \sup_{0 \le s \le 1} \| \int_i^{i+s} S(i+s-s')B(s')\, dW_{s'}^Q \|_{L^2} \tag{5.3.12}$$

for any $i \in \{0, \dots, T-1\}$, while for $i = -1$ we define $\Upsilon_B^{(i)} = 0$. An elementary computation allows us to bound $\mathcal{E}_B^{\mathrm{sh}}(t)$ in terms of at most two of this finite set of quantities.

Lemma 5.3.2. *Pick any integer $T \ge 2$ and assume that (hB) holds. Then for all $0 \le t \le T$ we have the bound*

$$\|\mathcal{E}_B^{\mathrm{sh}}(t)\|_{L^2} \le 2M\Upsilon_B^{(\lfloor t \rfloor - 1)} + \Upsilon_B^{(\lfloor t \rfloor)}. \tag{5.3.13}$$

Proof. Since the estimate is immediate for $0 \le t < 1$, we pick $t \ge 1$. Splitting the integral yields

$$\begin{aligned} \|\mathcal{E}_B^{\mathrm{sh}}(t)\|_{L^2} &\le \| \int_{t-1}^{\lfloor t \rfloor} S(t-s)B(s)dW_s^Q \|_{L^2} + \| \int_{\lfloor t \rfloor}^{t} S(t-s)B(s)dW_s^Q \|_{L^2} \\ &\le \| \int_{\lfloor t \rfloor - 1}^{\lfloor t \rfloor} S(t-s)B(s)dW_s^Q \|_{L^2} + \| \int_{\lfloor t \rfloor - 1}^{t-1} S(t-s)B(s)dW_s^Q \|_{L^2} \\ &\quad + \| \int_{\lfloor t \rfloor}^{t} S(t-s)B(s)dW_s^Q \|_{L^2}. \end{aligned} \tag{5.3.14}$$

Using Lemma 5.2.1 we obtain the estimate

$$\begin{aligned} \| \int_{\lfloor t \rfloor - 1}^{\lfloor t \rfloor} S(t-s)B(s)dW_s^Q \|_{L^2} &\le \|S(t - \lfloor t \rfloor)\|_{\mathcal{L}(L^2)} \| \int_{\lfloor t \rfloor - 1}^{\lfloor t \rfloor} S(\lfloor t \rfloor - s)B(s)dW_s^Q \|_{L^2} \\ &\le M \| \int_{\lfloor t \rfloor - 1}^{\lfloor t \rfloor} S(\lfloor t \rfloor - s)B(s)dW_s^Q \|_{L^2} \\ &\le M\Upsilon_B^{(\lfloor t \rfloor - 1)}, \end{aligned} \tag{5.3.15}$$

together with

$$\begin{aligned} \| \int_{\lfloor t \rfloor - 1}^{t-1} S(t-s)B(s)dW_s^Q \|_{L^2} &\le \|S(1)\|_{\mathcal{L}(L^2, L^2)} \| \int_{\lfloor t \rfloor - 1}^{t-1} S(t-1-s)B(s)dW_s^Q \|_{L^2} \\ &\le M\Upsilon_B^{(\lfloor t \rfloor - 1)}, \end{aligned} \tag{5.3.16}$$

from which the desired bound readily follows. □

Corollary 5.3.3. *Pick any integer $T \ge 2$ and assume that (hB) holds. Then we have the pathwise bound*

$$\sup_{0 \le t \le T} \|\mathcal{E}_B^{\mathrm{sh}}(t)\|_{L^2}^2 \le 9M^2 \max_{i \in \{0, \dots, T-1\}} \left(\Upsilon_B^{(i)}\right)^2. \tag{5.3.17}$$

The expectation of the right-hand side of (5.3.17) can be controlled using Corollary 5.2.4. We hence require moment bounds on $\Upsilon_B^{(i)}$, which can be obtained by applying the mild Burkholder-Davis-Gundy inequality. Here we use the crucial fact that $\mathcal{L}_{\mathrm{tw}}$ admits an H^∞-calculus.

Lemma 5.3.4. *Pick any integer $T \geq 2$ and assume that (hB) holds. Then for any integer $p \geq 1$ and any $i \in \{0, \ldots, T-1\}$ we have the bound*

$$E\left[\Upsilon_B^{(i)}\right]^{2p} \leq K_{\mathrm{cnv}}^{2p} p^p e^{\varepsilon p} \Theta_*^{2p}. \tag{5.3.18}$$

Proof. Applying Lemma 5.2.5, we readily compute

$$\begin{aligned} E\left[\Upsilon_B^{(i)}\right]^{2p} &\leq K_{\mathrm{cnv}}^{2p} p^p E\left[\int_i^{i+1} \|B(s)\|_{HS}^2 \, ds\right]^p \\ &\leq K_{\mathrm{cnv}}^{2p} p^p e^{\varepsilon p} E\left[\int_0^{i+1} e^{-\varepsilon(i+1-s)} \|B(s)\|_{HS}^2 \, ds\right]^p, \end{aligned} \tag{5.3.19}$$

which implies the stated bound on account of (5.3.5). □

Long-term bounds The goal here is to apply the chaining result from Theorem 5.2.7 to the long-term integral $\mathcal{E}_B^{\mathrm{lt}}$. To achieve this, we will use Lemma 5.2.2 to turn moment bounds for the increments of $\mathcal{E}_B^{\mathrm{lt}}$ into the desired tail bounds for the associated probability distribution.

For any pair $0 \leq t_1 \leq t_2 \leq T$, we split this increment into two parts

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$$\mathcal{E}_B^{\mathrm{lt}}(t_1) - \mathcal{E}_B^{\mathrm{lt}}(t_2) = \mathcal{I}_1(t_1, t_2) + \mathcal{I}_2(t_1, t_2) \tag{5.3.20}$$

that are defined by

$$\begin{aligned} \mathcal{I}_1(t_1, t_2) &= \int_0^{t_1 - 1} [S(t_2 - s) - S(t_1 - s)] \, B(s) \, dW_s^Q, \\ \mathcal{I}_2(t_1, t_2) &= \int_{t_1 - 1}^{t_2 - 1} S(t_2 - s) B(s) \, dW_s^Q. \end{aligned} \tag{5.3.21}$$

The first of these can be analysed by exploiting the regularity of the semigroup $S(t-s)$ for $t - s \geq 1$, while the second requires a supremum bound on the 'smoothened' process $S(1)B$, hence explaining the assumption in equation (5.3.5).

Lemma 5.3.5. *Pick any integer $T \geq 2$ and assume that (hB) holds. Then for any $1 \leq t_1 \leq t_2 \leq T$ and any integer $p \geq 1$ we have the bound*

$$E\|\mathcal{I}_1(t_1, t_2)\|_{L^2}^{2p} \leq p^p K_{\mathrm{cnv}}^{2p} M^{4p} \Theta_*^{2p} \left|t_2 - t_1\right|^{2p}. \tag{5.3.22}$$

Proof. Observe first that

$$\begin{aligned} E\|\mathcal{I}_1(t_1, t_2)\|_{L^2}^{2p} &\leq \|[S(t_2 - t_1) - I]S(1)\|_{\mathcal{L}(L^2)}^{2p} E\|\int_0^{t_1 - 1} S(t_1 - 1 - s)B(s) \, dW_s^Q\|_{L^2}^{2p} \\ &\leq M^{2p} \left|t_2 - t_1\right|^{2p} E\|\int_0^{t_1 - 1} S(t_1 - 1 - s)B(s) \, dW_s^Q\|_{L^2}^{2p}. \end{aligned} \tag{5.3.23}$$

Applying (5.2.25) with $T = t_1 - 1$, we find

$$\begin{aligned} E\|\mathcal{I}_1(t_1,t_2)\|_{L^2}^{2p} &\le p^p K_{\mathrm{cnv}}^{2p} M^{2p} \left|t_2 - t_1\right|^{2p} E\left[\int_0^{t_1-1} \|S(t_1-1-s)B(s)\|_{HS}^2\, ds\right]^p \\ &\le p^p K_{\mathrm{cnv}}^{2p} M^{4p} \left|t_2 - t_1\right|^{2p} E\left[\int_0^{t_1-1} e^{-2\beta(t_1-1-s)}\|B(s)\|_{HS}^2\, ds\right]^p, \end{aligned} \tag{5.3.24}$$

which yields the stated bound in view of (5.3.5). □

Lemma 5.3.6. *Pick any integer $T \ge 2$ and assume that (hB) holds. Then for any $1 \le t_1 \le t_2 \le T$ and any integer $p \ge 1$ we have the bound*

$$E\|\mathcal{I}_2(t_1,t_2)\|_{L^2}^{2p} \le p^p K_{\mathrm{cnv}}^{2p} M^{2p} \Theta_*^{2p} \left|t_2 - t_1\right|^p. \tag{5.3.25}$$

Proof. It suffices to compute

$$\begin{aligned} E\|\mathcal{I}_2(t_1,t_2)\|_{L^2}^{2p} &= E\left[\|\int_{t_1-1}^{t_2-1} S(t_2-1-s)S(1)B(s)\, dW_s^Q\|_{L^2}\right]^{2p} \\ &\le p^p K_{\mathrm{cnv}}^{2p} E\left[\int_{t_1-1}^{t_2-1} \|S(t_2-1-s)\|_{\mathcal{L}(L^2,L^2)}^2 \|S(1)B(s)\|_{HS}^2\, ds\right]^p \\ &\le p^p K_{\mathrm{cnv}}^{2p} M^{2p} \left|t_2 - t_1\right|^p E\left[\sup_{t_1-1\le s\le t_2-1} \|S(1)B(s)\|_{HS}^2\right]^p \end{aligned} \tag{5.3.26}$$

and apply (5.3.5). □

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The previous two results were tailored to handle small increments $|t_2 - t_1| \le 1$. For larger increments one can exploit the decay of the semigroup to show that $\mathcal{E}_B^{\mathrm{lt}}$ remains bounded in expectation.

