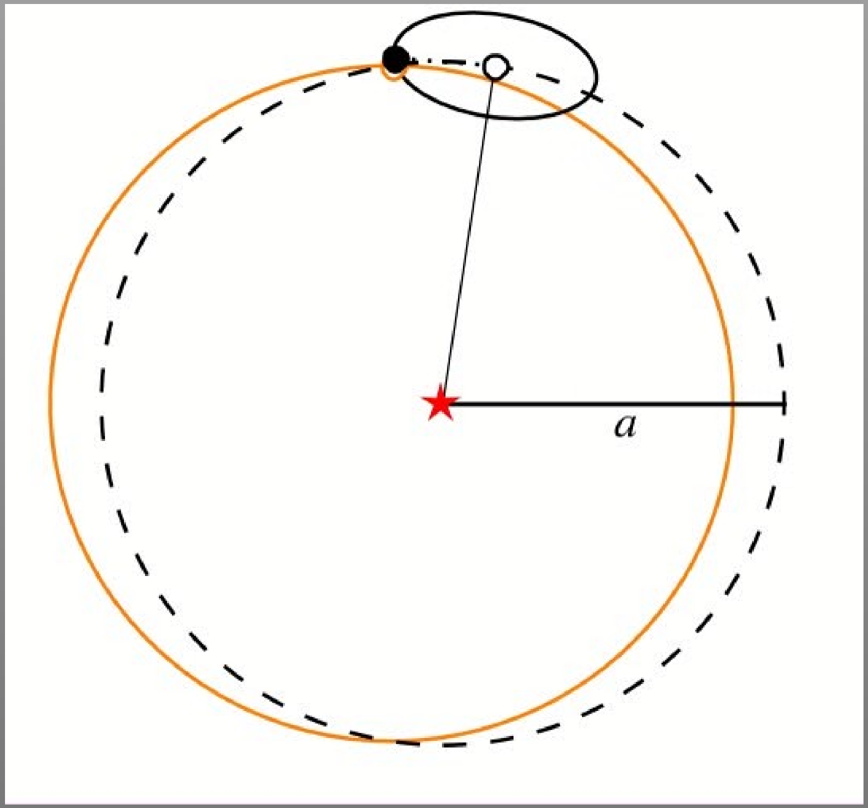
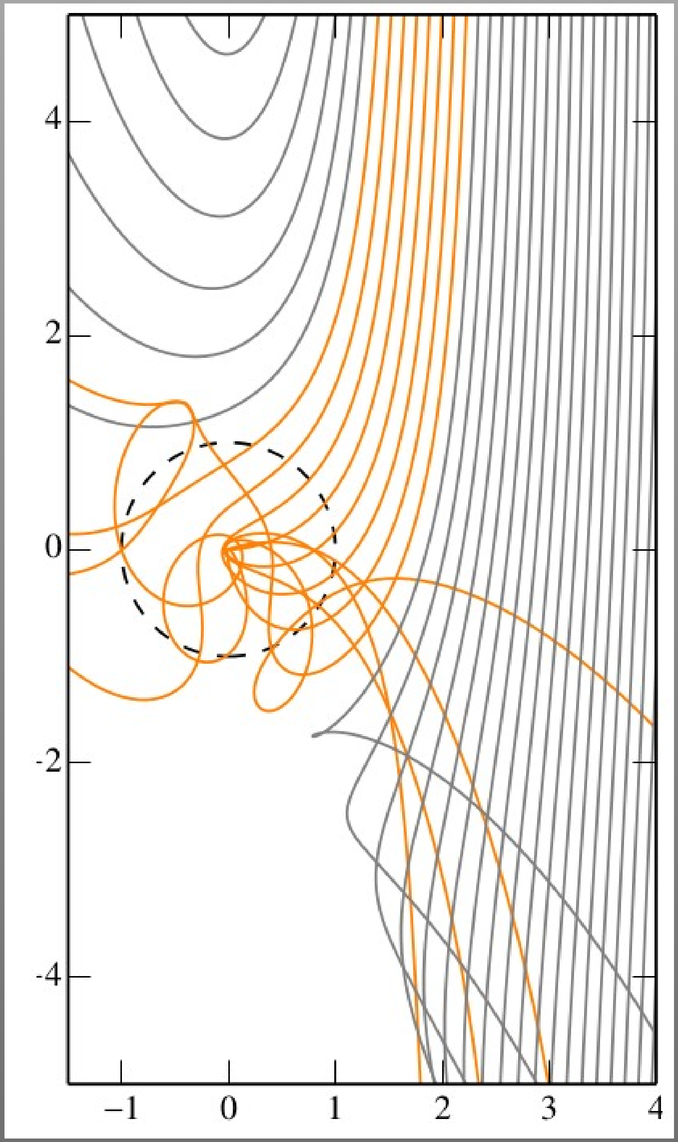
L10: The 2 and 3 body problems





