Part: II planet formation

L7: Aerodynamics

L8: pre-planetary growth

L9: Gravitational instability

L10: Two and three body systems

L11: Runaway & Oligarchic growth; Pebble accretion

L12: giant and terrestrial planet formation

L13: planet migration & resonant trapping
2.2 The 3-body problem

In the 3-body problem analytical (closed form) solutions are no longer possible. A simplification of the 3-body problem is that of a massless particle being perturbed by a secondary (e.g., planet) that moves on a circular orbit around the primary (star). This is known as the circular, restricted 3-body problem CR3BP. We will focus exclusively on this problem.

The equation of motion in a frame of reference rotating with angular frequency \( \omega \) is:

\[
\ddot{r} - \nabla \Phi = -2\omega \times \dot{r} - \omega \times (\omega \times r)
\]

(2.5)

In the CR3BP we will of course choose \( \omega = n_{PB} \), such that \( \Phi \) - the gravitational potential - is time-independent in the rotating frame. Equation (2.5) can be integrated to give an integration constant \( J^* \):

\[
J^* = \frac{1}{2} \dot{r}^2 + \Phi - \frac{1}{2}(\omega \times r)^2
\]

(2.6)

which is the Jacobi energy. In the 3-body problem it is the only integral of motion.

Exercise 2.2 [Jacobi integral]

(a) Converting Equation (2.6) back to the inertial frame, show that:

\[
\dot{E} = L \cdot \omega - I \dot{L} = E - n_{PB} L
\]

(2.7)

where \( E \) and \( L \) are the energy and angular momentum measured in the inertial frame. Hence, in the CR3BP interactions will exchange \( E \) and \( L \), while \( J^* \) stays constant.

(b) Express \( J^* \) in orbital elements:

\[
J^* = \frac{GM_2}{2} (e - 1) \cos \iota
\]

(2.8)

where \( e \) is the mean motion of the secondary and the other symbols refer to the test particle. Written in the form of Equation (2.8) (or analogous) the Jacobi integral is called the *toroidal relative*.

(c) Let \( e = e_0 + b \) with \( e_0 \) the semi-major axis corresponding to \( n^* \), and consider the limit where \( b/n^* \to \infty \), \( I \to 1 \), and \( e \to 1 \). Show that in that case:

\[
J^* \to \frac{GM_2}{2} \left( \frac{1}{e} - \frac{1}{e} \right) \cos \iota
\]

(2.9)

where we have discarded a constant term from \( J^* \).

It is instructive to redefine the potential in Equation (2.5), incorporating the centrifugal term:

\[
\Phi_{eff} = \Phi_1 + \Phi_2 + \frac{1}{2} r^2 \gamma^2 = \left( \frac{GM_2}{r_1} + \frac{1}{2} \frac{1}{r_f^3} + \Phi_2 \right)
\]

(2.10)

where we used the identity \( \sqrt{\gamma} \approx r \). Consider the motion of the test particle in the vicinity of \( \Phi_{eff} \). See Figure 2.4, and express the potential in local coordinates \((x, y)\) centered on \( n^* \). This amounts to expanding the inverse distance \( 1/r \) in terms of \( x \) and \( y \).

The result is (Hill's approximation):

\[
\Phi_{eff} = -\frac{3}{2} x^2 + \frac{1}{2} y^2 - \frac{GM_2}{r}
\]

(2.11)

with which the Jacobi energy is written:

\[
J = \frac{1}{2} \dot{r}^2 + \Phi_{eff}
\]

(2.12)

Contours of \( \Phi_{eff} \) are known as zero velocity curves; they define the regions where a particle of a certain \( L \) can move, since \( \Phi_{eff} = J - \frac{1}{2} L^2 \leq J \). Therefore, although the 3-body problem is not integrable, given \( J \) we can constrain the regions where particles can be found. Figure 2.5 shows contours of constant \( \Phi_{eff} \) with lighter contours having larger \( \Phi_{eff} \). The regions bounded by high \( \Phi_{eff} \) (the darker contours) are therefore not accessible for low-energy particles (i.e., \( J \)). In particular, the high \( \Phi_{eff} \) zero velocity curves have a horseshoe shape and the corresponding orbits are referred to as horseshoe orbits as they make a U-turn. It must be emphasized however that in general particles do not follow the zero velocity contours as \( J \) is a function of time. Figure 2.6 gives example of particle trajectories obtained from integrating Hill's equation of motion. Three types of orbits can be seen:

- **Horseshoe orbit**, which make a U-turn (impact parameter \( b \leq 1.7850 \))
- **Hill-perpetuating orbit**: They are strongly excited after they leave the Hill sphere (1.7850 \( \leq b \leq 2.52318 \))
- **Circulating orbits**, which are most generally excited (\( b \geq 2.52318 \)).

