Example: Letter Frequencies

i	a_i	p_i
1	a	0.0575
2	b	0.0128
3	c	0.0263
4	d	0.0285
5	e	0.0913
6	f	0.0173
7	g	0.0133
8	h	0.0313
9	i	0.0599
10	j	0.0006
11	k	0.0084
12	l	0.0335
13	m	0.0235
14	n	0.0596
15	o	0.0689
16	p	0.0192
17	q	0.0008
18	r	0.0508
19	s	0.0567
20	t	0.0706
21	u	0.0334
22	v	0.0069
23	w	0.0119
24	x	0.0073
25	y	0.0164
26	z	0.0007
27	–	0.1928

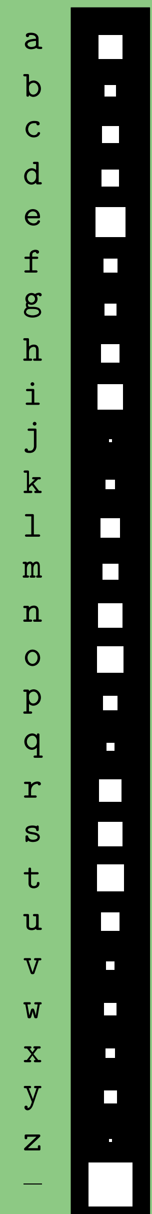

Figure 2.1. Probability distribution over the 27 outcomes for a randomly selected letter in an English language document (estimated from *The Frequently Asked Questions Manual for Linux*). The picture shows the probabilities by the areas of white squares.

