Oblivious Transfer and Linear Functions

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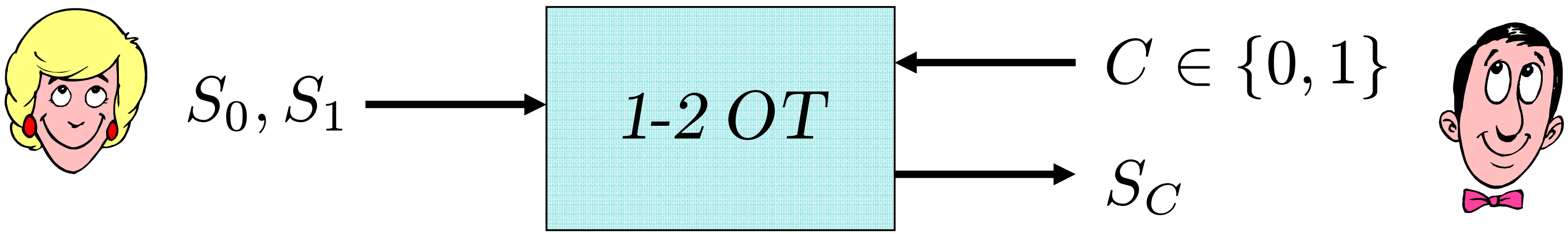
Agenda

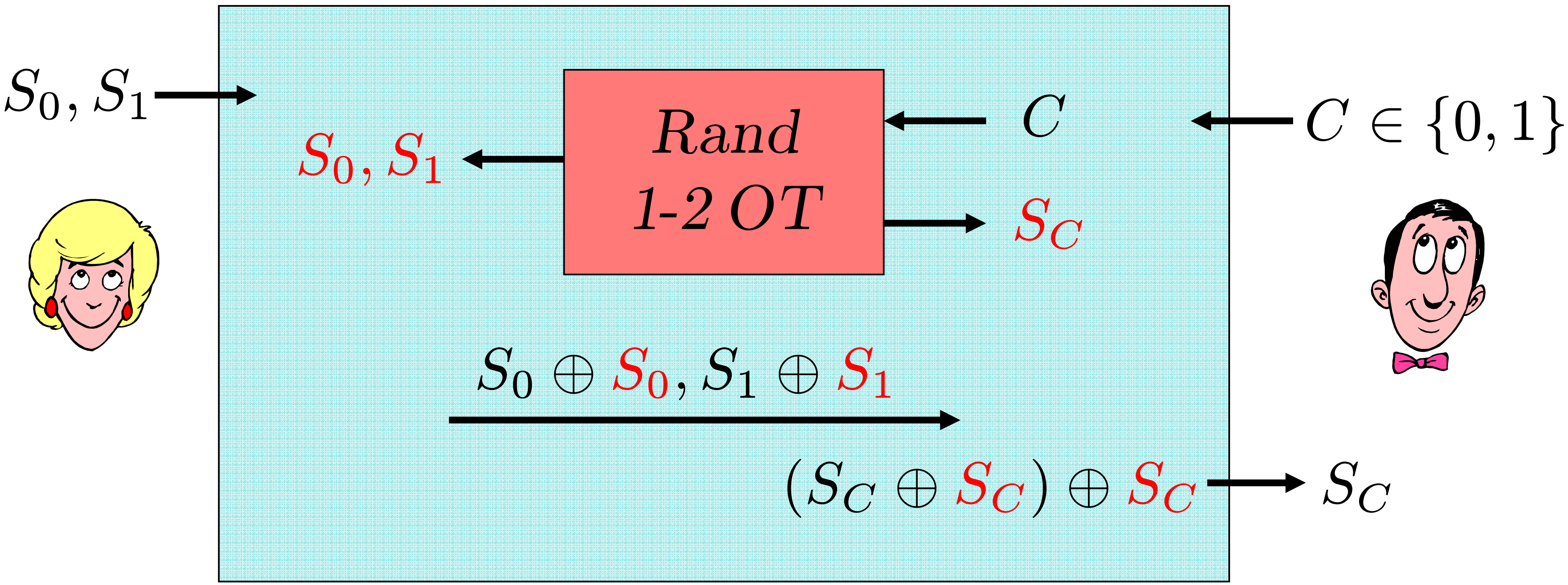
- OT and Randomized OT
- Characterisation of Sender-Security with Linear Functions
- Application: Universal OT
- Conclusion

1-2 Oblivious Transfer



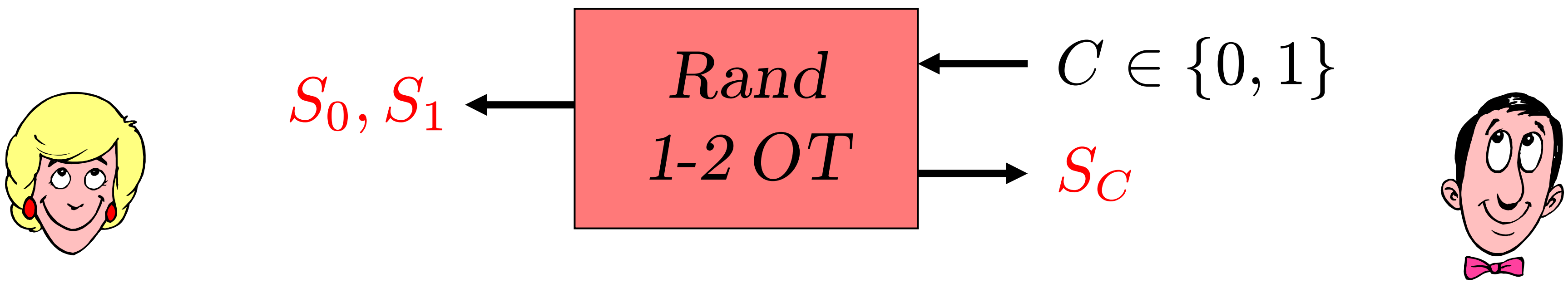
- **correctness:** works for honest players,
- **receiver-security:** $\forall \tilde{\mathsf{A}}$ dishonest senders, $\tilde{\mathsf{A}}$ should not learn C,
- **sender-security:** $\forall \tilde{\mathsf{B}}$ dishonest receivers, $\tilde{\mathsf{B}}$ should not learn S_{1-C}.