Exercise 2.3 [Hill's equations]

(a) Show that the equations of motion in Hill's approximation are:

\[
\ddot{x} = -\frac{GM_2}{r^2} x + 2\dot{x} \dot{y} + \frac{3}{2} \dot{y}^2 \left( \frac{1}{2} \frac{1}{r_f^3} \right) \cos \iota
\]

(2.13a)

\[
\ddot{y} = -\frac{GM_2}{r^2} y - 2\dot{x} \dot{y} + \frac{3}{2} \dot{x}^2 \left( \frac{1}{2} \frac{1}{r_f^3} \right) \cos \iota
\]

(2.13b)

where \( x^2 + y^2 = r^2 \) if we restrict the motion to the collinear plane.

(b) Show that zero eccentricity particles at distances far from the secondary obey \( \dot{x} = 2\dot{y} \dot{y} + \dot{y}^2 = 0 \). This (local) approximation of the Keplerian flow is known as the tangential flow.

(c) Equilibrium points are points where \( \dot{x} = \dot{y} = 0 \). Show that these Lagrange points are located at \((x, y) = (m_{PB}, 0)\) where \( m_{PB} \) is the Hill radius:

\[
R_{Hill} = \frac{m_{PB}}{m} \left( \frac{m}{M} \right)^{1/3}
\]

(2.14)

(d) Are these stable or unstable equilibrium points?

(e) What is the Jacobi constant at the Lagrange point \((x, y) = (m_{PB}, 0)\)? And what is the Jacobi constant far from the perturber \( n^* \), assuming \( e = 0 \)? What is the semi-major axis \( a_0 \) of the corresponding horseshoe orbit?

From this section it is clear that particles that enter the Hill sphere do so at a velocity \( v_{Hill} = \sqrt{\frac{GM_2}{R_{Hill}}} \) - the Hill velocity. This is therefore the minimum (relative) velocity at which the gravitational scattering takes
Planetary sizes

μm

10^4 km

Dust grain

1 μm

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Planetary sizes

- $10^4$ km
- $10 \mu m$
- $100 \mu m$

aggregates?
Planetary sizes

μm

10^4 km

cm

km

chondrules

meter
Planetary sizes

$\mu m$ --- $10^4$ km

cm

km

meter

boulder
Planetary sizes

\[ \mu \text{m} \quad \text{cm} \quad \text{meter} \]

\[ 10^4 \text{ km} \]

planetaryesimal
Planetary sizes

μm

10^4 km

_planetary embryo_

cm

km

meter
Planetary sizes

- μm
- cm
- km

10^4 km

rocky planet
Planetary sizes

μm

10^4 km

gas giant

Planetary sizes

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L7: Particle aerodynamics
Lecture 7: Particle Aerodynamics

- Intro particle sizes
- Disk review
  - MMSN, disk “headwind”
- Aerodynamics
  - gas drag laws, Reynolds numbers, stopping time, radial and azimuthal drift, orbital decay, meter-size problem, Brownian motion, random vs. systematic motions.
- Turbulence-induced velocities
  - Kolmogorov fully developed turbulence theory, large/small eddies
- Relative velocities
  - Turbulence, total
Review: the (idealized) gas disk

- properties PPD:
  - *thin*, \( h_{\text{gas}}/r \ll 1 \), and *flared*
  - scaleheight: \( h_{\text{gas}} = c_s/\Omega_K \)
  - isothermal & pressure-supported in \( z \)
  - *partly* pressure-supported in \( r \)
    \[ \rightarrow \text{gas rotates sub Keplerian: } u_{\text{gas}} \approx v_K - \eta v_K \]
  \[
  \rho_{\text{gas}}(z) = \frac{\Sigma_{\text{gas}}(r)}{\sqrt{2\pi h_{\text{gas}}}} \exp \left[ -\frac{1}{2} \left( \frac{z}{h_{\text{gas}}} \right)^2 \right]
  \]
  \[
  g_{hs} = \frac{1}{\rho_{\text{gas}}} \frac{dP}{dr}
  \]
Minimum-mass solar nebula
(Weidenschilling 1977, Hayashi et al. 1985)

Assume power-laws:

\[ \Sigma_{\text{gas}} = \Sigma_1 \left( \frac{r}{\text{AU}} \right)^{-p}, \quad T_{\text{gas}} = T_1 \left( \frac{r}{\text{AU}} \right)^{-q} \]

MMSN choices (Hayashi et al. 1985):
\[ p = 1.5, \quad q = 0.5, \]
\[ T_1 = 300 \text{ K}; \quad \Sigma_1 = 1700 \text{ g cm}^{-2} \]

Q: Criticism of MMSN model:
1. ....
2. ....