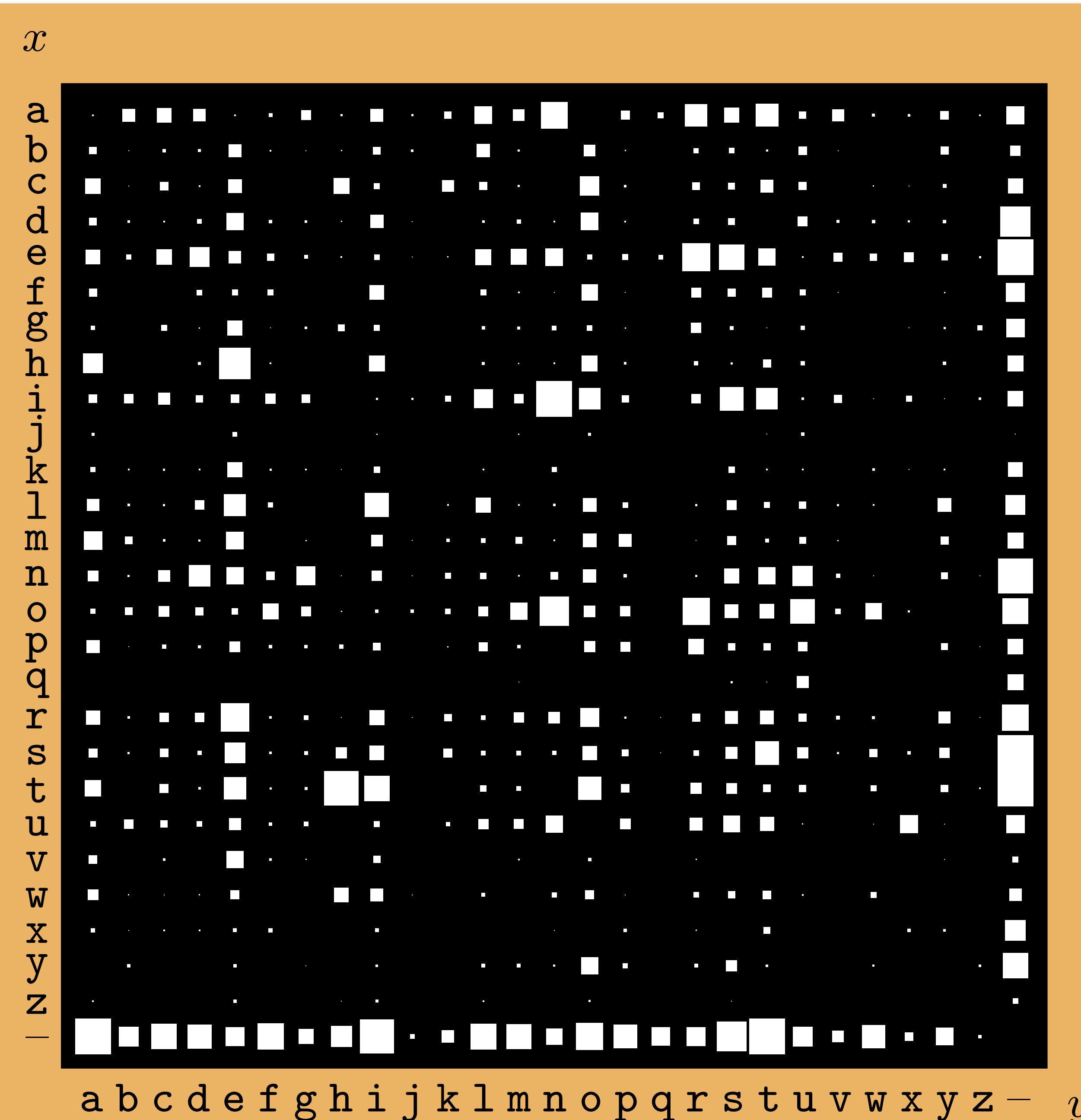

Figure 2.2. The probability distribution over the 27×27 possible bigrams xy in an English language document, *The Frequently Asked Questions Manual for Linux*.

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Example: Surprisal Values

from http://www.umsl.edu/~fraundorfp/egsurpri.html

situation	probability $p = 1/2^{\#bits}$	surprisal #bits = $\ln_2[1/p]$
one equals one	1	0 bits
wrong guess on a 4-choice question	3/4	$\ln_2[4/3]$ ~0.415 bits
correct guess on true-false question	1/2	$\ln_2[2]$ =1 bit
correct guess on a 4-choice question	1/4	$\ln_2[4]$ =2 bits
seven on a pair of dice	$6/6^2$ =1/6	$\ln_2[6]$ ~2.58 bits
snake-eyes on a pair of dice	$1/6^2$ =1/36	$\ln_2[36]$ ~5.17 bits
random character from the 8-bit ASCII set	1/256	$\ln_2[2^8]$ =8 bits =1 byte
N heads on a toss of N coins	$1/2^N$	$\ln_2[2^N]$ =N bits
harm from a smallpox vaccination	~1/1,000,000	~$\ln_2[10^6]$ ~19.9 bits
win the UK Jackpot lottery	1/13,983,816	~23.6 bits
RGB monitor choice of one pixel's color	$1/256^3$ ~5.9×10^{-8}	$\ln_2[2^{8*3}]$ =24 bits
gamma ray burst mass extinction event TODAY!	<$1/(10^9*365)$ ~2.7×10^{-12}	hopefully >38 bits
availability to reset 1 gigabyte of random access memory	$1/2^{8E9}$ ~$10^{-2.4E9}$	8×10^9 bits ~7.6×10^{-14} J/K
choices for 6×10^{23} Argon atoms in a 24.2L box at 295K	~$1/2^{1.61E25}$ ~$10^{-4.8E24}$	~1.61×10^{25} bits ~155 J/K
one equals two	0	∞ bits

i	a_i	p_i	$h(p_i)$
1	a	.0575	4.1
2	b	.0128	6.3
3	c	.0263	5.2
4	d	.0285	5.1
5	e	.0913	3.5
6	f	.0173	5.9
7	g	.0133	6.2
8	h	.0313	5.0
9	i	.0599	4.1
10	j	.0006	10.7
11	k	.0084	6.9
12	l	.0335	4.9
13	m	.0235	5.4
14	n	.0596	4.1
15	o	.0689	3.9
16	p	.0192	5.7
17	q	.0008	10.3
18	r	.0508	4.3
19	s	.0567	4.1
20	t	.0706	3.8
21	u	.0334	4.9
22	v	.0069	7.2
23	w	.0119	6.4
24	x	.0073	7.1
25	y	.0164	5.9
26	z	.0007	10.4
27	-	.1928	2.4
$\sum_i p_i \log_2 \frac{1}{p_i}$			4.1

Table 2.9. Shannon information contents of the outcomes a–z.