Lemma 5.3.7. *Pick any integer $T \ge 2$ and assume that (hB) holds. Then for any $0 \le t \le T$ and any integer $p \ge 1$ we have the bound*

$$E\|\mathcal{E}_B^{\mathrm{lt}}(t)\|_{L^2}^{2p} \le p^p K_{\mathrm{cnv}}^{2p} M^{2p} \Theta_*^{2p}. \tag{5.3.27}$$

Proof. Using Corollary 5.2.6, we find

$$\begin{aligned} E\|\mathcal{E}_B^{\mathrm{lt}}(t)\|_{L^2}^{2p} &\le p^p K_{\mathrm{cnv}}^{2p} E\left[\int_0^{t-1} \|S(t-s)\Pi\|_{\mathcal{L}(L^2,L^2)}^2 \|B(s)\|_{HS}^2\, ds\right]^p \\ &\le p^p K_{\mathrm{cnv}}^{2p} M^{2p} E\left[\int_0^{t-1} e^{-2\beta(t-1-s)}\|B(s)\|_{HS}^2 ds\right]^p \\ &\le p^p K_{\mathrm{cnv}}^{2p} M^{2p} \Theta_*^{2p}. \end{aligned} \tag{5.3.28}$$

□

Corollary 5.3.8. *Pick any integer $T \ge 2$ and assume that (hB) holds. Then for any $0 \le t_1 \le t_2 \le T$ and any integer $p \ge 1$ we have the bound*

$$E\|\mathcal{E}_B^{\mathrm{lt}}(t_1) - \mathcal{E}_B^{\mathrm{lt}}(t_2)\|_{L^2}^{2p} \le 2^{2p} p^p K_{\mathrm{cnv}}^{2p} M^{4p} \Theta_*^{2p} \min\{\left|t_2 - t_1\right|^{1/2}, 1\}^{2p}. \tag{5.3.29}$$

Proof. This follows from the standard inequality $(a+b)^{2p} \leq 2^{2p-1}(a^{2p}+b^{2p})$ and a combination of the estimates from Lemmas 5.3.5-5.3.7. □

Lemma 5.3.9. *Fix a sufficiently large constant $K_{\mathrm{lt}} \geq 1$. Then for any integer $T \geq 2$ and any process B that satisfies (hB), we have the supremum bound*

$$E \sup_{0\leq t\leq T} \|\mathcal{E}_B^{\mathrm{lt}}(t)\|_{L^2}^2 \leq K_{\mathrm{lt}}\Theta_*^2 \ln(T). \tag{5.3.30}$$

Proof. Upon writing $d_{\max} = 2\sqrt{e}K_{\mathrm{cnv}}M^2\Theta_*$ together with

$$d(t_1,t_2) = d_{\max}\min\{\sqrt{|t_2-t_1|},1\}, \tag{5.3.31}$$

an application of Lemma 5.2.2 to Corollary 5.3.8 provides the bound

$$P\left(\|\mathcal{E}_B^{\mathrm{lt}}(t_1)-\mathcal{E}_B^{\mathrm{lt}}(t_2)\|_{L^2} > \vartheta\right) \quad \leq \quad 2\exp\left[-\frac{\vartheta^2}{2d(t_1,t_2)^2}\right]. \tag{5.3.32}$$

Turning to the packing number $N(T,d,\nu)$ introduced in Theorem 5.2.7, we note that $N(T,d,\nu)=1$ whenever $\nu \geq d_{\max}$, while for smaller ν we have

$$N(T,d,\nu) \leq \frac{Td_{\max}^2}{\nu^2}. \tag{5.3.33}$$

In particular, the Dudley entropy integral can be bounded by

$$\begin{aligned}\int_0^\infty \sqrt{\ln(N(T,d,\nu))}\,d\nu &\leq \int_0^{d_{\max}} \sqrt{\ln(Td_{\max}^2/\nu^2)}\,d\nu \\ &= \int_0^{d_{\max}} \sqrt{-2\ln\left(\nu/(d_{\max}\sqrt{T})\right)}\,d\nu \\ &= d_{\max}\sqrt{T}\int_0^{1/\sqrt{T}} \sqrt{-2\ln(\nu)}\,d\nu \\ &= d_{\max}\left(\sqrt{2\ln(T)}+\sqrt{\pi}\sqrt{T}\mathrm{erfc}\big(\sqrt{\ln(T)}\big)\right).\end{aligned} \tag{5.3.34}$$

Since the function $\sqrt{T}\mathrm{erfc}(\sqrt{\ln(T)})$ is uniformly bounded for $T \geq 2$, the desired estimate now follows directly from Theorem 5.2.7. □

Proof of (5.3.8) in Proposition 5.3.1. Applying Corollary 5.2.4 to the estimates (5.3.17)-(5.3.18), we directly find

$$E \sup_{0\leq t\leq T} \|\mathcal{E}_B^{\mathrm{sh}}(t)\|_{L^2}^2 \leq 108M^2eK_{\mathrm{cnv}}^2e^{\varepsilon}\Theta_*^2\ln(T). \tag{5.3.35}$$

Combining this with the analogous long-term estimate (5.3.30) readily yields the result. □

5.3.2 Estimates for $\mathcal{I}_B^{\mathrm{s}}$

Our strategy here for controlling $\mathcal{I}_B^{\mathrm{s}}$ is to appeal to Corollary 5.2.4, which requires us to obtain moment bounds on $\mathcal{I}_B^{\mathrm{s}}(i)$. As a preparation, we switch the order of integration to find

$$\begin{aligned} \mathcal{I}_B^{\mathrm{s}}(i) &= \int_0^i e^{-\varepsilon(i-s)} \int_0^s \langle S(s-s')V(s'), S(s-s')B(s')\cdot\rangle_{H^1}\, dW_{s'}^Q\, ds \\ &= \int_0^i \int_{s'}^i e^{-\varepsilon(i-s)} \langle S(s-s')V(s'), S(s-s')B(s')\cdot\rangle_{H^1}\, ds\, dW_{s'}^Q. \end{aligned} \tag{5.3.36}$$

Corollary 5.2.6 hence yields

$$E[\mathcal{I}_B^{\mathrm{s}}(i)]^{2p} \le p^p K_{\mathrm{cnv}}^{2p} E\left[\int_0^i \sum_{k=0}^{\infty} \mathcal{K}_k^{(i)}(s')^2 ds'\right]^p, \tag{5.3.37}$$

in which we have introduced the expression[5]

$$\mathcal{K}_k^{(i)}(s') = \int_{s'}^i e^{-\varepsilon(i-s)} \langle S(s-s')V(s'), S(s-s')B(s')\sqrt{Q}e_k\rangle_{H^1} ds. \tag{5.3.38}$$

Motivated by (5.2.5), we split $\mathcal{K}_k^{(i)}$ into the two parts

$$\begin{aligned} \mathcal{K}_{I;k}^{(i)}(s') &= \int_{s'}^i e^{-\varepsilon(i-s)} \mathcal{J}_{\mathrm{tw}}[V(s'), B(s')\sqrt{Q}e_k] ds, \\ \mathcal{K}_{II;k}^{(i)}(s') &= -\frac{1}{2\rho} \int_{s'}^i e^{-\varepsilon(i-s)} \frac{d}{ds} \langle S(s-s')V(s'), S(s-s')B(s')\sqrt{Q}e_k\rangle_{L^2} ds. \end{aligned} \tag{5.3.39}$$

Performing an integration by parts, the second of these integrals can be further decomposed into the three terms

$$\begin{aligned} \mathcal{K}_{IIa;k}^{(i)}(s') &= -\frac{\varepsilon}{2\rho} \int_{s'}^i e^{-\varepsilon(i-s)} \langle S(s-s')V(s'), S(s-s')B(s')\sqrt{Q}e_k\rangle_{L^2} ds, \\ \mathcal{K}_{IIb;k}^{(i)}(s') &= -\frac{1}{2\rho} \langle S(i-s')V(s'), S(i-s')B(s')\sqrt{Q}e_k\rangle_{L^2}, \\ \mathcal{K}_{IIc;k}^{(i)}(s') &= \frac{1}{2\rho} e^{-\varepsilon(i-s')} \langle V(s'), B(s')\sqrt{Q}e_k\rangle_{L^2}. \end{aligned} \tag{5.3.40}$$

Lemma 5.3.10. *Pick a sufficiently large constant $K_{\mathcal{K}} > 0$. Then for any integer $T \ge 2$, any pair of processes (B, V) that satisfies (hB) and (hV) and any $i \in \{1, \dots, T\}$, we have the bound*

$$\sum_k \mathcal{K}_{\#;k}^{(i)}(s')^2 \le K_{\mathcal{K}} e^{-2\varepsilon(i-s')} \|V(s')\|_{L^2}^2 \|B(s')\|_{HS}^2 \tag{5.3.41}$$

[5] Note that this integral is an improper integral, as the integrand is not defined for the lower boundary $s = s'$. In Chapter 2 we show how this problem can be circumvented by replacing s by $s + \delta$ and subsequently sending $\delta \to 0$.

for all $0 \le s' \le i$ *and each of the symbols* $\# \in \{I, IIa, IIb, IIc\}$.

Proof. Upon introducing the expression

$$K(\varepsilon, \beta) = e^{\varepsilon(i-s')} \int_{s'}^{i} e^{-\varepsilon(i-s)} e^{-2\beta(s-s')} \left(1 + \rho^{-1}(s-s')^{-1/2}\right) ds \tag{5.3.42}$$

we may exploit (5.2.7) to obtain the bound

$$\begin{aligned} \sum_k \mathcal{K}^{(i)}_{I;k}(s')^2 &\le M^2 e^{-2\varepsilon(i-s')} \sum_k \|V(s')\|^2_{L^2} \|B(s')\sqrt{Q} e_k\|^2_{L^2} K(\varepsilon, \beta)^2 \\ &= M^2 e^{-2\varepsilon(i-s')} \|V(s')\|^2_{L^2} \|B(s')\|^2_{HS} K(\varepsilon, \beta)^2. \end{aligned} \tag{5.3.43}$$

The estimate for $\# = I$ hence follows from the computation

$$K(\varepsilon, \beta) \le \int_0^\infty e^{(\varepsilon - 2\beta)s} \left(1 + \rho^{-1} s^{-1/2}\right) ds = \frac{1}{2\beta - \varepsilon} + \frac{1}{\rho}\sqrt{\frac{\pi}{2\beta - \varepsilon}}. \tag{5.3.44}$$

The estimate for $\mathcal{K}^{(i)}_{IIa;k}$ can be obtained in the same fashion, but here the $(s - s')^{-1/2}$ term in (5.3.42) is not required. Finally, the estimates for $\mathcal{K}^{(i)}_{IIb;k}$ and $\mathcal{K}^{(i)}_{IIc;k}$ are immediate from Lemma 5.2.1 and the choice $\beta > \varepsilon$. □

Proof of (5.3.9) in Proposition 5.3.1. Applying Young's inequality to the decomposition above, we obtain the pathwise bound

$$\int_0^i \sum_k \mathcal{K}^{(i)}_k(s')^2 ds' \le 16 K_{\mathcal{K}} \int_0^i e^{-2\varepsilon(i-s')} \|V(s')\|^2_{L^2} \|B(s')\|^2_{HS}\, ds' \le 16 K_{\mathcal{K}} \Lambda_*^2 \Theta_*^2. \tag{5.3.45}$$

In view of (5.3.37) this implies

$$E[\mathcal{I}^{\mathrm{s}}_B(i)]^{2p} \le 2^{4p} p^p K^p_{\mathcal{K}} K^{2p}_{\mathrm{cnv}} \Lambda_*^{2p} \Theta_*^{2p}, \tag{5.3.46}$$

which leads to the desired bound by exploiting Corollary 5.2.4. □

5.4 Deterministic supremum bounds

Our goal here is to obtain pathwise bounds on the deterministic integrals

$$\begin{aligned} \mathcal{I}_F(t) &= \int_0^t e^{-\varepsilon(t-s)} \int_0^s \langle S(s-s')V(s'), S(s-s')F(s')\rangle_{H^1}\, ds'\, ds, \\ \mathcal{I}^{\mathrm{d}}_B(t) &= \int_0^t e^{-\varepsilon(t-s)} \int_0^s \|S(s-s')B(s')\|^2_{HS(L^2_Q, H^1)}\, ds'\, ds. \end{aligned} \tag{5.4.1}$$

We are using the process F in the first integral as a placeholder for various linear and nonlinear expressions in V that we will encounter in §5.5. The second integral arises as the Itô correction term coming from the integrated H^1-norm of V; see Lemma 5.5.5. Besides the assumptions (hB) and (hV) introduced in §5.5, we impose the following condition on the new function F.

