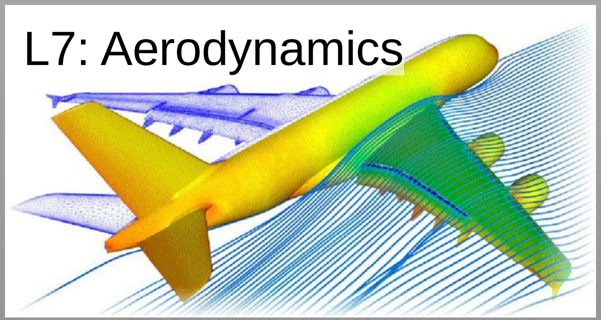
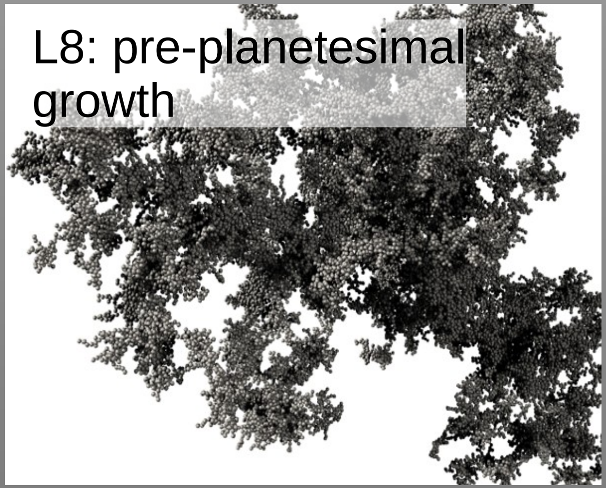
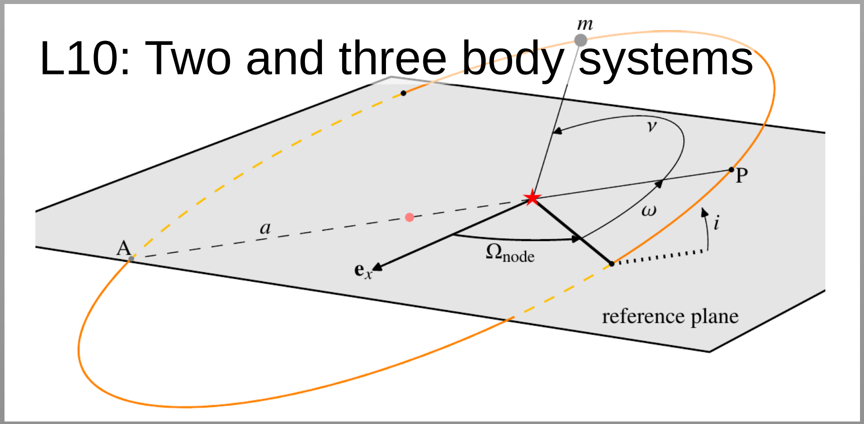
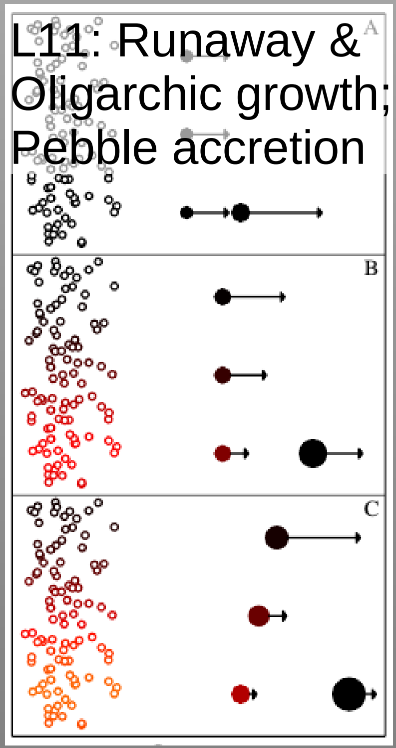
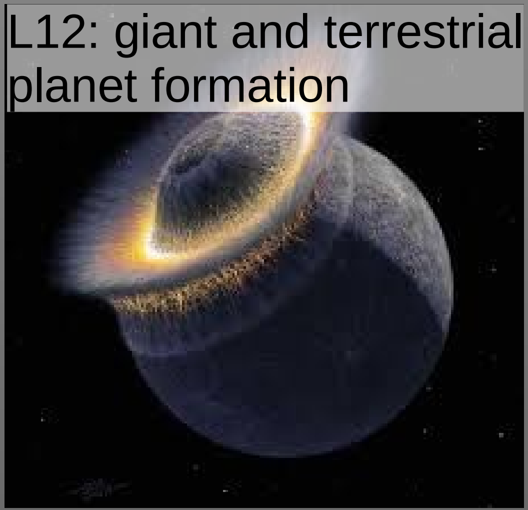
Part: II planet formation











Read the lecture notes!

2.2 *The 3-body problem*

In the 3-body problem analytical (closed-form) solutions are no longer possible. A simplification of the 3-body problem is that of a massless particle being perturbed by a secondary (*e.g.* planet) that moves on a circular orbit around the primary (star). This is known as the circular, restricted 3-body problem *CR3BP*. We will focus exclusively on this problem.

The equation of motion in a frame of reference rotating with angular frequency ω is:

$$\ddot{\boldsymbol{r}} = -\nabla\Phi - 2\boldsymbol{\omega}\times\dot{\boldsymbol{r}} - \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}) \quad (2.5)$$

In the CR3BP we will of course choose $\omega = n_p \boldsymbol{e}_z$ such that Φ – the gravitational potential – is time-independent in the rotating frame. Equation (2.5) can be integrated to give an integration constant J:*

$$J = \frac{1}{2}\dot{r}^2 + \Phi - \frac{1}{2}(\boldsymbol{\omega}\times\boldsymbol{r})^2 \quad (2.6)$$

which is the *Jacobi energy*. In the 3-body problem it is the only integral of motion.

* To see this, multiply Equation (2.5) by $\dot{\boldsymbol{r}}$ and write all terms as time-differentials (d/dt): $\frac{d}{dt}(\dot{r}^2/2) = \ddot{\boldsymbol{r}}\cdot\dot{\boldsymbol{r}}$, $\frac{d\Phi}{dt} = \dot{\boldsymbol{r}}\cdot\nabla\Phi$, and $\frac{d}{dt}(\boldsymbol{\omega}\times\boldsymbol{r})^2/2 = [\boldsymbol{\omega}\times\boldsymbol{\omega}\times\boldsymbol{r}]\cdot\dot{\boldsymbol{r}}$. Also, $\dot{\boldsymbol{r}}\cdot(2\boldsymbol{\omega}\times\dot{\boldsymbol{r}}) = 0$

Exercise 2.2 Jacobi integral:

(a) Converting Equation (2.6) back to the inertial frame, show that:

$$J = E - \boldsymbol{\omega}\cdot\boldsymbol{l} = E - n_p l_z \quad (2.7)$$

where E and l are the energy and angular momentum measured in the inertial frame. Hence, in the CR3BP interactions will exchange E and l, while J stays constant.

(b) Express J in orbital elements:

$$J = -\frac{Gm_*}{2a} - n_p\sqrt{Gm_*(1-e^2)a}\cos i \quad (2.8)$$

where n_p is the mean motion of the secondary and the other symbols refer to the test particle. Written in the form of Equation (2.8) (or analogous) the Jacobi integral is called the *Tisserand relation*.

(c) Let $a = a_p + b$ with a_p the semimajor axis corresponding to n_p and consider the limits where $b/a_0 \ll 1$, $i \ll 1$ and $e \ll 1$. Show that in that case:

$$J \approx \frac{Gm_*}{a_p}\left(-\frac{3}{8}\frac{b^2}{a_p^2} + \frac{e^2+i^2}{2}\right) \quad (2.9)$$

where we have discarded a constant term from J.

It is instructive to redefine the potential in Equation (2.5), incorporating the centrifugal term:†

$$\Phi_{\rm eff} \equiv \Phi_1 + \Phi_2 - \frac{1}{2}n_p^2 r^2 = -\left[\frac{Gm_*}{r_1} + \frac{1}{2}n_p^2 r_1^2\right] + \Phi_2 \quad (2.10)$$

where we used the identity $\frac{1}{2}\nabla r^2 = \boldsymbol{r}$. Consider the motion of the test particle in the vicinity of m_2, see Figure 2.4, and express the potential in local coordinates (x,y) centered on m_2. This amounts to expanding the inverse distance $1/r_1$ in terms of (the small) x and y.‡

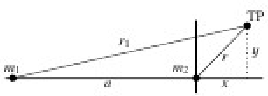


Figure 2.4: Definitions of x and y in Hill's approximation of the CR3BP.

† In celestial mechanics text books it is customary to define $\Phi_{\rm eff}$ with the opposite sign.

‡ this becomes $r_1^{-1} \approx 1/a_2 - x/a_2^2 + x^2/a_2^3 - \frac{1}{2}y^2/a_2^3 - \frac{1}{2}z^2/a_2^3$. The leading (constant) term of the expansion can be discarded from $\Phi_{\rm eff}$ as it is a potential function. In addition, in Equation (2.11) we assumed that $m_1 \gg m_2$ such that $Gm_1 \approx n_2^2 a_2^3$.

The result is (Hill's approximation):

$$\Phi_{\rm eff} = -\frac{3}{2}n_2^2 x^2 + \frac{1}{2}n_2^2 z^2 - \frac{Gm_2}{r} \quad (2.11)$$

with which the Jacobi energy is written:

$$J = \frac{1}{2}\dot{r}^2 + \Phi_{\rm eff} \quad (2.12)$$

Contours of $\Phi_{\rm eff}(x,y)$ are known as *zero velocity curves;* they define the region where a particle of a certain J can move, since $\Phi_{\rm eff} = J - \frac{1}{2}\dot{r}^2 \le J$. Therefore, although the 3-body problem is not integrable, given J, we can constrain the regions where particles can be found. Figure 2.5 shows contours of constant $\Phi_{\rm eff}$ with lighter contours having larger $\Phi_{\rm eff}$. The regions bounded by high $\Phi_{\rm eff}$ (the darker contours) are therefore not accessible for low-energy particles (low J). In particular, the high $\Phi_{\rm eff}$ zero velocity curves have a horseshoe shape and the corresponding orbits are referred to as *horseshoe orbits* as they make a U-turn. It must be emphasized however that in general particles do not follow the zero velocity contours as $\dot{r}$ is a function of time. Figure 2.6 gives examples of particle trajectories obtained from integrating Hill's equation of motion. Three types of orbits can be seen:

- *Horseshoe orbits,* which make a U-turn (impact parameter $b \lesssim 1.7R_{\rm Hill}$;
- *Hill-penetrating orbits.* They are strongly excited after they leave the Hill sphere ($1.7R_{\rm Hill} \lesssim b \lesssim 2.5R_{\rm Hill}$);
- *Circulating orbits,* which are only modestly excited. ($b \gtrsim 2.5R_{\rm Hill}$).

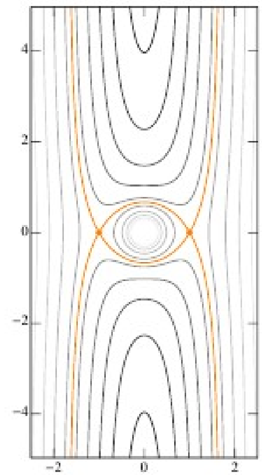


Figure 2.5: Zero velocity curves (contours of $\Phi_{\rm eff}$) in the $z = 0$ plane. Contours of larger $\Phi_{\rm eff}$ are darker. The Lagrange equilibrium points L1 and L2 are indicated by circles. Distances are in units of Hill sphere. Curves are *not* orbits.

Exercise 2.3 Hill's equations:

(a) Show that the equations of motion in Hill's approximation are:

$$\ddot{x} = -\frac{Gm_p}{r^3}x + 2n_p v_y + 3n_p^2 x \quad (2.13a)$$

$$\ddot{y} = -\frac{Gm_p}{r^3}y - 2n_p v_x \quad (2.13b)$$

where $r^2 = x^2 + y^2$ if we restrict the motion to the orbital plane.

(b) Show that zero eccentricity particles at distances far from the secondary obey $v_y = -\frac{3}{2}n_p x$ and $v_x = 0$. This (local) approximation of the Keplerian flow is known as the *shearing sheet*.

(c) Equilibrium points are points where $\ddot{\boldsymbol{r}} = \dot{\boldsymbol{r}} = 0$. Show that these *Lagrange points* are located at $(x,y) = (\pm R_{\rm Hill}, 0)$ where $R_{\rm Hill}$ is the *Hill radius:*

$$R_{\rm Hill} = a_p\left(\frac{m_p}{3m_*}\right)^{1/3} \quad (2.14)$$

(d) Are these stable or unstable equilibrium points?

(e) What is the Jacobi constant at the Lagrange point (J_L)? And what is the Jacobi constant far from the perturber (J_∞), assuming $e = 0$. What is the half-width $x_{\rm hs}$ of the corresponding horseshoe orbit?

From this section it is clear that particles that enter the Hill sphere do so at a velocity $\sim R_{\rm Hill}n_p$ – the *Hill velocity*. This is therefore the minimum (relative) velocity at which the gravitational scattering takes

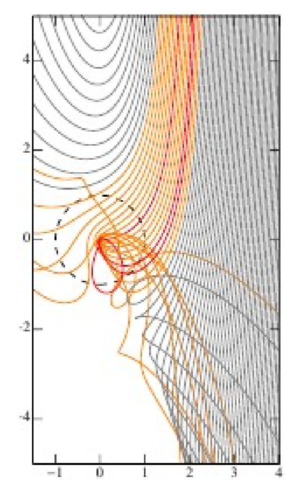
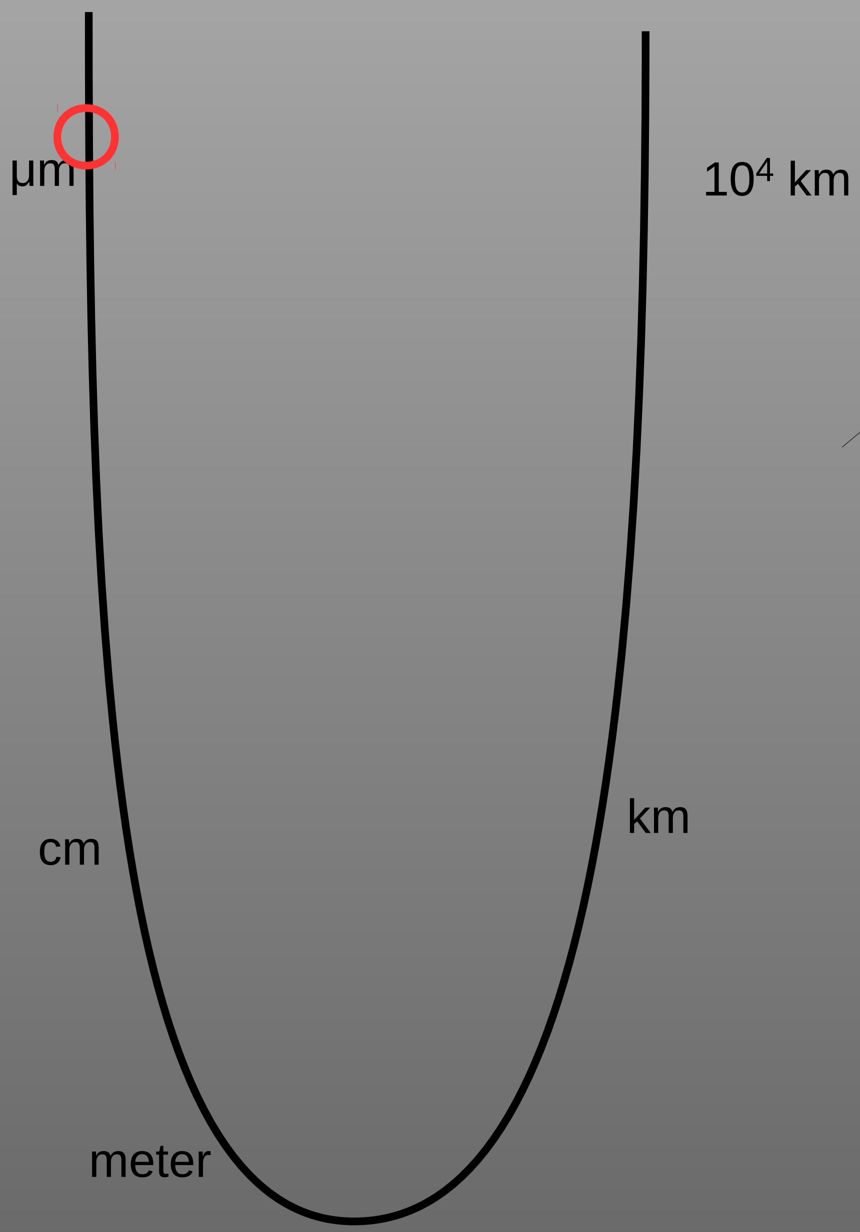
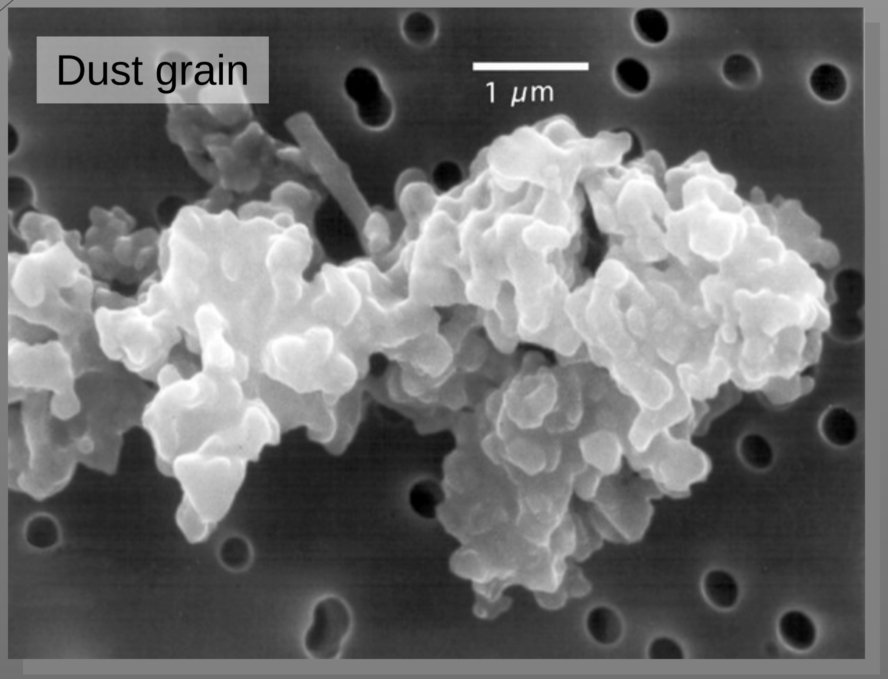