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MacKay's Mnemonic

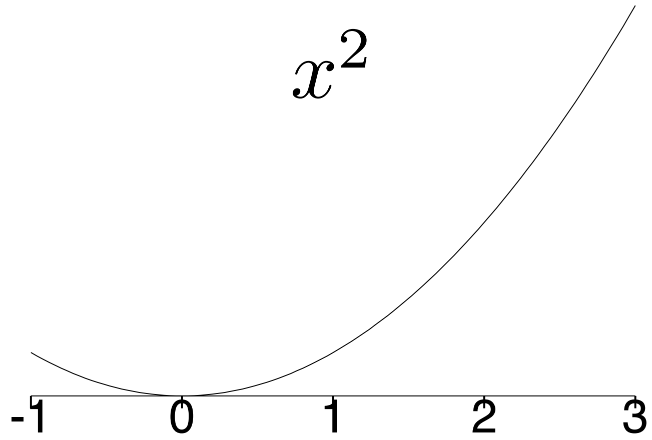
convex

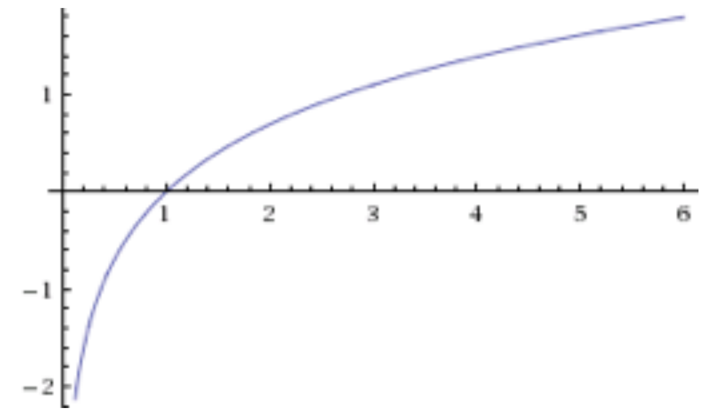
convec-smile

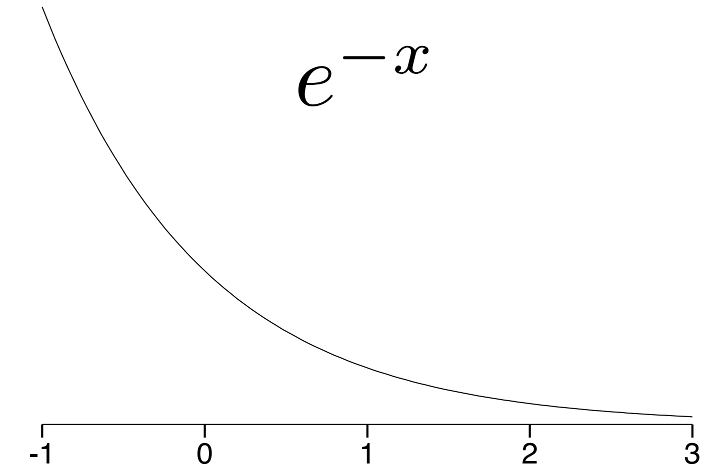
concave

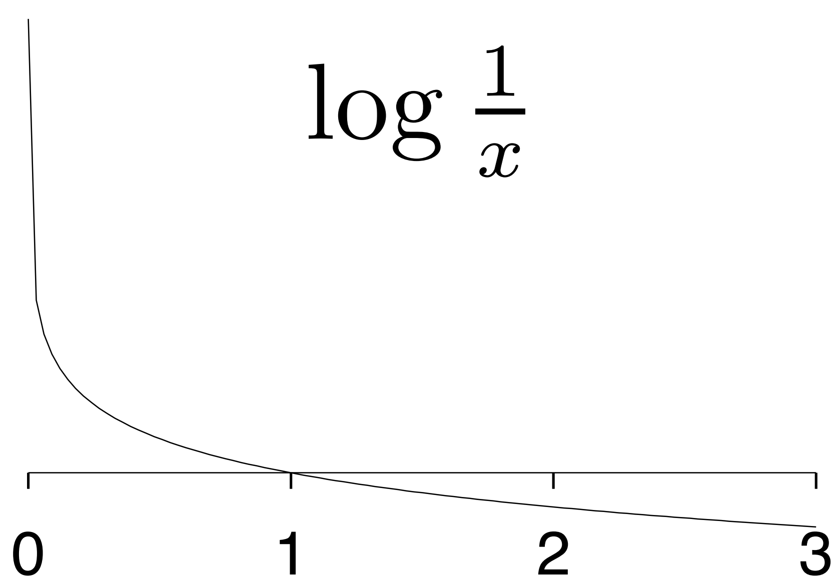
conca-frown

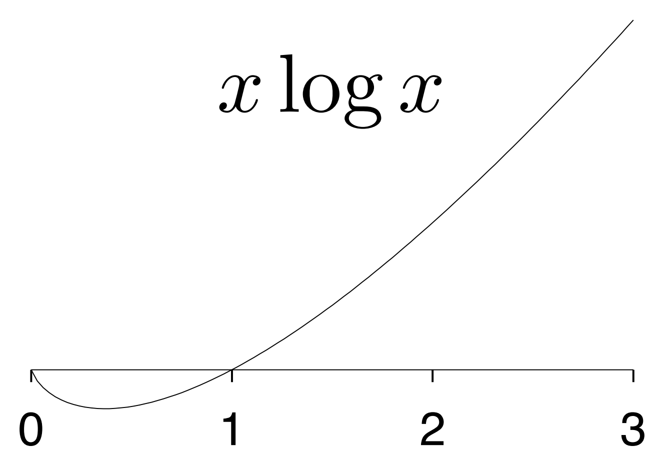
Examples: Convex & Concave Functions

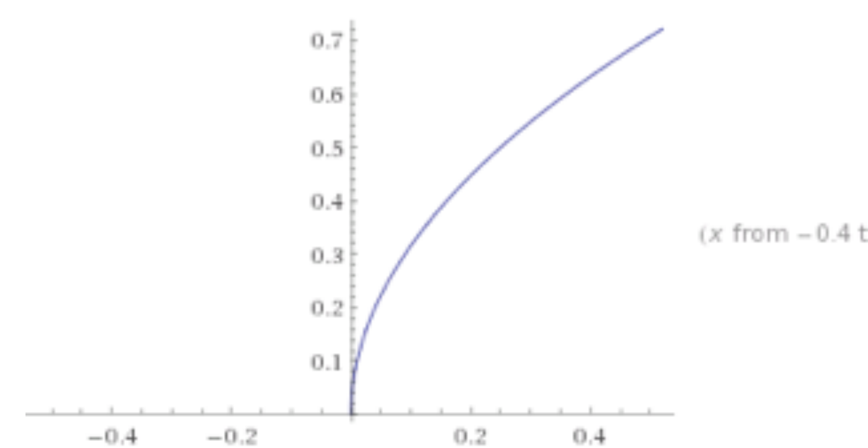












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Jensen's Inequality

Definition 1 *The function $f : \mathcal{D} \to \mathbb{R}$ is* convex *if for all $x_1, x_2 \in \mathcal{D}$ and for all $\lambda \in [0,1] \subset \mathbb{R}$:*

$$\lambda f(x_1) + (1-\lambda) f(x_2) \geq f(\lambda x_1 + (1-\lambda) x_2).$$

The function f is strictly convex *if equality only holds when $\lambda = 0$ or $\lambda = 1$, or when $x_1 = x_2$. The function f is* (strictly) concave *if the function $-f$ is (strictly) convex.*

Proposition 2 *(**Jensen's inequality**) Let the function $f : \mathcal{D} \to \mathbb{R}$ be convex, and let $n \in \mathbb{N}$. Then for any $p_1, \ldots, p_n \in \mathbb{R}_{\geq 0}$ such that $\sum_{i=1}^{n} p_i = 1$ and for any $x_1, \ldots, x_n \in \mathcal{D}$ it holds that*

$$\sum_{i=1}^{n} p_i f(x_i) \geq f\Big(\sum_{i=1}^{n} p_i x_i\Big).$$

If f is strictly convex and $p_1, \ldots, p_n > 0$, then equality holds iff $x_1 = \cdots = x_n$.