Randomized 1-2 Oblivious Transfer



Definition Rand 1-2 OT

[Crépeau Savvides Schaffner Wullschleger 06]

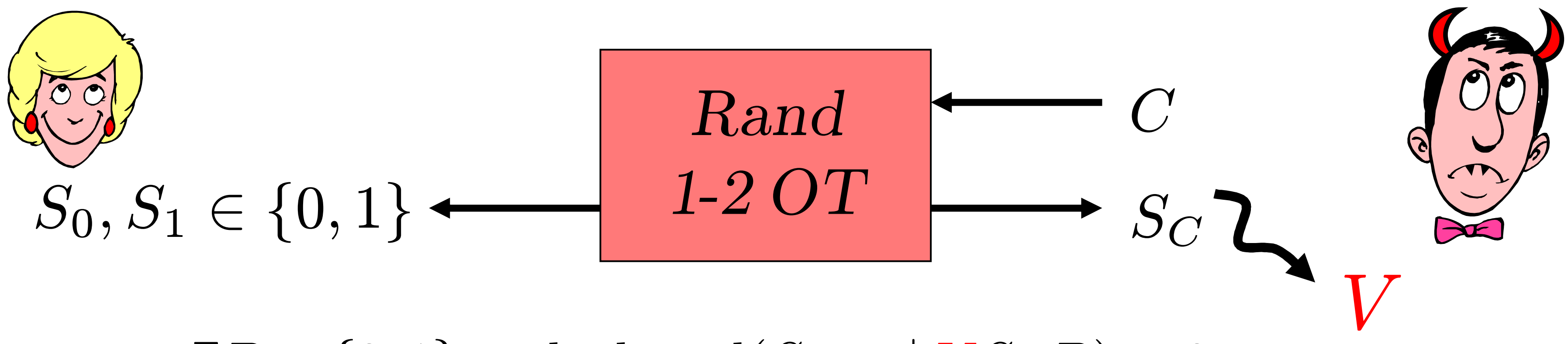


Protocol Π computes *Rand 1-2 OT* securely, if $\forall$ inputs C
Π produces outputs such that

- **correctness:** If A and B honest, A gets S_0, S_1, B gets S_C.
- **receiver-security:** $\forall \tilde{\mathsf{A}}$ dishonest senders with view U, $P_{UC} = P_U \cdot P_C$.
- **sender-security:** $\forall \tilde{\mathsf{B}}$ dishonest receivers with view V, $\exists D \in \{0, 1\}$ s.t. $d(S_{1-D} \mid V S_D D) = 0$.

$$\left\| P_{S_{1-D} V S_D D} - P_{\text{UNIF}} \cdot P_{V S_D D} \right\| = 0$$

Sender-Security for Rand OT of Bits



$$\exists D \in \{0,1\} \text{ such that } \mathrm{d}(S_{1-D} \mid V S_D D) = 0$$

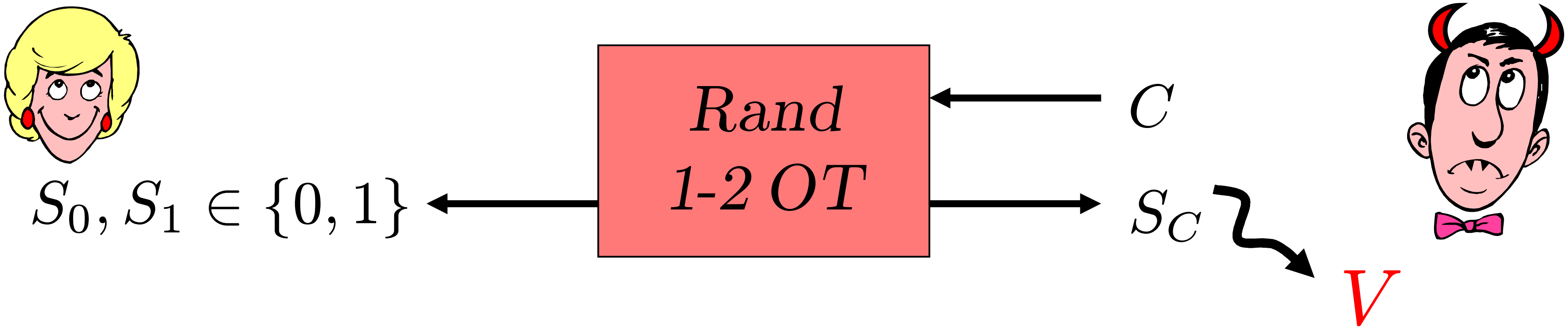
$$\Leftrightarrow \mathrm{d}(S_0 \oplus S_1 \mid V) = 0$$

S_0 \ S_1	0	1
0	½	¼
1	¼	0

$P_{S_0 S_1 V}(\cdot, \cdot, v)$

D ?