(hF) The process $F : [0,T] \times \Omega \to L^2$ has paths in $L^1([0,T];L^2)$ and satisfies

$$\langle F(t), \psi_{\mathrm{tw}} \rangle_{L^2} = 0 \tag{5.4.2}$$

for all $t \in [0,T]$.

In contrast to the stochastic setting of §5.3, pathwise bounds for the expressions (5.4.1) can be easily used to control their supremum expectations. Indeed, we do not need to use the Burkholder-Davis-Gundy inequalities, which allows us to take a far more direct approach to establish our two main results below. Notice that we are making no a priori assumptions on the size of F. This will be useful in §5.5 to obtain sharp estimates for the nonlinear terms.

Proposition 5.4.1. *Fix a sufficiently large constant $K > 0$. Then for any $T > 0$ and any pair of processes (F, V) that satisfies (hF) and (hV), we have the supremum bound*

$$E \sup_{0 \le t \le T} |\mathcal{I}_F(t)| \le K \Lambda_* E \sup_{0 \le t \le T} \int_0^t e^{-\varepsilon(t-s)} \|F(s)\|_{L^2} \, ds. \tag{5.4.3}$$

Proposition 5.4.2. *Fix a sufficiently large constant $K > 0$. Then for any $T > 0$ and any process B that satisfies (hB), we have the supremum bound*

$$E \sup_{0 \le t \le T} \mathcal{I}_B^{\mathrm{d}}(t) \le K \Theta_*^2. \tag{5.4.4}$$

5.4.1 Estimates for $\mathcal{I}_F$ and $\mathcal{I}_B^{\mathrm{d}}$

Upon introducing the expressions

$$\begin{aligned} \mathcal{K}_F(t,s') &= \int_{s'}^t e^{-\varepsilon(t-s)} \langle S(s-s')V(s'), S(s-s')F(s') \rangle_{H^1} ds, \\ \mathcal{K}_{B;k}^{\mathrm{d}}(t,s') &= \int_{s'}^t e^{-\varepsilon(t-s)} \langle S(s-s')B(s')\sqrt{Q}e_k, S(s-s')B(s')\sqrt{Q}e_k \rangle_{H^1} ds, \end{aligned} \tag{5.4.5}$$

we may reverse the order of integration to find

$$\mathcal{I}_F(t) = \int_0^t \mathcal{K}_F(t,s') ds', \qquad \mathcal{I}_B^{\mathrm{d}}(t) = \int_0^t \sum_k \mathcal{K}_{B;k}^{\mathrm{d}}(t,s') ds'. \tag{5.4.6}$$

Lemma 5.4.3. *Pick a sufficiently large constant $K_F > 0$. Then for any $T > 0$, any pair of processes (V, F) that satisfies (hV) and (hF) and any $t \in [0,T]$, we have the bound*

$$\mathcal{K}_F(t,s') \le K_F e^{-\varepsilon(t-s')} \|V(s')\|_{L^2} \|F(s')\|_{L^2} \tag{5.4.7}$$

for all $0 \le s' \le t$.

Proof. Observe that $\mathcal{K}_F(t,s')$ is identical to (5.3.38) after making the replacement $B(s')\sqrt{Q}e_k \mapsto F(s')$. We can hence use the same decomposition as in §5.3.2 and follow the proof of Lemma 5.3.10 to obtain the stated bound. □

Lemma 5.4.4. *Pick a sufficiently large constant* $K_B^{\mathrm{d}} > 0$. *Then for any* $T > 0$, *any process* B *that satisfies (hB) and any* $t \in [0, T]$, *we have the bound*

$$\sum_k \mathcal{K}_{B;k}^{\mathrm{d}}(t, s') \quad \leq \quad K_B^{\mathrm{d}}\, e^{-\varepsilon(t-s')} \|B(s')\|_{HS}^2 \tag{5.4.8}$$

for all $0 \leq s' \leq t$.

Proof. Observe that $\mathcal{K}_{B;k}^{\mathrm{d}}(t, s')$ is identical to (5.3.38) after making the replacement $V(s') \mapsto B(s')\sqrt{Q}e_k$. We can hence use the same decomposition as in §5.3.2 and follow the proof of Lemma 5.3.10 to obtain the stated bound. □

Proof of Proposition 5.4.1. Combining the identity (5.4.6) with the bound (5.4.7), we readily obtain the pathwise bound

$$|\mathcal{I}_F(t)| \leq K_F \Lambda_* \int_0^t e^{-\varepsilon(t-s)} \|F(s)\|_{L^2} ds. \tag{5.4.9}$$

The result hence follows by taking the expectation of the supremum. □

Proof of Proposition 5.4.2. Combining the identity (5.4.6) with the estimate (5.4.8), we readily obtain the pathwise bound

$$\left|\mathcal{I}_B^{\mathrm{d}}(t)\right| \leq K_B^{\mathrm{d}}\, \Theta_*^2, \tag{5.4.10}$$

which of course survives taking the expectation of the supremum. □

5.5 Nonlinear stability

With the results from the previous sections under our belt, we now set out to prove the estimates in Theorem 5.1.1 and hence establish the stochastic stability of the travelling wave on exponentially long timescales. Our starting point will be the computations in Chapters 2 and 4, which use a time transformation to construct a mild integral equation for the perturbation $V(t)$ that contains no dangerous second order derivatives.

The arguments in Proposition 2.6.4 and §5.5 indicate that this time transformation only affects the constants in the final estimate (5.1.10). For presentation purposes, we therefore simply reuse t for the transformed time and leave the definition (5.1.9) for the stopping time $t_{\mathrm{st}}(\eta)$ intact. The mild representation for V can now be written in the generic form

$$V(t) = V(0) + \int_0^t S(t-s)[\sigma^2 F_{\mathrm{lin}}\big(V(s)\big) + F_{\mathrm{nl}}\big(V(s)\big)]ds + \sigma \int_0^t S(t-s)B\big(V(s)\big)dW_s^Q, \tag{5.5.1}$$

where the maps

$$F_{\mathrm{lin}} : H^1 \to L^2, \quad F_{\mathrm{nl}} : H^1 \to L^2, \quad B : H^1 \to HS(L_Q^2, L^2) \tag{5.5.2}$$

satisfy the bounds

$$\begin{aligned} \|F_{\rm lin}(v)\|_{L^2} &\leq K_{\rm lin}\|v\|_{H^1}, \\ \|F_{\rm nl}(v)\|_{L^2} &\leq K_{\rm nl}\|v\|_{H^1}^2(1+\|v\|_{L^2}^3), \\ \|B(v)\|_{HS} &\leq K_B(1+\|v\|_{H^1}), \\ \|S(1)B(v)\|_{HS} &\leq K_B M(1+\|v\|_{L^2}). \end{aligned} \tag{5.5.3}$$

In addition, whenever $\|v\|_{L^2}$ is sufficiently small, we have the identities

$$\langle \sigma^2 F_{\rm lin}(v) + F_{\rm nl}(v), \psi_{\rm tw}\rangle_{L^2} = 0, \qquad \langle B(v)[w], \psi_{\rm tw}\rangle_{L^2} = 0 \tag{5.5.4}$$

for every $w \in L^2_Q$. Notice that $v = V(t)$ automatically satisfies this condition for $t \leq t_{\rm st}(\eta)$ provided that $\eta < \eta_0$ for some sufficiently small η_0.

In order to state the main result of this section, we write

$$N(t) = \|V(t)\|_{L^2}^2 + \int_0^t e^{-\varepsilon(t-s)}\|V(s)\|_{H^1}^2\, ds \tag{5.5.5}$$

for the size of the solution V, which also features in the definition (5.1.9) for the stopping time $t_{\rm st} = t_{\rm st}(\eta)$. The various supremum bounds derived in §5.3 and §5.4 can now be used to obtain a similar bound for $N(t)$. This result can be seen as a significantly sharpened version of its counterpart Proposition 2.9.1, which allows Theorem 5.1.1 to be established in a standard fashion.

Proposition 5.5.1. *Pick a constant $0 < \varepsilon < \beta$, together with two sufficiently small constants $\delta_\eta > 0$ and $\delta_\sigma > 0$. Then there exists a constant $K > 0$ so that for any integer $T \geq 2$, any $0 < \eta < \delta_\eta$ and any $0 \leq \sigma \leq \delta_\sigma \ln(T)^{-1/2}$ we have the bound*

$$E\left[\sup_{0\leq t\leq t_{\rm st}} N(t)\right] \leq K\left[\|V(0)\|_{L^2}^2 + \sigma^2\ln(T) + \sigma\sqrt{\eta}\sqrt{\ln(T)}\right]. \tag{5.5.6}$$

Proof of Theorem 5.1.1. The arguments in Corollary 2.9.4 and the proof of Theorem 2.2.4 can be used almost verbatim to derive (5.1.10) from (5.5.6). □

5.5.1 Supremum bounds in L^2

In this subsection we establish the following bound on the supremum of the L^2-norm of $V(t)$. Notice that we are imposing less restrictions on σ and η here, but $N(t)$ still appears on the right-hand side of our estimate.

Lemma 5.5.2. *Pick a constant $0 < \varepsilon < \beta$. Then there exists a constant $K > 0$ so that for any integer $T \geq 2$, any $0 < \eta < \eta_0$ and any $0 \leq \sigma \leq 1$ we have the bound*

$$E \sup_{0\leq t\leq t_{\rm st}} \|V(t)\|_{L^2}^2 \leq K\Big[\|V(0)\|_{L^2}^2 + \sigma^2\ln(T) + (\sigma^4+\eta)E\sup_{0\leq t\leq t_{\rm st}} N(t)\Big]. \tag{5.5.7}$$

In order to streamline the proof, we recall the definition of Π from Lemma 5.2.1 and define the functions

$$\begin{aligned}\mathcal{E}_0(t) =&S(t)V(0),\\ \mathcal{E}_{\text{lin}}(t) =& \int_0^t S(t-s)\Pi F_{\text{lin}}\big(V(s)\big)\mathbf{1}_{s\le t_{\text{st}}}ds,\\ \mathcal{E}_{\text{nl}}(t) =& \int_0^t S(t-s)\Pi F_{\text{nl}}\big(V(s)\big)\mathbf{1}_{s\le t_{\text{st}}}ds,\\ \mathcal{E}_B(t) =& \int_0^t S(t-s)B\big(V(s)\big)\mathbf{1}_{s\le t_{\text{st}}}dW^Q_s.\end{aligned} \tag{5.5.8}$$

The three deterministic expressions can be controlled in a direction fashion, while the final stochastic integral was analysed in §5.4.

