Gravitational instability & planetesimal formation

67P/Churyumov–Gerasimenko

HR8799
Marois et al. 2010
Gravitational instability and planetesimal formation

- Dispersion relation
  - for thin disks, Toomre-Q
  - giant planet formation
  - planetesimal formation: Goldreich-Ward (GW) mechanism

- Collective effects
  - Collective particle velocities, Kelvin-Helmholtz instability, Streaming instability
From last week...

Sticking of **micron**-size grains ✓
- low $\Delta v$
- large $v_{\text{stick}}$

Sticking of **mm/cm**-size pebbles ?
- $\Delta v$ increases (turbulence, drift)
- $v_{\text{stick}}$ decrease

**Meter-size** boulders unlikely to stick
- “meter size” ($\tau_p \sim 1$) barrier
- caveat: fractal growth (?)

Perhaps growth by sticking stalls (bouncing, fragmentation)

However: (even small) particles can settle into a very thin midplane

The *dust-dominated* midplane may become gravitationally-unstable and collapses (fragments) into planetesimals!
– dispersion relation for thin disks:

\[ \omega^2 = \kappa^2 - 2\pi G\Sigma k + k^2 c_s^2 \]

– giant planet formation
– planetesimal formation
## Dispersion relation results

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<th>Gas</th>
<th>Solids</th>
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<td>Disk instability</td>
<td>Goldreich-Ward mechanism</td>
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<td><strong>Important scales</strong></td>
<td>$\lambda_c = 2c_s^2/G\Sigma_{\text{gas}}$ (most unstable $\lambda$)</td>
<td>$\lambda_c = 4\pi^2\Sigma_p/\Omega^2$ ($\lambda &gt; \lambda_c$ unstable)</td>
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<td><strong>Condition instability</strong></td>
<td>$Q_T &lt; 1$</td>
<td>$h_p &lt; \lambda_c$</td>
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<td>Also: cooling gas</td>
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<td><strong>Problem:</strong></td>
<td>– Need massive disk</td>
<td>$\textit{Kelvin-Helmholtz (KH) turbulence}$</td>
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<td>– rapid cooling</td>
<td></td>
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<td></td>
<td>– too massive planets?</td>
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Kelvin-Helmholtz turbulence
Dust-dominant layer

- Settling: Particles end up near $z \approx 0$.
- Collectively their density exceeds that of the gas: $\rho_p > \rho_{\text{gas}}$ (in midplane)
- Dust dominates the dynamics $\rightarrow$ midplane tends to Keplerian rotation and drags gas along.
- Difference of $\sim \eta v_K$ triggers KH-instability.
Driving equations
(Sometimes referred to as NSH-solutions, after Nagakawa et al. 1986)

\[
\begin{align*}
\text{solids: } & \quad \frac{Dv}{Dt} = \frac{v - u}{t_{\text{stop}}} - 2\Omega_K \times v + F_{\text{Euler-dust}} \\
\text{gas: } & \quad \frac{Du}{Dt} = \frac{\rho_p}{\rho_g} \frac{v - u}{t_{\text{stop}}} - 2\Omega_K \times u + F_{\text{Euler-gas}} + F_{\text{pres}}
\end{align*}
\]

(1.34a)
(1.34b)

Back reaction (Newton's 3\textsuperscript{rd} law)

Pressure gradient (involves $\eta v_K$)

4 equations, 4 unknowns
→ solve $u_r$, $u_\phi$, $v_r$, $v_\phi$
as function of $\rho_p$, $t_{\text{stop}}$

For the KH-instability we are interested in $u_\phi$ at midplane

Q: Instability when:
A) $u_\phi = 0$
B) $u_\phi = -\eta v_K$
Solution

Solids

\[ v_r = -\frac{2\tau_p}{\tau_p^2 + (1 + Z)^2}\eta v_K \]
\[ v_\phi = -\frac{1 + Z}{\tau_p^2 + (1 + Z)^2}\eta v_K \]

Gas

\[ u_r = \frac{2Z\tau_p}{\tau_p^2 + (1 + Z)^2}\eta v_K \]
\[ u_\phi = -\frac{1 + Z + \tau_p^2}{\tau_p^2 + (1 + Z)^2}\eta v_K \]

**Z**: dust-to-gas ratio “metallicity”

**\(\tau_p\)**: dimensionless tstop

**\(\eta\)**: pressure gradient parameter

**HW 1.10**

interpret limits

\[ Z \to 0 \]
\[ \tau_p \to 0, \infty \]