Sender-Security for Rand OT of Bits



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S_0 \ S_1	0	1
0	½	¼
1	¼	0

$P_{S_0 S_1 V}(\cdot,\cdot,v)$

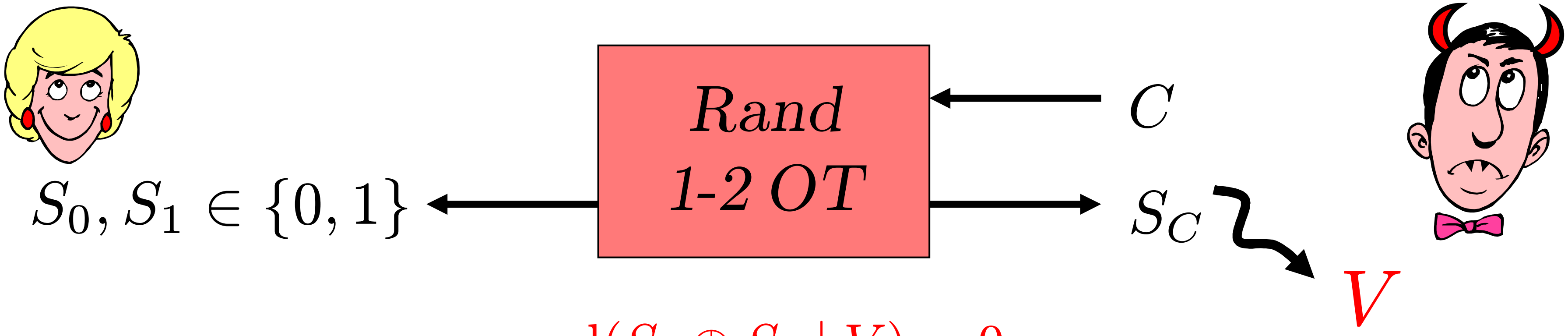
S_0 \ S_1	0	1
0	¼	¼
1	0	0

$P_{S_0 S_1 D V}(\cdot,\cdot,0,v)$

S_0 \ S_1	0	1
0	¼	0
1	¼	0

$P_{S_0 S_1 D V}(\cdot,\cdot,1,v)$

Sender-Security for Rand OT of Bits



$$d(S_0 \oplus S_1 \mid V) = 0$$

$\Rightarrow$ $\exists D \in \{0,1\}$ such that $d(S_{1-D} \mid V S_D D) = 0$

S_0 \ S_1	0	1
0	a	b
1	c	d

$P_{S_0 S_1 V}(\cdot, \cdot, v)$

S_0 \ S_1	0	1
0	a	a
1	c	c

$P_{S_0 S_1 D V}(\cdot, \cdot, 0, v)$

S_0 \ S_1	0	1
0	0	b-a
1	0	b-a

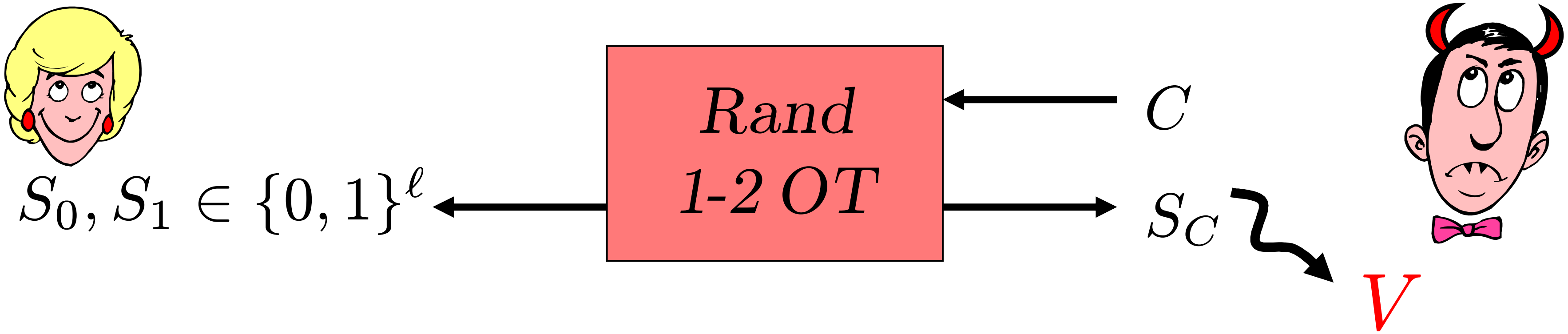
$P_{S_0 S_1 D V}(\cdot, \cdot, 1, v)$

$a + d = c + b$

$c + (b - a) = d$

wlog: $b \geq a$

Characterisation of Sender-Security



$$\exists D \in \{0,1\} \text{ such that } \mathrm{d}(S_{1-D} \mid V S_D D) \leq \varepsilon.$$

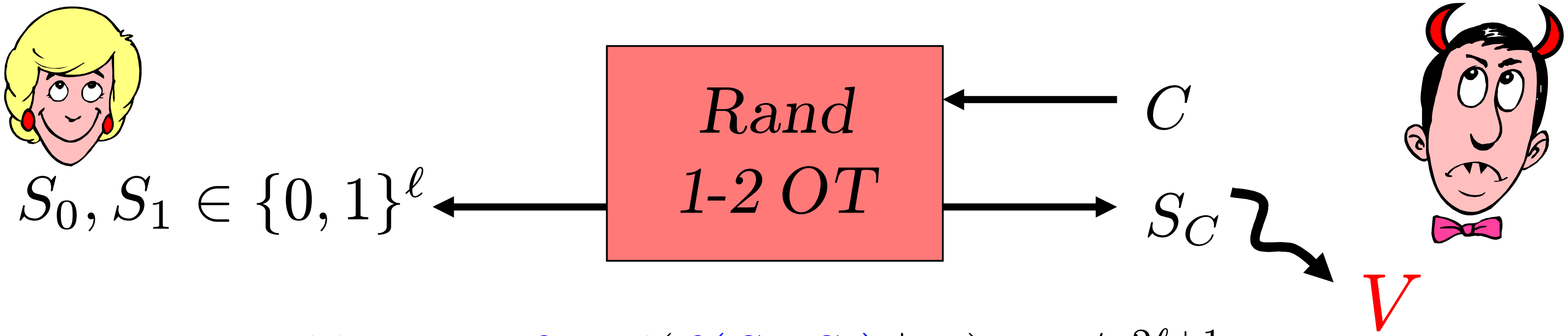
$$\Rightarrow \forall \text{ NDLF } \beta \;:\; \mathrm{d}(\beta(S_0, S_1) \mid V) \leq \varepsilon$$

Def: *A non-degenerate linear function (NDLF) is a function*

$$\beta : \{0,1\}^\ell \times \{0,1\}^\ell \to \{0,1\}$$
$$(s_0, s_1) \mapsto \langle a_0, s_0 \rangle \oplus \langle a_1, s_1 \rangle$$

for non-zero $a_0, a_1 \in \{0,1\}^\ell$, *i.e., it is linear and non-trivially depends on both inputs.*

Characterisation of Sender-Security



$$\forall \text{ NDLF } \beta \; : \; \mathrm{d}(\beta(S_0, S_1) \mid V) \leq \varepsilon / 2^{2\ell+1}$$

$$\Rightarrow \exists D \in \{0,1\} \text{ such that } \mathrm{d}(S_{1-D} \mid V S_D D) \leq \varepsilon.$$

Theorem: If for all NDLF β, $\mathrm{d}(\beta(S_0, S_1) \mid V) \leq \varepsilon / 2^{2\ell+1}$ holds, then there exists $D \in \{0,1\}$ such that $\mathrm{d}(S_{1-D} \mid V S_D D) \leq \varepsilon$.

Proof for $\ell = 2$, perfect case

Theorem: If for all NDLF β, $\mathrm{d}(\beta(S_0, S_1) \mid V) = 0$ holds, then there exists $D \in \{0,1\}$ such that $\mathrm{d}(S_{1-D} \mid V S_D D) = 0$.

