Channel Coding: Zero-error case

Sander Bet & Ismani Nieuweboer

January 2015

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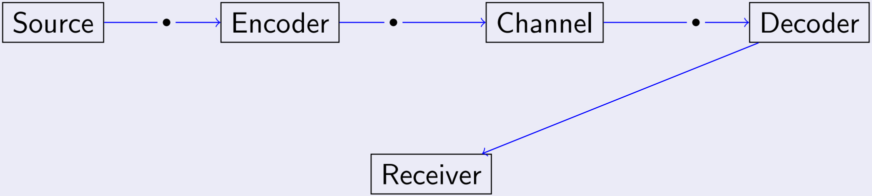
Channel Definition

What is a Channel?



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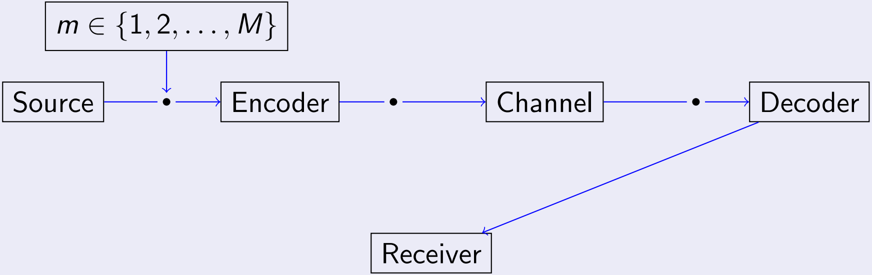

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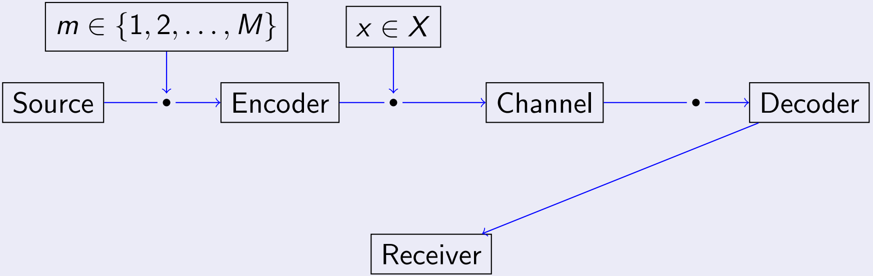

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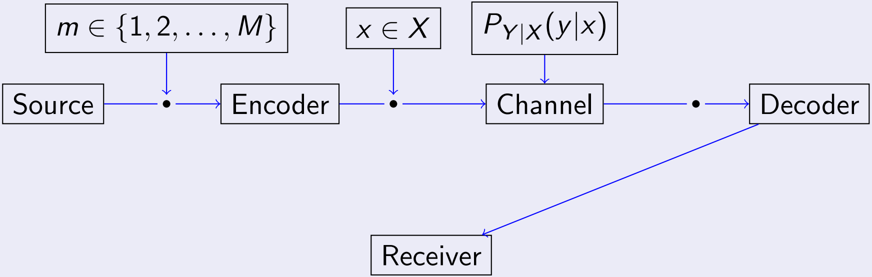

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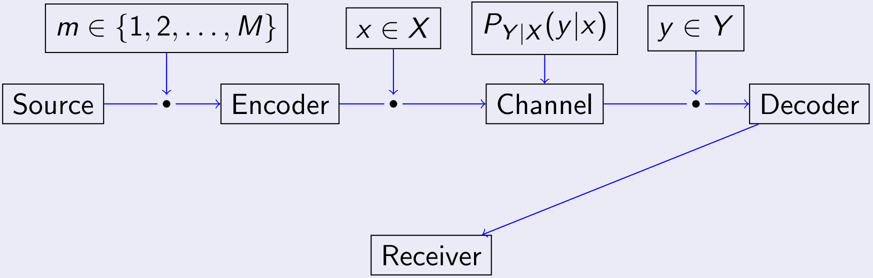

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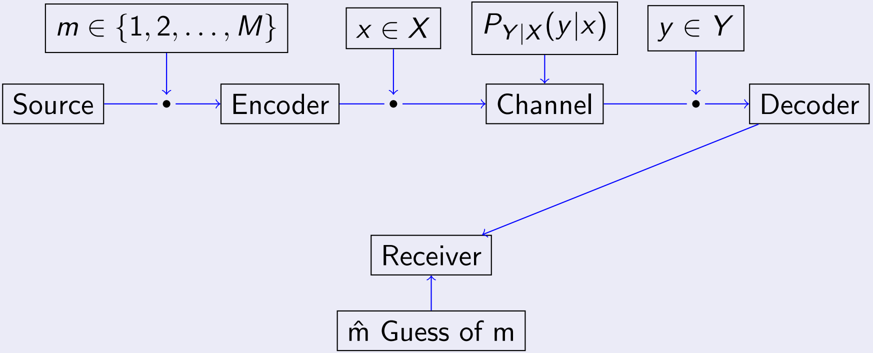