In particular, if X is a real random variable whose image $\mathcal{X}$ is contained in $\mathcal{D}$, then

$$E[f(X)] \geq f(E[X]),$$

and, if f is strictly convex, equality holds iff there is $c \in \mathcal{X}$ such that $X = c$ with probability 1.

Binary Entropy Function

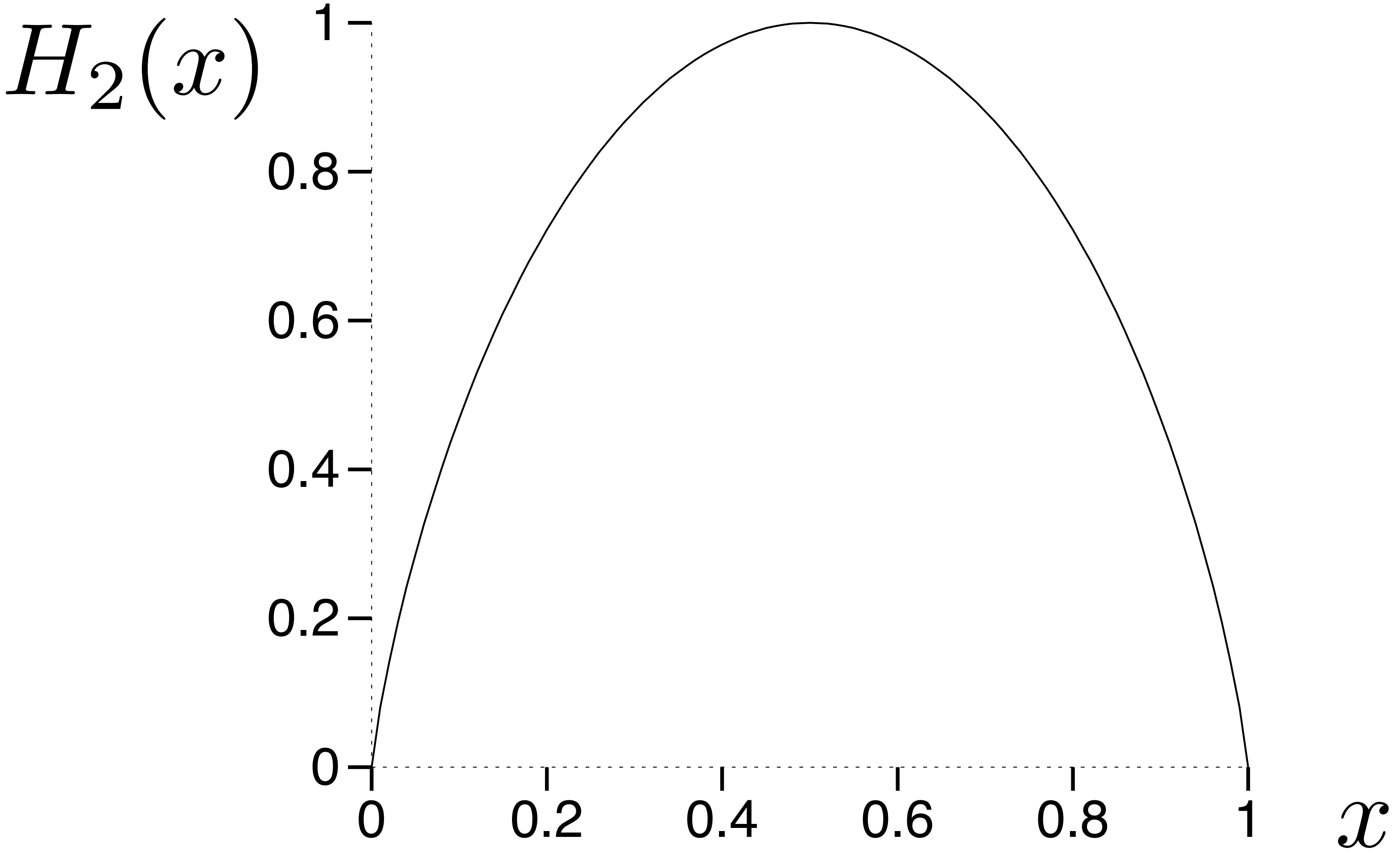


Figure 1.3. The binary entropy function.