Lemma 5.5.3. *For any $0 < \eta < \eta_0$, any $0 \le \sigma \le 1$ and any $T > 0$, we have the bounds*

$$\begin{aligned} E \sup_{0\le t\le T} \|\mathcal{E}_0(t)\|^2_{L^2} &\le M^2\|V(0)\|^2_{L^2},\\ E \sup_{0\le t\le T} \|\mathcal{E}_{\text{lin}}(t)\|^2_{L^2} &\le M^2K^2_{\text{lin}}E \sup_{0\le t\le t_{\text{st}}} \|V(t)\|^2_{L^2},\\ E \sup_{0\le t\le T} \|\mathcal{E}_{\text{nl}}(t)\|^2_{L^2} &\le M^2K^2_{\text{nl}}(1+\eta^3)^2\eta E \sup_{0\le t\le t_{\text{st}}} \int_0^t e^{-\varepsilon(t-s)}\|V(t)\|^2_{H^1}ds.\end{aligned} \tag{5.5.9}$$

Proof. These results follow directly from Lemmas 2.9.8-2.9.11, where they were established using straightforward direct norm estimates. □

Lemma 5.5.4. *Pick a constant $0 < \varepsilon < \beta$. Then there is a $K > 0$ such that the bound*

$$E \sup_{0\le t\le T} \|\mathcal{E}_B(t)\|^2_{L^2} \le K \ln(T) \tag{5.5.10}$$

holds for any integer $T \ge 2$, any $0 < \eta < \eta_0$ and any $0 \le \sigma \le 1$.

Proof. We will prove this by appealing to Proposition 5.3.1. In order to verify (hB), we simply compute

$$\begin{aligned}\int_0^t e^{-\varepsilon(t-s)}\|B\big(V(s)\big)\mathbf{1}_{s\le t_{\text{st}}}\|^2_{HS}\,ds \le& K_B^2 \int_0^t e^{-\varepsilon(t-s)}(1+\|V(s)\|^2_{H^1})\mathbf{1}_{s\le t_{\text{st}}}\,ds\\ \le& K_B^2\left(\varepsilon^{-1} + \int_0^{\min\{t,t_{\text{st}}\}} e^{-\varepsilon(t-s)}\|V(s)\|^2_{H^1}\,ds\right)\\ \le& K_B^2(\varepsilon^{-1}+\eta),\end{aligned} \tag{5.5.11}$$

together with

$$\|S(1)B\big(V(s)\big)\mathbf{1}_{s\le t_{\text{st}}}\|^2_{HS} \le M^2K_B^2(1+\|V(s)\mathbf{1}_{s\le t_{\text{st}}}\|^2_{L^2}) \le M^2K_B^2(1+\eta), \tag{5.5.12}$$

which allows us to take $\Theta_*^2 = M^2K_B^2(\varepsilon^{-1}+\eta)$. □

Proof of Lemma 5.5.2. We directly find that

$$E \sup_{0 \le t \le t_{\mathrm{st}}} \|V(t)\|_{L^2}^2 \le 4E \sup_{0 \le t \le T} \left[\|\mathcal{E}_0(t)\|_{L^2}^2 + \sigma^4 \|\mathcal{E}_{\mathrm{lin}}(t)\|_{L^2}^2 + \|\mathcal{E}_{\mathrm{nl}}(t)\|_{L^2}^2 + \sigma^2 \|\mathcal{E}_B(t)\|_{L^2}^2 \right]. \tag{5.5.13}$$

Collecting the results from Lemmas 5.5.3 and 5.5.4 now proves the result. □

5.5.2 Supremum bounds in H^1

In this subsection we control the H^1-norm of V by establishing a supremum bound for the integrated expression

$$\mathcal{I}(t) = \int_0^t e^{-\varepsilon(t-s)} \|V(s)\|_{H^1}^2 ds. \tag{5.5.14}$$

In particular, we set out to obtain the following estimate.

Lemma 5.5.5. *Pick a constant $0 < \varepsilon < \beta$. Then there exists a constant $K > 0$ so that for any integer $T \ge 2$, any $0 < \eta < \eta_0$ and any $0 \le \sigma \le 1$ we have the bound*

$$E \sup_{0 \le t \le T} \mathcal{I}(t) \le K \Big[\|V(0)\|_{H^1}^2 + \eta\sigma^2 + \sqrt{\eta} E \big[\sup_{0 \le t \le t_{\mathrm{st}}} N(t) \big] + \sigma^2 + \sigma\sqrt{\eta}\sqrt{\ln(T)} \Big]. \tag{5.5.15}$$

Compared to §5.5.1 and §2.9, our approach here is rather indirect. First of all, we exploit the fact that T is an integer to compute

$$\begin{aligned} \sup_{0 \le t \le T} \mathcal{I}(t) &= \max_{i \in \{1,\ldots,T\}} \sup_{i-1 \le t \le i} \int_0^t e^{-\varepsilon(t-s)} \|V(s)\|_{H^1}^2 ds \\ &\le \max_{i \in \{1,\ldots,T\}} e^{\varepsilon} \int_0^i e^{-\varepsilon(i-s)} \|V(s)\|_{H^1}^2 ds \\ &= e^{\varepsilon} \max_{i \in \{1,\ldots,T\}} \mathcal{I}(i). \end{aligned} \tag{5.5.16}$$

We continue by applying a mild Itô formula [27] to $\|V(s)\|_{H^1}^2$, which yields

$$\begin{aligned} \|V(s)\|_{H^1}^2 =& \|S(s)V(0)\|_{H^1}^2 + 2\sigma^2 \int_0^s \langle S(s-s')V(s'), S(s-s')F_{\mathrm{lin}}(V(s')) \rangle_{H^1} ds' \\ &+ 2 \int_0^s \langle S(s-s')V(s'), S(s-s')F_{\mathrm{nl}}\big(V(s')\big) \rangle_{H^1} ds' \\ &+ 2\sigma \int_0^s \langle S(s-s')V(s'), S(s-s')B\big(V(s')\big) dW_{s'}^Q \rangle_{H^1} \\ &+ \sigma^2 \int_0^s \|S(s-s')B\big(V(s')\big)\|_{HS(L_Q^2, H^1)}^2 ds'. \end{aligned} \tag{5.5.17}$$

In particular, upon introducing the components

$$
\begin{aligned}
\mathcal{I}_0(t) &= \int_0^t e^{-\varepsilon(t-s)} \|S(s)V(0)\|_{H^1}^2 ds, \\
\mathcal{I}_{\rm lin}(t) &= \int_0^t e^{-\varepsilon(t-s)} \int_0^s \langle S(s-s')V(s'), S(s-s')\Pi F_{\rm lin}\big(V(s')\big)\mathbf{1}_{s'\le t_{\rm st}}\rangle_{H^1} ds' ds, \\
\mathcal{I}_{\rm nl}(t) &= \int_0^t e^{-\varepsilon(t-s)} \int_0^s \langle S(s-s')V(s'), S(s-s')\Pi F_{\rm nl}\big(V(s')\big)\mathbf{1}_{s'\le t_{\rm st}}\rangle_{H^1} ds' ds, \\
\mathcal{I}_B^{\rm s}(t) &= \int_0^i e^{-\varepsilon(t-s)} \int_0^s \langle S(s-s')V(s'), S(s-s')B\big(V(s')\big)\mathbf{1}_{s'\le t_{\rm st}} dW_{s'}^Q\rangle_{H^1} ds, \\
\mathcal{I}_B^{\rm d}(t) &= \int_0^i e^{-\varepsilon(t-s)} \int_0^s \|S(s-s')B\big(V(s')\big)\mathbf{1}_{s'\le t_{\rm st}}\|_{HS(L_Q^2,H^1)}^2 ds' ds,
\end{aligned}
\tag{5.5.18}
$$

we obtain the bound

$$
\begin{aligned}
E \sup_{0\le t\le t_{\rm st}} \mathcal{I}(t) &\le e^{\varepsilon} E \max_{i\in\{1,..,T\}} \big[\mathcal{I}_0(i) + 2\sigma^2\mathcal{I}_{\rm lin}(i) + 2\mathcal{I}_{\rm nl}(i) + 2\sigma\mathcal{I}_B^{\rm s}(i) + \sigma^2\mathcal{I}_B^{\rm d}(i)\big] \\
&\le e^{\varepsilon} E \sup_{0\le t\le T} \big[\mathcal{I}_0(t) + 2\sigma^2\mathcal{I}_{\rm lin}(t) + 2\mathcal{I}_{\rm nl}(t) + \sigma^2\mathcal{I}_B^{\rm d}(t)\big] + 2e^{\varepsilon}\sigma E \max_{i\in\{1,..,T\}} \mathcal{I}_B^{\rm s}(i).
\end{aligned}
\tag{5.5.19}
$$

This decomposition highlights the fact that supremum bounds over deterministic integrals are easily obtained, while the stochastic integral needs to be handled with care.

Lemma 5.5.6. *Pick a constant* $0 < \varepsilon < \beta$*. Then there exists a constant* $K > 0$ *so that for any integer* $T \ge 2$*, any* $0 < \eta < \eta_0$ *and any* $0 \le \sigma \le 1$ *we have the bounds*

$$
\begin{aligned}
&E \sup_{0\le t\le T} \mathcal{I}_{\rm lin}(t) \le K\eta, \\
&E \sup_{0\le t\le T} \mathcal{I}_{\rm nl}(t) \le K\sqrt{\eta} E \sup_{0\le t\le T} \int_0^t e^{-\varepsilon(t-s)} \|V(s)\|_{H^1}^2 \mathbf{1}_{s\le t_{\rm st}} ds.
\end{aligned}
\tag{5.5.20}
$$

Proof. In order to exploit Proposition 5.4.1, we first note that the orthogonality conditions (5.3.6) and (5.4.2) hold true by virtue of the stopping time. In particular, (hF) and (hV) are both satisfied, with $\Lambda_* = \sqrt{\eta}$. The stated bounds can hence be obtained by using the computation

$$
\begin{aligned}
\int_0^t e^{-\varepsilon(t-s)} \|F_{\rm lin}(V(s))\mathbf{1}_{s\le t_{\rm st}}\|_{L^2} ds &\le K_{\rm lin} \int_0^t e^{-\varepsilon(t-s)} \|V(s)\|_{H^1} \mathbf{1}_{s\le t_{\rm st}} ds \\
&\le K_{\rm lin} \frac{1}{\sqrt{\varepsilon}} \sqrt{\int_0^t e^{-\varepsilon(t-s)} \|V(s)\|_{H^1}^2 \mathbf{1}_{s\le t_{\rm st}} ds} \qquad (5.5.21) \\
&\le K_{\rm lin} \sqrt{\frac{\eta}{\varepsilon}},
\end{aligned}
$$

together with

$$\begin{aligned}\int_0^t e^{-\varepsilon(t-s)}\|F_{\mathrm{nl}}(V(s))\mathbf{1}_{s\le t_{\mathrm{st}}}\|_{L^2}ds \le& K_{\mathrm{nl}}\int_0^t e^{-\varepsilon(t-s)}\|V(s)\|_{H^1}^2(1+\|V(s)\|_{L^2}^3)\mathbf{1}_{s\le t_{\mathrm{st}}}ds\\ \le& K_{\mathrm{nl}}(1+\eta^3)\int_0^t e^{-\varepsilon(t-s)}\|V(s)\|_{H^1}^2\mathbf{1}_{s\le t_{\mathrm{st}}}ds\end{aligned} \tag{5.5.22}$$

to evaluate the right-hand side of (5.4.3). □

Lemma 5.5.7. *Pick a constant* $0<\varepsilon<\beta$. *Then there exists a constant* $K>0$ *so that for any integer* $T\ge 2$, *any* $0<\eta<\eta_0$ *and any* $0\le\sigma\le 1$ *we have the bounds*

$$\begin{aligned}E\sup_{0\le t\le T}\mathcal{I}_B^{\mathrm{d}}(t)&\le K,\\ E\max_{i\in\{1,\dots,T\}}\mathcal{I}_B^{\mathrm{s}}(t)&\le K\sqrt{\eta}\sqrt{\ln(T)}.\end{aligned} \tag{5.5.23}$$