$S_0 \backslash S_1$	00	01	10	11
00	a	b	c	d
01	e	f	g	h
10	i	j	k	l
11	m	n	o	p

$P_{S_0 S_1 V}(\cdot, \cdot, v)$

$S_0 \backslash S_1$	00	01	10	11
00	a	a	a	a
01	e	e	e	e
10	i	i	i	i
11	m	m	m	m

$P_{S_0 S_1 D V}(\cdot, \cdot, 0, v)$

$S_0 \backslash S_1$	00	01	10	11
00	0	b-a	c-a	d-a
01	0	b-a	c-a	d-a
10	0	b-a	c-a	d-a
11	0	b-a	c-a	d-a

$P_{S_0 S_1 D V}(\cdot, \cdot, 1, v)$

Proof for $\ell = 2$, perfect case

Theorem: If for all NDLF β, $d(\beta(S_0, S_1) \mid V) = 0$ holds, then there exists $D \in \{0,1\}$ such that $d(S_{1-D} \mid V S_D D) = 0$.

$S_0 \backslash S_1$	00	01	10	11
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10	i	j	k	l
11	m	n	o	p

$P_{S_0 S_1 V}(\cdot, \cdot, v)$

$S_0 \backslash S_1$	00	01	10	11
00	a	a	a	a
01	e	e	e	e
10	i	i	i	i
11	m	m	m	m

$P_{S_0 S_1 D V}(\cdot, \cdot, 0, v)$

$S_0 \backslash S_1$	00	01	10	11
00	0	b-a	c-a	d-a
01	0	b-a	c-a	d-a
10	0	b-a	c-a	d-a
11	0	b-a	c-a	d-a

$P_{S_0 S_1 D V}(\cdot, \cdot, 1, v)$

$b+e=a+f$

Proof for $\ell = 2$, perfect case

Theorem: If for all NDLF β, $\mathrm{d}(\beta(S_0, S_1) \mid V) = 0$ holds, then there exists $D \in \{0,1\}$ such that $\mathrm{d}(S_{1-D} \mid V S_D D) = 0$.

$S_0 \backslash S_1$	00	01	10	11
00	a	b	c	d
01	e	f	g	h
10	i	j	k	l
11	m	n	o	p

$P_{S_0 S_1 V}(\cdot, \cdot, v)$

$S_0 \backslash S_1$	00	01	10	11
00	a	a	a	a
01	e	e	e	e
10	i	i	i	i
11	m	m	m	m

$P_{S_0 S_1 D V}(\cdot, \cdot, 0, v)$

$S_0 \backslash S_1$	00	01	10	11
00	0	b-a	c-a	d-a
01	0	b-a	c-a	d-a
10	0	b-a	c-a	d-a
11	0	b-a	c-a	d-a

$P_{S_0 S_1 D V}(\cdot, \cdot, 1, v)$

$b + e = a + f$

$c + e = a + g$

Proof for $\ell = 2$, perfect case

Theorem: If for all NDLF β, $\mathrm{d}(\beta(S_0, S_1) \mid V) = 0$ holds, then there exists $D \in \{0, 1\}$ such that $\mathrm{d}(S_{1-D} \mid V S_D D) = 0$.

$S_0 \backslash S_1$	00	01	10	11
00	a	b	c	d
01	e	f	g	h
10	i	j	k	l
11	m	n	o	p

$P_{S_0 S_1 V}(\cdot, \cdot, v)$

$b + e = a + f$

$S_0 \backslash S_1$	00	01	10	11
00	a	a	a	a
01	e	e	e	e
10	i	i	i	i
11	m	m	m	m

$P_{S_0 S_1 D V}(\cdot, \cdot, 0, v)$

$c + e = a + g$

$S_0 \backslash S_1$	00	01	10	11
00	0	b-a	c-a	d-a
01	0	b-a	c-a	d-a
10	0	b-a	c-a	d-a
11	0	b-a	c-a	d-a

$P_{S_0 S_1 D V}(\cdot, \cdot, 1, v)$

$d + e = a + h$

Proof for $\ell = 2$, perfect case

Theorem: If for all NDLF β, $\mathrm{d}(\beta(S_0, S_1) \mid V) = 0$ holds, then there exists $D \in \{0,1\}$ such that $\mathrm{d}(S_{1-D} \mid V S_D D) = 0$.

$S_0 \backslash S_1$	00	01	10	11
00	a	b	c	d
01	e	f	g	h
10	i	j	k	l
11	m	n	o	p

$P_{S_0 S_1 V}(\cdot,\cdot,v)$

$S_0 \backslash S_1$	00	01	10	11
00	a	a	a	a
01	e	e	e	e
10	i	i	i	i
11	m	m	m	m

$P_{S_0 S_1 D V}(\cdot,\cdot,0,v)$

$S_0 \backslash S_1$	00	01	10	11
00	0	b-a	c-a	d-a
01	0	b-a	c-a	d-a
10	0	b-a	c-a	d-a
11	0	b-a	c-a	d-a

$P_{S_0 S_1 D V}(\cdot,\cdot,1,v)$

$+ \quad b + e = a + f$
$- \quad b + i = a + j$
$+ \quad b + m = a + n$

$- \quad c + e = a + g$
$+ \quad c + i = a + k$
$- \quad c + m = a + o$

$+ \quad d + e = a + h$
$- \quad d + i = a + l$
$+ \quad d + m = a + p$

$$b + d + e + g + j + l + m + o = a + c + f + h + i + k + n + p$$

Proof for $\ell = 2$, perfect case

Theorem: If for all NDLF β, $d(\beta(S_0, S_1) \mid V) = 0$ holds, then there exists $D \in \{0, 1\}$ such that $d(S_{1-D} \mid V S_D D) = 0$.

$S_0 \backslash S_1$	00	01	10	11
00	a	b	c	d
01	e	f	g	h
10	i	j	k	l
11	m	n	o	p

$P_{S_0 S_1 V}(\cdot, \cdot, v)$

$\beta(s_0, s_1) := s_0^2 \oplus s_1^2$

$P_{S_0 S_1 D V}(\cdot, \cdot, 0, v)$

01	10	11
b-a	c-a	d-a
b-a	c-a	d-a
b-a	c-a	d-a
b-a	c-a	d-a

$P_{S_0 S_1 D V}(\cdot, \cdot, 1, v)$

$+\ b + e = a + f$	$-\ c + e = a + g$	$+\ d + e = a + h$
$-\ b + i = a + j$	$+\ c + i = a + k$	$-\ d + i = a + l$
$+\ b + m = a + n$	$-\ c + m = a + o$	$+\ d + m = a + p$

$$b + d + e + g + j + l + m + o = a + c + f + h + i + k + n + p$$

Proof for $\ell = 2$, perfect case

Theorem: If for all NDLF β, $\mathrm{d}(\beta(S_0, S_1) \mid V) = 0$ holds, then there exists $D \in \{0,1\}$ such that $\mathrm{d}(S_{1-D} \mid V S_D D) = 0$.