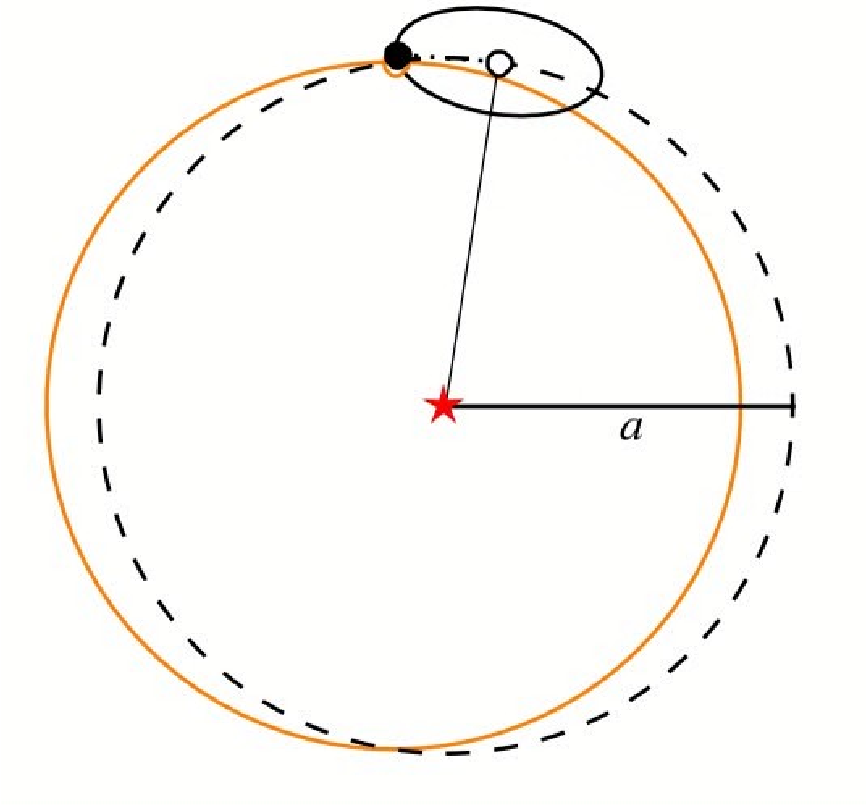
Lecture 10: two and three body problem

- Two body problem
  Relative motion, Integrals of motion, orbit solution, anomalies, guiding center approximation, the orbit in space, orbital elements
- Three body problem
  - Circular, restricted three body problem, Jacobi energy, zero velocity curves, Tisserand relation, Hill's equations

Blackboard

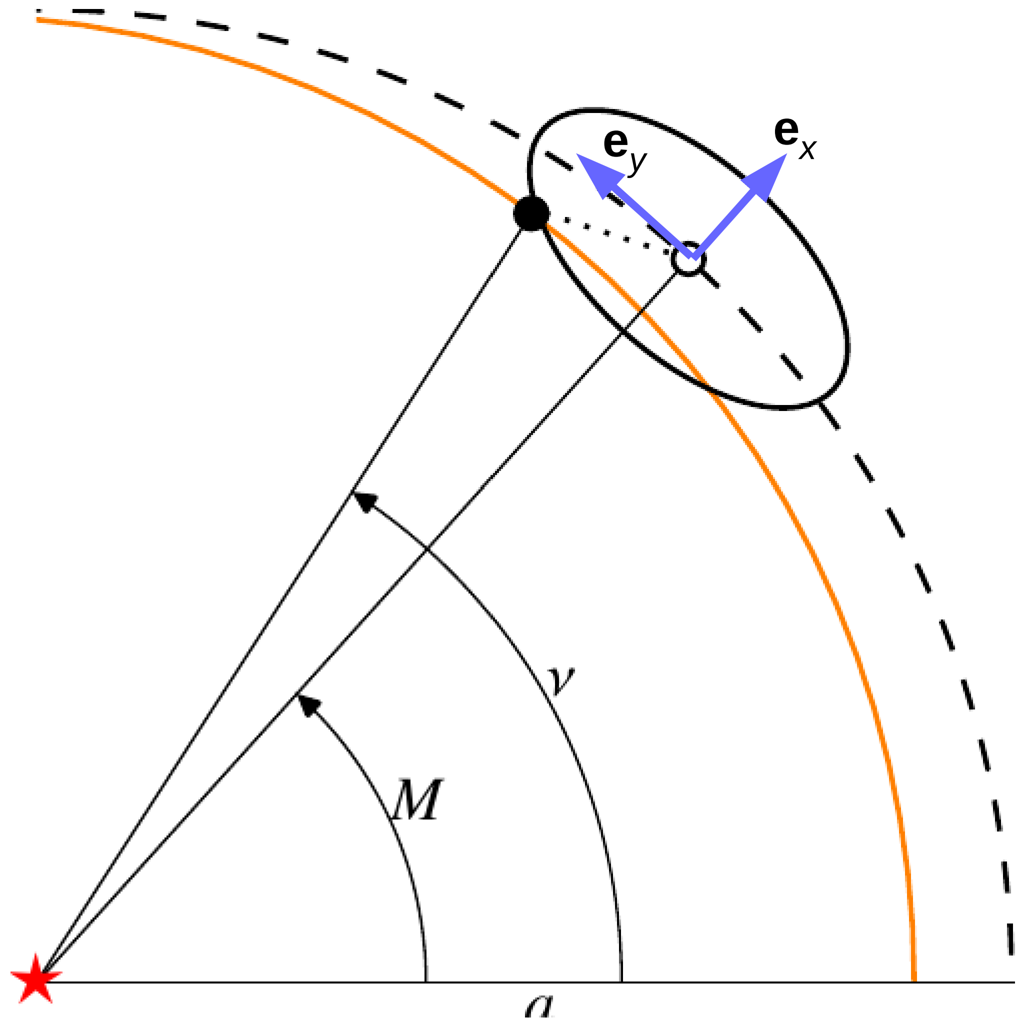
- Angular momentum, energy conservation, eccentricity vector, true anomaly, mean anomaly
- Guiding center approximation
- Circular, restricted three body problem: Jacobi integral