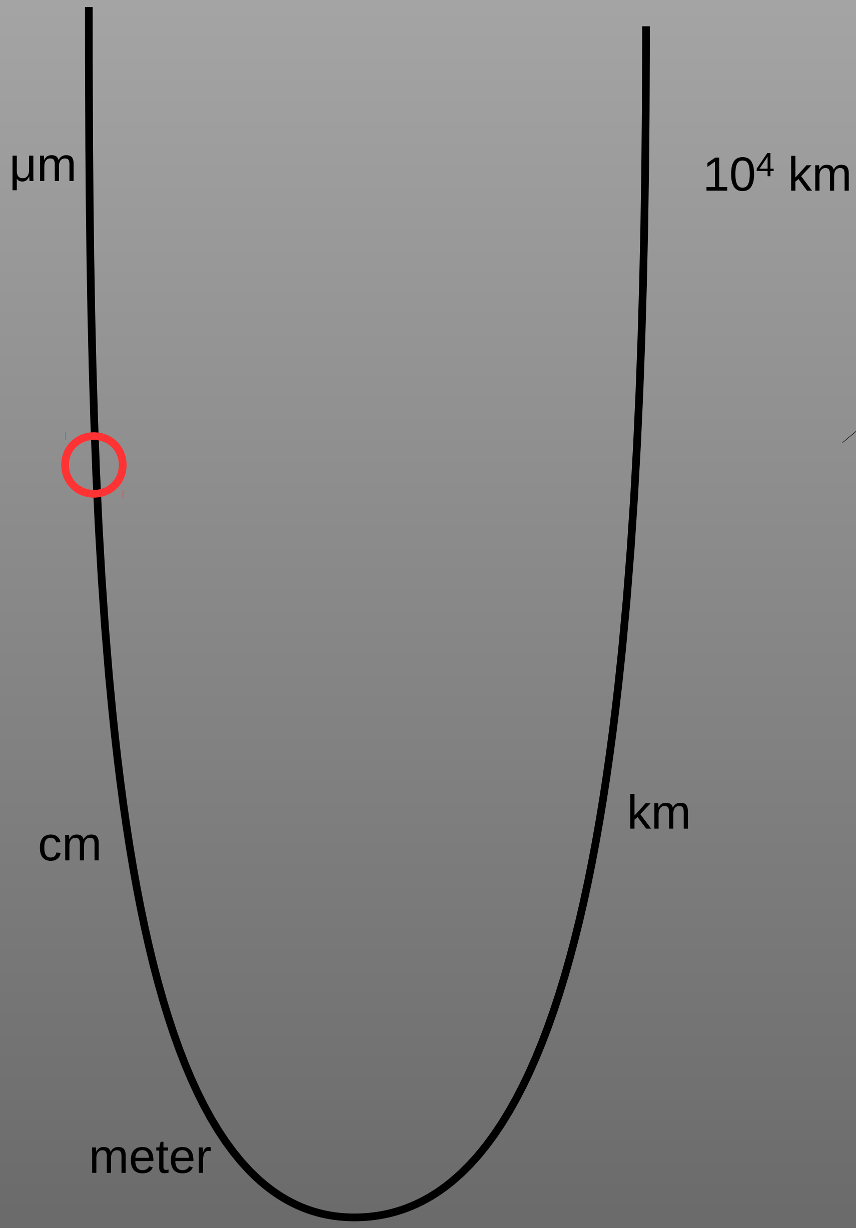
Figure 2.6: Examples of particle trajectories (initially on circulating orbits) in the CR3BP, obtained by integrating Equation (2.13). Particles that enter the Hill sphere (dashed circle) are highlighted. Red streams hit the planet ($R < R_p = 5\times10^{-3}R_{\rm Hill}$).

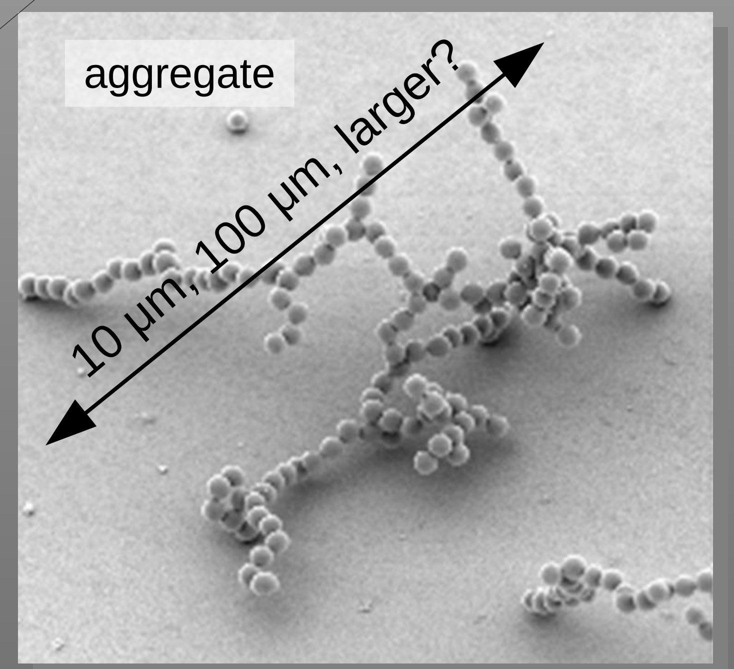
Planetary sizes



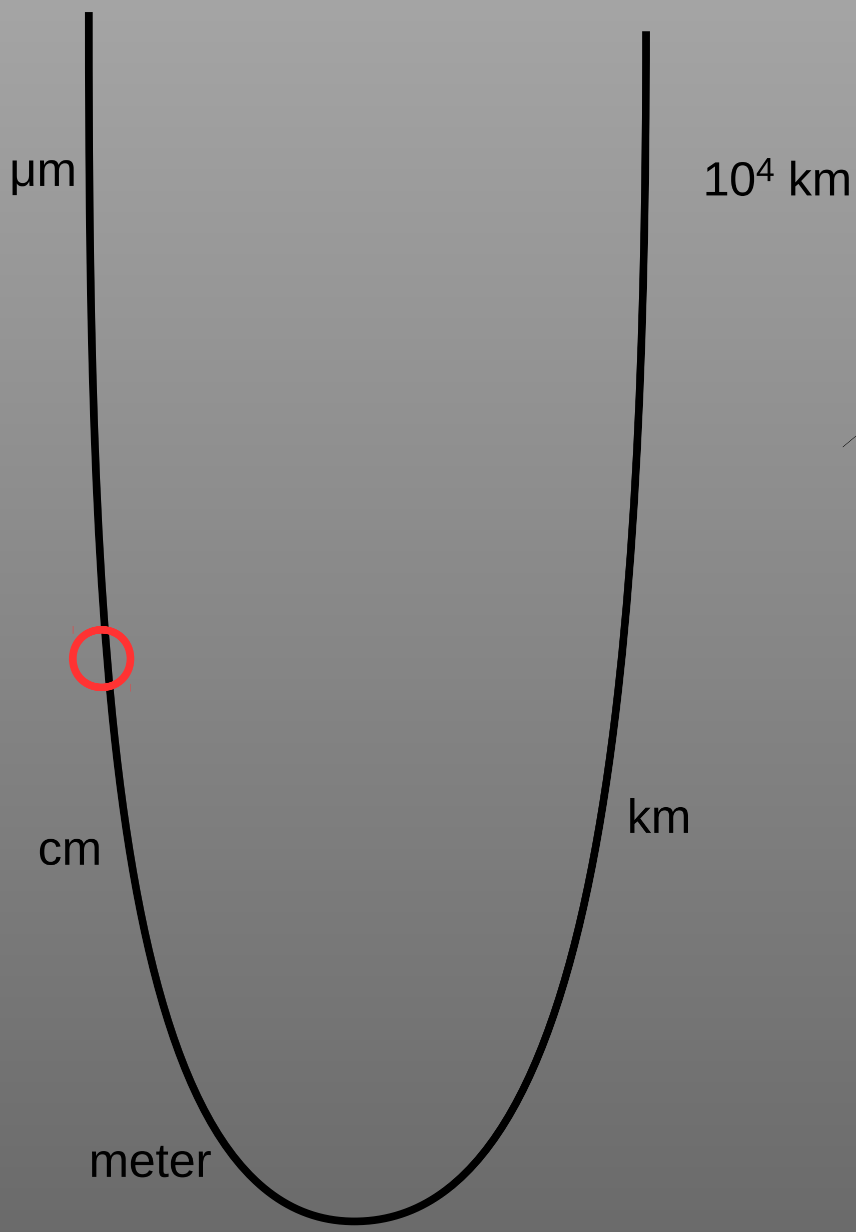


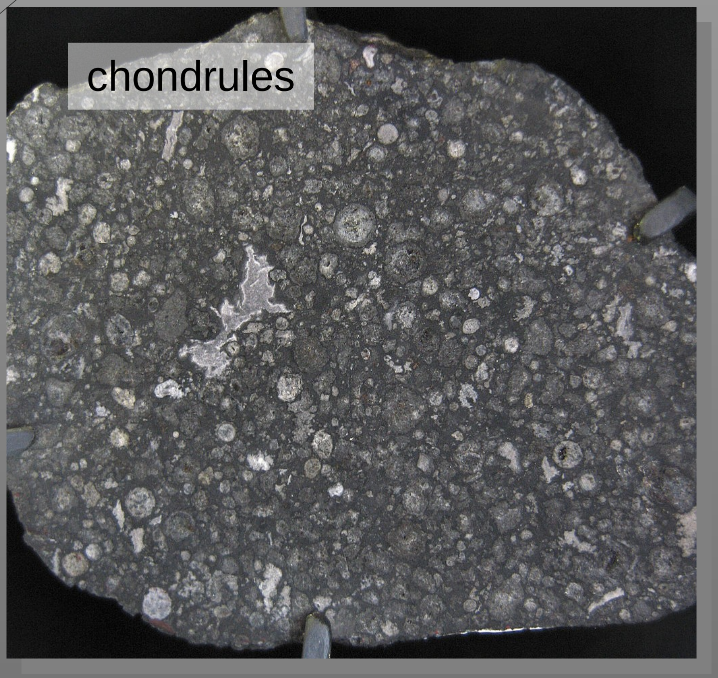
Planetary sizes





Planetary sizes





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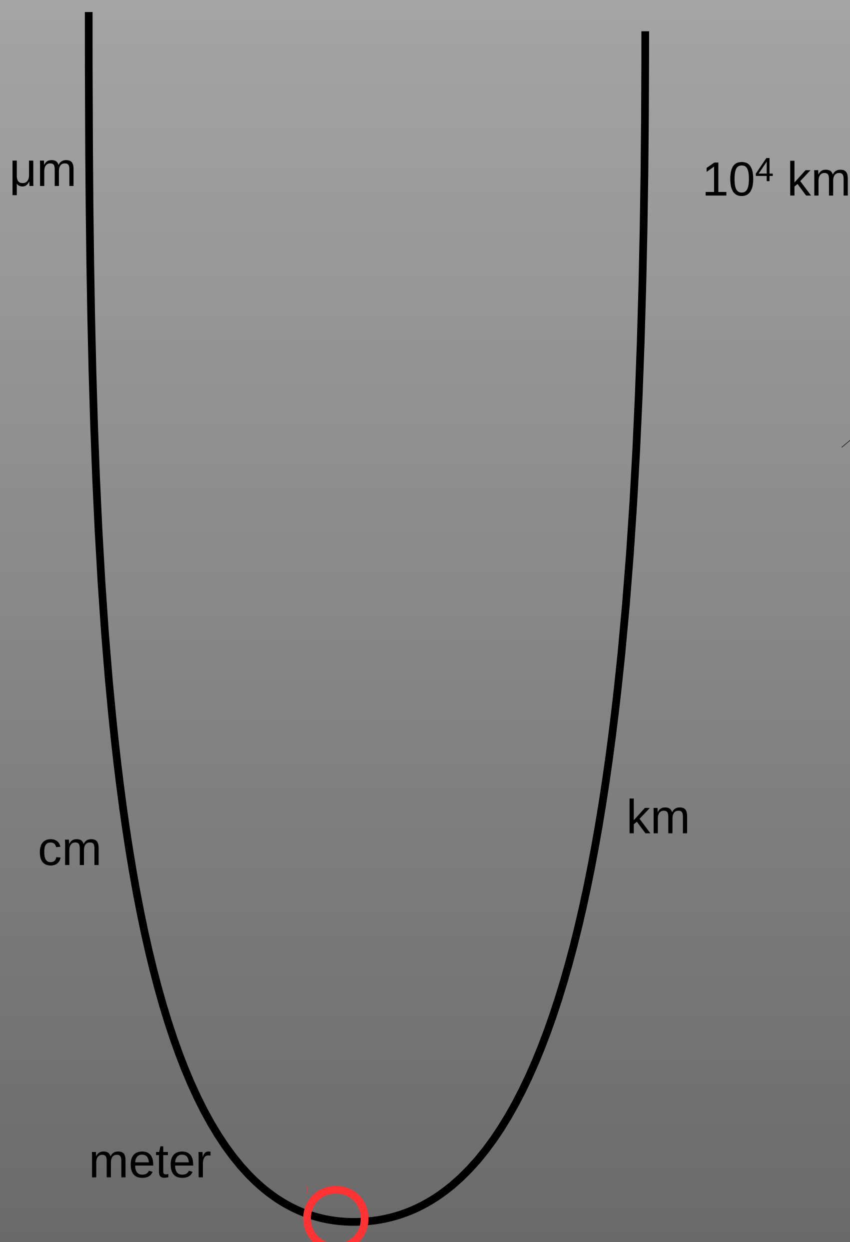

