Information Theory Exercise Sheet #5

University of Amsterdam, Master of Logic, Fall 2014
Lecturer: Christian Schaffner
TA: Philip Schulz

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Previous Exam Questions

At this point in the course, you should be able to solve the following exercises which were exam questions in previous editions of the course:

Given a random variable X with the following distribution

x	1	2	3	4	5	6
$P_X(x)$	0.1	0.1	0.3	0.1	0.25	0.15

1. Draw a binary Huffman tree which is optimal in average codeword length, and give the corresponding codewords.
2. Draw a ternary Huffman tree which is optimal in average codeword length, and give the corresponding codewords.
3. Draw a 4-ary Huffman tree which is optimal in average codeword length, and give the corresponding codewords.

To be solved in Class

1. Consider a random variable X that takes on four values with probabilities $\frac{1}{3}, \frac{1}{3}, \frac{1}{4}, \frac{1}{12}$. Show that there exist two different sets of optimal length for the (binary) Huffman codewords.
2. Describe an arithmetic coding algorithm to encode random bit strings of length 5 and weight 2 (i.e., 5-bit strings with 2 ones and 3 zeroes). What are the intervals corresponding to the source-substrings 0 and 1? What are the intervals corresponding to 00, 01, 10, 11? What are the intervals corresponding to 000, 001, 010, 011, 100, 101, 110, 111? What are the intervals corresponding to source-substrings of length 4 and 5?
3. How many bits are needed to specify a selection of k objects from n objects? How might such a selection be made at random without being wasteful of random bits?

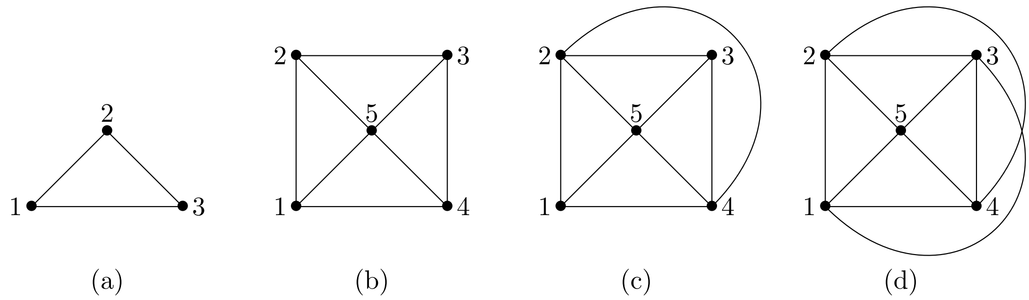
Homework

1. For each of the channels below, give the corresponding confusability graph.

   (a) [1 point] $\mathcal{X} = \{1, 2, 3, 4, 5\}$, $\mathcal{Y} = \{a, b, c\}$, $P_{Y|X}(a|1) = P_{Y|X}(b|1) = P_{Y|X}(a|2) = P_{Y|X}(b|2) = \frac{1}{2}$, $P_{Y|X}(b|3) = \frac{1}{3}$, $P_{Y|X}(c|3) = \frac{2}{3}$, $P_{Y|X}(c|4) = P_{Y|X}(c|5) = 1$.

   (b) [1 point] $\mathcal{X} = \{1, 2, 3, 4, 5\}$, $\mathcal{Y} = \{a, b, c, d\}$, $P_{Y|X}(a|2) = P_{Y|X}(b|2) = P_{Y|X}(c|2) = P_{Y|X}(a|4) = P_{Y|X}(c|4) = P_{Y|X}(d|4) = \frac{1}{3}$, $P_{Y|X}(b|3) = P_{Y|X}(c|3) = \frac{1}{2}$, $P_{Y|X}(a|1) = P_{Y|X}(d|5) = 1$.

2. For each of the confusability graphs below, describe one of the possible corresponding channels. Try to minimize the number of output symbols you are using.



   (a) [1 point]

   (b) [1 point]

   (c) [1 point]

   (d) [1 point]

   (e) [2 points] Can you argue that you reached the minimal number of outputs in (a), (b), (c), (d) ?

   (f) [1 point] Show that for any confusability graph G with no isolated vertices, there exists a corresponding channel with $|E(G)|$ output symbols.

3. *Shannon capacity of the complete graph.* A graph G with n vertices $V(G) = \{1, 2, \ldots, n\}$ is called *complete* if it has edges between any two vertices, i.e. $\forall i \neq j : ij \in E(G)$.

   (a) [2 points] Compute $\alpha(K_n)$, the independence number of the complete graph.

   (b) [2 points] Show that $K_n \boxtimes K_n = K_{n^2}$.

   (c) [2 points] Use (a) and (b) to prove that the Shannon capacity of K_n is 0. Note that this result formally confirms the intuition that channels whose confusability graphs are complete are useless for zero-error communication, because all symbols can possibly be confused with each other.

4. *Disjoint graphs.* For two graphs G and H, the graph $G + H$ is defined as the disjoint union of the two graphs[1]. Formally, assuming without loss of generality that $V(G) \cap V(H) = \emptyset$, then $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H)$.

   For a graph G, the disjoint union of t copies of G is denoted as G^{+t}. Similarly, we write $G^{\boxtimes t}$ for the t-time strong product of G with itself.

   (a) [2 points] Prove that $\alpha(G + H) = \alpha(G) + \alpha(H)$.

   (b) [Bonus: +2 points] Prove that for any three graphs G, H, L, it holds that

   $$(G + H) \boxtimes L = (G \boxtimes L) + (H \boxtimes L)$$

   and for the same reason, it also holds that

   $$G \boxtimes (H + L) = (G \boxtimes H) + (G \boxtimes L)$$

   .

   (c) [4 points] Use (b) to derive that for any natural number $k \in \mathbb{N}$, $(G + G)^{\boxtimes k} = (G^{\boxtimes k})^{+2^k}$.

5. Let $\mathcal{X} = \mathcal{Y} = \{1, 2, 3, 4, 5, 6\}$. In this exercise, we compute the zero-error Shannon capacity of the noisy channel with transition probabilities $P_{Y|X}(y|x) = 1/3$ if and only if $x \equiv y \mod 2$.

   (a) [2 points] Give the confusability graph G of the noisy channel $P_{Y|X}$ described above.

   (b) [4 points] Use 4.(c) and 3.(a) and 3.(b) to show that the Shannon capacity of G is 1.

[1] You can think of $G + H$ as G and H "next to each other".