$S_0 \backslash S_1$	00	01	10	11
00	a	b	c	d
01	e	f	g	h
10	i	j	k	l
11	m	n	o	p

$P_{S_0 S_1 V}(\cdot,\cdot,v)$

$S_0 \backslash S_1$	00	01	10	11
00				
01				
10				
11				

$P_{S_0 S_1 D V}(\cdot,\cdot,0,v)$

$S_0 \backslash S_1$	00	01	10	11
		b-a	c-a	d-a
		b-a	c-a	d-a
		b-a	c-a	d-a
		b-a	c-a	d-a

$P_{S_0 S_1 D V}(\cdot,\cdot,1,v)$

$\beta(s_0, s_1) := s_0^2 \oplus s_1^2$

$\beta(s_0, s_1) = 1$

$\beta(s_0, s_1) = 0$

$+ \quad b + e = a + f$	$-\quad c + e = a + g$	$+\quad d + e = a + h$
$-\quad b + i = a + j$	$+\quad c + i = a + k$	$-\quad d + i = a + l$
$+\quad b + m = a + n$	$-\quad c + m = a + o$	$+\quad d + m = a + p$

$$\underbrace{b + d + e + g + j + l + m + o}_{\beta(s_0,s_1)=1} = \underbrace{a + c + f + h + i + k + n + p}_{\beta(s_0,s_1)=0}$$

Proof for $\ell = 2$, perfect case

Theorem: If for all NDLF β, $\mathrm{d}(\beta(S_0, S_1) \mid V) = 0$ holds, then there exists $D \in \{0,1\}$ such that $\mathrm{d}(S_{1-D} \mid V S_D D) = 0$.

$S_0 \backslash S_1$	00	01	10	11
00	a	b	c	d
01	e	f	g	h
10	i	j	k	l
11	m	n	o	p

$P_{S_0 S_1 V}(\cdot,\cdot,v)$

$\beta(s_0, s_1) := s_0^1 \oplus s_1^2$

$\beta(s_0, s_1) = 1$

$\beta(s_0, s_1) = 0$

$P_{S_0 S_1 D V}(\cdot,\cdot,0,v)$

$S_0 \backslash S_1$	00	01	10	11
[illegible]	[illegible]	b-a	c-a	d-a
[illegible]	[illegible]	b-a	c-a	d-a
[illegible]	[illegible]	b-a	c-a	d-a
[illegible]	[illegible]	b-a	c-a	d-a

$P_{S_0 S_1 D V}(\cdot,\cdot,1,v)$

$b+e=a+f$	$c+e=a+g$	$d+e=a+h$
$b+i=a+j$	$c+i=a+k$	$d+i=a+l$
$b+m=a+n$	$c+m=a+o$	$d+m=a+p$

$$b+d+e+g+j+l+m+o = a+c+f+h+i+k+n+p$$

$$b+d+f+h+i+k+m+o = a+c+e+g+j+l+n+p$$

Proof for $\ell = 2$, perfect case

Theorem: If for all NDLF β, $\mathrm{d}(\beta(S_0, S_1) \mid V) = 0$ holds, then there exists $D \in \{0,1\}$ such that $\mathrm{d}(S_{1-D} \mid V S_D D) = 0$.

$S_0 \backslash S_1$	00	01	10	11
00	a	b	c	d
01	e	f	g	h
10	i	j	k	l
11	m	n	o	p

$P_{S_0 S_1 V}(\cdot,\cdot,v)$

$S_0 \backslash S_1$	00	01	10	11
00	a	a	a	a
01	e	e	e	e
10	i	i	i	i
11	m	m	m	m

$P_{S_0 S_1 D V}(\cdot,\cdot,0,v)$

$S_0 \backslash S_1$	00	01	10	11
00	0	b-a	c-a	d-a
01	0	b-a	c-a	d-a
10	0	b-a	c-a	d-a
11	0	b-a	c-a	d-a

$P_{S_0 S_1 D V}(\cdot,\cdot,1,v)$

$$\begin{cases} b+e=a+f & c+e=a+g & d+e=a+h \\ b+i=a+j & c+i=a+k & d+i=a+l \\ b+m=a+n & c+m=a+o & d+m=a+p \end{cases}$$

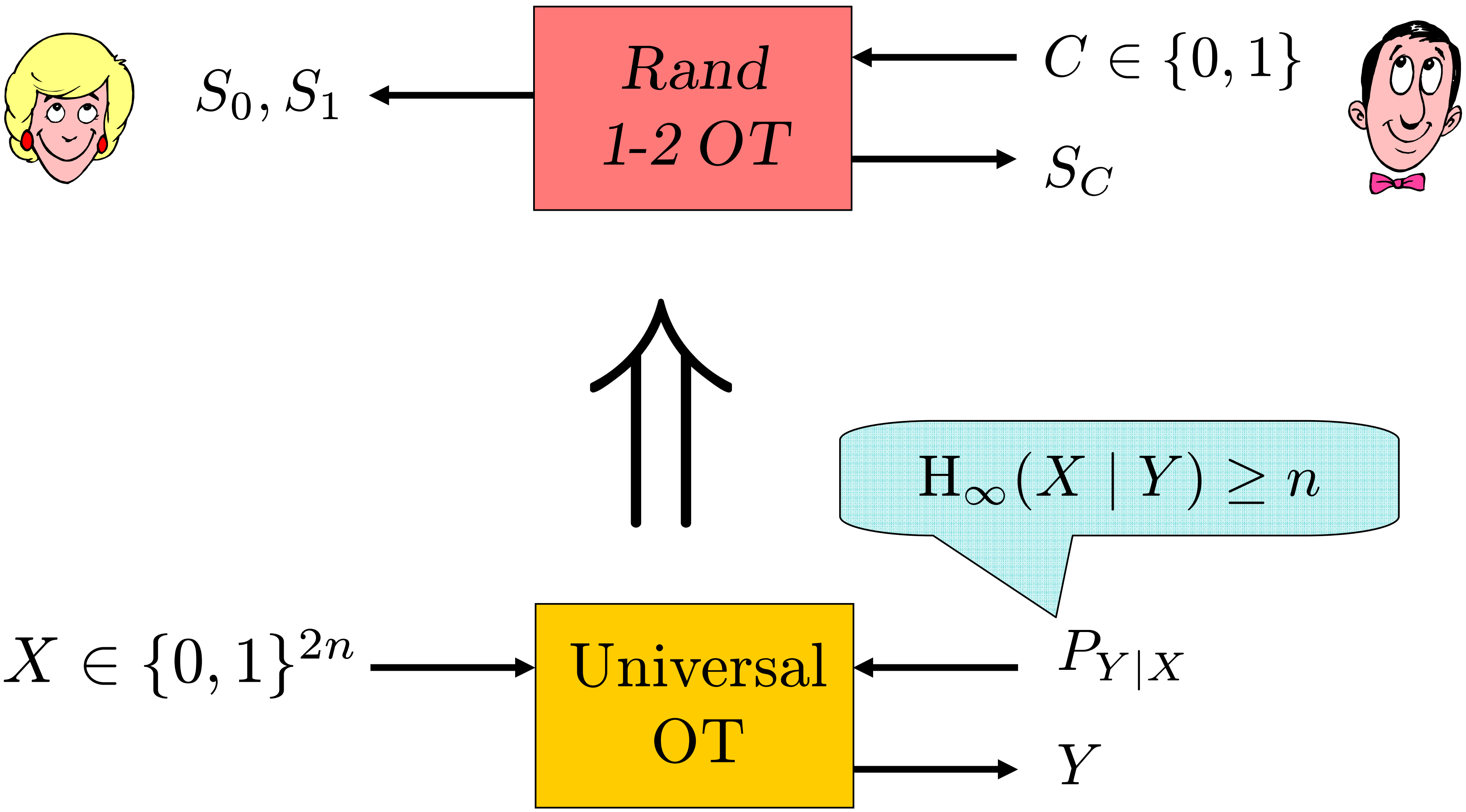
$$\begin{cases} b+d+e+g+j+l+m+o=a+c+f+h+i+k+n+p \\ b+d+f+h+i+k+m+o=a+c+e+g+j+l+n+p \\ \vdots \end{cases}$$