Proof. Recall from the proof of Lemma 5.5.2 that (hB) holds with $\Theta_*^2=M^2K_B^2(\varepsilon^{-1}+\eta)$. The first estimate now follows directly from Proposition 5.4.2, while the second can be obtained from Proposition 5.3.1 using the fact that (hV) is satisfied with $\Lambda_*=\sqrt{\eta}$. □

Proof of Lemma 5.5.5. The bound follows immediately from the decomposition (5.5.19) and Lemmas 5.5.6-5.5.7. □

Proof of Proposition 5.5.1. Summing the estimates from Lemmas 5.5.2 and 5.5.5 yields the bound

$$E\sup_{0\le t\le t_{\mathrm{st}}}N(t)\le K\Big[\|V(0)\|_{L^2}^2+\sigma^2\ln(T)+\sigma^2+\sigma\sqrt{\eta}\sqrt{\ln(T)}+(\sigma^4+\eta+\sqrt{\eta})E\sup_{0\le t\le t_{\mathrm{st}}}N(t)\Big]. \tag{5.5.24}$$

Upon restricting the size of $\sigma^4+\eta+\sqrt{\eta}$, the result readily follows. □

Bibliography

[1] J. Alexander, R. Gardner and C. K. R. T. Jones (1990), A topological invariant arising in the stability analysis of travelling waves. *J. reine angew. Math* **410**(167-212), 143.

[2] L. Alili, P. Patie and J. L. Pedersen (2005), Representations of the first hitting time density of an Ornstein-Uhlenbeck process. *Stochastic Models* **21**(4), 967–980.

[3] J. Armero, J. M. Sancho, J. Casademunt, A. M. Lacasta, L. Ramirez-Piscina, and F. Sagués (1996), External fluctuations in front propagation. *Physical review letters* **76**(17), 3045.

[4] D. G. Aronson and H. F. Weinberger (1975), Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation. In: *Partial differential equations and related topics.* Springer, pp. 5–49.

[5] D. G. Aronson and H. F. Weinberger (1978), Multidimensional nonlinear diffusion arising in population genetics. *Adv. in Math.* **30**(1), 33–76.

[6] M. Beck, H. J. Hupkes, B. Sandstede and K. Zumbrun (2010), Nonlinear Stability of Semidiscrete Shocks for Two-Sided Schemes. *SIAM J. Math. Anal.* **42**, 857–903.

[7] M. Beck, B. Sandstede and K. Zumbrun (2010), Nonlinear stability of time-periodic viscous shocks. *Archive for rational mechanics and analysis* **196**(3), 1011–1076.

[8] H. Berestycki, F. Hamel and H. Matano (2009), Bistable traveling waves around an obstacle. *Comm. Pure Appl. Math.* **62**(6), 729–788.

[9] N. Berglund and B. Gentz (2013), Sharp estimates for metastable lifetimes in parabolic SPDEs: Kramers' law and beyond. *Electron. J. Probab.* **18**, 58 pp.

[10] E. Bernitt and H.-G. Döbereiner (2017), Spatiotemporal Patterns of Noise-Driven Confined Actin Waves in Living Cells. *Physical review letters* **118**(4), 048102.

[11] W.-J. Beyn and V. Thümmler (2004), Freezing solutions of equivariant evolution equations. *SIAM Journal on Applied Dynamical Systems* **3**(2), 85–116.

[12] L. A. Bianchi, D. Blömker and P. Wacker (2017), Pattern size in Gaussian fields from spinodal decomposition. *SIAM Journal on Applied Mathematics* **77**(4), 1292–1319.

[13] G. Birzu, O. Hallatschek and K. S. Korolev (2018), Fluctuations uncover a distinct class of traveling waves. *Proceedings of the National Academy of Sciences* **115**(16), E3645–E3654.

[14] S. Brassesco, A. De Masi and E. Presutti (1995), Brownian fluctuations of the interface in the D=1 Ginzburg-Landau equation with noise. *Ann. Inst. H. Poincaré Probab. Statist* **31**(1), 81–118.

[15] P. C. Bressloff and Z. P. Kilpatrick (2015), Nonlinear Langevin equations for wandering patterns in stochastic neural fields. *SIAM Journal on Applied Dynamical Systems* **14**(1), 305–334.

[16] P. C. Bressloff and M. A. Webber (2012), Front propagation in stochastic neural fields. *SIAM Journal on Applied Dynamical Systems* **11**(2), 708–740.

[17] E. Brunet and B. Derrida (1997), Shift in the velocity of a front due to a cutoff. *Physical Review E* **56**(3), 2597.

[18] G. A. Carpenter (1977), A geometric approach to singular perturbation problems with applications to nerve impulse equations. *Journal of Differential Equations* **23**(3), 335–367.

[19] M. Cartwright and G. A. Gottwald (2019), A collective coordinate framework to study the dynamics of travelling waves in stochastic partial differential equations. *Physica D: Nonlinear Phenomena.*

[20] S. Cerrai (2005), Stabilization by noise for a class of stochastic reaction-diffusion equations. *Probability theory and related fields* **133**(2), 190–214.

[21] S. Cerrai and M. Röckner (2004), Large deviations for stochastic reaction-diffusion systems with multiplicative noise and non-Lipshitz reaction term. *The Annals of Probability* **32**(1B), 1100–1139.

[22] C.-N. Chen and Y. S. Choi (2015), Traveling pulse solutions to FitzHugh–Nagumo equations. *Calculus of Variations and Partial Differential Equations* **54**(1), 1–45.

[23] P.-L. Chow (2014), *Stochastic partial differential equations.* CRC Press.

[24] P. Cornwell (2017), Opening the Maslov Box for Traveling Waves in Skew-Gradient Systems. *arXiv preprint arXiv:1709.01908.*

[25] P. Cornwell and C. K. R. T. Jones (2018), On the Existence and Stability of Fast Traveling Waves in a Doubly Diffusive FitzHugh–Nagumo System. *SIAM Journal on Applied Dynamical Systems* **17**(1), 754–787.

[26] G. Da Prato, A. Jentzen and M. Röckner (2010), A mild Ito formula for SPDEs. *arXiv preprint arXiv:1009.3526.*

[27] G. Da Prato, A. Jentzen and M. Röckner (2019), A mild Itô formula for SPDEs. *Transactions of the American Mathematical Society.*

[28] G. Da Prato, S. Kwapień and J. Zabczyk (1988), Regularity of solutions of linear stochastic equations in Hilbert spaces. *Stochastics: An International Journal of Probability and Stochastic Processes* **23**(1), 1–23.

[29] R. C. Dalang and L. Quer-Sardanyons (2011), Stochastic integrals for spde's: a comparison. *Expositiones Mathematicae* **29**(1), 67–109.

[30] G. di Nunno and B. Oksendal (eds.) (2011), *Advanced Mathematical Methods for Finance.* Springer.

[31] K. Eichinger, M. V. Gnann and C. Kuehn (2020), Multiscale analysis for traveling-pulse solutions to the stochastic FitzHugh-Nagumo equations. *arXiv preprint arXiv:2002.07234.*

[32] L. C. Evans (1998), *Partial differential equations.* American Mathematical Society, Providence, R.I.

[33] L. C. Evans (2012), *An introduction to stochastic differential equations*, Vol. 82. American Mathematical Soc.

[34] P. C. Fife and J. B. McLeod (1977), The approach of solutions of nonlinear diffusion equations to travelling front solutions. *Arch. Ration. Mech. Anal.* **65**(4), 335–361.

[35] C. L. E. Franzke, T. J. O'Kane, J. Berner, P. D. Williams and V. Lucarini (2015), Stochastic climate theory and modeling. *Wiley Interdisciplinary Reviews: Climate Change* **6**(1), 63–78.

[36] M. I. Freidlin and A. D. Wentzell (1998), Random perturbations. In: *Random perturbations of dynamical systems.* Springer, pp. 15–43.

[37] T. Funaki (1995), The scaling limit for a stochastic PDE and the separation of phases. *Probability Theory and Related Fields* **102**(2), 221–288.

[38] J. García-Ojalvo, F. Sagués, J. M. Sancho and L. Schimansky-Geier (2001), Noise-enhanced excitability in bistable activator-inhibitor media. *Physical Review E* **65**(1), 011105.

[39] J. García-Ojalvo and J. Sancho (2012), *Noise in spatially extended systems.* Springer Science & Business Media.

[40] R. A. Gardner (1982), Existence and stability of travelling wave solutions of competition models: a degree theoretic approach. *Journal of Differential equations* **44**(3), 343–364.

[41] E. Gautier (2005), Uniform large deviations for the nonlinear Schrödinger equation with multiplicative noise. *Stochastic processes and their applications* **115**(12), 1904–1927.

[42] L. Gawarecki and V. Mandrekar (2010), *Stochastic differential equations in infinite dimensions: with applications to stochastic partial differential equations.* Springer Science & Business Media.

[43] K. Gowda and C. Kuehn (2015), Early-warning signs for pattern-formation in stochastic partial differential equations. *Communications in Nonlinear Science and Numerical Simulation* **22**(1), 55–69.

[44] M. Hairer (2009), An Introduction to Stochastic PDEs. `http://www.hairer.org/notes/SPDEs.pdf`.

[45] M. Hairer and É. Pardoux (2015), A Wong-Zakai theorem for stochastic PDEs. *Journal of the Mathematical Society of Japan* **67**(4), 1551–1604.

[46] J. Hale, L. A. Peletier and W. C. Troy (1999), Stability and instability in the Gray-Scott model: the case of equal diffusivities. *Applied mathematics letters* **12**(4), 59–65.

[47] C. H. S. Hamster and H. J. Hupkes (2018), Stability of Travelling Waves for Systems of Reaction-Diffusion Equations with Multiplicative Noise. *To Appear in SIAM Journal on Mathematical Analysis.*

[48] C. H. S. Hamster and H. J. Hupkes (2019), Stability of Traveling Waves for Reaction-Diffusion Equations with Multiplicative Noise. *SIAM Journal on Applied Dynamical Systems* **18**(1), 205–278.