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Definition: Discrete Channel

A discrete channel is denoted by $(\mathcal{X}, P_{Y|X}(y|x), \mathcal{Y})$. Where $\mathcal{X}$ is a finite non-empty input set, $\mathcal{Y}$ a finite output set. And $P_{Y|X}(y|x)$ is a conditional probability distribution that satisfies the following properties;

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$$P_{Y|X}(y|x) \geq 0 : \forall x \in \mathcal{X}, \forall y \in \mathcal{Y}$$

$$\sum_{y \in \mathcal{Y}} P_{Y|X}(y|x) = 1 : \forall x \in \mathcal{X}$$

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Definition: Memory-less Channel

A memory-less channel is a channel the probability distribution $P_{Y|X}(y|x)$ is independent of previous channel inputs and outputs.

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Channel Definition

Example

$(\mathcal{X} = \{0, 1\},$
$P_{Y|X}(0|0) = p, P_{Y|X}(1|1) = p, P_{Y|X}(1|0) = 1 - p, P_{Y|X}(0|1) = 1 - p,$
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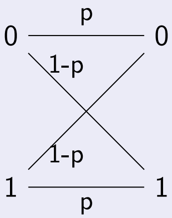

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Channel Definition

Multiple uses of a memory-less channel

n uses of the memory-less channel $(\mathcal{X}, P_{Y|X}(y|x), \mathcal{Y})$ corresponds to the memory-less channel $(\mathcal{X}^n, P_{Y^n|X^n}(y^n|x^n), \mathcal{Y}^n)$

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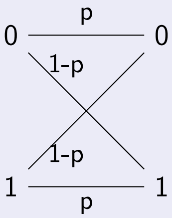

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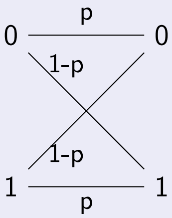

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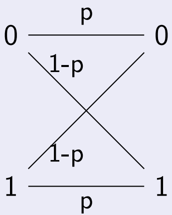

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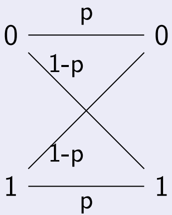

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Is this a channel?

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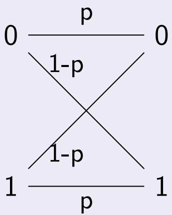

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$P_{Y^2|X^2}(00|11) = P_{Y|X}(0|1) \cdot P_{Y|X}(0|1) = (1-p)^2$
Is this a channel? Yes, all the probabilities are positive and
$\sum_{y \in \mathcal{Y}} P_{Y^2|X^2}(y|11) = p^2 + 2 \cdot p(1-p) + (1-p)^2 = (p + (1-p))^2 = 1$

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Code Definition

Definition: (M, n)-code

A (M, n)-code for channel $(\mathcal{X}, P_{Y|X}(y|x), \mathcal{Y})$ with;

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A decoding function d: $\mathcal{Y}^n \rightarrow \{1, 2, \ldots, M\}$

Definition: Transmission Rate

The transmission rate R of a (M, n)-code is $R = \frac{\log M}{n}$.

Zero-error Problem

Given a channel how many bits of information can we send through it without any errors?

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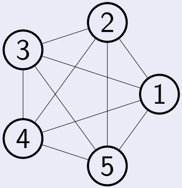
Given a channel how many bits of information can we send through it without any errors? In other words what is the maximal transmission rate of the channel? We use graph theory to clarify the problem.

Graph theory

Graph definition

A graph G is a set of vertices $V(G)$ and a set of edges $E(G)$. Example (full graph with 5 vertices):



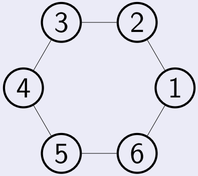
$V = \{1, 2, 3, 4, 5\}$ and $E = \{12, 13, 14, \ldots, 45\}$

Graph theory

Independent set

Definition

For a graph G, an independent set is a set of vertices $I \subset V(G)$ such that no edge $e \in E(G)$ contains two vertices from I



I can be $\{1\}, \{2\}, \{1, 2\}, \{1, 3, 5\}$, etc.

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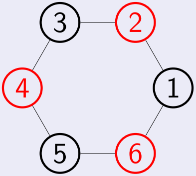

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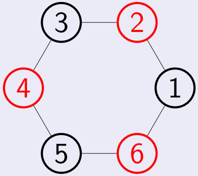


$\alpha(G) = |\{1, 3, 5\}| = |\{2, 4, 6\}| = 3$

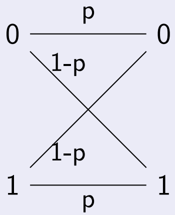
Confusability graph

Definition

Given a discrete channel $(\mathcal{X}, P_{Y|X}(y|x), \mathcal{Y})$, the confusability graph G is defined by $V(G) = \mathcal{X}$ and
$E(G) = \{vw : \exists y : P_{Y|X}(y, v) \neq 0 \wedge P_{Y|X}(y|w) \neq 0\}$ i.e. vertices are connected when they can get confused with each other

Confusability graph

Example



Zero-error codes

Given a (discrete memoryless) channel, how much information can you perfectly send through it?

