Information & Communication Exercise Sheet #3

University of Amsterdam, Bachelor of Computer Science, January 2016
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Out: Tuesday, 11 January 2016
(due: Friday, 15 January 2016, 14:00, by email or in my ILLC post box)

To be solved in Class

1. **Huffman** [CT, 5.38] Find the Huffman D-ary code for $(p_1, p_2, p_3, p_4, p_5, p_6) = (\frac{6}{25}, \frac{6}{25}, \frac{4}{25}, \frac{4}{25}, \frac{3}{25}, \frac{2}{25})$ and the expected codeword length
   (a) for $D = 2$,
   (b) for $D = 4$.

2. **Error Penalty** Suppose that an engineer believes that a source X can be described by the distribution Q_X given by the following table:

x	a	b	c
$Q_X(x)$	$1/2$	$1/4$	$1/4$

   In fact, however, the source follows the distribution P_X:

x	a	b	c
$P_X(x)$	$1/4$	$1/2$	$1/4$

   (a) Design a code for X based on the wrong distribution Q_X.
   (b) Design a code for X based on the correct distribution P_X.
   (c) Compute the expected number of bits per symbol used by each of these codes when X is sampled from P_X. How big is the difference?
   (d) Explain how this number relates to the Kullback-Leibler divergence

$$D(P_X \,||\, Q_X) \;=\; \sum_x P_X(x) \log \frac{P_X(x)}{Q_X(x)}.$$

3. Let X, Y, Z be *binary* random variables such that $I(X;Y) = 0$ and $I(X;Z) = 0$.
   (a) Does it follow that $I(X;Y,Z) = 0$? If yes, prove it. If no, give a counterexample.
   Hint: Consider the case where X and Y are two independent uniform bits and $Z = X \oplus Y$.
   (b) Does it follow that $I(Y;Z) = 0$? If yes, prove it. If no, give a counterexample.

4. For the Markov chain $X \leftrightarrow Y \leftrightarrow \hat{X}$, show that $H(X|\hat{X}) \geq H(X|Y)$.

5. [Cover-Thomas 2.32]. We are given the following joint distribution of $X \in \{1,2,3\}$ and $Y \in \{a,b,c\}$:

$$P_{XY}(1,a) = P_{XY}(2,b) = P_{XY}(3,c) = 1/6$$
$$P_{XY}(1,b) = P_{XY}(1,c) = P_{XY}(2,a) = P_{XY}(2,c) = P_{XY}(3,a) = P_{XY}(3,b) = 1/12.$$

   Let $\hat{X}(Y)$ be an estimator for X (based on Y) and let $p_e = P(\hat{X} \neq X)$.
   (a) Find an estimator $\hat{X}(Y)$ for which the probability of error p_e is as small as possible.
   (b) Evaluate Fano's inequality for this problem and compare.

Homework

1. Let X, Y, Z be arbitrary random variables, and let f be any deterministic function acting on $\mathcal{Y}$. In the following, replace "?" by "$\geq$" or "$\leq$" to obtain the correct inequalities, and reason each time with the help of an entropy diagram. **Hint:** $H(f(Y)|Y) = 0$.

   (a) $H(f(Y)) \,?\, H(Y)$ 2 p.

   (b) $H(X|f(Y)) \,?\, H(X|Y)$ 2 p.

   (c) $I(X;Z|Y) = 0$ implies $I(X;Z) \,?\, I(X;Y)$ and $I(X;Z) \,?\, I(Y;Z)$. 2 p.

2. For each statement below, specify a (different) joint distribution P_{XYZ} of random variables X, Y and Z such that the inequalities hold.

   (a) There exists a y, such that $H(X|Y = y) > H(X)$ 2 p.

   (b) $I(X;Y) > I(X;Y|Z)$ 2 p.

   (c) $I(X;Y) < I(X;Y|Z)$ 2 p.

   Note that the distributions have to be different from the ones seen as examples during the lecture.

3. *Bottleneck.* Suppose a Markov chain starts in one of n states, necks down to $k < n$ states, and then fans back to $m > k$ states. Thus $X_1 \to X_2 \to X_3$, i.e.,

$$P_{X_1X_2X_3}(x_1, x_2, x_3) = P_{X_1}(x_1) \cdot P_{X_2|X_1}(x_2|x_1) \cdot P_{X_3|X_2}(x_3|x_2)$$

   for all $x_1 \in \{1, 2, ..., n\}$, $x_2 \in \{1, 2, ..., k\}$, $x_3 \in \{1, 2, ..., m\}$.

   (a) Show that the (unconditional) dependence of X_1 and X_3 is limited by the bottleneck by proving that $I(X_1;X_3) \leq \log k$. 3 p.

   (b) Evaluate $I(X_1;X_3)$ for $k = 1$, and explain why no dependence can survive such a bottleneck. 1 p.

4. Let A, B, C be random variables such that

$$I(A;B) = 0\,, \tag{1}$$
$$I(A;C|B) = I(A;B|C)\,, \tag{2}$$
$$H(A|BC) = 0\,. \tag{3}$$

   Which of the three relations $\leq$, $\geq$, $=$ holds between the quantities $H(A)$ and $H(C)$? Prove your answer. 3 p.