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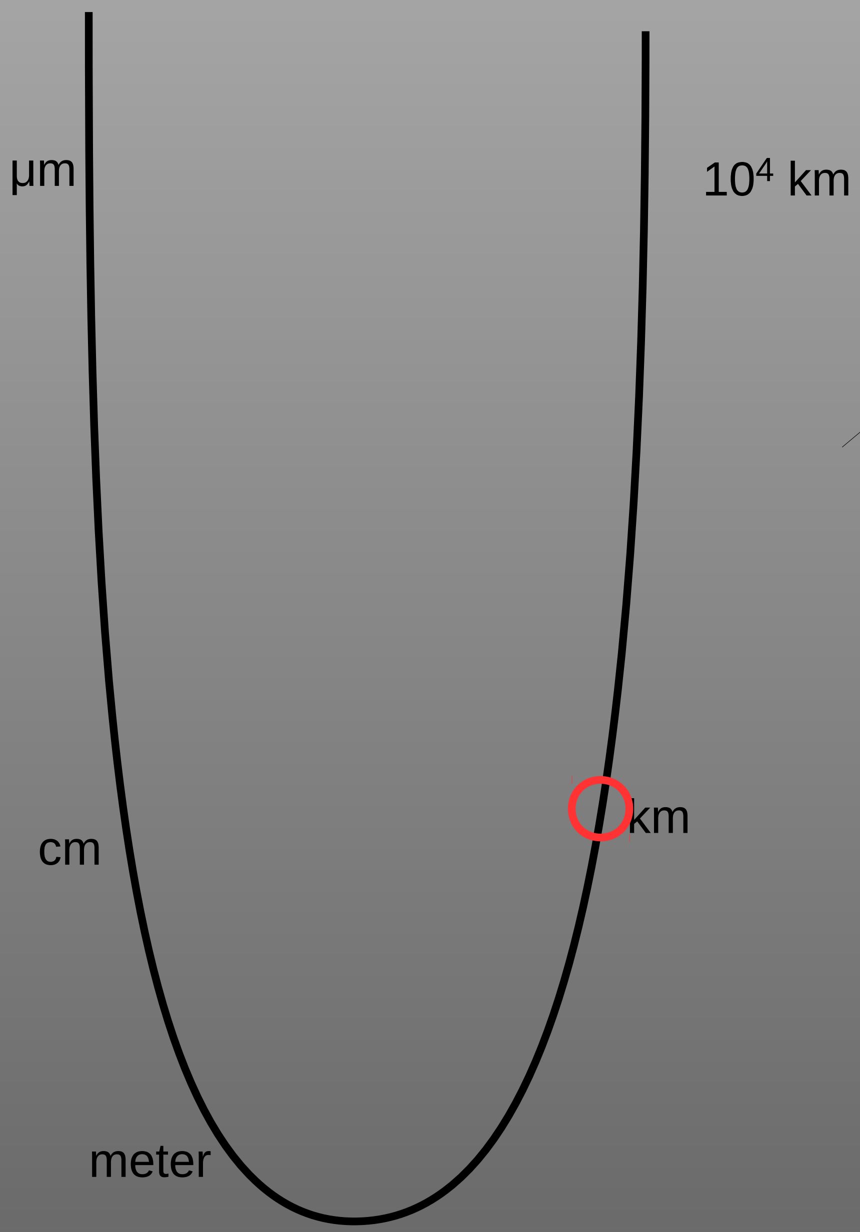

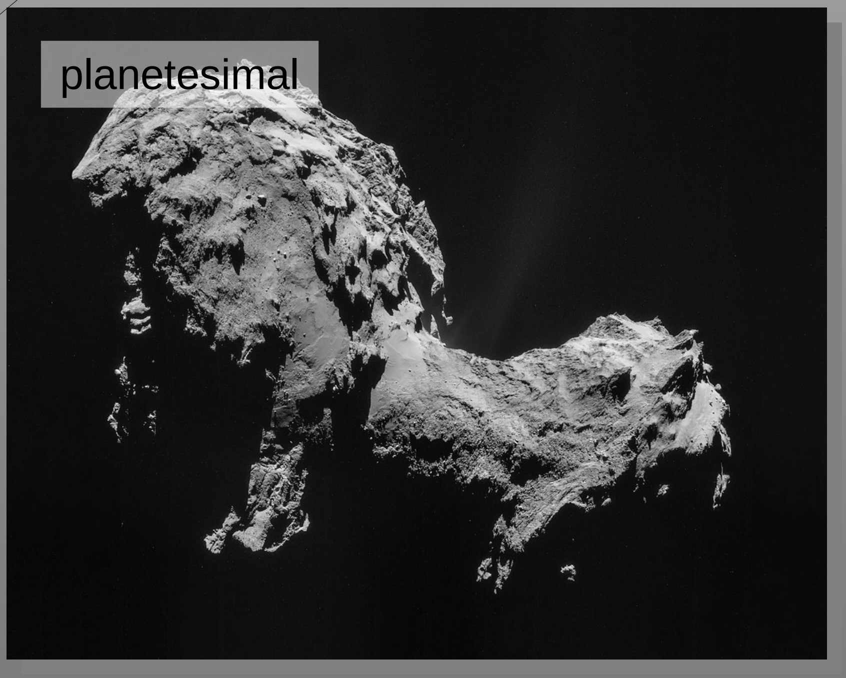

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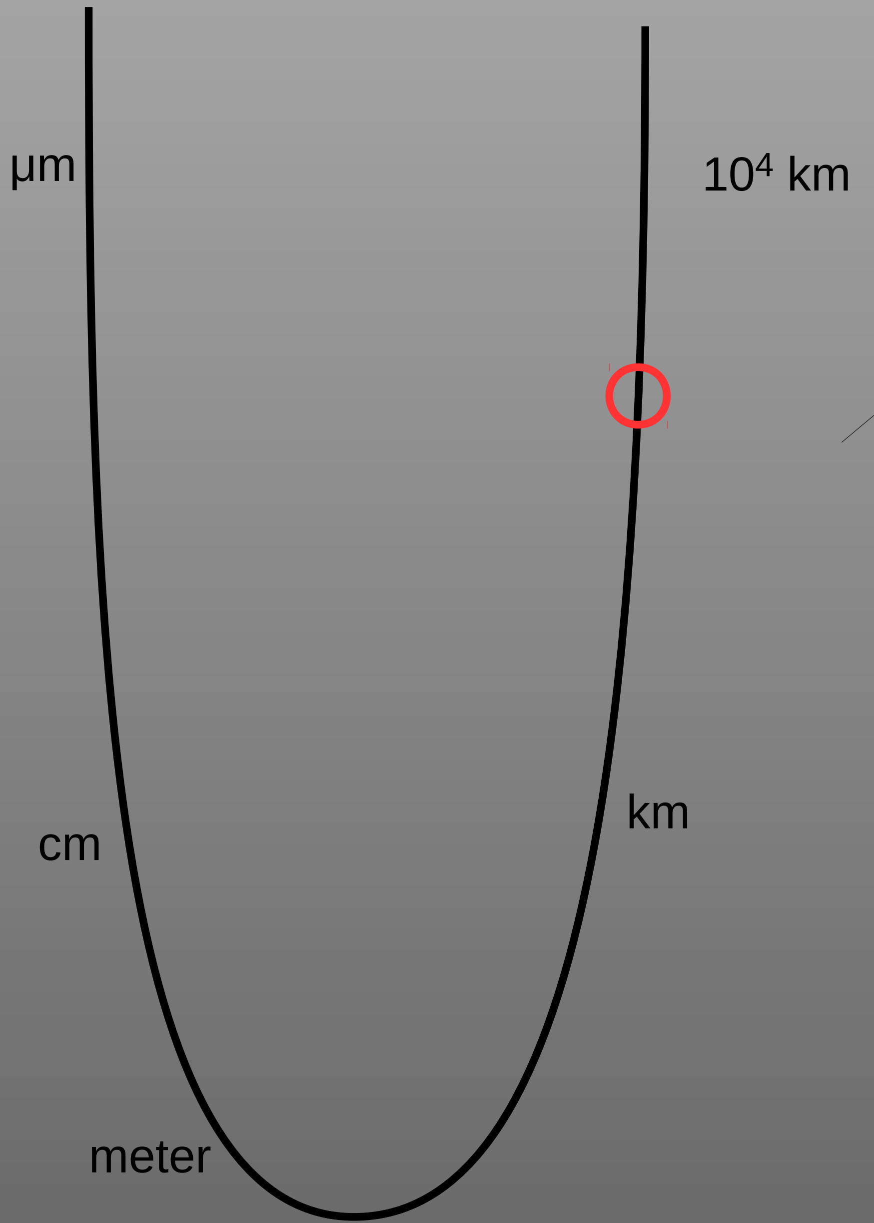

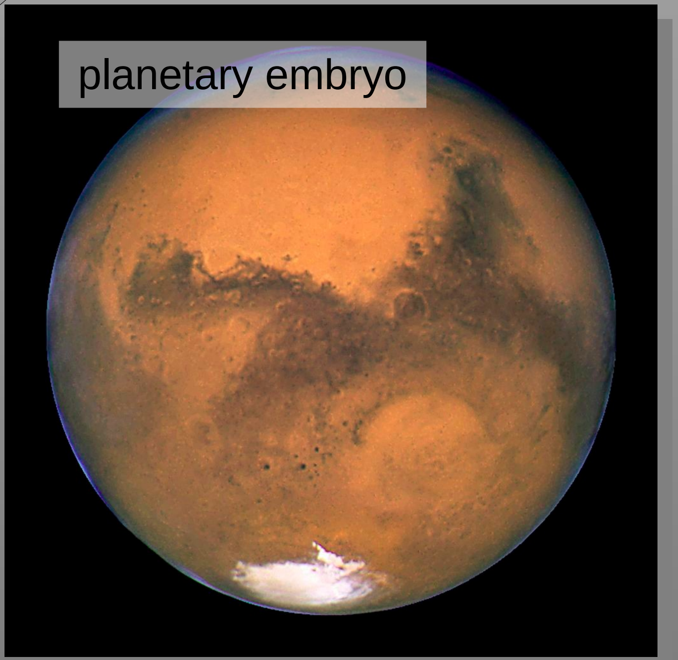

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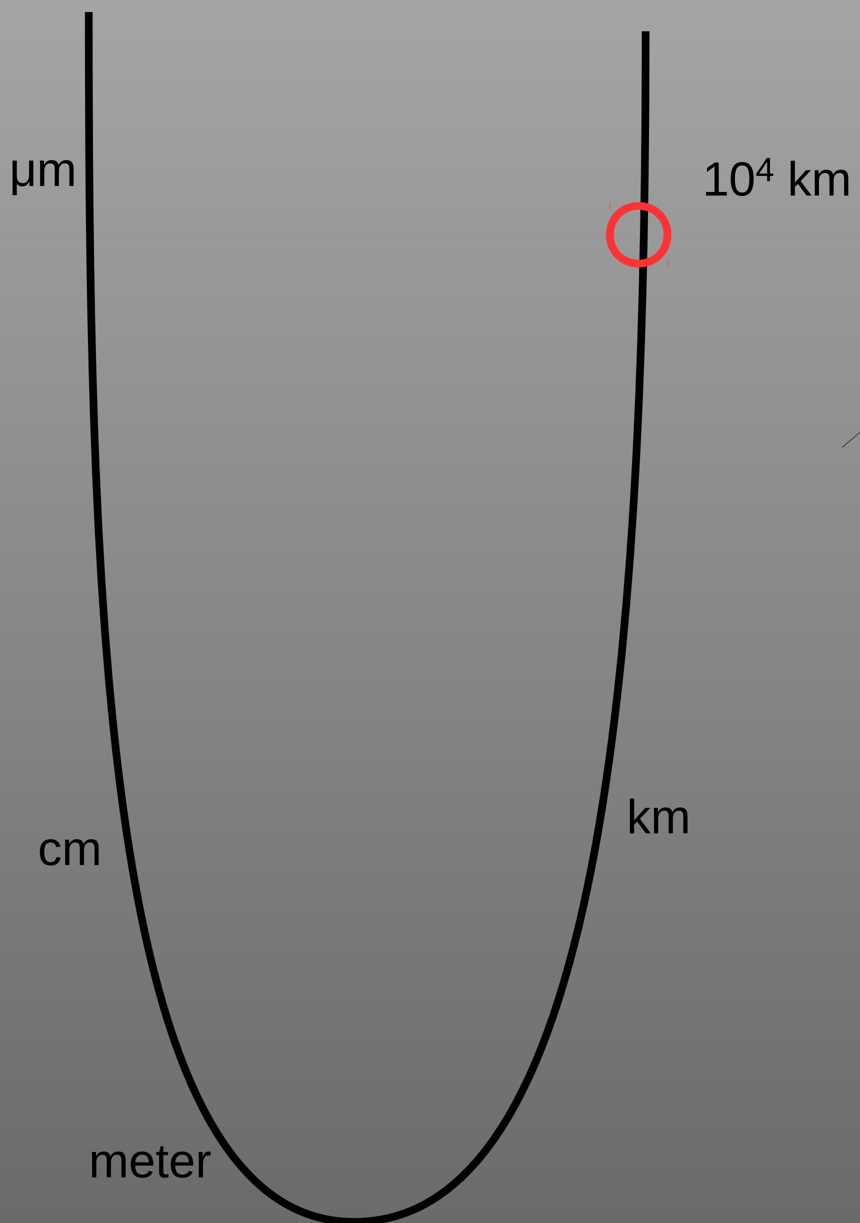