Other choices for \( \Sigma(r), T(r) \) possible and arguably physically (or observationally) more plausible!
Minimum-mass solar nebula
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Q: Criticism of MMSN model:
1. ....
2. ....

Other choices for \( \Sigma(r), \ T(r) \) possible and arguably physically (or observationally) more plausible!
- gas drag low & (dimensionless) stopping time, particle Reynolds number
  Epstein/Stokes regimes
- headwind derivation $\eta v_K$;
  formal derivation for drift velocities
  orbital decay timescale
- Brownian motion
  systematic vs. random motions
Drag regimes

Epstein drag
\((s < 9l_{mfp}/4)\)

Stokes drag
\((\text{Re}_p < 1)\)

Quadratic drag
\((\text{Re}_p > 800)\)

Free molecular flow

\[ t_{\text{stop}} = \frac{\rho_o s}{\rho_{\text{gas}} v_{\text{th}}} \]

\[ t_{\text{stop}} \propto s^2 \]

\[ t_{\text{stop}} \propto \rho_{\text{gas}} s v^2 \]

Increasing particle size
Gas drag/ Flow past sphere/cylinder

- \( \text{Re} < 1 \) (laminar)
- \( 1 < \text{Re} < 10 \) (Bound vortex)
- \( 10 < \text{Re} < 10^5 \) (Vortex shedding)
- \( \text{Re} > 10^5 \) (Turbulent BL)
Dimensionless stopping time $\tau_p = t_{\text{stop}} \Omega_K$

Aerodynamical definition:

- pebble ($\tau_p < 1$)
- planetesimal ($\tau_p >> 1$)
Dimensionless stopping time $\tau_p = t_{\text{stop}} \Omega_K$

Aerodynamical definition:

- pebble ($\tau_p < 1$)
- planetesimal ($\tau_p \gg 1$)

Q: why these inflections?
Dimensionless stopping time \( \tau_p = t_{\text{stop}} \Omega_K \)

Aerodynamical definition:
- pebble \( (\tau_p < 1) \)
- planetesimal \( (\tau_p >> 1) \)

Q: why these inflections?

Q: what happens to \( t_{\text{stop}} \) when the gas disappears?
Turbulence
Eddies...
Fully developed turbulence
(Kolmogorov 1941)

Energy dissipation
\[ \varepsilon = \nu_i^2 / t_i = \text{cnst} \]

Driving scale:
\[ L, \nu_L, t_L \]

Dissipation scale:
\[ \nu_{Kol} I_{Kol} = \nu_{mol} \]

Scale (eddy size)

\[ L \nu_L = \nu_{turb} \]
Fully developed turbulence
(Kolmogorov 1941)

Energy dissipation
\( \varepsilon = \nu l^2 / t_l = \text{cnst} \)

Driving scale:
\( L, \nu_L, t_L \)

Dissipation scale:
\( \nu_{Kol} l_{Kol} = \nu_{mol} \)

Scale (eddy size)

\( \nu_l = (\varepsilon l)^{1/3} \)

\( t_l = (l^2 / \varepsilon)^{1/3} \)
Fully developed turbulence

\[ v_f = (\varepsilon t)^{1/2} \]

\[ t_{Kol} \quad t_L \]

eddy turnover time

eddy velocity \( v_f \)
Fully developed turbulence

\[ t_{Kol} \quad t_{stop} \quad t_L \]

\[ v_I = (\epsilon t)^{1/2} \]

eddy turnover time
Fully developed turbulence

\[ v_f = (\varepsilon t)^{1/2} \]

- **small eddies** (random)
- **large eddies** (systematic)

**Eddy turnover time**

**Eddy velocity** \( v_f \)

**Velocity**

**t_{Kol}**, **t_{stop}**, **t_L**
Fully developed turbulence

\[ t_{L} = (\varepsilon t_{L})^{1/2} \]

small eddies (random)
large eddies (systematic)

eddy turnover time

eddy velocity \( v_{l} \)

wrt eddy

\( t_{Kol} \)
\( t_{stop} \)
Small eddies: Random kicks

Eddy kicks a particle by \( v_1 \sim v_{\perp} t_{\perp}/t_{\text{stop}} \) in a random direction.