[49] C. H. S. Hamster and H. J. Hupkes (2020), Stability of Travelling Waves on Exponentially Long Timescales in Stochastic Reaction-Diffusion Equations. *Preprint.*

[50] C. H. S. Hamster and H. J. Hupkes (2020), Travelling waves for reaction–diffusion equations forced by translation invariant noise. *Physica D: Nonlinear Phenomena* **401**, 132233.

[51] S. P. Hastings (1976), On travelling wave solutions of the Hodgkin-Huxley equations. *Archive for Rational Mechanics and Analysis* **60**(3), 229–257.

[52] E. Hausenblas, T. A. Randrianasolo and M. Thalhammer (2020), Theoretical study and numerical simulation of pattern formation in the deterministic and stochastic Gray–Scott equations. *Journal of Computational and Applied Mathematics* **364**, 112335.

[53] A. Hoffman, H. J. Hupkes and E. Van Vleck (2015), Multi-dimensional stability of waves travelling through rectangular lattices in rational directions. *Transactions of the American Mathematical Society* **367**(12), 8757–8808.

[54] A. Hoffman, H. J. Hupkes and E. Van Vleck (2017), Entire solutions for bistable lattice differential equations with obstacles. *American Mathematical Society* **250**(1188).

[55] T. Hytönen, J. Van Neerven, M. Veraar and L. Weis (2018), *Analysis in Banach Spaces: Volume II: Probabilistic Methods and Operator Theory*, Vol. 67. Springer.

[56] M. Iannelli and A. Pugliese (2015), *An Introduction to Mathematical Population Dynamics: Along the Trail of Volterra and Lotka*, Vol. 79. Springer.

[57] J. Inglis and J. MacLaurin (2016), A general framework for stochastic traveling waves and patterns, with application to neural field equations. *SIAM Journal on Applied Dynamical Systems* **15**(1), 195–234.

[58] J. Jacod (2006), *Calcul stochastique et problemes de martingales*, Vol. 714. Springer.

[59] M. Jeanblanc, M. Yor and M. Chesney (2009), *Mathematical methods for financial markets.* Springer Science & Business Media.

[60] C. K. R. T. Jones (1984), Stability of the travelling wave solution of the FitzHugh-Nagumo system. *Transactions of the American Mathematical Society* **286**(2), 431–469.

[61] C. K. R. T. Jones (1995), Geometric singular perturbation theory. In: *Dynamical systems.* Springer, pp. 44–118.

[62] C. K. R. T. Jones, N. Kopell and R. Langer (1991), Construction of the FitzHugh-Nagumo pulse using differential forms. In: *Patterns and dynamics in reactive media.* Springer, pp. 101–115.

[63] S. Kadar, J. Wang and K. Showalter (1998), Noise-supported travelling waves in sub-excitable media. *Nature* **391**(6669), 770.

[64] T. Kapitula (1997), Multidimensional Stability of Planar Travelling Waves. *Trans. Amer. Math. Soc.* **349**, 257–269.

[65] T. Kapitula and K. Promislow (2013), *Spectral and dynamical stability of nonlinear waves.* Springer.

[66] A. Karczewska (2005), Stochastic integral with respect to cylindrical Wiener process. *arXiv preprint math/0511512.*

[67] C. Knoche and K. Frieler (2001), Solutions of stochastic differential equations in infinite dimensional Hilbert spaces and their dependence on initial data. Diplomarbeit, BiBoS-Preprint E02-04-083, Bielefeld University.

[68] J. Krüger and W. Stannat (2017), A multiscale-analysis of stochastic bistable reaction–diffusion equations. *Nonlinear Analysis* **162**, 197–223.

[69] C. Kuehn (2019), Travelling Waves in Monostable and Bistable Stochastic Partial Differential Equations. *Jahresbericht der Deutschen Mathematiker-Vereinigung* pp. 1–35.

[70] C. Kuehn and J. M. Tölle (2019), A gradient flow formulation for the stochastic Amari neural field model. *Journal of Mathematical Biology* **79**(4), 1227–1252.

[71] R. Kuske, C. Lee and V. Rottschäfer (2017), Patterns and coherence resonance in the stochastic Swift-Hohenberg equation with Pyragas control: The Turing bifurcation case. *Physica D: Nonlinear Phenomena.*

[72] E. Lang (2016), A multiscale analysis of traveling waves in stochastic neural fields. *SIAM Journal on Applied Dynamical Systems* **15**(3), 1581–1614.

[73] E. Lang (2016), Traveling waves in stochastic neural fields. Ph.D. thesis, TU Berlin.

[74] E. Lang and W. Stannat (2016), L^2-stability of traveling wave solutions to nonlocal evolution equations. *Journal of Differential Equations* **261**(8), 4275–4297.

[75] M. R. Leadbetter, G. Lindgren and H. Rootzén (2012), *Extremes and related properties of random sequences and processes*. Springer Science & Business Media.

[76] K.-J. Lee, W. D. McCormick, J. E. Pearson and H. L. Swinney (1994), Experimental observation of self-replicating spots in a reaction–diffusion system. *Nature* **369**(6477), 215.

[77] W. Liu and M. Röckner (2010), SPDE in Hilbert space with locally monotone coefficients. *Journal of Functional Analysis* **259**(11), 2902–2922.

[78] G. J. Lord, C. E. Powell and T. Shardlow (2014), *An introduction to computational stochastic PDEs*. Cambridge University Press.

[79] G. J. Lord and V. Thümmler (2012), Computing stochastic traveling waves. *SIAM Journal on Scientific Computing* **34**(1), B24–B43.

[80] L. Lorenzi, A. Lunardi, G. Metafune and D. Pallara (2004), Analytic semigroups and reaction-diffusion problems. In: *Internet Seminar*, Vol. 2005. p. 127.

[81] J. N. MacLaurin and P. C. Bressloff (2020), Wandering bumps in a stochastic neural field: A variational approach. *Physica D: Nonlinear Phenomena* p. 132403.

[82] C. Mascia and K. Zumbrun (2002), Pointwise Green's function bounds and stability of relaxation shocks. *Indiana Univ. Math. J.* **51**(4), 773–904.

[83] H. Matano, Y. Mori and M. Nara (2019), Asymptotic behavior of spreading fronts in the anisotropic Allen–Cahn equation on $\mathbb{R}^n$. *Annales de l'Institut Henri Poincaré C, Analyse non linéaire* **36**(3), 585 – 626.

[84] G. F. Mazenko, O. T. Valls and P. Ruggiero (1989), Front propagation into an unstable state in the presence of noise. *Physical Review B* **40**(1), 384.

[85] C. Mueller, L. Mytnik and J. Quastel (2011), Effect of noise on front propagation in reaction-diffusion equations of KPP type. *Inventiones mathematicae* **184**(2), 405–453.

[86] C. Mueller and R. B. Sowers (1995), Random travelling waves for the KPP equation with noise. *Journal of Functional Analysis* **128**(2), 439–498.

[87] A. G. Nobile, L. M. Ricciardi and L. Sacerdote (1985), Exponential trends of Ornstein–Uhlenbeck first-passage-time densities. *Journal of Applied Probability* **22**(2), 360–369.

[88] B. Oksendal (2013), *Stochastic differential equations: an introduction with applications.* Springer Science & Business Media.

[89] M. Or-Guil, M. Bode, C. P. Schenk and H. G. Purwins (1998), Spot Bifurcations in Three-Component Reaction-Diffusion Systems: The Onset of Propagation. *Physical Review E* **57**(6), 6432.

[90] S. Peszat and J. Zabczyk (1997), Stochastic evolution equations with a spatially homogeneous Wiener process. *Stochastic Processes and their Applications* **72**(2), 187–204.

[91] J. Pickands (1969), Asymptotic properties of the maximum in a stationary Gaussian process. *Transactions of the American Mathematical Society* **145**, 75–86.

[92] G. Prato and J. Zabczyk (1992), *Stochastic equations in infinite dimensions.* Cambridge University Press, Cambridge New York.

[93] C. Prévôt and M. Röckner (2007), *A concise course on stochastic partial differential equations*, Vol. 1905. Springer.

[94] M. Reed (1980), *Methods of modern mathematical physics.* Academic Press, New York.

[95] D. Revuz and M. Yor (2013), *Continuous martingales and Brownian motion*, Vol. 293. Springer Science & Business Media.

[96] L. M. Ricciardi and S. Sato (1988), First-passage-time density and moments of the Ornstein-Uhlenbeck process. *Journal of Applied Probability* **25**(1), 43–57.

[97] J. Rinzel and J. B. Keller (1973), Traveling wave solutions of a nerve conduction equation. *Biophysical journal* **13**(12), 1313–1337.

[98] M. Salins and K. Spiliopoulos (2019), Metastability and exit problems for systems of stochastic reaction-diffusion equations. *arXiv preprint arXiv:1903.06038.*

[99] B. Sandstede (2002), Stability of travelling waves. In: *Handbook of dynamical systems*, Vol. 2. Elsevier, pp. 983–1055.

[100] D. H. Sattinger (1976), On the stability of waves of nonlinear parabolic systems. *Advances in Mathematics* **22**(3), 312–355.

[101] C. P. Schenk, M. Or-Guil, M. Bode and H. G. Purwins (1997), Interacting Pulses in Three-component Reaction-Diffusion Systems on Two-Dimensional Domains. *Physical Review Letters* **78**(19), 3781.

[102] L. Schimansky-Geier and C. Zülicke (1991), Kink propagation induced by multiplicative noise. *Zeitschrift für Physik B Condensed Matter* **82**(1), 157–162.

[103] T. Shardlow (2005), Numerical simulation of stochastic PDEs for excitable media. *Journal of computational and applied mathematics* **175**(2), 429–446.

[104] W. Stannat (2013), Stability of travelling waves in stochastic Nagumo equations. *arXiv preprint arXiv:1301.6378.*

[105] W. Stannat (2014), Stability of travelling waves in stochastic bistable reaction-diffusion equations. *arXiv preprint arXiv:1404.3853.*

[106] M. Talagrand (2006), *The generic chaining: upper and lower bounds of stochastic processes.* Springer Science & Business Media.

[107] A. Tonnelier and W. Gerstner (2003), Piecewise linear differential equations and integrate-and-fire neurons: insights from two-dimensional membrane models. *Physical Review E* **67**(2), 021908.

[108] K. Twardowska (1996), Wong-Zakai approximations for stochastic differential equations. *Acta Applicandae Mathematica* **43**(3), 317–359.

[109] P. van Heijster, A. Doelman, T. J. Kaper and K. Promislow (2010), Front interactions in a three-component system. *SIAM Journal on Applied Dynamical Systems* **9**(2), 292–332.

[110] N. Van Kampen (1981), Itô versus Stratonovich. *Journal of Statistical Physics* **24**(1), 175–187.

[111] M. Veraar and L. Weis (2011), A note on maximal estimates for stochastic convolutions. *Czechoslovak mathematical journal* **61**(3), 743.

[112] J. Viñals, E. Hernández-García, M. San Miguel and R. Toral (1991), Numerical study of the dynamical aspects of pattern selection in the stochastic Swift-Hohenberg equation in one dimension. *Physical Review A* **44**(2), 1123.