Decomposability of Entropy

$$H(\mathbf{p}) = H(p_1, 1-p_1) + (1-p_1) H\left(\frac{p_2}{1-p_1}, \frac{p_3}{1-p_1}, \ldots, \frac{p_I}{1-p_1}\right). \qquad (2.43)$$

When it's written as a formula, this property looks regrettably ugly; nevertheless it is a simple property and one that you should make use of.

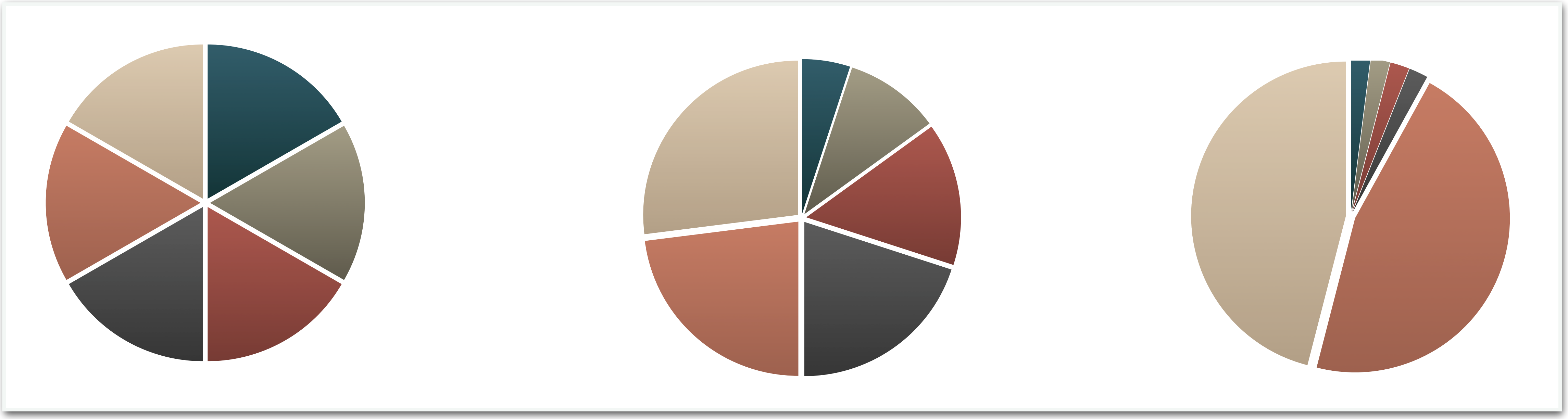
Generalizing further, the entropy has the property for any m that

$$\begin{aligned} H(\mathbf{p}) \;=\; & H\left[(p_1 + p_2 + \cdots + p_m), (p_{m+1} + p_{m+2} + \cdots + p_I)\right] \\ & + (p_1 + \cdots + p_m) H\left(\frac{p_1}{(p_1 + \cdots + p_m)}, \ldots, \frac{p_m}{(p_1 + \cdots + p_m)}\right) \\ & + (p_{m+1} + \cdots + p_I) H\left(\frac{p_{m+1}}{(p_{m+1} + \cdots + p_I)}, \ldots, \frac{p_I}{(p_{m+1} + \cdots + p_I)}\right). \end{aligned} \qquad (2.44)$$

