Information & Communication Exercise Sheet #2

University of Amsterdam, Bachelor of Computer Science, January 2015
Lecturer: Christian Schaffner

Out: Wednesday, 7 January 2015
(due: Monday, 12 January 2015, 13:00)

To be solved in Class

1. ([Yeung]) Let X and Y be random variables over alphabets $\mathcal{X} = \mathcal{Y} = \{1, 2, 3, 4, 5\}$ and joint distribution P_{XY} given by the following matrix (where the entry in row i and column j is the probability $P_{XY}(i, j)$)

$$\frac{1}{25}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 0 & 0 \\ 2 & 0 & 1 & 1 & 1 \\ 0 & 3 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 3 \end{bmatrix}$$

Calculate $H(X)$, $H(Y)$, $H(X|Y)$, $H(Y|X)$, and $I(X;Y)$.

2. ([MacKay], Example 2.13:) A source produces a character x from alphabet $\mathcal{A} = \{\mathtt{0}, \mathtt{1}, \mathtt{2}, \ldots, \mathtt{9}, \mathtt{a}, \mathtt{b}, \mathtt{c}, \ldots, \mathtt{z}\}$. With probability 1/3, x is a uniformly random numeral $\mathtt{0}, \mathtt{1}, \mathtt{2}, \ldots, \mathtt{9}$, with probability 1/3, x is a random vowel $\{\mathtt{a}, \mathtt{e}, \mathtt{i}, \mathtt{o}, \mathtt{u}\}$ and with probability 1/3, x is one of the 21 consonants. Estimate the entropy of X.

3. ([MacKay], Exercise 2.29) An unbiased coin is flipped until one head is thrown. What is the entropy of the random variable $X \in \{1, 2, 3, \ldots\}$, the number of flips? Repeat the calculation for the case of a biased coin with probability p of coming up heads.
Hint: solve the problem both directly and by using the decomposability of the entropy, i.e. that for a probability distribution $\mathbf{p} = \{p_1, p_2, \ldots, p_n\}$, it holds that

$$H(\mathbf{p}) = H(p_1, 1 - p_1) + (1 - p_1) H\left(\frac{p_2}{1 - p_1}, \frac{p_3}{1 - p_1}, \ldots, \frac{p_n}{1 - p_1}\right).$$

4. Let X, Y, Z be *binary* random variables such that $I(X;Y) = 0$ and $I(X;Z) = 0$.
   (a) Does it follow that $I(X;Y,Z) = 0$? If yes, prove it. If no, give a counterexample.
   Hint: Consider the case where X and Y are two independent uniform bits and $Z = X \oplus Y$.
   (b) Does it follow that $I(Y;Z) = 0$? If yes, prove it. If no, give a counterexample.

5. *Maximal conditional entropy implies independence.* Let $n = \log(|\mathcal{X}|)$.
   (a) Prove that $H(X|Y) = n$ implies that X and Y are independent.
   (b) Give a joint distribution P_{XY} where $H(X) = n$, but X and Y are dependent.

6. For two distributions P and Q over $\mathcal{X}$, the *relative entropy* or *Kullback-Leibler divergence* is defined as

$$D(P||Q) := \sum_{\substack{x \in \mathcal{X} \\ P(x) > 0}} P(x) \log \frac{P(x)}{Q(x)}.$$

Note that if $Q(x) = 0$ for some x, then $D(P||Q) = \infty$. Prove that $D(P||Q) \geq 0$, and that equality holds if and only if $P = Q$.
Hint: Use Jensen's inequality.

Homework

1. Entropy of functions of a random variable. Let X be a discrete random variable. Show that the entropy of a function g of X is less than or equal to the entropy of X by justifying the following steps: 3 p.

$$\begin{aligned} H(X) &= H(X) + H(g(X)|X) && (1)\\ &= H(X, g(X)) && (2)\\ &= H(g(X)) + H(X|g(X)) && (3)\\ &\geq H(g(X)) && (4)\end{aligned}$$

2. Consider the following random experiment with two fair (regular six-sided) dice. First, the first die is thrown, and let the outcome be A. Then, the second die is thrown until the outcome has the same parity (even, odd) as A. Let this final outcome of the second die be B. The random variables X, Y and Z are defined as follows:

$$X = (A + B) \mod 2, \qquad Y = (A \cdot B) \mod 2, \qquad Z = |A - B|.$$

   (a) Find the joint distribution P_{AB}. 1 p.
   (b) Determine $H(X)$, $H(Y)$ and $H(Z)$. 3 p.
   (c) Compute $H(Z|A = 1)$. 1 p.
   (d) Compute $H(AB)$, i.e. the joint entropy of A and B. 1 p.
   (e) A random variable M describes whether the sum $A + B$ is strictly larger than seven, between five and seven (both included), or strictly smaller than five. How much entropy is present in this random variable M? 2 p.

3. Let X, Y, Z be arbitrary random variables, and let f be any deterministic function acting on $\mathcal{Y}$. In the following, replace "?" by "$\geq$" or "$\leq$" to obtain the correct inequalities, and reason each time with the help of an entropy diagram. **Hint:** $H(f(Y)|Y) = 0$.

   (a) $H(f(Y))\, ?\, H(Y)$ 2 p.
   (b) $H(X|f(Y))\, ?\, H(X|Y)$ 2 p.
   (c) $I(X; Z|Y) = 0$ implies $I(X; Z)\, ?\, I(X; Y)$ and $I(X; Z)\, ?\, I(Y; Z)$. 2 p.

4. For each statement below, specify a (different) joint distribution P_{XYZ} of random variables X, Y and Z such that the inequalities hold.

   (a) There exists a y, such that $H(X|Y = y) > H(X)$ 2 p.
   (b) $I(X; Y) > I(X; Y|Z)$ 2 p.
   (c) $I(X; Y) < I(X; Y|Z)$ 2 p.

   Note that the distributions have to be different from the ones seen as examples during the lecture.

5. The mutual information between two random variables X and Y is defined as $I(X; Y) := H(X) - H(X|Y)$

   (a) Show that the mutual information can be expressed in terms of the relative entropy, i.e. that $I(X; Y) = D(P_{XY}||P_X P_Y)$ 3 p.
   (b) Use (a) and Class exercise 6. to prove that $H(X|Y) \leq H(X)$. 1 p.