The particle “remembers” \( N = t_{\text{stop}}/t_{\perp} \) of these kicks.

\[ v_{\text{ran}} = N^{1/2} \; v_1 = (t_{\perp}/t_{\text{stop}})^{1/2} v_{\perp} \]
Large eddies: drift

Large eddies: $t_l > t_{\text{stop}}$

Particles obtain eddy velocity $v_l$

Experience pressure forces $g_l \sim v_l/t_l$ [Weidenschilling 1984]

$\rightarrow$ drift velocity: $v_{\text{sys}} \sim g_l t_{\text{stop}}$

with respect to eddy
Relative velocity

Relative velocity $v_1 \propto t_1^{1/2}$

- **Large** eddies (systematic): subtract velocities (vanish equal $t_{\text{stop}}$)
- **Small** eddies (random): add velocities
Relative velocities:
(Brownian motion, rad+azi drift, turbulence)
Relative velocities:
(Brownian motion, rad+azi drift, turbulence)
Relative velocities:
(Brownian motion, rad+azi drift, turbulence)
Exercise 1.2 particle scaleheight: When disks are turbulent, particles will diffuse along with the gas. A popular model for the turbulent viscosity is the Shakura & Sunyaev (1973) alpha-model, which parameterizes the turbulent viscosity as $\nu_T = \alpha c_s h_{\text{gas}}$. When particles are small ($\tau_p < 1$) this is also the particles diffusivity $D_p$. The definition of $D_p$ is such that particles move a distance $\sqrt{D_p t}$ in time $t$.

(a) Find the equilibrium distance $h_p$: the height where the settling timescale (from $h_p$ to the midplane) equals the time to diffuse the particles from the midplane to $h_p$. Express the result in terms of $\alpha$ and $\tau_p$.

(b) Naively, when $\tau_p \to 0$, one obtains $h_p > h_{\text{gas}}$. Why is this result incorrect?
**Exercise 1.3**

**Exercise 1.3– individual drift velocities:** These two equations allow us to solve for the two unknowns \((v_r, v_\phi)\). But we can greatly simplify the procedure by using that \(v_r, v_\phi\), and the disk headwind \(\eta v_K\) (see Equation (1.2)) are small with respect to the Keplerian velocity \(v_K\). Expressions as \((u_{\text{gas}} + v_\phi)^2\) can then be approximated as \(u_{\text{gas}}^2 + 2u_{\text{gas}}v_\phi\). In the same gist, \(u_{\text{gas}}^2 = (1-\eta)^2v_K^2 \approx (1-2\eta)v_K^2\), and \(d/dt(u_{\text{gas}} + v_\phi) \approx dv_K/dt\). This linearization allows the expressions to be put in matrix form:

\[
A\begin{pmatrix} v_r \\ v_\phi \end{pmatrix} = b. \tag{1.10}
\]

where \(A\) is a 2x2 matrix and \(b\) a 2x1 vector. Inverting this system of equations, show that the radial drift velocity becomes:

\[
v_r = -2\eta v_K \frac{\tau_p}{1 + \tau_p^2} \tag{1.11}
\]

and the azimuthal velocity:

\[
v_\phi = \eta v_K \frac{\tau_p^2}{1 + \tau_p^2}. \tag{1.12}
\]

(remark again that \(v_r, v_\phi\) are with respect to the gas velocity.)

---

1. Force balance
2. Angular momentum loss

**Warning:**
\(v_r, v_\phi\) are here defined relative to the gas velocity (!) and are therefore small w.r.t. to \(v_K\) and \(u_{\text{gas}}\).

\(v_K = \) Keplerian velocity
\(u_{\text{gas}} = (1-\eta) = \) sub-Keplerian vel. gas

---

This is not necessary; plain substitution is easier
Exercise 1.4 turbulent velocities: Consider driving scales of $t_L = 1 \text{yr}$, $c_s = 1 \text{km s}^{-1}$ and a turbulence Mach number of $\approx 0.1$, so that $v_L = 0.1c_s$. Take a Reynolds number of $Re = 10^8$.

(a) What are the values at the inertial scale, $\ell_{Kol}$, $t_{Kol}$, and $v_{Kol}$?

(b) Given the toy model for the velocity excitation of particles above, as summarized in Figure 1.6, we can derive expressions for the relative velocities of particles. For example, for two particles of stopping times $t_{s1} \leq t_{s2} \leq t_{Kol}$ (where $t_{s1} = t_{\text{stop}}$ of particle #1 and $t_{s2}$ of #2) all eddies are large (top panel). In that case, argue that the relative velocity becomes $\Delta v \sim |t_{s2} - t_{s1}|v_{Kol}/t_{Kol}$. (The minus sign is important: the velocity will vanish for $t_{s1} = t_{s2}$. Why?).

(c) In the case where the largest particle (2) has a stopping time in the inertial range, $t_{Kol} \leq t_{s2} \leq t_L$, argue that the relative velocity becomes $\Delta v \sim v_L \sqrt{t_{s2}/t_L}$. Why does this expression not depend on $t_{s1}$?

(d) If both particles are large, $t_L < t_{s1} < t_{s2}$, argue that it is the particle of the shortest stopping time that determines the relative motions and give $\Delta v$.

(e) Between a very small particle ($t_{s1} < t_{Kol}$) and a very big one ($t_{s2} > t_L$) the relative collision velocity is $\Delta v \sim v_L$. Why?