Mutual Information of Multiple Transmissions

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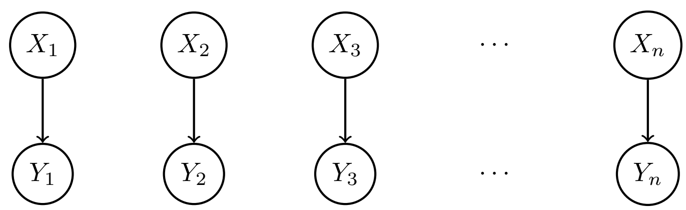
Multiple uses of a memoryless channel can be described in terms of the following model:



This model expresses that the input symbols are independent, and that the output symbol Y_i is independent of everything except its parent input, X_i. Using these assumptions, we can now compute:

1. **The Unconditional Output Entropy:** According to the model, the Y's are independent. Hence,

$$H(Y_1, Y_2, \dots, Y_n) \;=\; nH(Y).$$

2. **The Conditional Output Entropy:** The Y's are independent, so

$$\begin{aligned} H(Y_1, Y_2, \dots, Y_n \,|\, X_1, X_2, \dots, X_n) \;&=\; H(Y_1 \,|\, X_1, X_2, \dots, X_n) + \\ &\;\; H(Y_2 \,|\, X_1, X_2, \dots, X_n) + \\ &\;\; \vdots \\ &\;\; H(Y_n \,|\, X_1, X_2, \dots, X_n). \end{aligned}$$

Moreover, Y_i depends only on X_i, so

$$H(Y_i \,|\, X_1, X_2, \dots, X_n) \;=\; H(Y_i \,|\, X_i).$$

Hence,

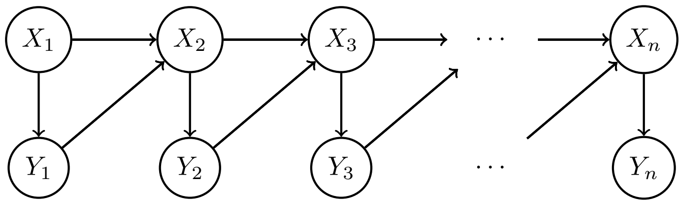
$$H(Y_1, Y_2, \dots, Y_n \,|\, X_1, X_2, \dots, X_n) \;=\; nH(Y \,|\, X).$$

3. **The Mutual Information:** Together, these two facts give us that

$$I(X_1, X_2, \dots, X_n; Y_1, Y_2, \dots, Y_n) \;=\; nI(X; Y).$$

So for the memoryless channel, the information conveyed by multiple channel uses is simply the sum of the information conveyed by each channel use.

Consider now the memoryless channel with feedback:



Under this model, the Y's are not independent in absolute terms, but they are conditionally independent given the X's. We therefore still have

$$H(Y_1, Y_2, \ldots, Y_n \,|\, X_1, X_2, \ldots, X_n) \;=\; nH(Y \,|\, X).$$

On the other hand, we no longer have (unconditional) independence, so

$$H(Y_1, Y_2, \ldots, Y_n) \;\le\; nH(Y),$$

without equality. We can therefore prove

$$I(X_1, X_2, \ldots, X_n; Y_1, Y_2, \ldots, Y_n) \;\le\; nI(X; Y),$$

but not the equality we had in the i.i.d. case.