For the KHI \(u_\phi\) is the most relevant

\[ u_\phi \to 0: \text{midplane rotates Keplerian, vertical shear (KHI)} \]

\[ u_\phi \to -\eta v_K: \text{midplane rotates subKeplerian, no vertical shear} \]
NSH solution

Whether or not the KHI is triggered depends on the *Richardson number*:

\[
\text{Ri} = \frac{-g_z/\rho \left( \partial \rho / \partial z \right)}{\left( \partial u_\phi / \partial z \right)^2} < \text{Ri}_{\text{crit}}
\]

- **nominator**: buoyancy (stabilizing)
- **denominator**: shear (destabilizing)
- **Ri}_{\text{crit}}**: critical Richardson number; around unity; \( \text{Ri} > \text{Ri}_{\text{crit}} \) for stability
Recapitulate...

**Goal:** planetesimal formation

- **particle collisions**
  - uncertain ($\Delta v \sim 10$ m/sec)

- **GI of a dense solid layer**
  - a.k.a. Goldreich-Ward mechanism

**Requires:**
- a very thin particle disk ($h_p < \lambda_c$)

**Problem:**
- very thin particle layers will force the gas to move Keplerian and can trigger KHI when $Ri < 1$

- **Ri**
  \[
  Ri = \frac{-(g_z/\rho)(\partial \rho/\partial z)}{(\partial u_\phi/\partial z)^2} < R_{icrit}
  \]

- **No KHI:**
  - Disks avoid triggering KHI
  - $\rightarrow$ GW-mechanism viable
  - Planetesimals!

- **KHI triggered:**
  - Turbulence lofts particle back up ($h_p > \lambda_c$); no GI
  - No planetesimals
Streaming instability

**KH-stable?**
In HW 1.11 you will assess w/r or not the GW-mechanism is viable

A: only for very massive disks

**Alternative:**
One can conduct a linear perturbation analysis to the (KH-stable) NSH-solutions for the 2-fluid (dust+gas) mixture!

It turns out that the 2-fluid harbors exponentially-growing modes for $\rho_p$, especially for $\tau_p \sim 1$ particles. This is known as the **streaming instability** (Youdin & Goodman 2005)

\[
Ri = \frac{- (g_z / \rho) (\partial \rho / \partial z)}{(\partial u_\phi / \partial z)^2} < Ri_{\text{crit}}
\]
Streaming instability

**Streaming instability (SI)**

*Linear* perturbation analysis (Youdin & Goodman 2005) quite technical.

SI occurs even in absence of self-gravity!

Best analogies are clusters of cyclists or geese that organize themselves in the optimal way to deal with the headwind!

*Nonlinear* effects occur when perturbations gets large; can best be investigated by hydrodynamical simulations

... and *bound* clumps when gravity is accounted for
Streaming instability

Initial condition: dense-layer of pebble-size particles: $\tau_p \sim 0.25-1$

Unit of mass is Ceres (already 1,000 km)

Very big planetesimals form, but this may be a question of resolution

Johansen, Klahr, & Henning (2011)
Streaming instability

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Johansen, Klahr, & Henning (2011)
Homework set changes

- New deadline: **Tuesday 13:00** (sharp! no delays/exceptions!)
- No more scans!
- Bonus questions (updated problem set on BB)
Project (2 more weeks)

Communicate!
Exercise 1.9

(a) From Equation (1.31), show that the scale most ‘vulnerable’ to instability is \( \lambda_c = \frac{2 \cs^2}{G \Sigma} \) (where we switched to a spatial scale \( \lambda = 2\pi/k \)) and that instability is triggered when

\[
Q_T = \frac{c_s \Omega}{\pi G \Sigma} < 1, \tag{1.32}
\]

where \( Q_T \) is the Toomre-Q parameter.

(b) A physically-intuitive way to obtain \( Q_T \) approximately is to compare the total internal, rotational, and gravitational energies. When

\[
\left| \frac{E_{\text{therm}}}{E_{\text{grav}}} \right| \times \left| \frac{E_{\text{rot}}}{E_{\text{grav}}} \right| < 1 \tag{1.33}
\]

the gravitational energy dominates over the combined rotational and thermal energies, leading to instability. Show that the above estimate results in \( Q_T \), barring a factor of unity.