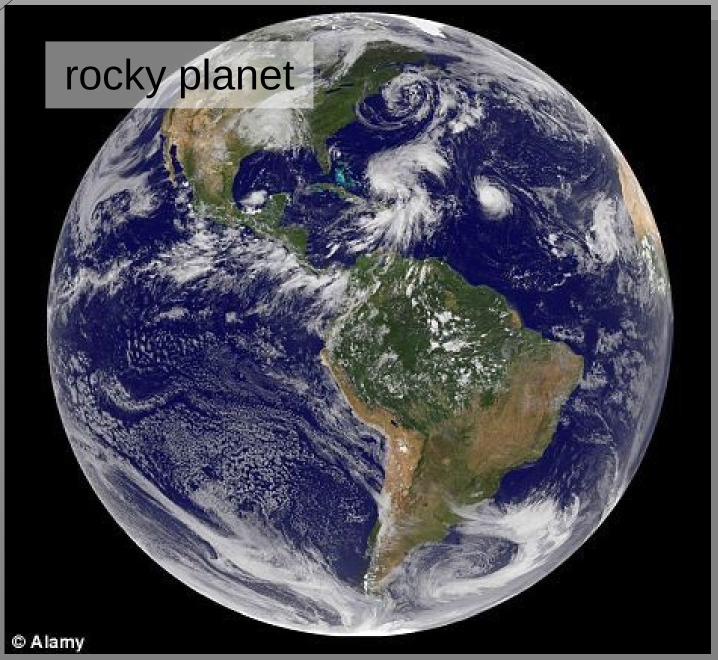

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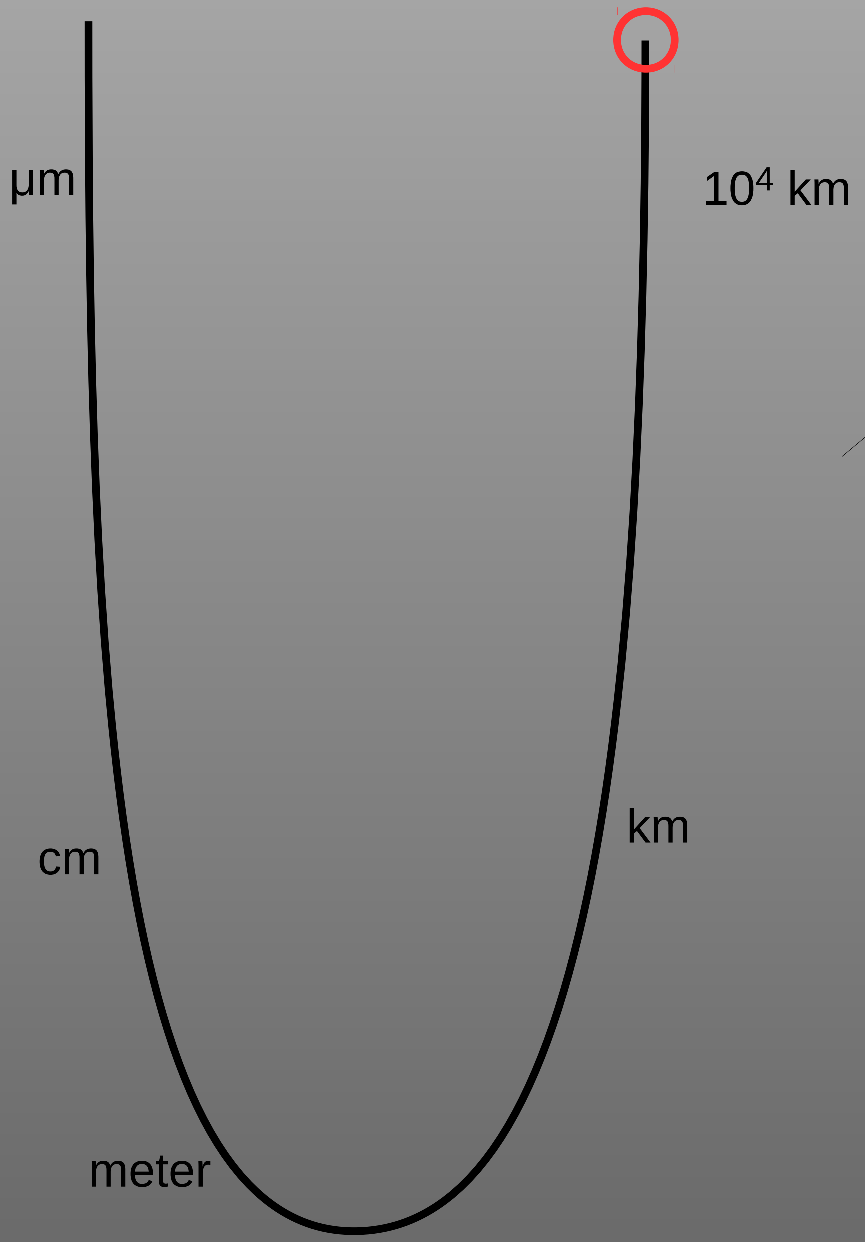
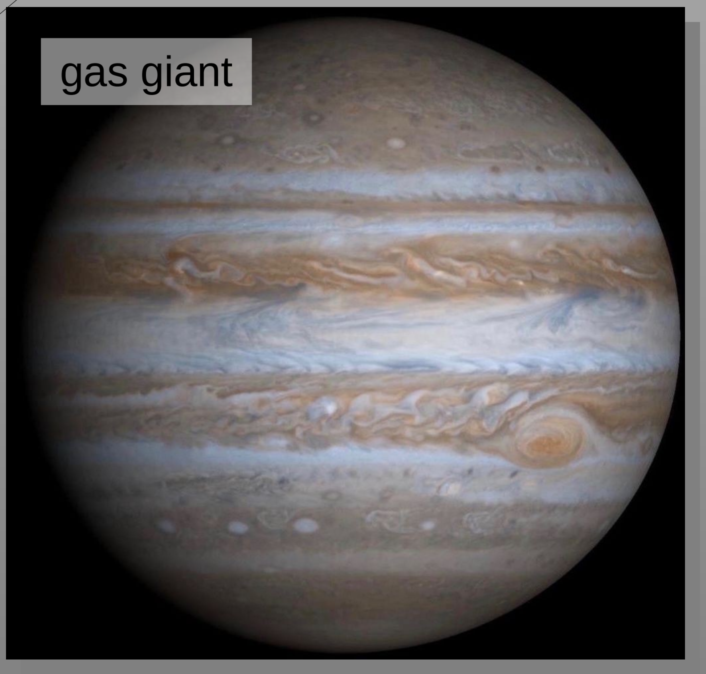
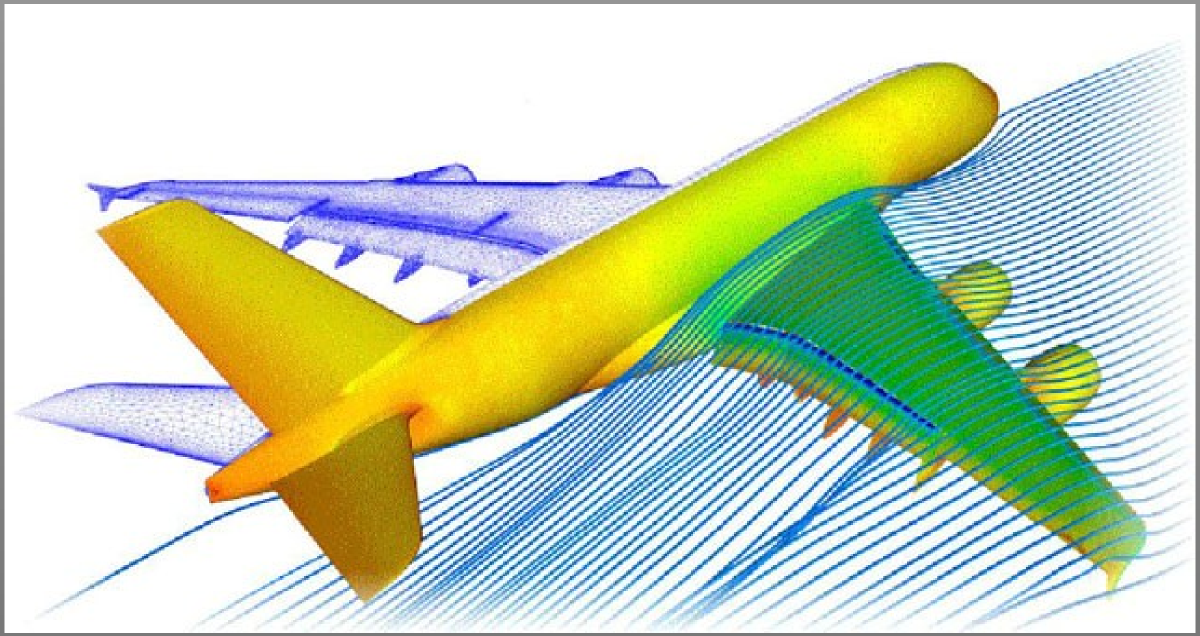




L7: Particle aerodynamics

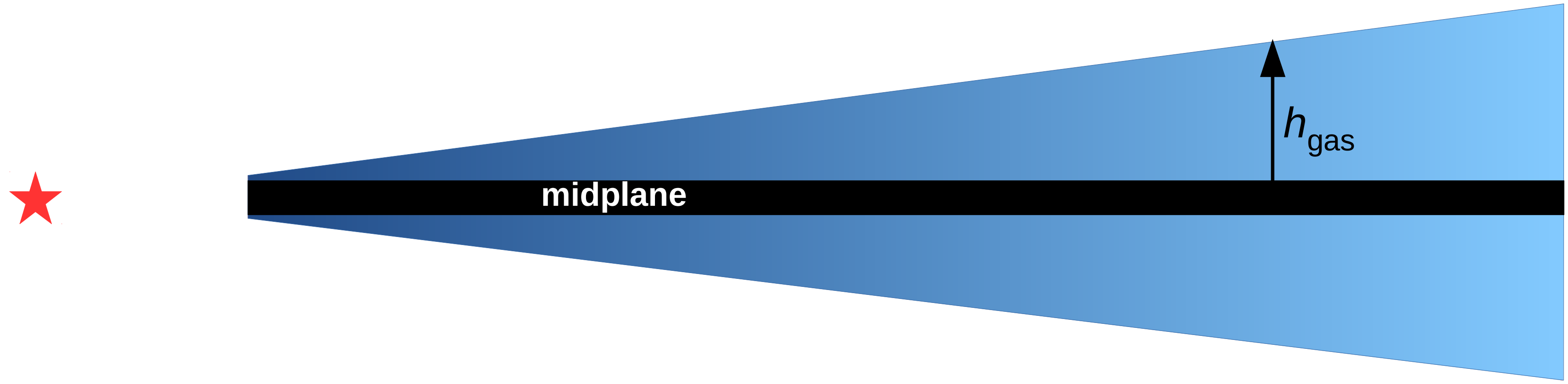


Lecture 7: Particle Aerodynamics

- Intro particle sizes
- Disk review

  MMSN, disk “headwind”
- Aerodynamics
  - gas drag laws, Reynolds numbers, stopping time, radial and azimuthal drift, orbital decay, meter-size problem, Brownian motion, random vs. systematic motions.
- Turbulence-induced velocities
  - Kolmogorov fully developed turbulence theory, large/small eddies
- Relative velocities
  - Turbulence, total

Review: the (idealized) gas disk



- properties PPD:
  - *thin*, $h_{gas}/r \ll 1$, and *flared*
  - scaleheight: $h_{gas} = c_s/\Omega_K$
  - isothermal & pressure-supported in z
  - *partly* pressure-supported in r
    → gas rotates sub Keplerian: $u_{gas} \approx v_K - \eta v_K$

$$\rho_{gas}(z) = \frac{\Sigma_{gas}(r)}{\sqrt{2\pi} h_{gas}} \exp\left[-\frac{1}{2} \left(\frac{z}{h_{gas}} \right)^2 \right]$$

$$g_{hs} = \frac{1}{\rho_{gas}} \frac{dP}{dr}$$

Minimum-mass solar nebula

(Weidenschilling 1977, Hayashi et al. 1985)

Assume power-laws:

$$\Sigma_{\rm gas} = \Sigma_1 \left(\frac{r}{\rm AU}\right)^{-p}, \qquad T_{\rm gas} = T_1 \left(\frac{r}{\rm AU}\right)^{-q}$$

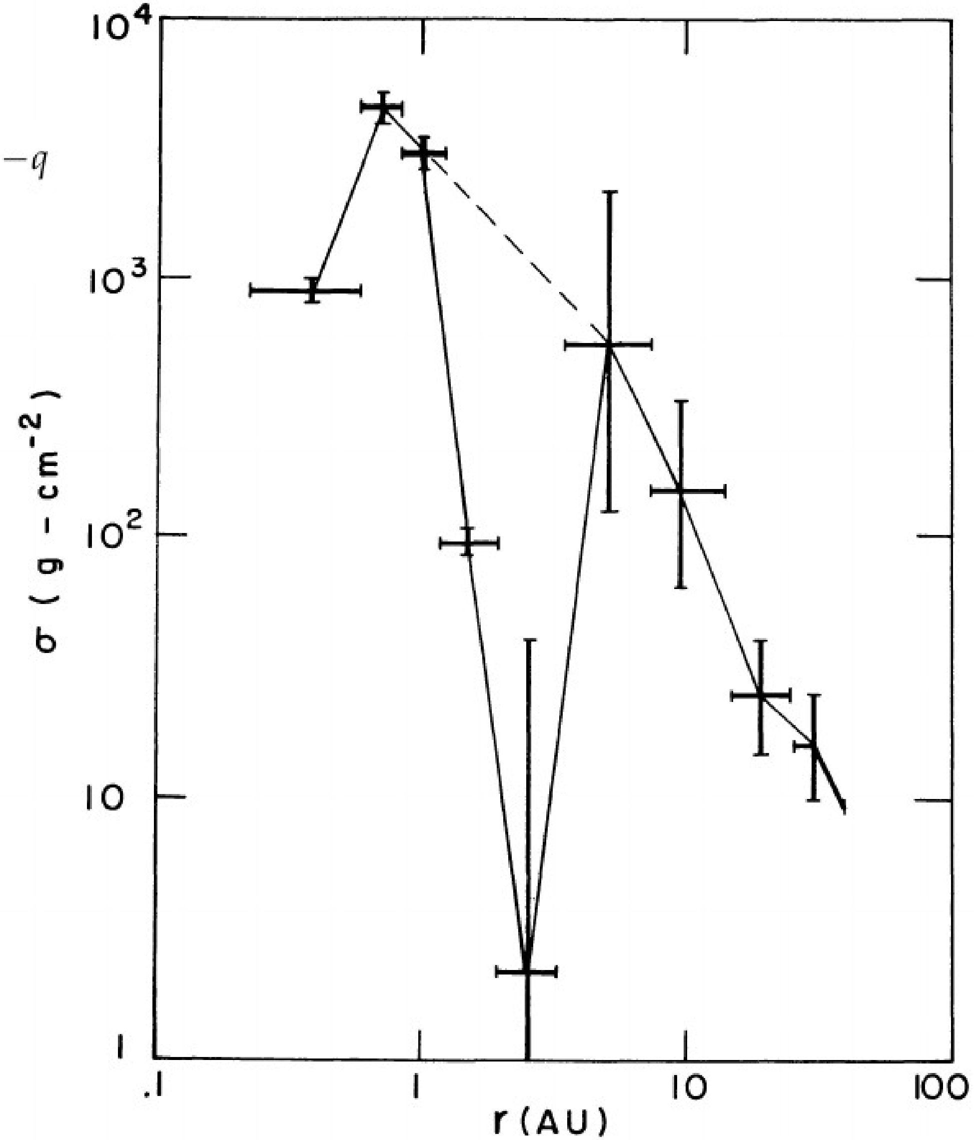
MMSN choices (Hayashi et al. 1985):

$p = 1.5$, q = 0.5,
$T_1 = 300$ K; $\Sigma_1 = 1700$ g cm^{-2}

Q: Criticism of MMSN model:

1. ….
2. ….

Other choices for $\Sigma(r)$, $T(r)$ possible and arguably physically (or observationally) more plausible!



[Weidenschilling 1977]

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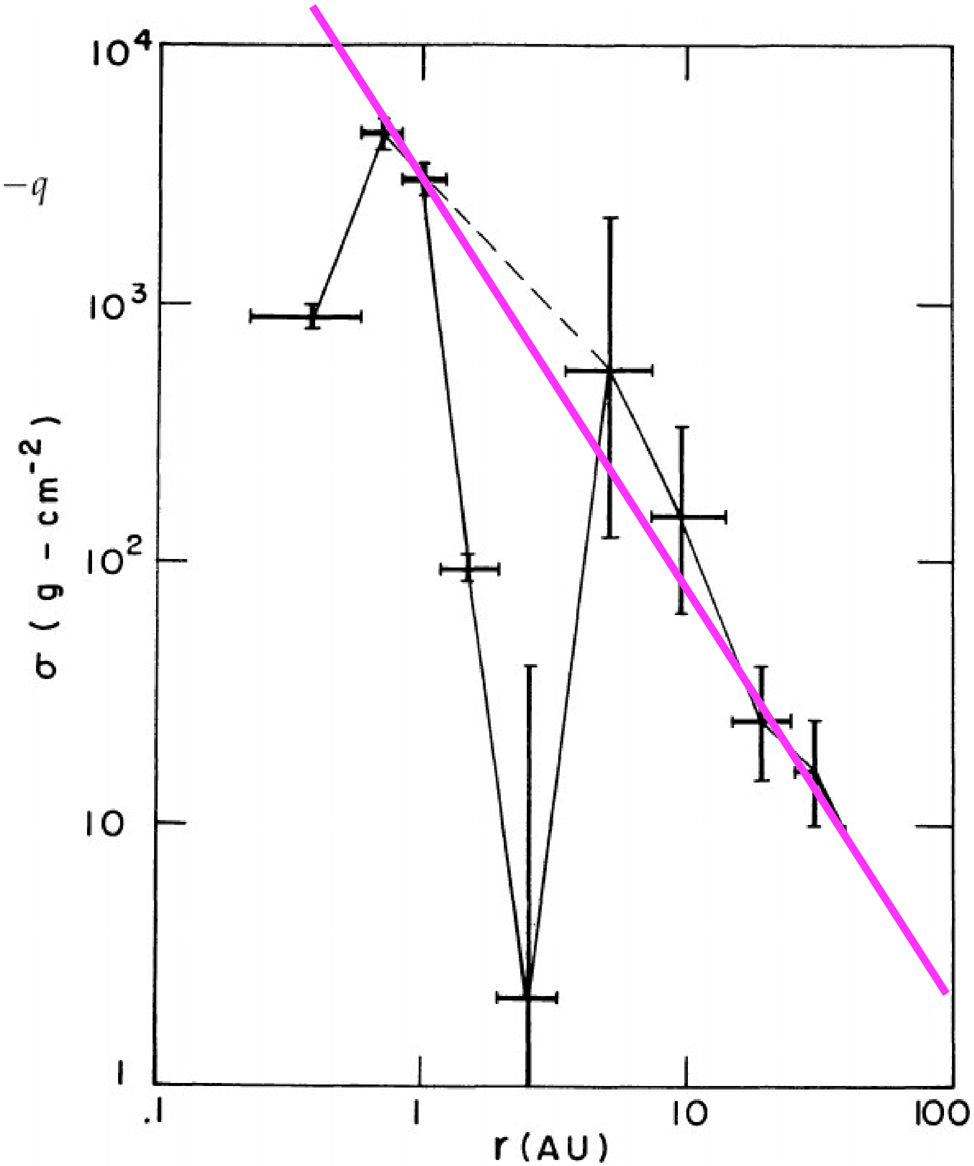


[Weidenschilling 1977]