□

Agenda

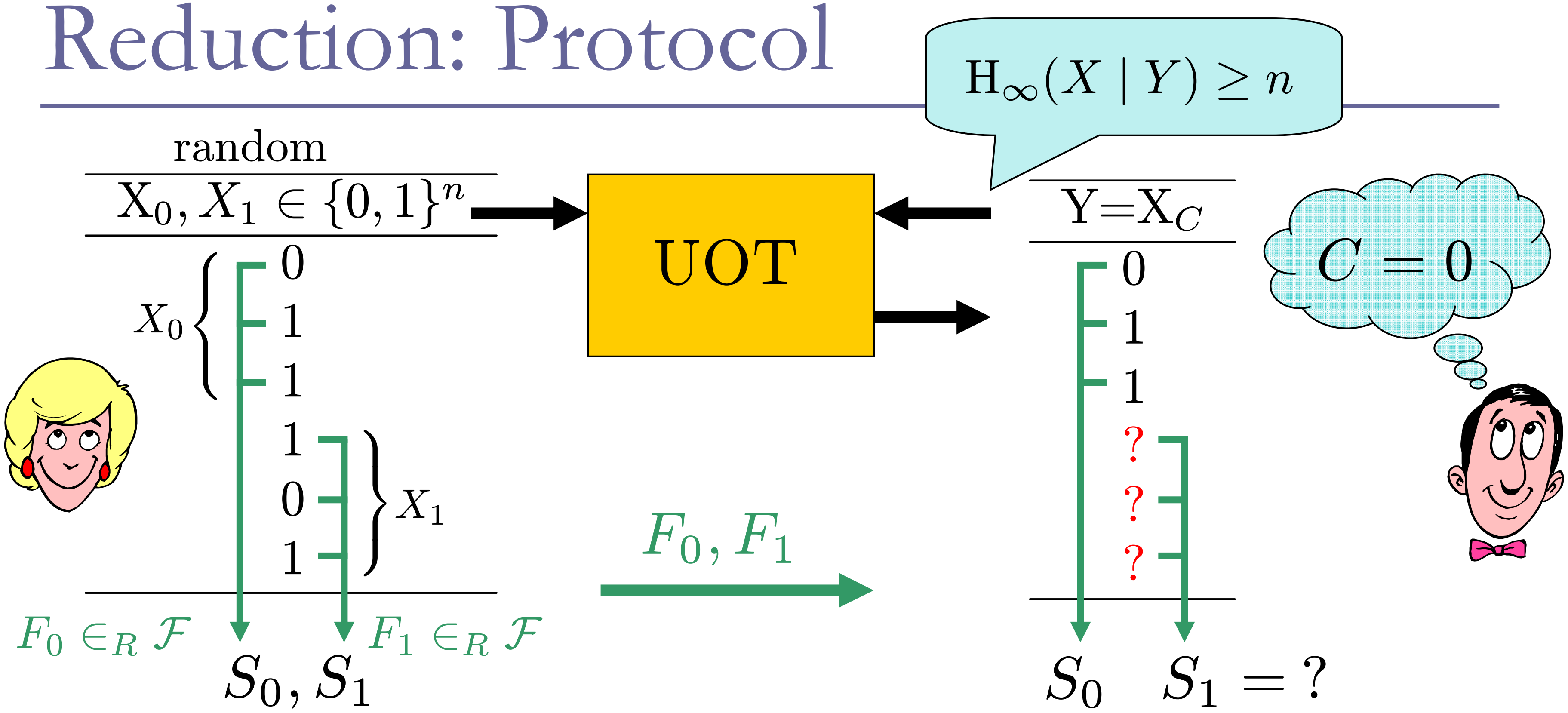
- ✓ OT and Randomized OT
- ✓ Characterisation of Sender-Security with Linear Functions
- **Application: Universal OT**
- Conclusion