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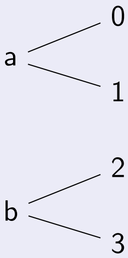
When using the channel once, the independence number $\alpha(G)$ of the confusability graph G tells you the maximum rate: $R = \log \alpha(G)$.

Zero-error codes

An ideal situation

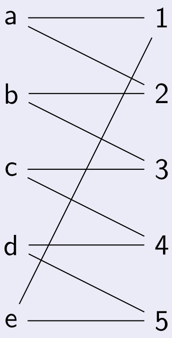
If there is no overlapping output, the maximum independent set is $\mathcal{X}$ itself:



Zero-error codes

Noisy typewriter

A more interesting case:



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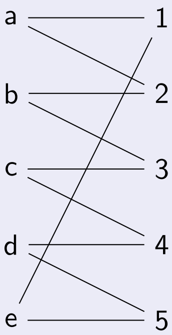
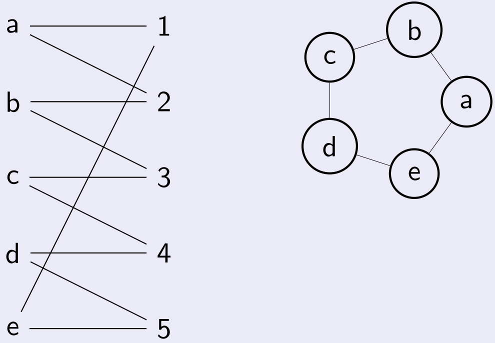


What is the confusability graph of this code?

Zero-error codes

Noisy typewriter



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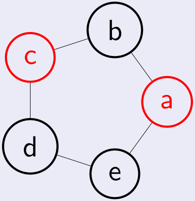

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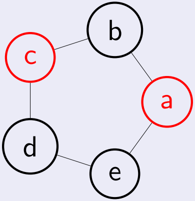

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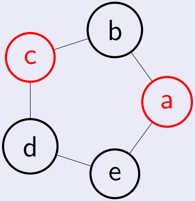

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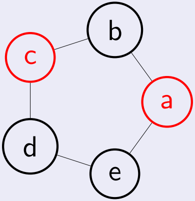

Index set: $\{1, 2\}$ Encoding function: $e(1) = a$ and $e(2) = c$ Decoding function: $d(1) = d(2) = a$ and $d(3) = d(4) = c$

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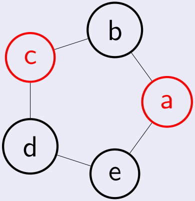

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Zero-error codes

Noisy typewriter

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We can do just as well using $\{aa, ac, ca, cc\}$; still no overlap.
Then again $R = \frac{\log M}{n} = \frac{\log 4}{2} = 1$.

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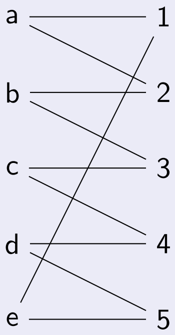

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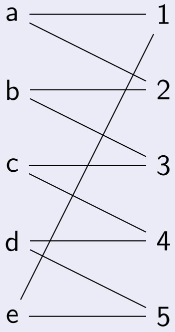

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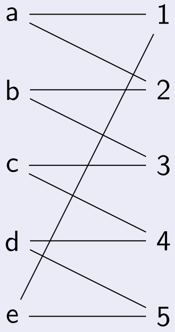

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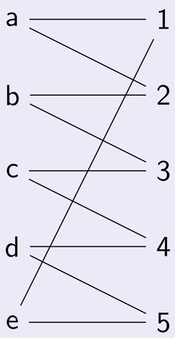

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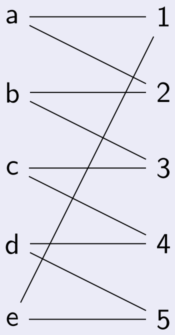

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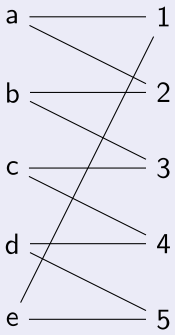

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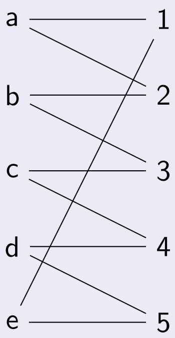

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What happens for bigger graphs? No one knows...

Extra

Multiple channel confusability

We saw an increase in transmission rate, when we used the channel multiple times.

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Reference

Lecture content of Information Theory given by Christian Schaffner (course at University of Amsterdam):
`http://homepages.cwi.nl/~schaffne/courses/inftheory/2014/`
(Blackboard photos from 24 and 26 November)
Many thanks to Christian for letting us use his lecture content!