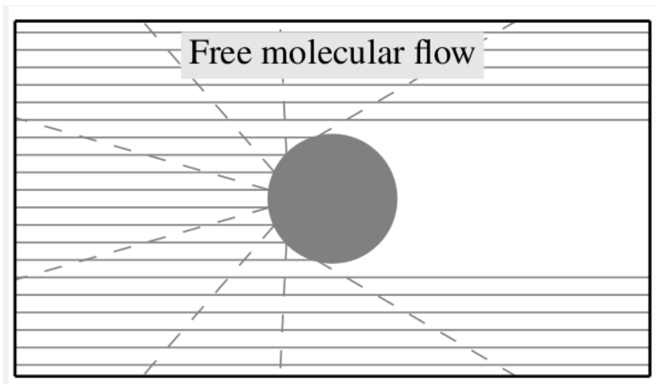
Blackboard

– gas drag low & (dimensionless) stopping time,
  particle Reynolds number
  Epstein/Stokes regimes

– headwind derivation ηv_K;
  formal derivation for drift velocities
  orbital decay timescale

– Brownian motion
  systematic vs. random motions

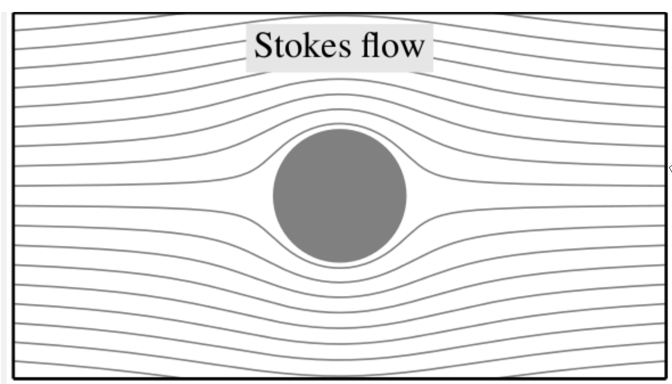
Drag regimes



Epstein drag
($s < 9l_{mfp}/4$)

$$t_{stop} = \frac{\rho_o s}{\rho_{gas} v_{th}}$$

increasing particle size



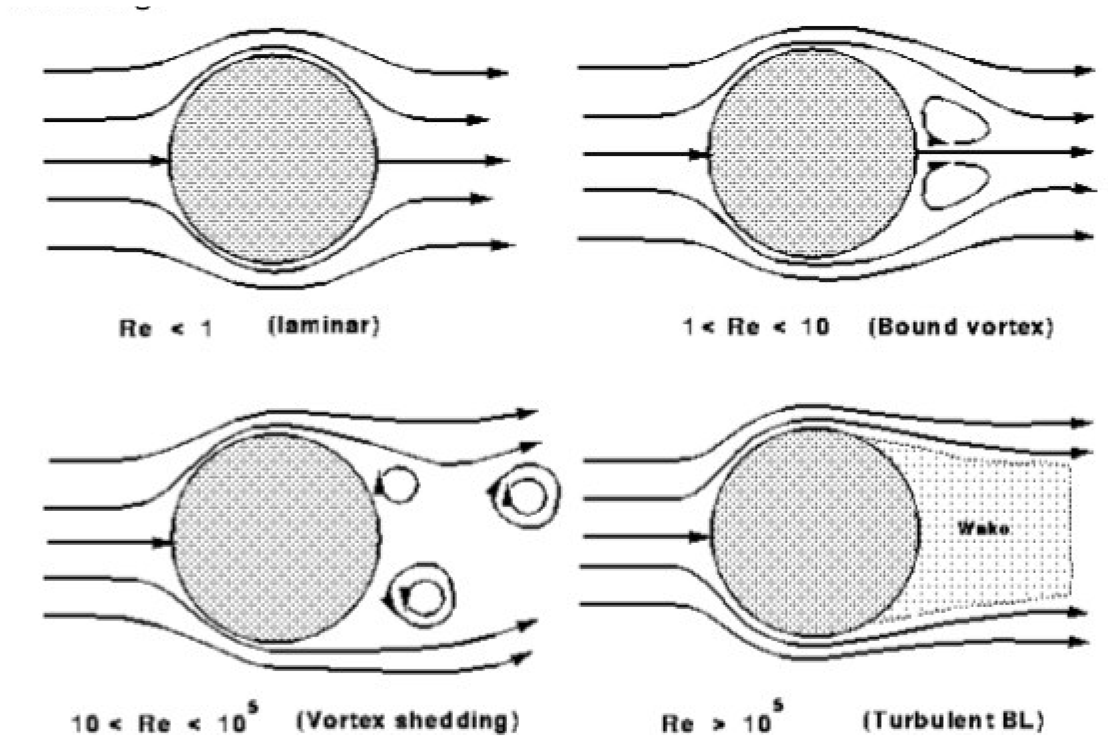
Stokes drag
($Re_p < 1$)

$$t_{stop} \propto s^2$$

Quadratic drag
($Re_p > 800$)

$$t_{stop} \propto \rho_{gas} s v^2$$

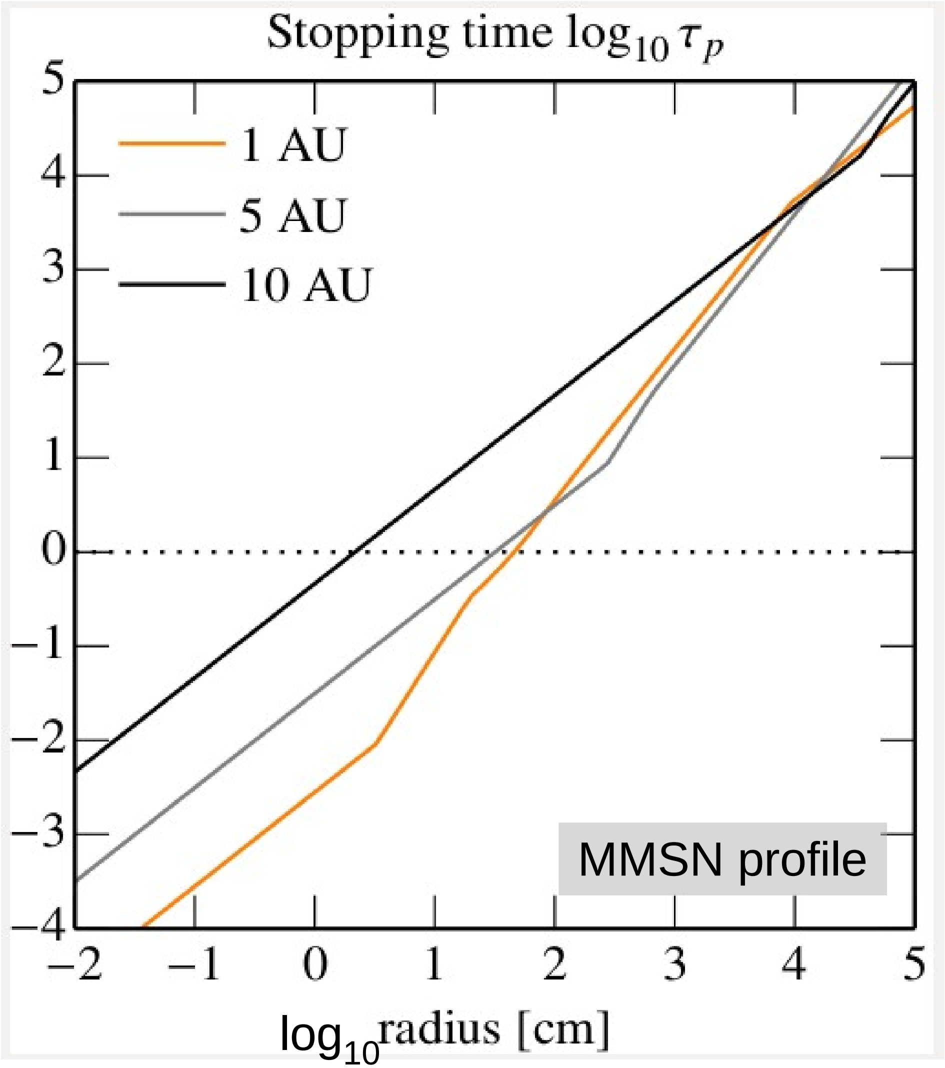
Gas drag/ Flow past sphere/cylinder



Dimensionless stopping time $\tau_p = t_{stop}\Omega_K$

Aerodynamical definition:

- – pebble (τ_p <1)
- – planetesimal (τ_p>>1)



Dimensionless stopping time $\tau_p = t_{stop}\Omega_K$

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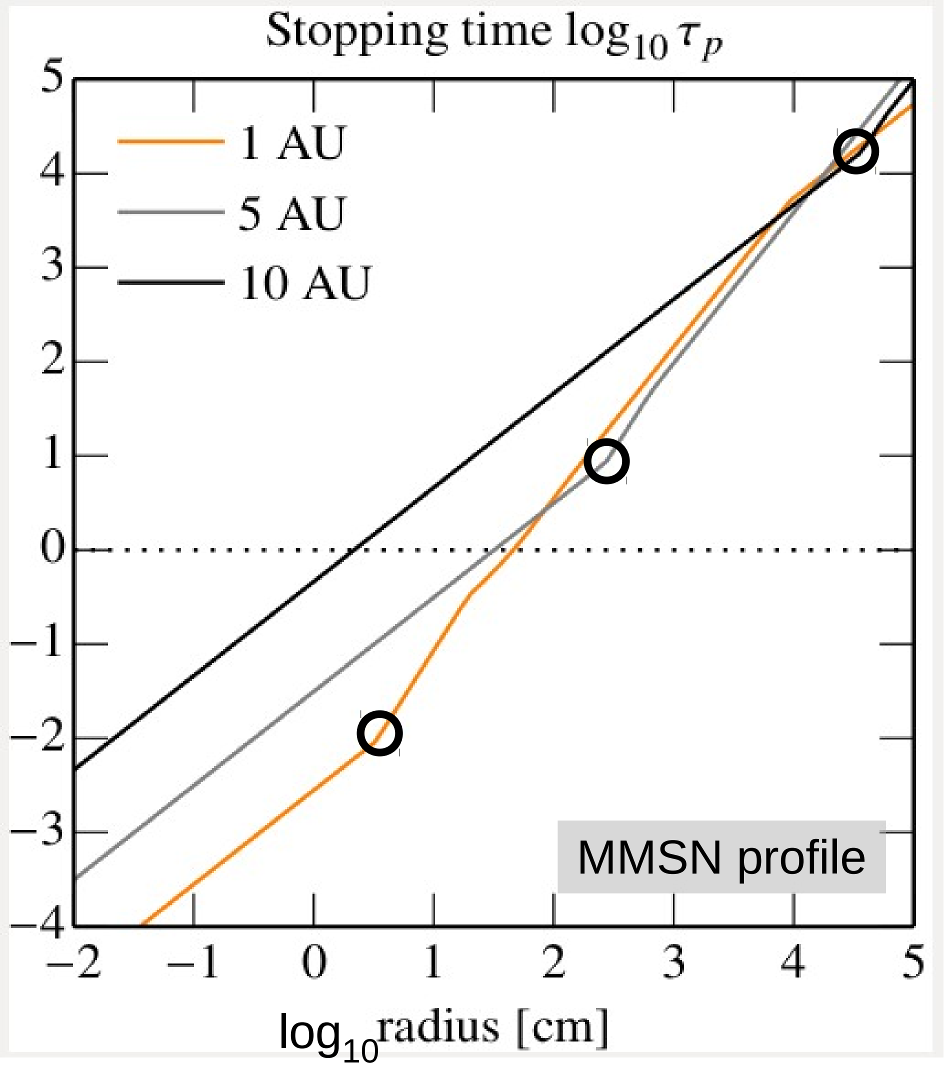
Q: why these inflections?



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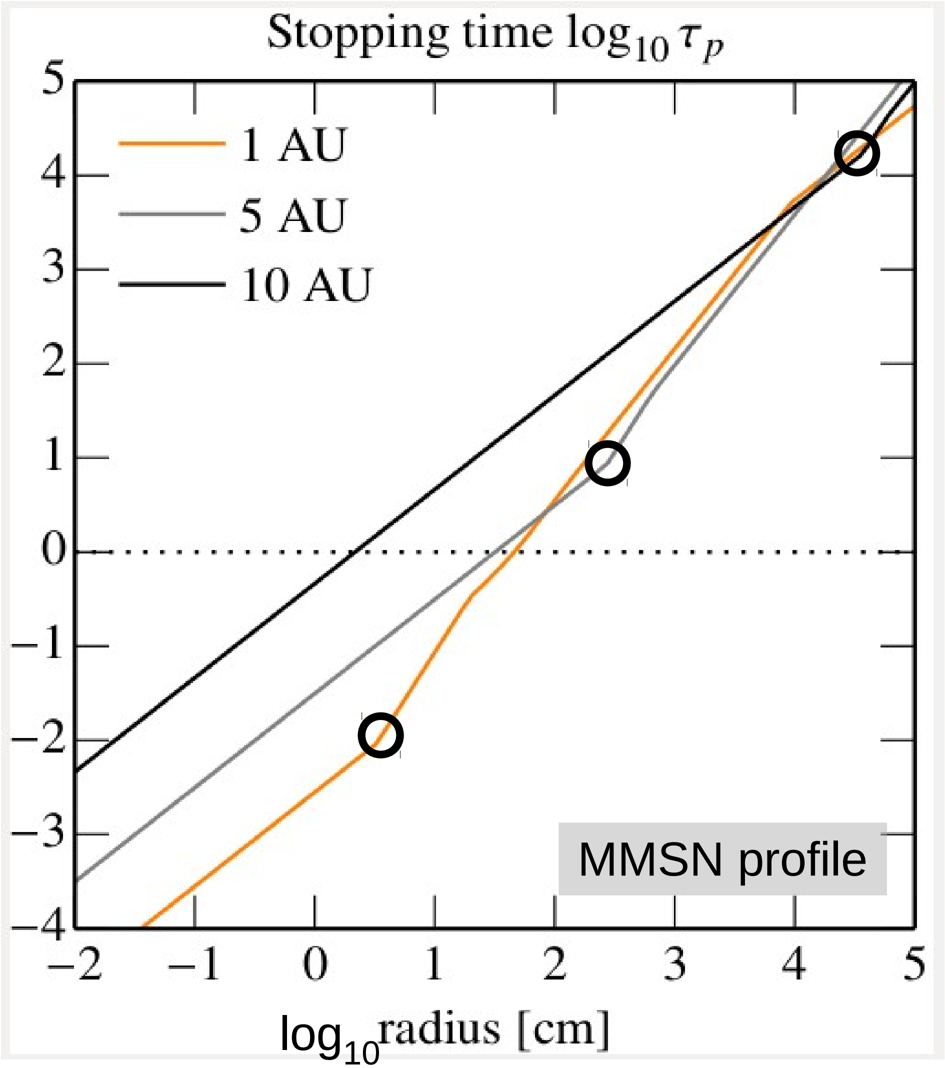
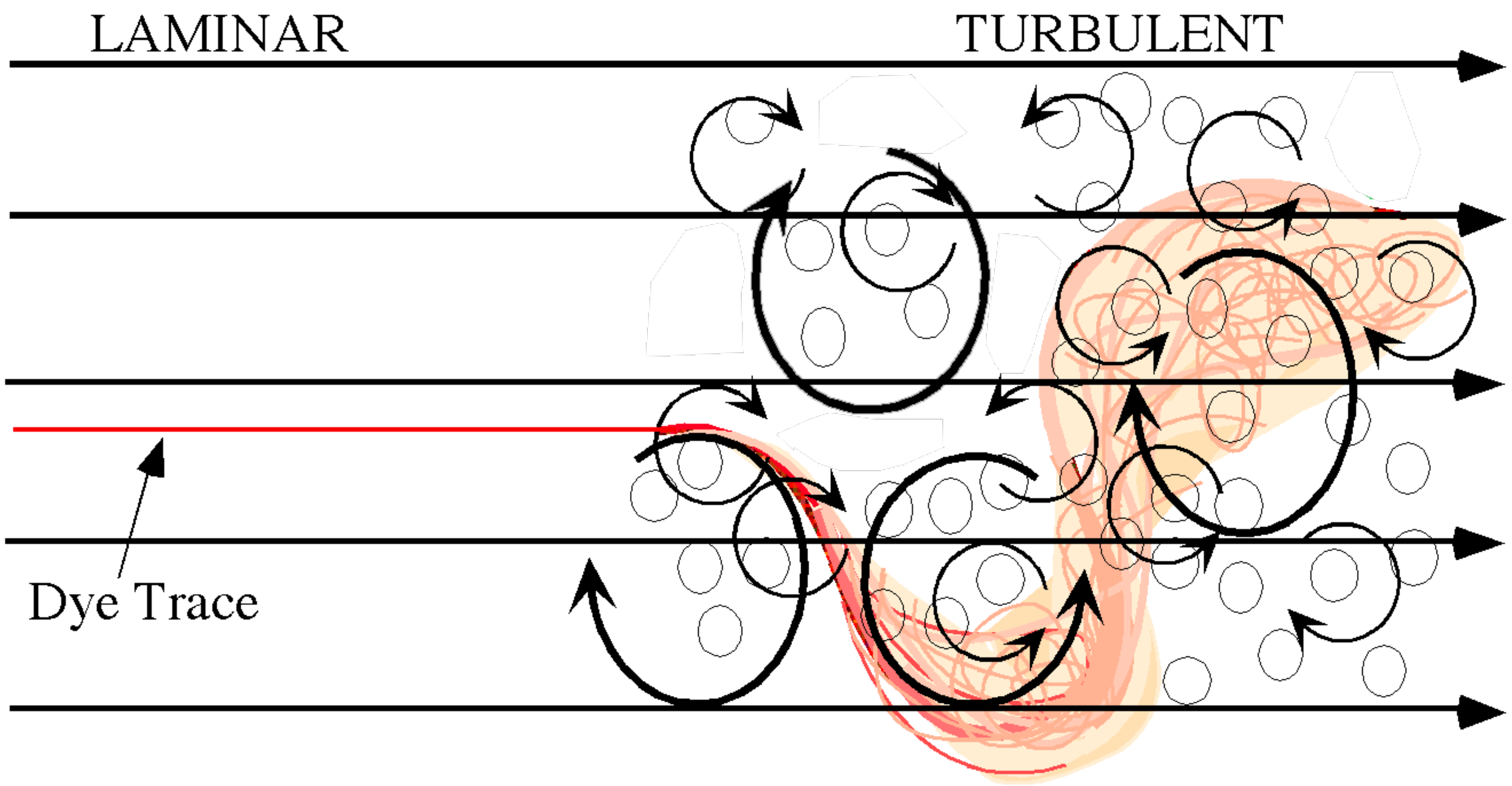


Q: why these inflections?