Book by David MacKay

Order These in Terms of Entropy

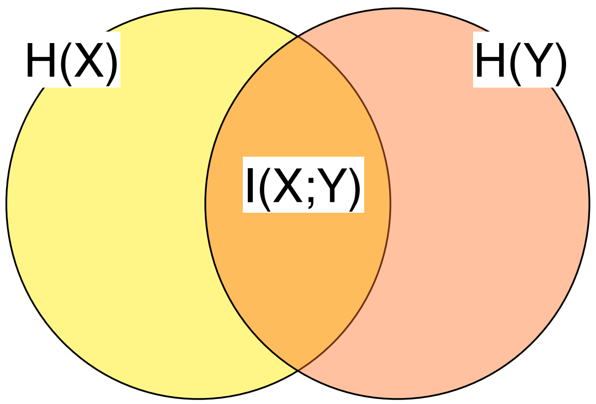
Order These in Terms of Entropy

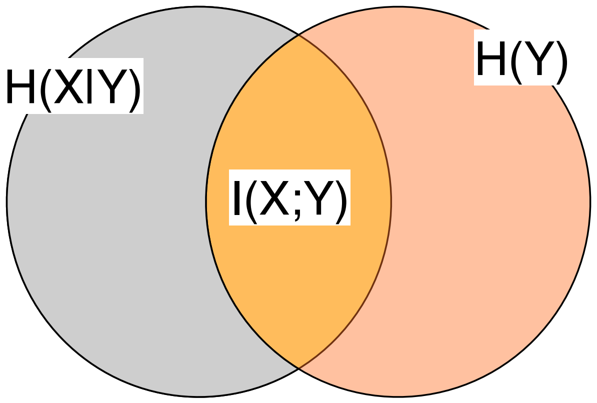


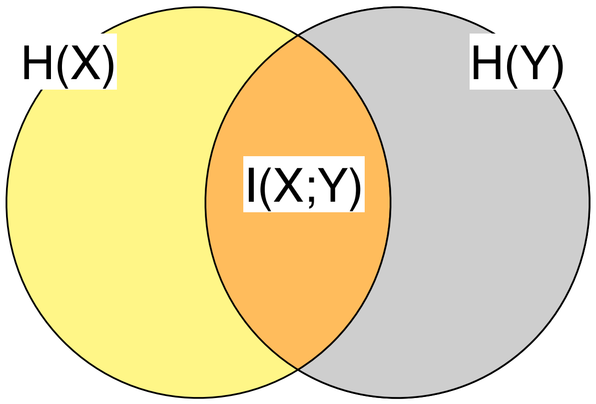
Mutual Information and Entropy

Theorem: Relationship between mutual information and entropy.

$$
\begin{aligned}
I(X;Y) &= H(X) - H(X|Y) \\
I(X;Y) &= H(Y) - H(Y|X) \\
I(X;Y) &= H(X) + H(Y) - H(X,Y) \\
I(X;Y) &= I(Y;X) \quad \text{(symmetry)} \\
I(X;X) &= H(X) \quad \text{(``self-information'')}
\end{aligned}
$$