[113] A. I. Volpert, V. A. Volpert and V. A. Volpert (1994), *Traveling wave solutions of parabolic systems*, Vol. 140. American Mathematical Soc.

[114] L. Weis (2006), The H^∞ holomorphic functional calculus for sectorial operators - a survey. In: *Partial differential equations and functional analysis.* Springer, pp. 263–294.

[115] E. Zemskov and I. Epstein (2010), Wave propagation in a FitzHugh-Nagumo-type model with modified excitability. *Physical Review E* **82**(2), 026207.

[116] J. Q. Zhang, A. V. Holden, O. Monfredi, M. R. Boyett and H. Zhang (2009), Stochastic vagal modulation of cardiac pacemaking may lead to erroneous identification of cardiac "chaos". *Chaos: An Interdisciplinary Journal of Nonlinear Science* **19**(2), 028509.

[117] K. Zumbrun (2011), Instantaneous Shock Location and One-Dimensional Nonlinear Stability of Viscous Shock Waves. *Quarterly of applied mathematics* **69**(1), 177–202.

[118] K. Zumbrun and P. Howard (1998), Pointwise semigroup methods and stability of viscous shock waves. *Indiana University Mathematics Journal* **47**(3), 741–872.

Samenvatting

Wiskundigen mogen graag vertellen dat de kracht van de wiskunde ligt in de abstractie. We strippen elk probleem, ongeacht of het komt uit de natuurkunde, biologie, scheikunde, logistiek, of welke tak van sport dan ook, tot er iets overblijft dat we kunnen vangen in definities, stellingen en bewijzen. De terugkoppeling naar de echte wereld is echter vaak lastig en de meeste wiskundigen zullen dan ook opmerken dat hun werk 'misschien ooit een echte toepassing[1] zal krijgen.' Dit betekent echter niet dat we als wiskundigen niet druk bezig zijn met 'ooit' dichterbij te brengen.

Stochastiek

Als eerste zullen we ons moeten realiseren dat de buitenwereld niet perfect is zoals onze wiskundige modellen, maar dat er juist vaak ruis op de lijn zit. Stel je de volgende situatie voor: je hebt een baksteen in je ene hand, een veer in de andere, en laat ze tegelijkertijd vallen. Allebei vallen ze volgens de wetten van Newton, allebei ondervinden ze dezelfde versnelling door de zwaartekracht (per definitie $1g$), maar het resultaat is totaal anders. De stroming van de lucht heeft nauwelijks invloed op de baksteen, maar des te meer op de veer. Het is dan ook gemakkelijk uit te rekenen waar de baksteen terecht komt (recht onder de plek waar je de steen losliet), maar datzelfde doen voor de veer is praktisch onmogelijk. Daarvoor zouden we de luchtstromen zeer gedetailleerd moeten kennen wat (nog) niet haalbaar is.

We kunnen dus de echte plek waar de veer landt niet bepalen, maar we kunnen onszelf wel de volgende vraag stellen: kunnen we voorspellen waar de veer waarschijnlijk terecht komt? Hiervoor moeten we dus bepalen hoe de lucht de veer beïnvloedt, maar welke kant de lucht de veer op duwt en hoe sterk weten we niet. Daarom zullen we de invloed van de lucht modelleren als een zogenaamde stochastische[2] variabele. Een stochastische variabele ligt, in tegenstelling tot een 'normale' variabele, of deterministische variabele, niet vast. We weten de massa van de veer, we kennen de temperatuur in ons laboratorium en hoewel deze waarden natuurlijk kunnen veranderen (het zijn immers variabelen) kunnen we de waarden op elk tijdstip zo precies mogelijk meten als onze meetapparatuur toelaat. Bij een stochastische variabele is dat anders, je weet alleen welke waarden een stochastische variabele *kan* aannemen.

Een voorbeeld van een stochastische variabele dat iedereen zal kennen is de dobbelsteen: voor elke worp komt er 1, 2, 3, 4, 5 of 6 tevoorschijn, maar je weet van te voren niet welke er boven komt. Wat we wel weten van de (eerlijke) dobbelsteen is de

[1] In de woorden van een meer nihilistisch ingestelde wiskundige: "Een toepassing, het kan de beste overkomen."

[2] Afgeleid van het Oudgriekse $\sigma\tau o\chi\alpha\zeta o\mu\alpha\iota$, *stochadzomai*, gokken.

kansverdeling: de kans dat je een specifiek getal gooit is 1/6. Nu wordt het tijd voor wat wiskunde. Eerst geven we de stochastische variabele 'gooi een dobbelsteen' een naam of symbool, laten we X nemen. Verder introduceren we de letter E voor het nemen van het gemiddelde. De zin 'het gemiddelde van een dobbelsteenworp' kan nu dus compact worden geschreven als $E[X]$. Dit getal kunnen we uitrekenen: er zijn 6 uitkomsten mogelijk die allemaal een gelijke kans hebben van 1/6, dus we vinden

$$E[X] = \frac{\frac{1}{6}\cdot 1 + \frac{1}{6}\cdot 2 + \frac{1}{6}\cdot 3 + \frac{1}{6}\cdot 4 + \frac{1}{6}\cdot 5 + \frac{1}{6}\cdot 6}{6} = 3{,}5.$$

Nota Bene: Je kan dus in een enkele worp nooit het gemiddelde gooien want er staat geen 3,5 op de dobbelsteen.

Terug naar de veer. We behandelen de invloed van de lucht op de veer dus ook als een stochastische variabele en we introduceren hiervoor de notatie[3] $F_L(t)$. We nemen voor het gemak aan dat op elk tijdstip de sterkte en de richting van de kracht willekeurig zijn. Dit betekent dus dat de netto kracht die de lucht uitoefent op de veer 0 is, of in wiskundige notatie:

$$E[F_L(t)] = 0.$$

Het is hier dus essentieel om op te merken dat, alhoewel de gemiddelde bijdrage van de lucht op 0 uitkomt, het effect op een individuele veer duidelijk niet nul is. De vraag is nu: als je het gemiddelde neemt van alle plekken waar de veer landt, is dat wel recht onder de plek waar je hem losliet? Een veer is nooit perfect symmetrisch, dus het zou zomaar kunnen dat een klein duwtje naar links meer effect heeft dan een klein duwtje naar rechts. Dit zou dan als resultaat kunnen hebben dat een kracht met netto sterkte nul, een niet-nul effect heeft op de veer, maar hoe zou je dat kunnen uitrekenen?

Dynamische systemen

Voordat we verder gaan met de stochastiek, keren we weer terug naar de klassieke tweede wet van Newton:

$$\text{kracht} = \text{massa} \times \text{versnelling}.$$

Het is hier belangrijk om op te merken dat de kracht van veel factoren afhangt. Als we opnieuw het voorbeeld van de vallende baksteen nemen, weten we bijvoorbeeld dat de luchtweerstand afhangt van hoe snel de baksteen valt. Hoe sneller de steen valt, hoe groter de luchtweerstand. De luchtweerstand hangt ook af van de dichtheid van de lucht. Als je de steen laat vallen vanuit een vliegtuig, waar de lucht erg ijl is, zal hij sneller vallen dan vanaf een flat. De snelheidsverandering van de steen hangt dus af van een veranderende kracht die weer afhangt van de valsnelheid en de hoogte, die beide weer constant veranderen omdat de steen valt. Kortom, alle parameters in het systeem veranderen, niet alleen door een oorzaak van buitenaf, maar juist ook door de beweging van de steen zelf. De vergelijkingen die dit samenspel beschrijven noemen we een dynamisch systeem.

[3] F voor kracht (force), het subscript L om de afhankelijkheid van de lucht aan te geven (i.t.t. bijvoorbeeld de zwaartekracht) en de t om de afhankelijkheid van de tijd aan te geven.

(a) (b)

Afbeelding 1: *Twee voorbeelden van patronen die kunnen worden verklaard met behulp van RDV's. In afbeelding (a) zien we strepen op de kop van een zebra. In afbeelding (b) zien we cirkelpatronen in een Belousov–Zhabotinsky-reactie. Afbeelding (b) met dank aan Stephen Morris, Toronto, Canada.*

In bijna alle situaties in de natuur waar verandering plaats vindt, of het nu het weer is of het groeien van een plant, kan men beschrijven met zogenoemde dynamische systemen. Slechts in zeldzame simpele gevallen kan je de oplossingen van dit type vergelijkingen met pen en papier uitrekenen. Indien men kwantitatieve resultaten over zo'n systeem wil hebben, wordt er vaak naar de computer gegrepen om het rekenwerk te doen. Je kan hierbij bijvoorbeeld denken aan uitgebreide simulaties voor de dagelijkse weersvoorspellingen. Indien meer kwalitatieve resultaten gewenst zijn, is de hulp van wiskundigen vaak onontbeerlijk. Nog los van het feit dat ook bij computersimulaties wiskundigen onontbeerlijk zijn, omdat de vraag "hoe stop ik vergelijkingen voor het weer in een computer?" niet kan worden beantwoord zonder wiskundigen.

Stochastische reactie-diffusievergelijkingen

In Leiden besteden we veel aandacht aan een specifiek type dynamisch systeem genaamd reactie-diffusievergelijkingen (RDV's). Oorspronkelijk werd dit type vergelijking geïntroduceerd door Alan Turing (die om andere redenen beroemd is geworden) om strepen op zebra's te kunnen verklaren. Bijna zeventig jaar later zijn RDV's hét middel geworden om allerhande patroonvorming in de natuur te begrijpen, variërend van de vorm van golven op het water tot de patronen die optreden tijdens hartritmestoornissen (tachycardia).

Het basisidee van RDV's is als volgt. Stel je een scheikunde-experiment voor waarbij meerdere chemicaliën zijn opgelost in water. De chemicaliën kunnen twee dingen doen: diffunderen en reageren. Denk bij diffunderen aan een druppel inkt in een bak water: ook als je niet roert zal de inkt vanzelf worden verspreid in het water. Dit proces noemen we diffusie. Als je meerdere stoffen toevoegt, kunnen deze gaan reageren met de inkt. Deze reactie zal het sterkst zijn op de plek waar van beide stoffen het meest aanwezig is. Als er nu een chemische reactie mogelijk is waardoor het reactieproduct ook weer terug kan naar de stoffen waar het uit is opgebouwd, kan de volgende keten ontstaan: stel je voegt een druppel inkt toe aan de bak met chemicaliën, dan zal op de plek waar de druppel landt én een sterke reactie optreden én veel diffusie van de inkt plaatsvinden. Dit betekent dat er na korte tijd geen inkt meer is op de oorspronkelijke plek, maar wel veel reactieproduct, en dat er in een kring omheen juist veel inkt is. Vervolgens zal er dus op de oorspronkelijke plek het reactieproduct weer terug kunnen gaan naar inkt, terwijl de reactie die de inkt wegneemt nu in een kring om de plek waar de druppel landde heen het sterkst zal zijn. Nu gaat het proces zich herhalen. Dit kan dus patronen opleveren met ringen waar veel inkt of weinig inkt aanwezig is. Zoek maar eens op YouTube naar de Belousov–Zhabotinsky-reactie of zie afbeelding 1(b).