Guiding center approximation

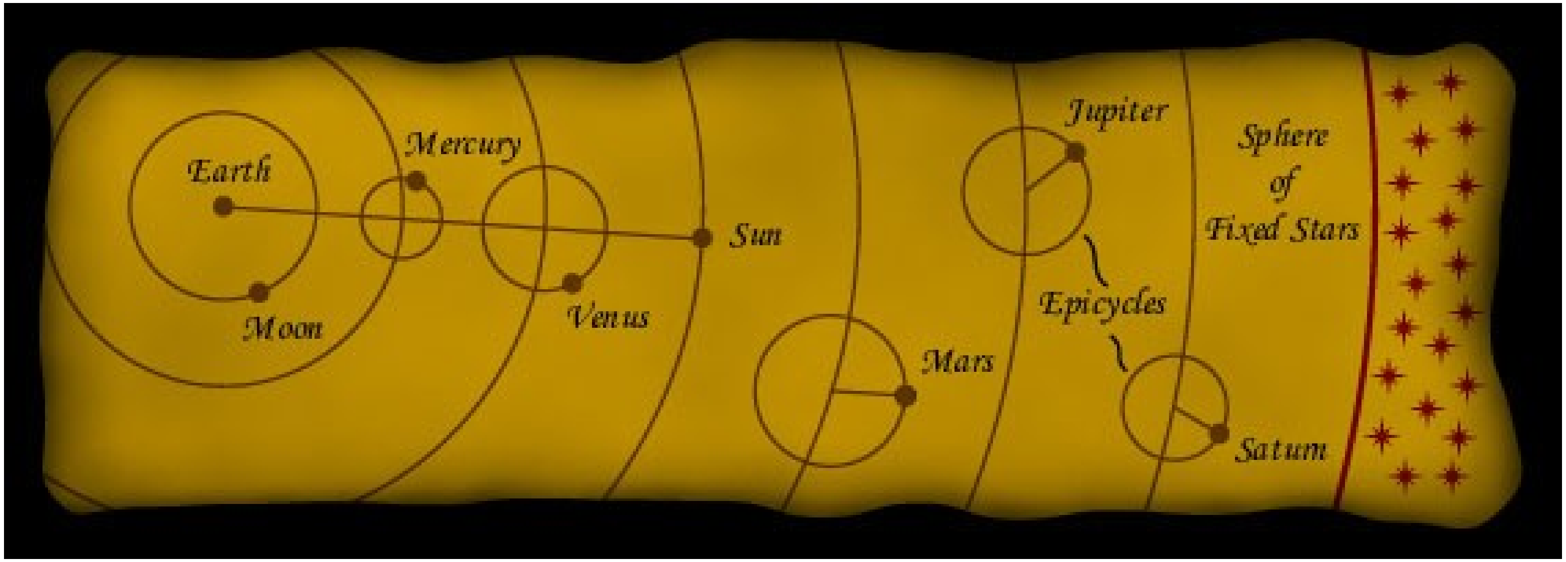


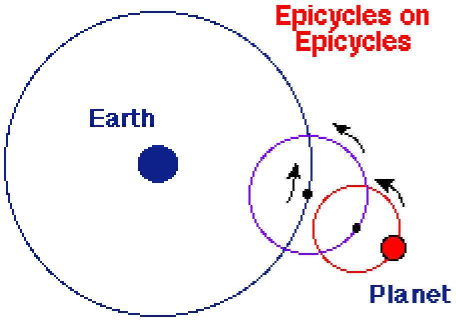
$$x = -a\,e\cos M$$

$$y = 2\,a\,e\sin M$$