Q: what happens to t_{stop} when the gas disappears?

Turbulence

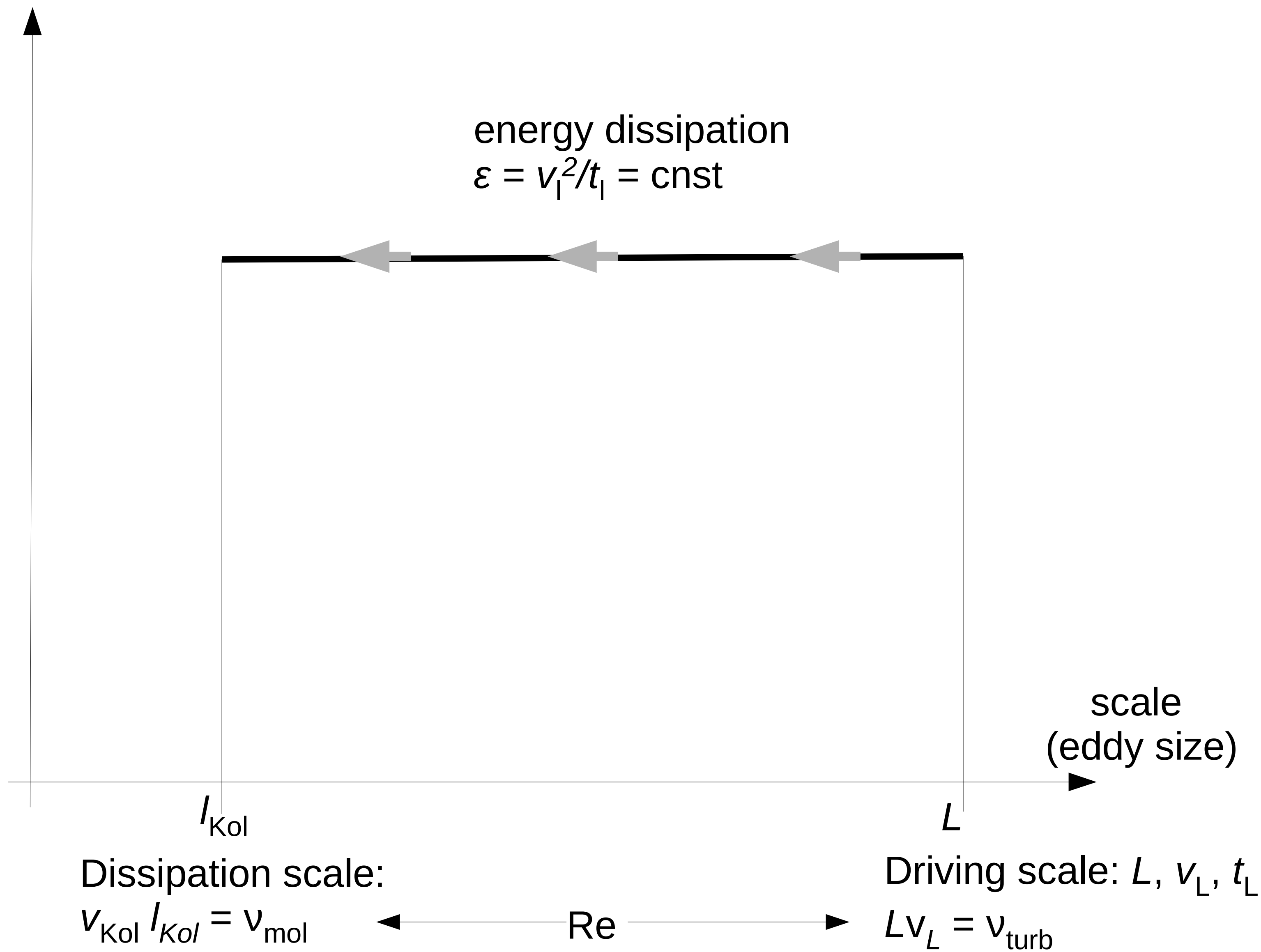
Eddies...



[MIT]

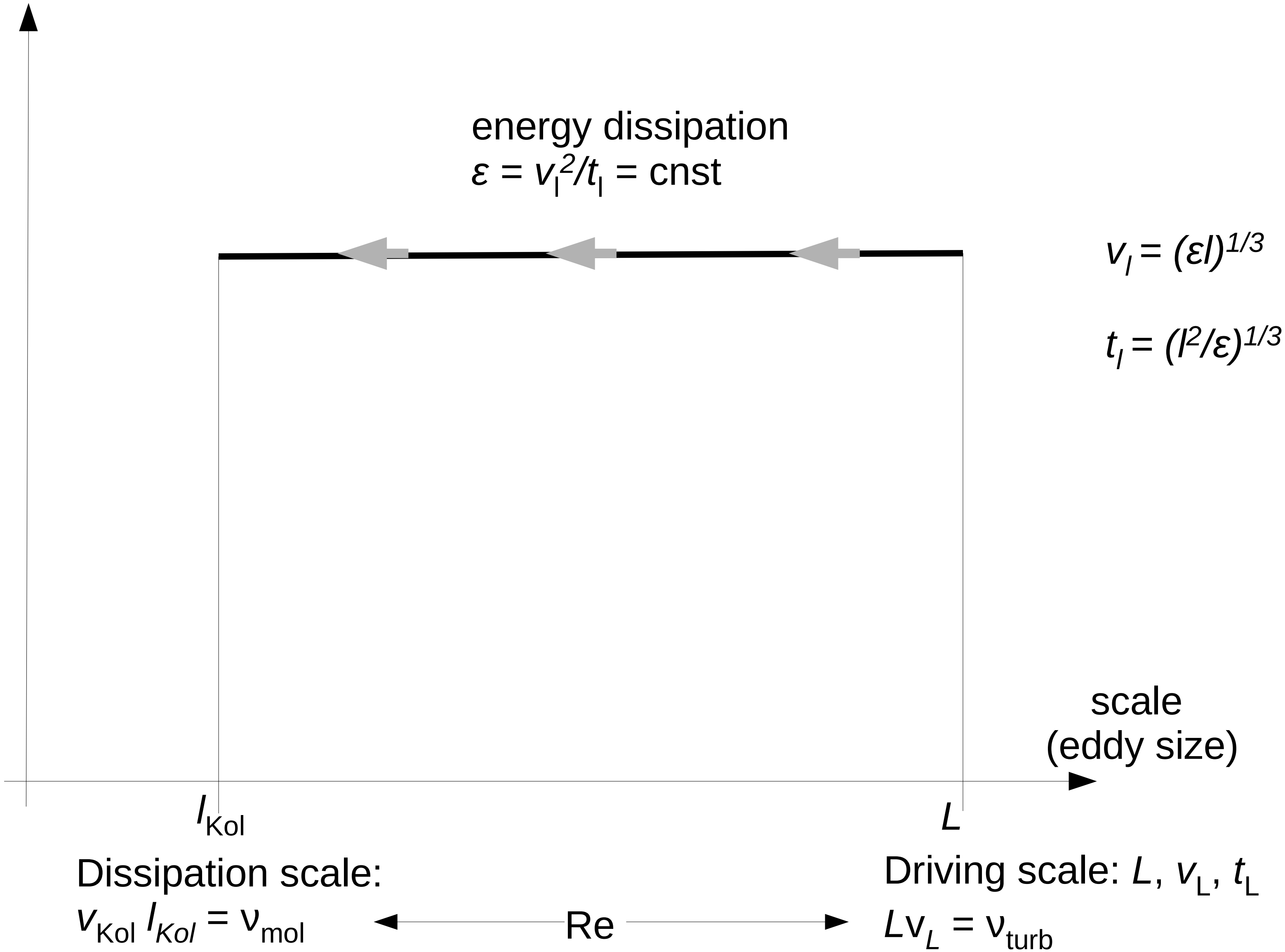
Fully developed turbulence

(Kolmogorov 1941)

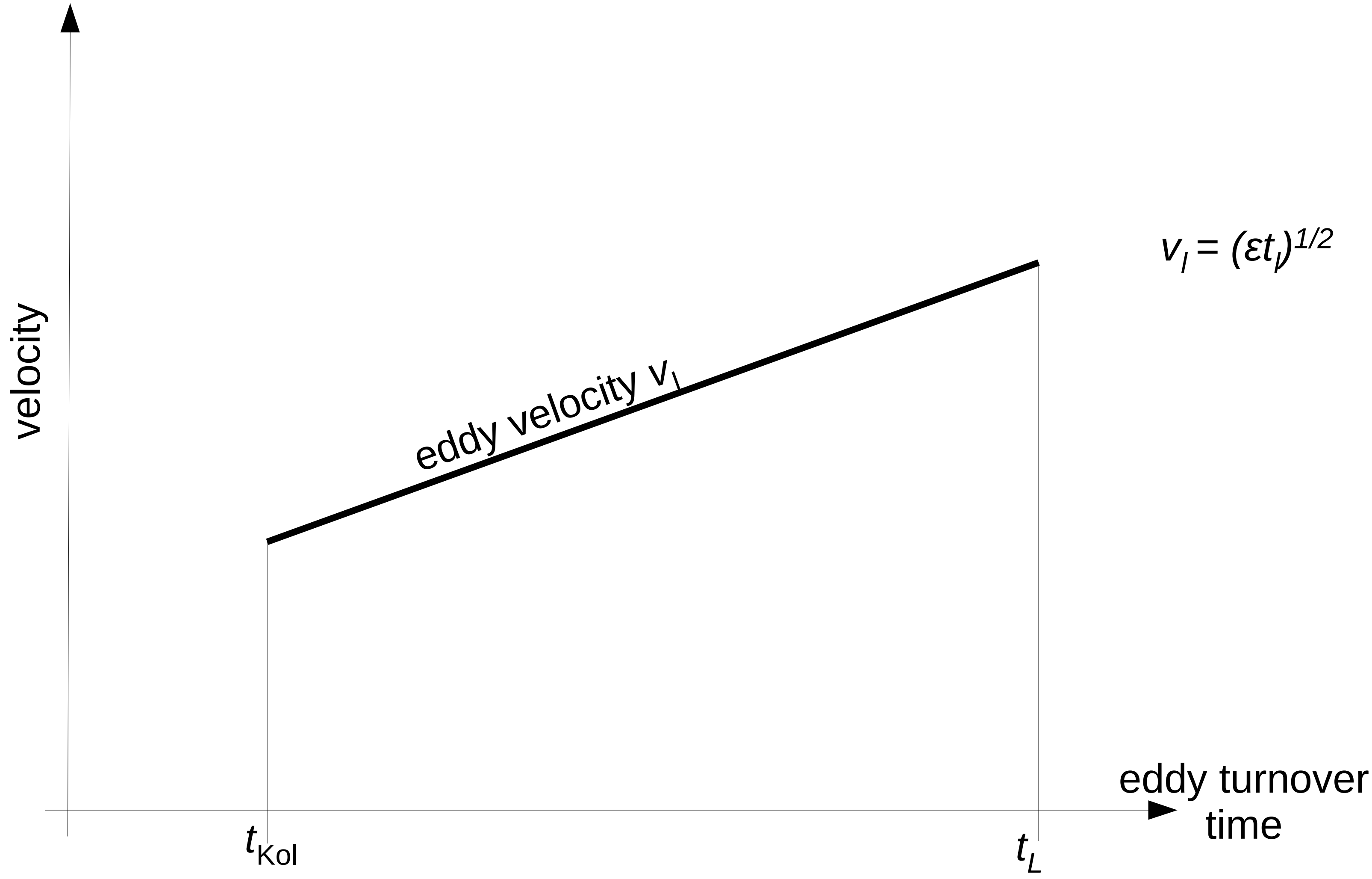


Fully developed turbulence

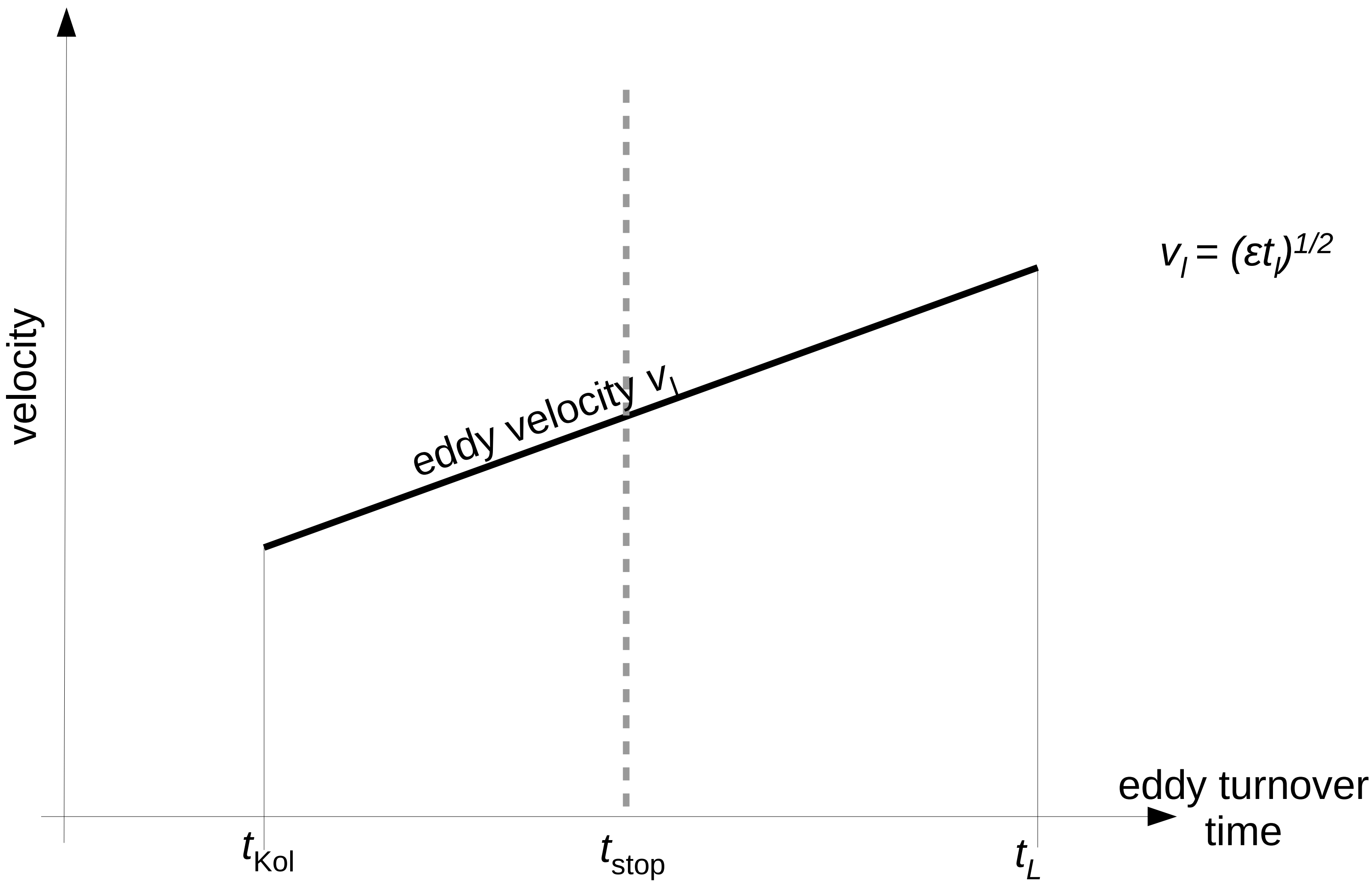
(Kolmogorov 1941)