(c) Take a disk with a solar-mass star and, sound speed \( \cs = 1 \text{ km s}^{-1} \times r_{\text{AU}}^{-1/4} \) and \( \Sigma = 10^3 \text{ g cm}^{-2} \times r_{\text{AU}}^{-1} \). Where does the disk become unstable and what is the corresponding disk mass?

(d) What is the mass associated with the scale \( \lambda_c \)?

Bonus HW
compare the (specific)
energy across a scale \( \lambda \)
e.g. \( E_{\text{therm}} \sim \lambda^2 \cs^2 \)

only Toomre-Q criterion
Exercise 1.10 (HW)

\[ v_r = -\frac{2\tau_p}{\tau_p^2 + (1 + Z)^2 \eta v_K} \]
\[ u_r = \frac{2Z\tau_p}{\tau_p^2 + (1 + Z)^2 \eta v_K} \]
\[ v_\phi = -\frac{1 + Z}{\tau_p^2 + (1 + Z)^2 \eta v_K} \]
\[ u_\phi = -\frac{1 + Z + \tau_p^2}{\tau_p^2 + (1 + Z)^2 \eta v_K} \]

\textbf{Exercise 1.10:} It is interesting to consider the limiting expressions of Equation (1.35). Verify that:

(a) \( \tau_p \gg 1 \) (big rocks): \( v_r = u_r = v_\phi = 0 \) and \( u_\phi = -\eta v_K \).

(b) \( Z = 0 \) (negligible dust): solutions are the same as the individual solutions of Equations (1.11) and (1.12)

(c) \( Z \gg 1 \) (dust-dominated): \( v_r = v_\phi = u_r = u_\phi = 0 \)

(d) \( \tau_p \ll 1 \) and \( Z \ll 1 \) (tracer dust): \( v_r = u_r = 0 \) and a reduced gas headwind velocity with an "effective" \( \eta \rightarrow \eta/(1 + Z) \).

Explain these limits physically.

(a)–(c) You get kudos only for the \textit{physical interpretation!} (“What does it mean”)

(d) More challenging. Take \( \tau_p = 0 \) and show that this effectively lowers \( \eta \). Consider the definition of \( \eta \) to see why you get a reduced headwind?
Exercise 1.11: Give an order-of-magnitude expression for $R_i$, by evaluating Equation (1.36) at $z = h_p$, the particle scaleheight. Assume that the dust has settled such that dust dominates $\rho$ up to $\sim h_p$. For $g_z$ consider both the self-gravity limit ($i.e.$ $g_z$ is determined by the dust) and the stellar gravity limit ($g_z = g_{*z}$). Taking a critical Richardson number of $R_{i_{\text{crit}}} = \frac{1}{4}$, what is the smallest scaleheight into which the dust can settle? How does this height compare to the critical wavelength $\lambda_c$ of the GW-model? (Note that $h_p < \lambda_c$ is required for the GW-instability.)

\[
R_i = \frac{-(g_z/\rho)(\partial \rho/\partial z)}{(\partial u_\phi/\partial z)^2} < R_{i_{\text{crit}}}
\]

This is a challenging exercise. See HW-notes for instructions!

You should “estimate” the gradients involved in the definition of $R_i$, e.g.,

\[
\begin{align*}
\Delta \rho & = \ldots \\
\Delta z & = \ldots \\
\Delta u_\phi & = \ldots
\end{align*}
\]
Exercise 1.12 (HW)

Exercise 1.12: Consider a test body of mass $M$ immersed in a sea of smaller bodies of mass $m$. Assume that the $M$-body is on a circular orbit at semi-major axis $a$ and orbital frequency $\Omega_K$, while the $m$-bodies are in Kepler orbits with eccentricity $e$ and inclinations $i \approx e$.

(a) Ignoring (for the moment) gravitational focusing, show that the growth timescale of the $M$-body is:

$$t_{\text{growth}} = \frac{M}{dM/dt} \approx \frac{R \rho_*}{\Sigma_m \Omega_K} \quad \text{(no focusing)} \quad (1.37)$$

where $R$ is the radius corresponding to $M$, $\rho_*$ the internal density of the material and $\Sigma_m$ the surface density in $m$-bodies.

(b) How long would it take to form an Earth-mass planet at: 0.1, 1, and 10 AU?

(c) How much shorter are growth times in the 2D case ($i = 0$)?

This is another $n*\sigma*\Delta v$ exercise.

You can assume that $\Delta v$ is given by the eccentricity of the planetesimals (dispersion-dominated regime)