Application: Universal OT



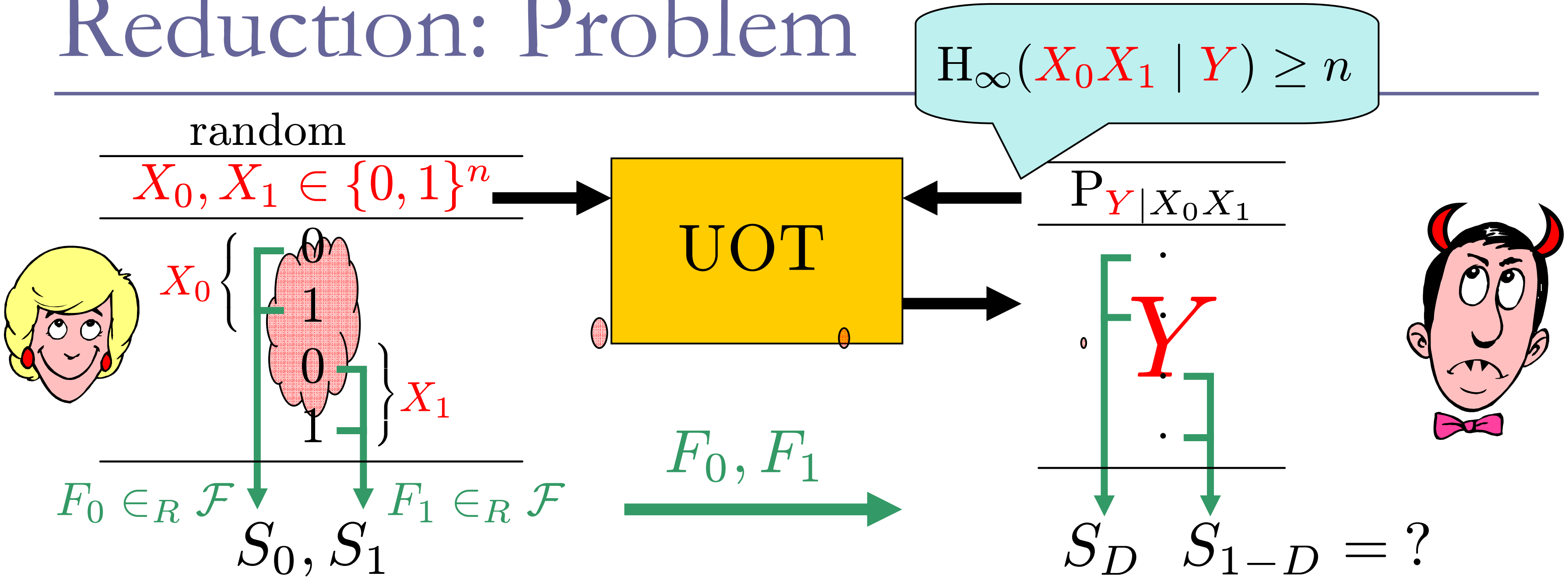
[Cachin 98]

Reduction: Protocol



- **correctness:** If A and B honest, A gets S_0, S_1, B gets S_C. ✓
- **receiver-security:** $\forall \tilde{A}$ dishonest senders with view U, $P_{UC} = P_U \cdot P_C$. ✓

Reduction: Problem



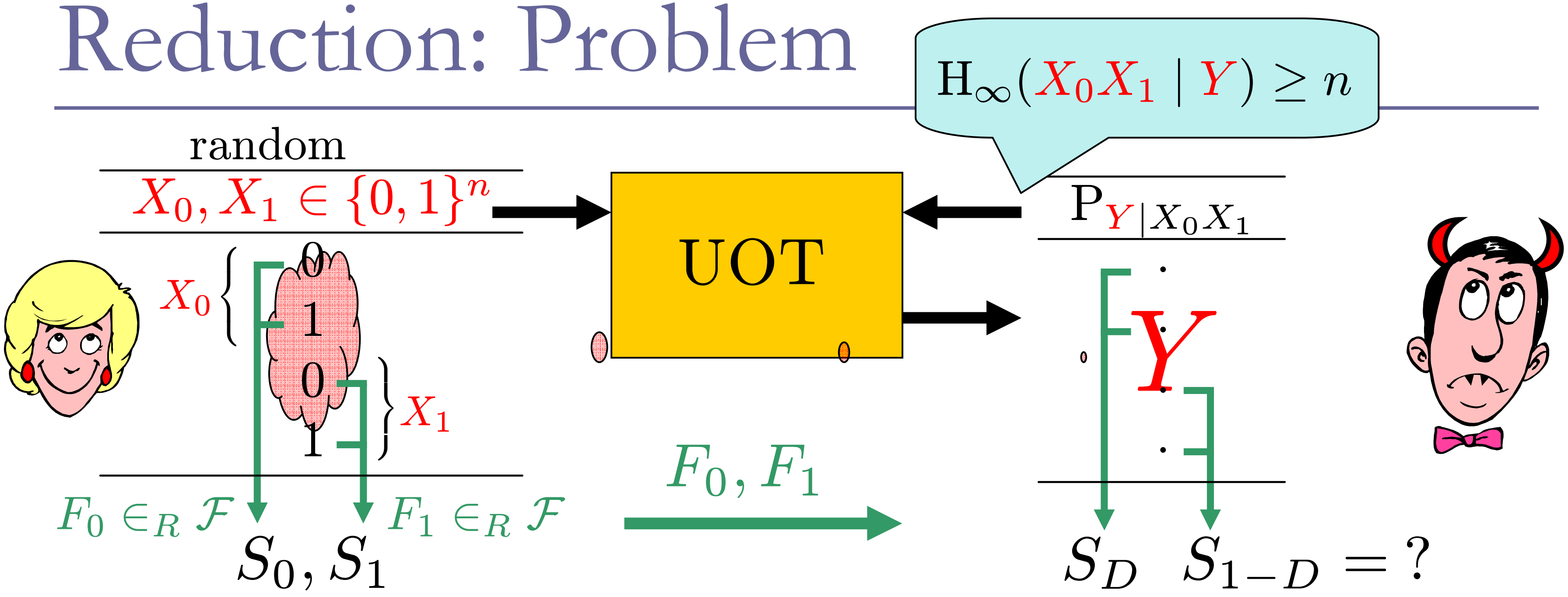
sender-security: $\exists D$ s.t. $d(S_{1-D} \mid V D S_D) \leq \varepsilon$

$X_0 \in \{0,1\}^n \xrightarrow{F_0 \in_R \mathcal{F}} S_0 \in \{0,1\}^\ell$

$x \neq x' \Rightarrow F(x), F(x')$ independent and uniform.

$H_\infty(X_0 \mid V)$ big $\Rightarrow d(S_0 \mid V F)$ small **Privacy Amplification**

Reduction: Problem



Theorem: $\forall$ NDLF β : $d(\beta(S_0, S_1) \mid V)$ small

$\Rightarrow$ **sender-security:** $\exists D$ s.t. $d(S_{1-D} \mid VDS_D) \leq \varepsilon$

$X_0 \in \{0,1\}^n \xrightarrow{F_0 \in_R \mathcal{F}} S_0 \in \{0,1\}^\ell$

$X_1 \in \{0,1\}^n \xrightarrow{F_1 \in_R \mathcal{F}} S_1 \in \{0,1\}^\ell$

$\Rightarrow \beta(S_0, S_1) \in \{0,1\}$

Observe: The class $\mathcal{F}^\beta := \{\beta(F_0(\cdot), F_1(\cdot)) \mid F_0 \in \mathcal{F}, F_1 \in \mathcal{F}\}$ is strongly two-universal.