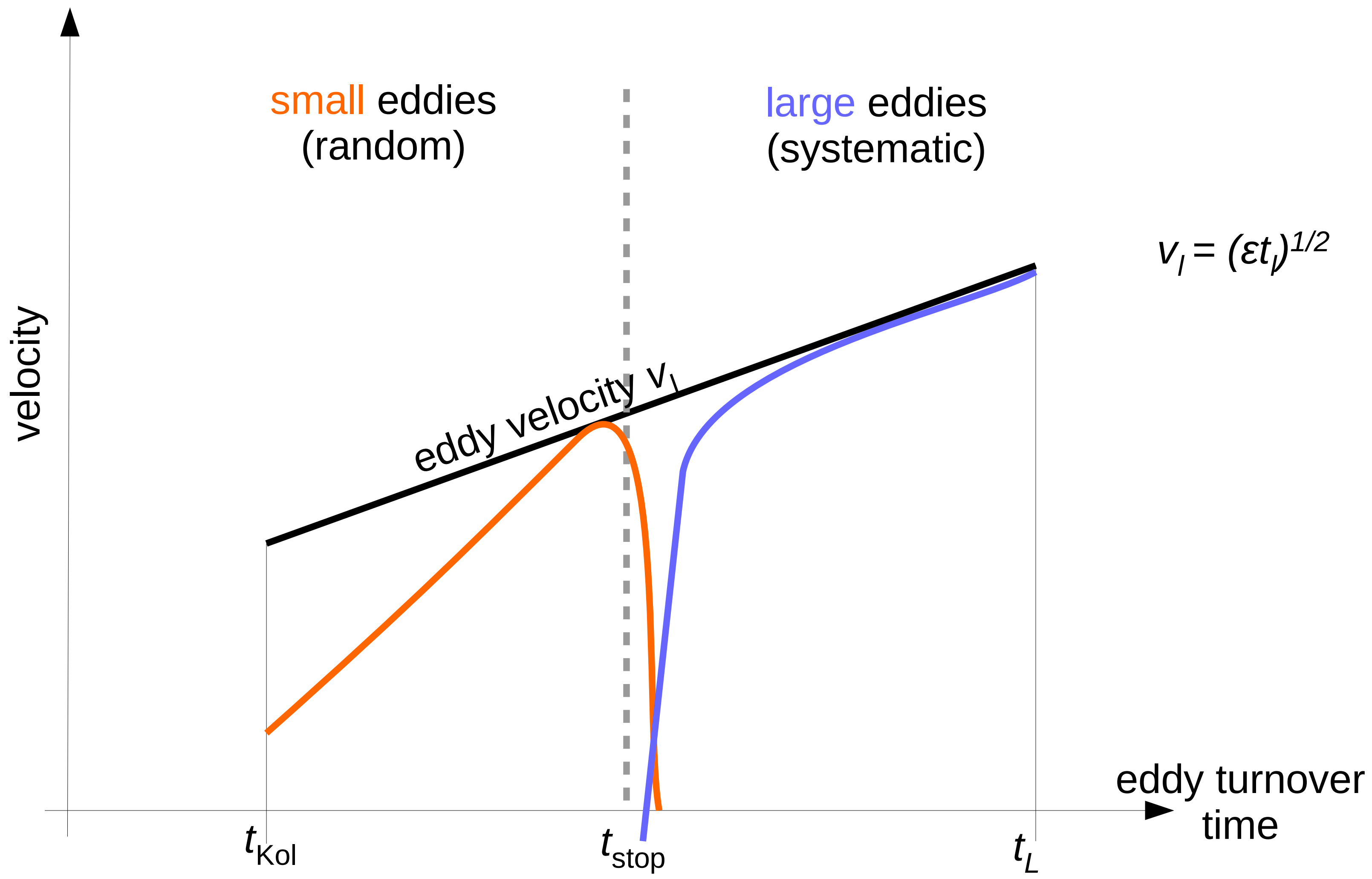
Fully developed turbulence



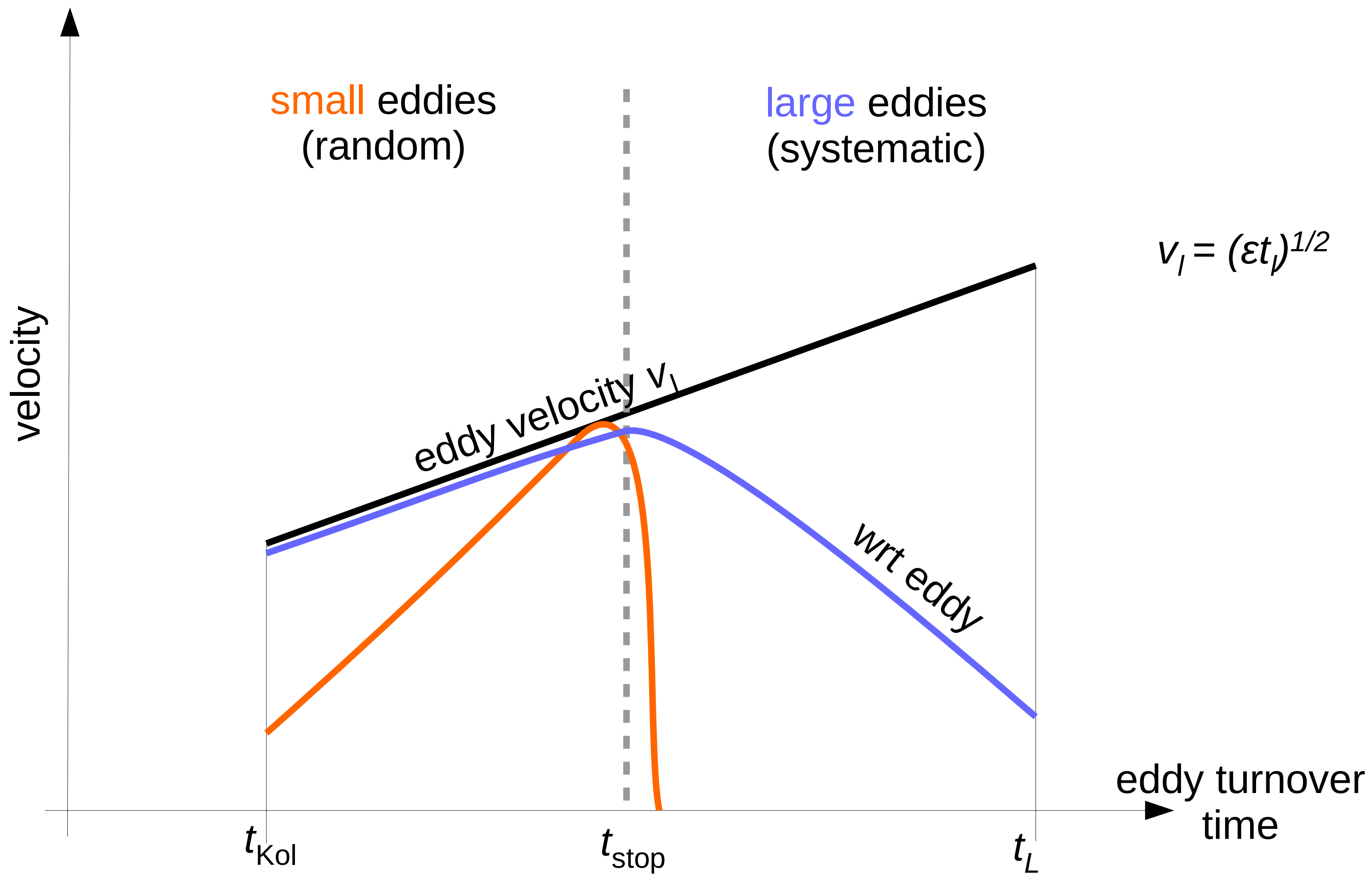
Fully developed turbulence



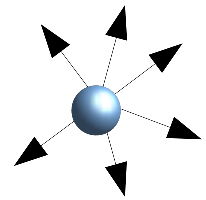
Fully developed turbulence

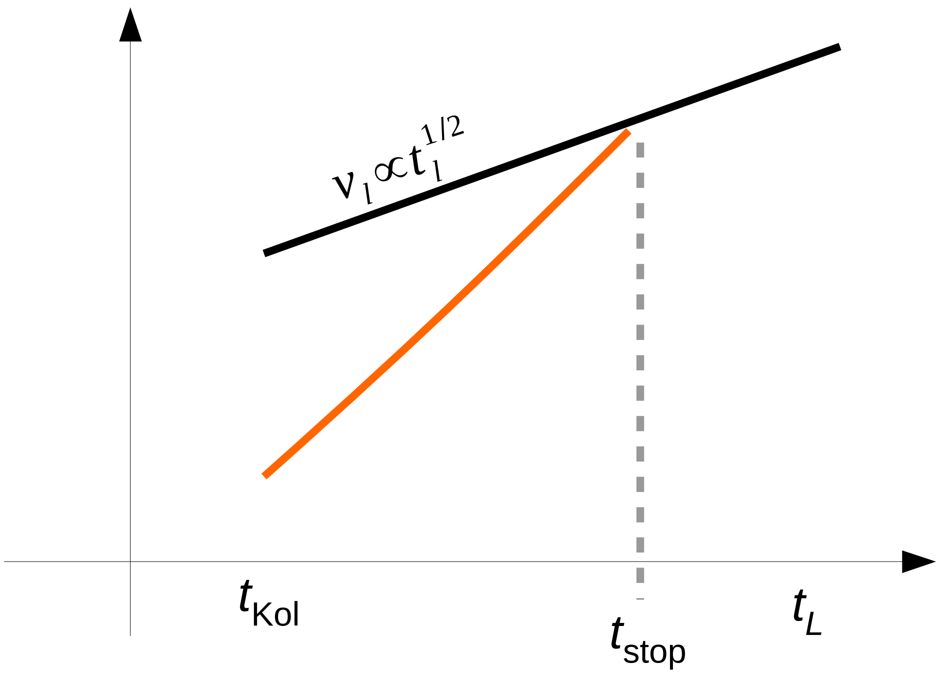


Fully developed turbulence



Small eddies: Random kicks





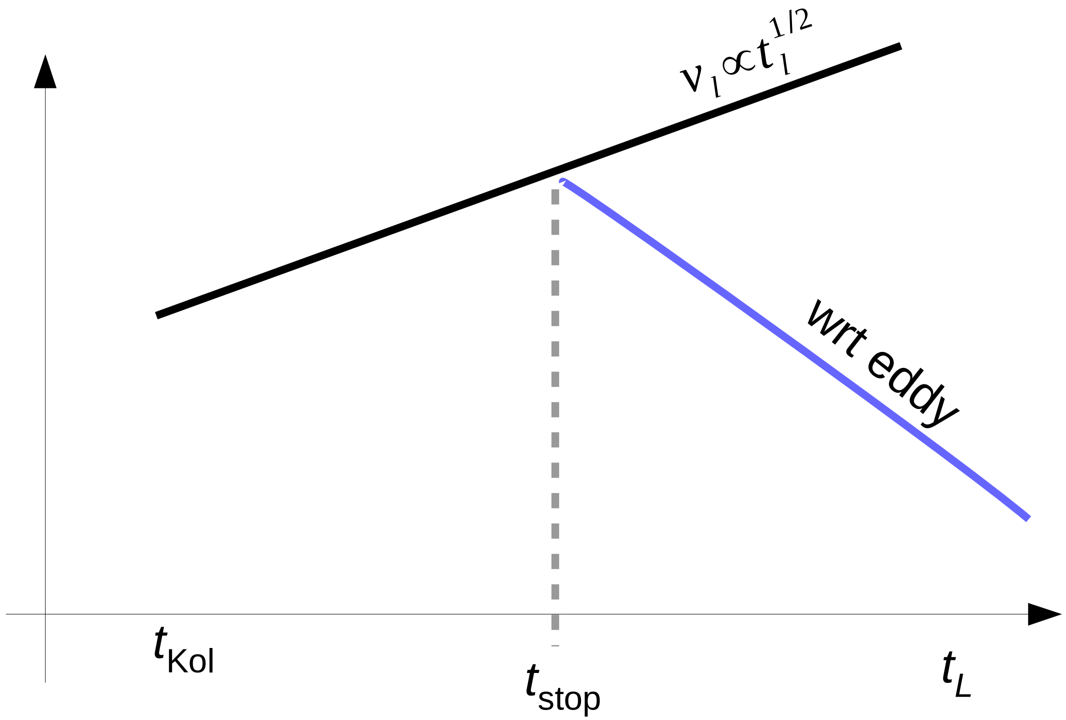
Small eddies: $t_l < t_{stop}$

Eddy kicks a particle by $v_1 \sim v_l\, t_l/t_{stop}$ in a *random* direction

The particle "remembers" $N = t_{stop}/t_l$ of these kicks

→ $v_{ran} = N^{1/2}\, v_1 = (t_l/t_{stop})^{1/2} v_l$

Large eddies: drift



Large eddies: $t_l > t_{stop}$

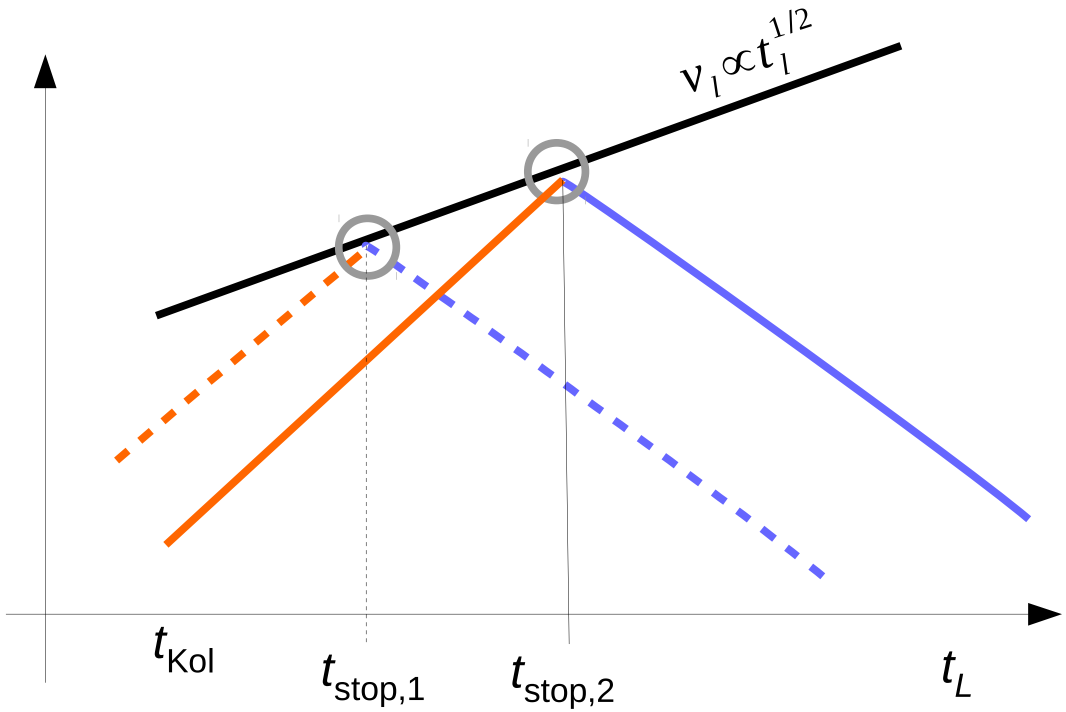
Particles obtain eddy velocity v_l

Experience pressure forces $g_l \sim v_l/t_l$ [Weidenschilling 1984]