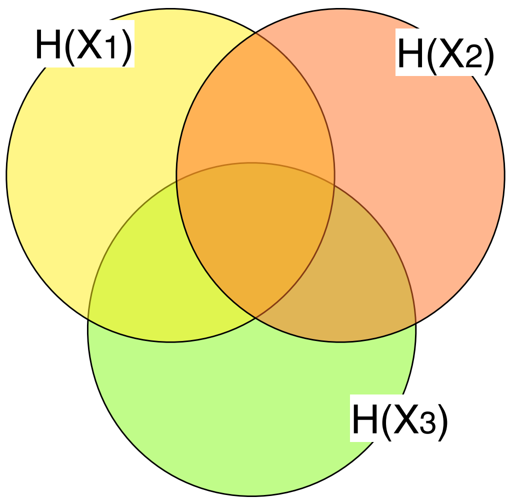




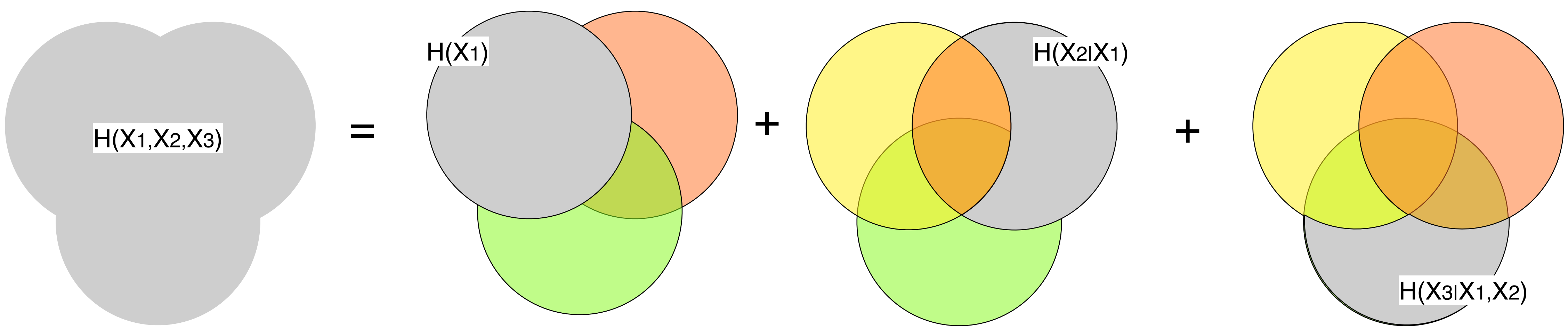


Chain Rule for Entropy

Theorem: (Chain rule for entropy): $(X_1, X_2, ..., X_n) \sim p(x_1, x_2, ..., x_n)$



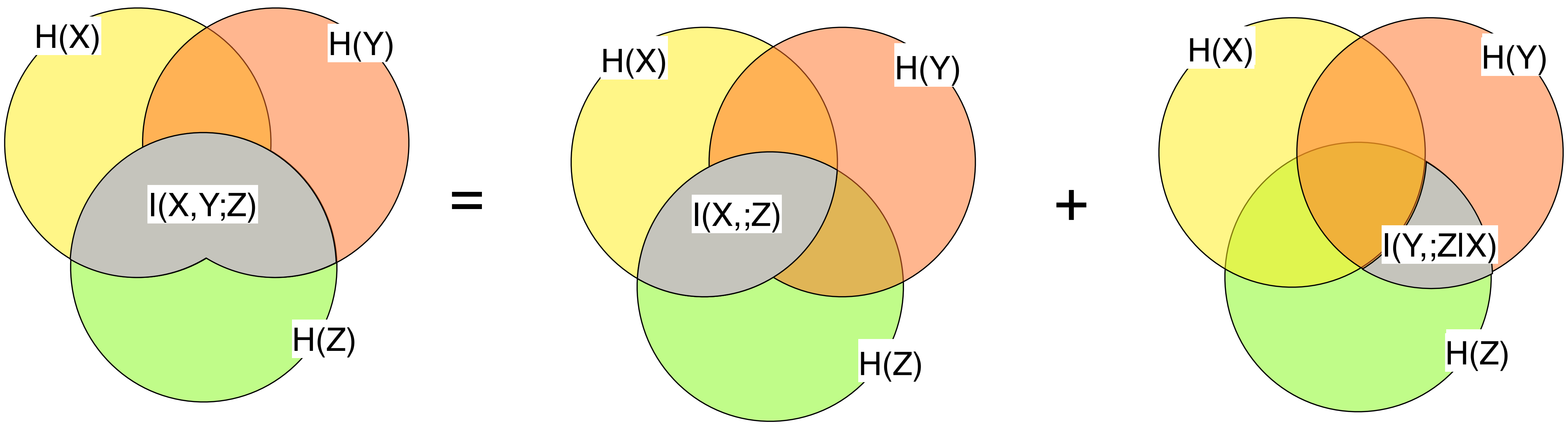
$$H(X_1, X_2, ..., X_n) = \sum_{i=1}^{n} H(X_i | X_{i-1}, ..., X_1)$$



Chain Rule for Mutual Information

Theorem: (Chain rule for mutual information)

$$I(X_1, X_2, ..., X_n; Y) = \sum_{i=1}^{n} I(X_i; Y | X_{i-1}, X_{i-2}, ..., X_1)$$



What are the Grey Regions?