$H_\infty(X_0 \mid V)$ big $\Rightarrow d(S_0 \mid V, F)$ small **Privacy Amplification**

$H_\infty(X_0X_1 \mid V)$ big $\Rightarrow d(\beta(S_0, S_1) \mid V)$ small **PA**

Conclusion

- **Characterisation of sender-privacy:**
  A protocol for Rand-OT is secure against a cheating receiver, iff he gets no information about any non-degenerate linear function.
- **Observation:**
  NDLF compose well with strongly two-universal hash functions.
- **yields powerful technique:**
  Enough min-entropy suffices to get an OT via (strongly) two-universal hashing,
  also works in quantum settings.
- **Reductions:**
  Simpler analyses of reductions from String-OT to weaker primitives, better parameters.

Thank you

The Bottom Line

Alice   Bob

$X \in \{0,1\}^n$

view $V = v$.

01100101010110

$F_0 \in_R \mathcal{F}$ $F_1 \in_R \mathcal{F}$

F_0, F_1

101 , 001

$S_0, S_1 \in \{0,1\}^\ell$

$H_\infty(X \mid V = v) \geq \alpha n$

$\exists C'$ s.t.

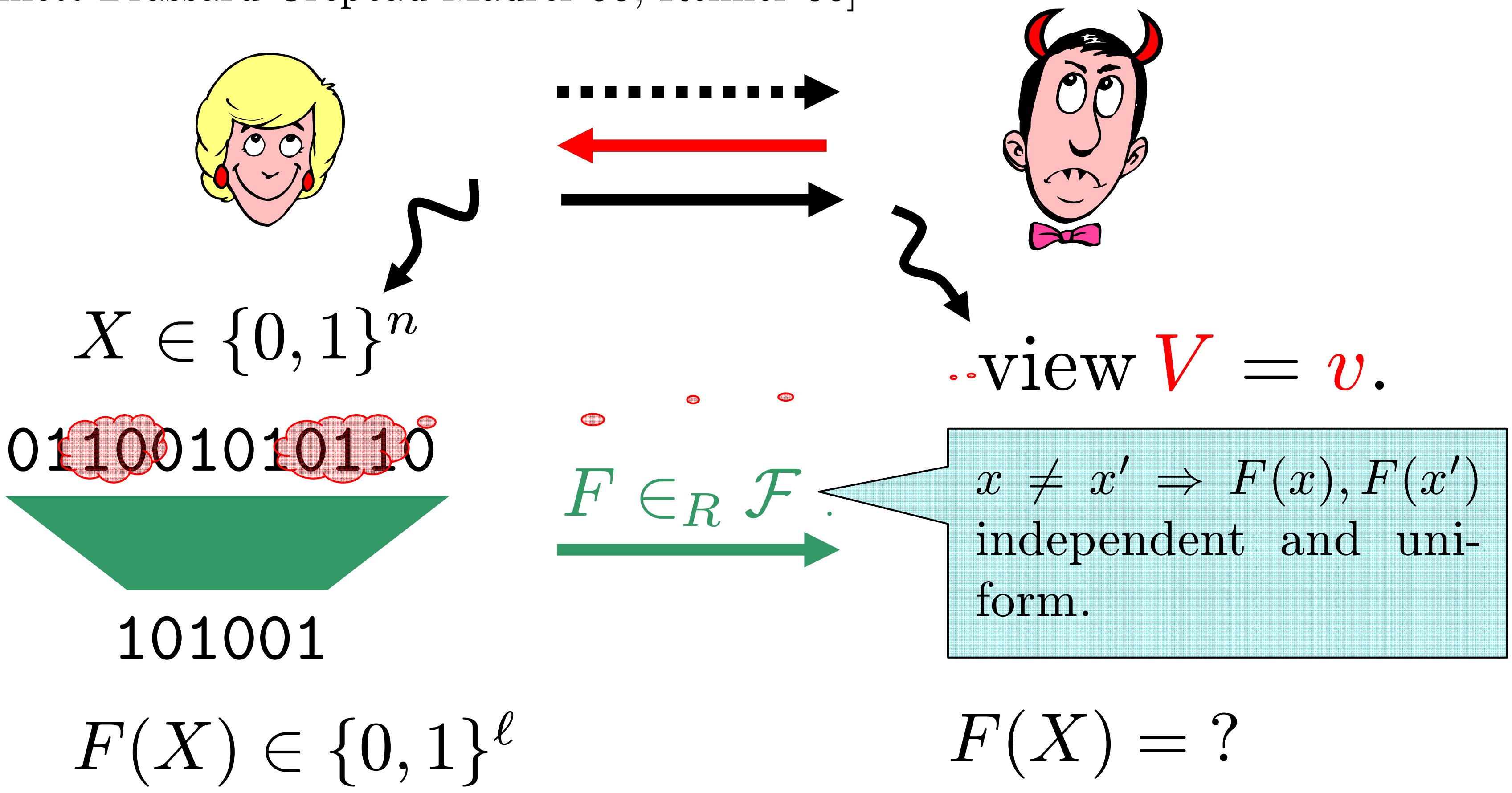
$d(S_{1-C'} \mid V C' S_{C'}) \leq \varepsilon$

$F \in_R \mathcal{F}^\beta := \{\beta(F_0(\cdot), F_1(\cdot)) \mid F_0 \in \mathcal{F}_0, F_1 \in \mathcal{F}_1\}$

$$\text{PA: } d(\underbrace{F(X)}_{\beta(S_0,S_1)} \mid \underbrace{F, V = v}_{V}) \leq 2^{-\frac{1}{2}\left(H_\infty(X|V=v)-1\right)} \leq \varepsilon \cdot 2^{-2\ell-1}$$

Privacy Amplification (PA)

[Impagliazzo Levin Luby 89, Håstad ILL 99,
Bennett Brassard Crépeau Maurer 95, Renner 06]



Thm: $\mathrm{d}(F(X) \mid F, V = v) \leq 2^{-\frac{1}{2}\left(\mathrm{H}_\infty(X|V=v)-\ell\right)}$