→ drift velocity: $v_{sys} \sim g_l\, t_{stop}$

with respect to eddy

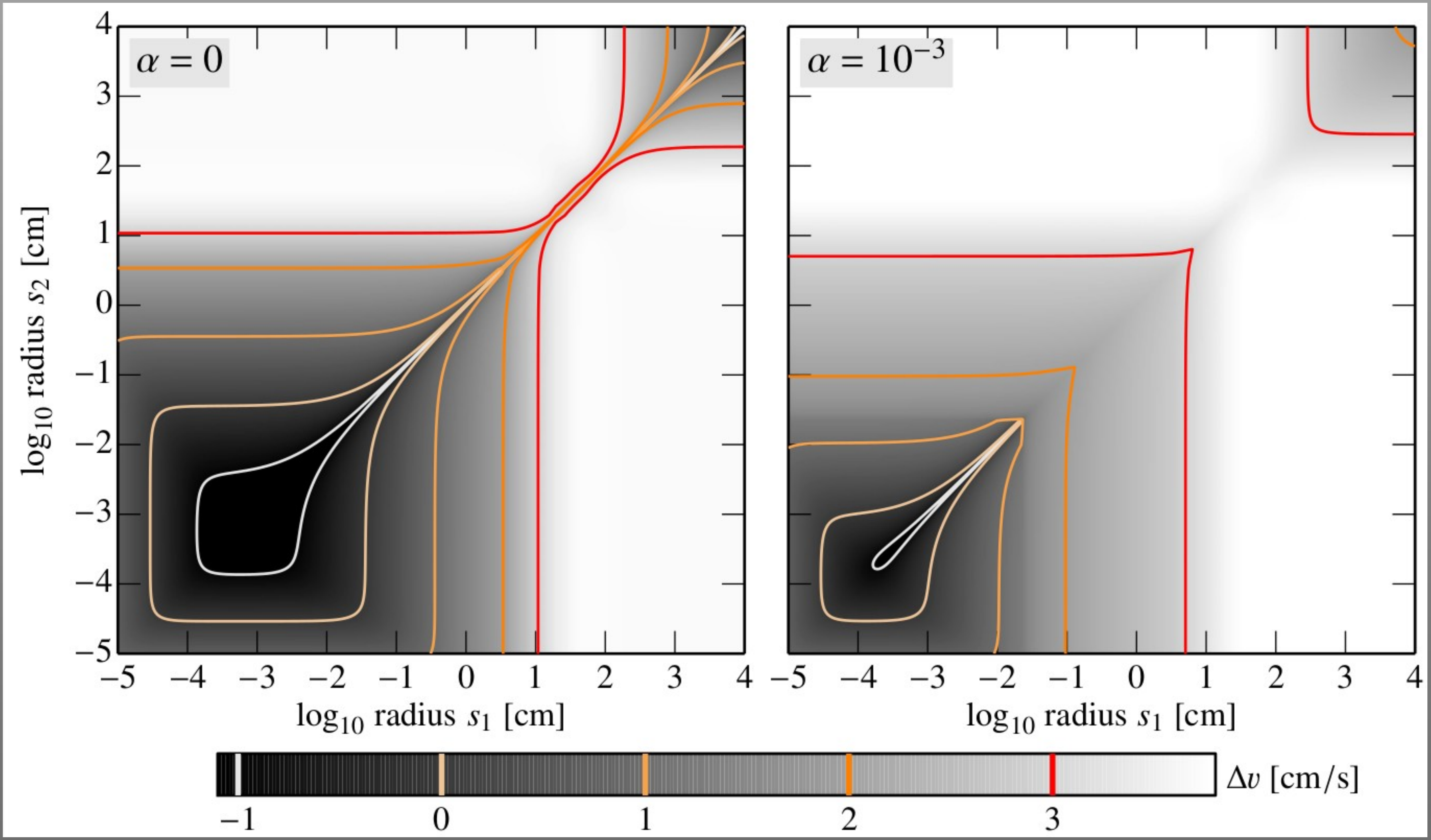
Relative velocity



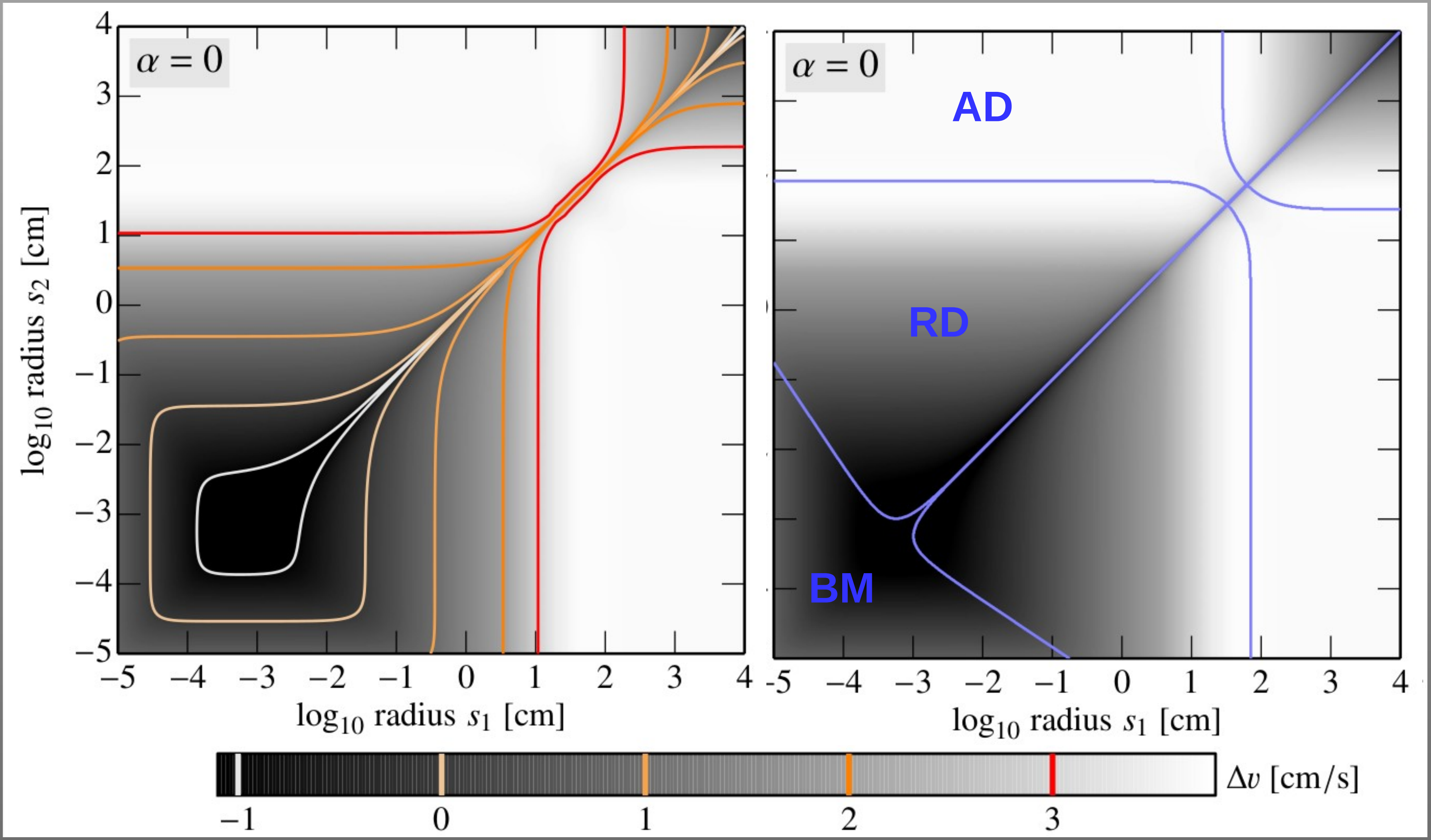
Large eddies (systematic):
→ *subtract* velocities
(vanish equal t_{stop})

Small eddies (random):
→ *add* velocities

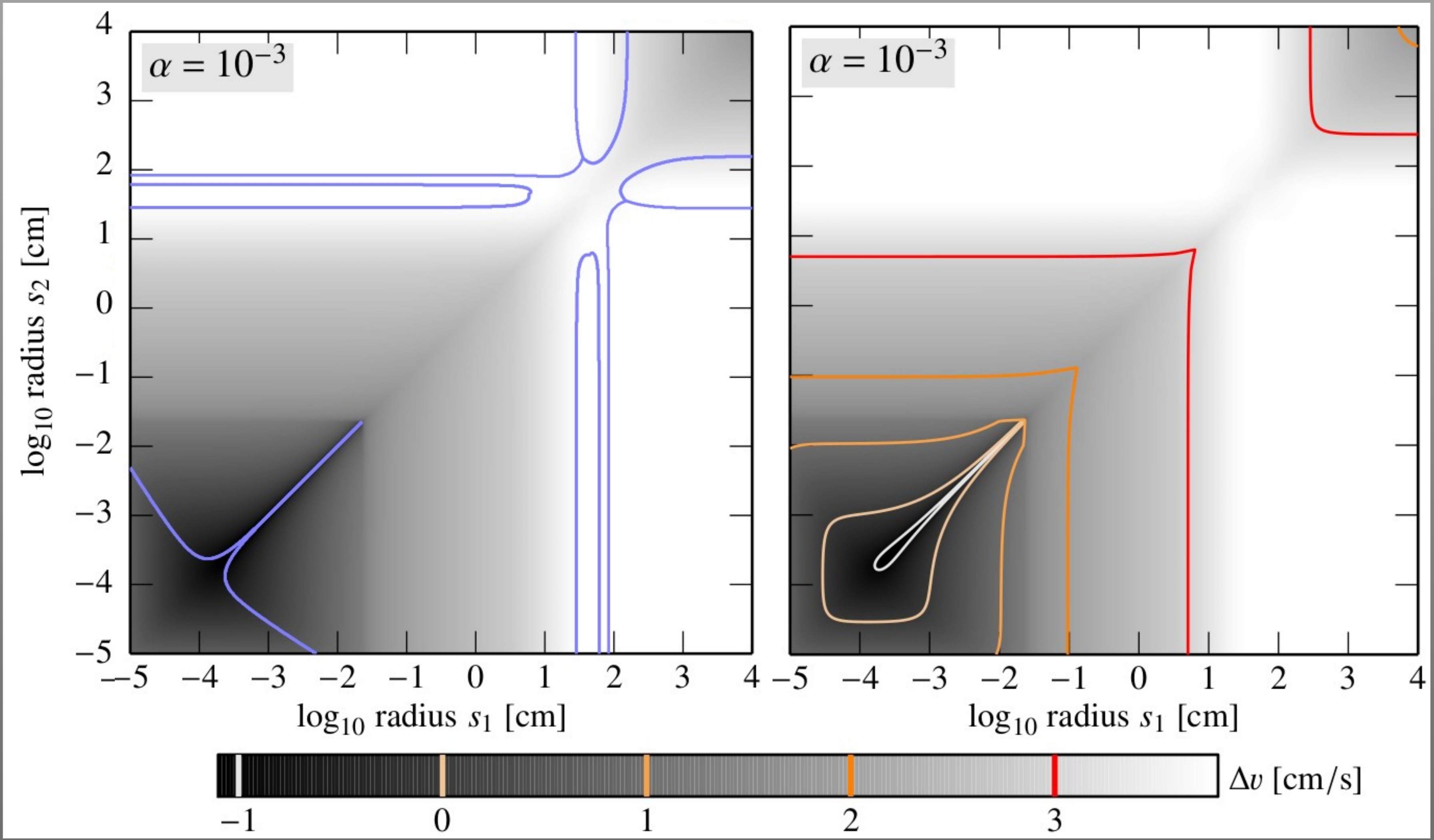
Relative velocities:
(Brownian motion, rad+azi drift, turbulence)



Relative velocities:
(Brownian motion, rad+azi drift, turbulence)



Relative velocities:
(Brownian motion, rad+azi drift, turbulence)



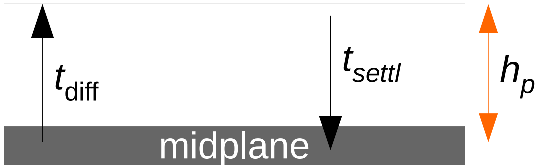
Exercise 1.2 (HW)

Exercise 1.2 particle scaleheight: When disks are turbulent, particles will diffuse along with the gas. A popular model for the turbulent viscosity is the Shakura & Sunyaev (1973) *alpha-model*, which parameterizes the turbulent viscosity as $\nu_T = \alpha c_s h_{\rm gas}$. When particles are small ($\tau_p < 1$) this is also the particles diffusivity D_p . The definition of D_p is such that particles move a distance $\sqrt{D_p t}$ in time t.

(a) Find the equilibrium distance h_p: the height where the settling timescale (from h_p to the midplane) equals the time to diffuse the particles from the miplane to h_p. Express the result in terms of α and τ_p.

(b) Naively, when $\tau_p \to 0$, one obtains $h_p > h_{\rm gas}$. Why is this result incorrect?

"random walk"; distance = rms average



Exercise 1.3

Exercise 1.3– individual drift velocities: These two equations allow us to solve for the two unknowns (v_r and v_ϕ). But we can greatly simplify the procedure by using that v_r, v_ϕ, and the disk headwind ηv_K (see Equation (1.2)) are small with respect to the Keplerian velocity v_K. Expressions as $(u_{\rm gas} + v_\phi)^2$ can then be approximated as $u_{\rm gas}^2 + 2u_{\rm gas}v_\phi$. In the same gist, $u_{\rm gas}^2 = (1-\eta)^2 v_K^2 \approx (1-2\eta)v_K^2$, and $d/dt(u_{\rm gas} + v_\phi) \approx dv_K/dt$. This linearization allows the expressions to be put in matrix form:

$$A \begin{pmatrix} v_r \\ v_\phi \end{pmatrix} = \boldsymbol{b}. \tag{1.10}$$

where A is a 2x2 matrix and $\boldsymbol{b}$ a 2x1 vector. Inverting this system of equations, show that the radial drift velocity becomes:

$$v_r = -2\eta v_K \frac{\tau_p}{1+\tau_p^2} \tag{1.11}$$

and the azimuthal velocity:

$$v_\phi = \eta v_K \frac{\tau_p^2}{1+\tau_p^2}. \tag{1.12}$$

(remark again that v_r, v_ϕ are with respect to the gas velocity.)

1. Force balance
2. Angular momentum loss

Warning:
v_r, v_φ are here defined relative to the gas velocity (!) and are therefore small w.r.t. to v_K and $u_{\rm gas}$.
v_K = Keplerian velocity
$u_{\rm gas}$ = (1-η) = sub-Keplerian vel. gas

This is not necessary; plain substitution is easier

Exercise 1.4 (HW)

Exercise 1.4 turbulent velocities: Consider driving scales of $t_L = 1$ yr, $c_s = 1$ km s^{-1} and a turbulence Mach number of $\approx$0.1, so that $v_L = 0.1 c_s$. Take a Reynolds number of $\mathrm{Re} = 10^8$.

(a) What are the values at the inertial scale, ℓ_{Kol}, t_{Kol}, and v_{Kol}?

(b) Given the toy model for the velocity excitation of particles above, as summarized in Figure 1.6, we can derive expressions for the relative velocities of particles. For example, for two particles of stopping times $t_{s1} \le t_{s2} \le t_{\mathrm{Kol}}$ (where $t_{s1} = t_{\mathrm{stop}}$ of particle #1 and t_{s2} of #2) all eddies are large (top panel). In that case, argue that the relative velocity becomes $\Delta v \sim |t_{s2} - t_{s1}| v_{\mathrm{Kol}} / t_{\mathrm{Kol}}$. (The minus sign is important: the velocity will vanish for $t_{s1} = t_{s2}$. Why?).

(c) In the case where the largest particle (2) has a stopping time in the inertial range, $t_{\mathrm{Kol}} \le t_{s2} \le t_L$, argue that the relative velocity becomes $\Delta v \sim v_L \sqrt{t_{s2}/t_L}$. Why does this expression not depend on t_{s1}?

(d) If both particles are large, $t_L < t_{s1} < t_{s2}$, argue that it is the particle of the shortest stopping time that determines the relative motions and give Δv.

(e) Between a very small particle ($t_{s1} < t_{\mathrm{Kol}}$) and a very big one ($t_{s2} > t_L$) the relative collision velocity is $\Delta v \sim v_L$. Why?

Give order-of-magnitude expressions! (no numerical prefactors)