De centrale vraag die we stellen in dit proefschrift is nu als volgt: stel we hebben een reactie-diffusievergelijking waarvan we de bijbehorende patronen kennen en technieken hebben om die te bestuderen, wat kunnen we dan zeggen over de patronen als we ruis toevoegen aan de vergelijking? Om terug te keren naar de veer: kunnen we begrijpen hoe de veer valt door de lucht in vergelijking met hoe de veer valt in het vacuüm (namelijk recht naar beneden). Of voor de patronen in afbeelding 1(b): kunnen we begrijpen hoe de patronen zich gedragen als de tafel, waar het petrischaaltje op ligt trilt, ten opzichte van de patronen als de tafel niet trilt? Is het gedrag slechts een kleine afwijking van het deterministische gedrag of zal er kwalitatief ander gedrag optreden?

Nieuwe resultaten

In dit proefschrift leggen we een basis om met pen en papier stochastische RDV's te kunnen bestuderen en ze te kunnen vergelijken met hun deterministische tegenhanger. We doen dit niet voor complexe systemen zoals de vallende veer of de Belousov–Zhabotinsky-reactie, maar voor simpelere systemen die zonder ruis goed begrepen zijn, zogenaamde 'toy models'. Voorbeelden hiervan zijn vergelijkingen met namen als de Nagumovergelijking of de FitzHugh-Nagumovergelijking, die veelvuldig voorkomen in dit proefschrift. De pulsen die overal onderaan de oneven bladzijdes in dit proefschrift staan, vormen een oplossing van de stochastische FitzHugh-Nagumovergelijking. Zonder ruis zou de afbeelding op elke pagina identiek zijn. De puls uit de inhoudsopgave zou stil staan en niet van vorm veranderen. We zien echter dat de puls wél beweegt en wél van vorm verandert. Met de technieken in dit proefschrift kunnen we deze veranderingen van de vorm en snelheid begrijpen en nauwkeurig berekenen.

De basis is gelegd, maar er is ook heel veel wat we nog niet kunnen op dit moment. We hebben één klasse van toy models bestudeerd, zogenaamde bistabiele vergelijkingen en we hebben de eerste stappen gezet om monostabiele vergelijkingen te bestuderen. Echter, andere belangrijke type vergelijkingen zoals de Korteweg-De Vriesvergelijking of Burgersvergelijking vragen nog om veel tijd en aandacht. Bovendien zijn al onze resultaten alleen nog maar in 1D; vergelijkbare problemen in 2D en 3D, zoals de Barkleyvergelijking op de cover, zijn nog onontgonnen terrein. Wij zijn niet de eersten die aan deze onderwerpen werken, maar dus ook zeker niet de laatsten. Samengevat, om 'ooit', het moment waarop dit onderzoek daadwerkelijk toepassingen op zal leveren, te bereiken, zullen nog vele promovendi aan de slag moeten.

Dankwoord

In de eerste plaats gaat mijn dank uit naar de man die waarschijnlijk de meeste slapeloze nachten heeft gehad van dit proefschrift, en nee, ik begin niet bij mezelf. Hermen Jan, in vier jaar tijd ben ik erachter gekomen dat er in jouw brein een schier onuitputtelijke voorraad aan mij onbekende wiskunde ligt opgeslagen. Als je me vraagt wat ik anders had willen doen tijdens mijn promotie, dan is het wel dat ik ook in mijn eerste jaar op de hoogte had willen zijn van dit feit, want de samenwerking in het eerste jaar was niet optimaal. Misschien kwam dat omdat we allebei geen idee hadden waar we mee bezig waren, maar waarschijnlijk had alleen ik geen idee waar ik mee bezig was. Ik weet niet of het oorspronkelijk de bedoeling was om het onderzoek uit het onderzoeksvoorstel te gaan doen, maar ik denk dat je daarvoor een nieuwe promovendus moet gaan zoeken. Na een traag eerste jaar kwam de samenwerking goed op gang en heb ik ontzettend veel van je geleerd, zowel over wiskunde als over hoe je over wiskunde moet schrijven (en dan met name dat ik mijn literaire carrière niet in mijn artikelen moest opstarten). Je hebt grootse plannen om de resultaten uit dit proefschrift verder te ontwikkelen en ik zal zeker in de gaten blijven houden hoe dat gaat verlopen.

Tijdens een promotietraject gaat natuurlijk niet altijd alles even soepel en soms kan het fijn zijn om even alles eruit te gooien tegen iemand die niet Hermen Jan is. Martina, hier heb jij een belangrijke rol gespeeld. Het ging helaas vaak in het Engels in plaats van het afgesproken Nederlands of Duits, desalniettemin hebben we alsnog goed kunnen praten over het leven als beginnend wiskundige. Op wiskundig gebied heb je misschien niet de rol gespeeld die je had gewild, maar het was fijn om je in de buurt te hebben.

Verder gaat mijn dank natuurlijk uit naar alle fascinerende wezens die de afgelopen 4 jaar kamer 202 hebben bevolkt. Olfa, Robbin, Esmée, Leonie, David, Willem en Hans, kamer 202 mag dan al de gezelligste zijn geweest van het Snellius door de rode vloerbedekking en de gele stoelen, jullie hebben er voor gezorgd dat er ook nog wat leven in de brouwerij kwam. Robbin, na een bachelor-, master-, en promotie-onderzoek in dezelfde groep te hebben gedaan, zijn onze wegen nu echt gescheiden, al heb ik nog mijn best gedaan wéér in dezelfde groep terecht te komen. Ik zal me de rest van mijn leven blijven verbazen over jouw vermogen om wiskunde op papier te doen zonder krassen en doorhalen. Olfa, elke ochtend hoorden we je al van verre aankomen, het geslof op de gang was een voorbode van het feit dat de serene rust van de vroege morgen voorbij zou zijn. Omvallende theevazen (vaak gevuld), heavy metal[1], een laptop die de tijd in blikkerig Russisch door de kamer schreeuwt; je wist ons altijd weer wakker te krijgen. Ook zorgde jij voor de sociale cohesie, je spoorde ons altijd aan iets te gaan drinken in de FooBar of de laatste tijd om online af te spreken. Willem, het was altijd erg

[1] Dit is sowieso de verkeerde muzikale term.

geruststellend om jou in de buurt te hebben, omdat jij dezelfde dingen als ik meemaakt met jouw begeleider.

Hermen Jans schatkamer aan wiskunde heb ik vaak geplunderd, maar er is nog iemand anders die ik vaak heb lastig gevallen als Google weer eens geen antwoord had op mijn vragen. Onno, je was altijd bereid om mee te denken als ik me op functionaal-analytisch gebied weer eens in de nesten had gewerkt. En eerlijk gezegd, niet alleen de afgelopen vier jaar, maar de afgelopen tien jaar heb je geholpen met de meest uiteenlopende vragen. Veel dank daarvoor.

Onno is een goed voorbeeld van waar het Mathematisch Instituut zo trots op is: een informeel opendeurenbeleid waar studenten met al hun vragen terecht kunnen. Als student heb ik het MI van binnen uit leren kennen als lid van de opleidingscommissie en heb dus kunnen meemaken hoe er altijd wordt meegedacht met de studenten. Ook over studenten met ingewikkelde vakkenlijsten (dubbele bachelor, half jaar buitenland, vakken van ander universiteiten) werd nooit moeilijk gedaan. Om een handtekening te krijgen voor mijn dubbele bachelor wis- en natuurkunde hoefde ik slechts naar een speech te luisteren over het feit dat natuurkundigen nooit ergens zullen komen tot het moment dat ze gedegen definities hebben. Ook toen ik wegens schouderklachten[2] geen opdrachten kon inleveren, waren alle docenten bereid er een mouw aan te passen en kon ik door zonder enige vertraging.

Verder zijn er natuurlijk ook veel mensen buiten de wiskunde belangrijk geweest. Pauline, meer dan eens heb ik conferenties als excuus gebruikt om afwezig te zijn tijdens belangrijke momenten op onze kalender, al leg ik de verantwoordelijkheid hiervoor bij de kwade geesten van SIAM. Ondanks het feit dat je lang niet altijd begreep waar ik mee bezig was, zorgde je er altijd voor dat de dingen die wel te begrijpen waren (filmpjes, het publiceren van papers, conferenties) uitgebreid werden verspreid onder familie en vrienden. Het enthousiasme waarmee jij reclame maakte voor mijn onderzoek was altijd erg fijn.

Ook die andere belangrijke vrouw, mama, heeft vaak wat moeite gehad met mijn onderwerp. Eerlijk is eerlijk, stochastische reactie-diffusievergelijkingen onthouden en snel uitspreken is best lastig. Het was wel heerlijk om tegen je te klagen over onderwerpen waar je je beter in kon inleven. Met name frustraties over papers die 95% af waren, maar ook niet verder wilden, was iets wat je erg goed begreep. En natuurlijk veel dank aan mijn broers, die als dappere paranimfen achter me staan.

Ik heb het geluk dat er onder mijn muzikale vrienden ook veel promovendi zijn. Er was natuurlijk de PhDriehoek waar ik deel van uit maakte met Sierk en Petra, maar ook Floris en Maurits, het is altijd goed om ook het wel en wee van promovendi buiten wiskunde te horen.

Verder natuurlijk iedereen van Sempre, BLOQ en daarbuiten, die altijd vriendelijk vroeg hoe het ging met mijn promotie ook al vreesde hij/zij het antwoord, bedankt daarvoor.

[2] Veel dank aan Don de fysiotherapeut voor het regelmatig redden van mijn rug en schouders.

Curriculum Vitae

Christian Hamster was born on the first of August 1992 in Apeldoorn, the Netherlands. From 2004 to 2010 he attended Gymnasium Apeldoorn, whereafter he received his gymnasium diploma. In the fall of 2010 Christian came to Leiden to study physics and mathematics (in that order) and completed both bachelor degrees after 3 years in 2013. However, during these 3 years, his taste developed more towards mathematics, so he started the master Applied Mathematics in Leiden in the fall of 2013. As part of his masters degree, Christian spend half a year at the Technical University München.

After his return to Leiden, he started working on his master thesis under the supervision of dr. Vivi Rottschäfer. This resulted in a thesis called 'Two scale dependent feedback as a model for pattern formation in ecology' and Christian graduated cum laude in the spring of 2016. During his studies, Christian was also active within the Mathematical Institute. He was a member of the 'opleidingscommissie' and promoted mathematics in Leiden as a member of the 'voorlichtingscommissie'.

Shortly after his graduation, he started with a PhD at Leiden University under the supervision of Hermen Jan Hupkes on a project called 'Noisy Patterns', which also gave the name to this thesis. The papers that followed form the heart of this thesis. During his PhD, he was invited to present his work at major SIAM conferences like 'Analysis of PDEs', 'Nonlinear Waves and Coherent Structures', and 'Snowbird'.

Apart from his studies, Christian has also enjoyed student life in Leiden. For many years, he played the clarinet in the student orchestra 'Sempre Crescendo' and he was a member of student association SSR-Leiden.

CV