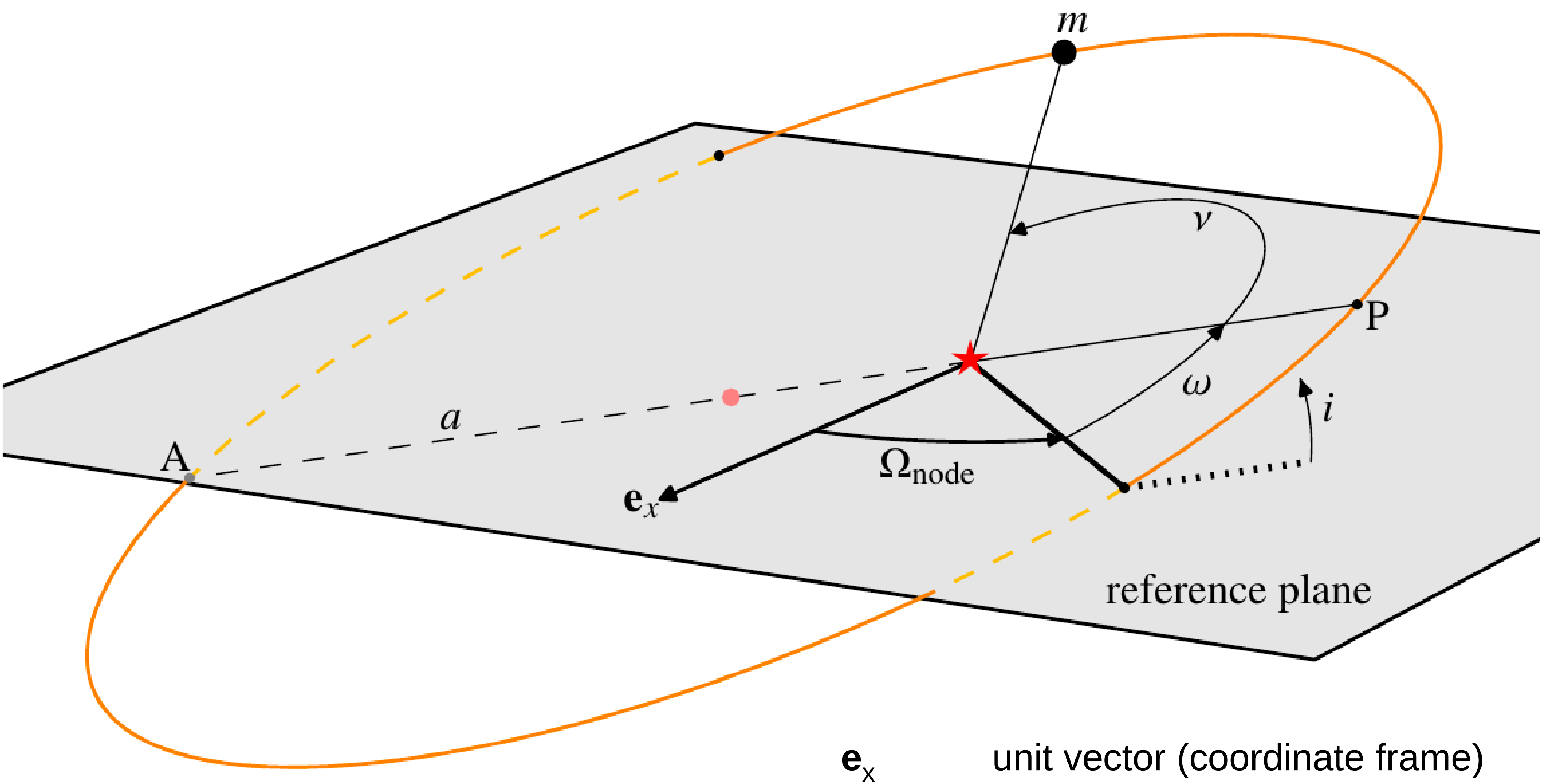
Historical epicycles





Ptolemaic model
This matches observations very precisely! (but is wrong)

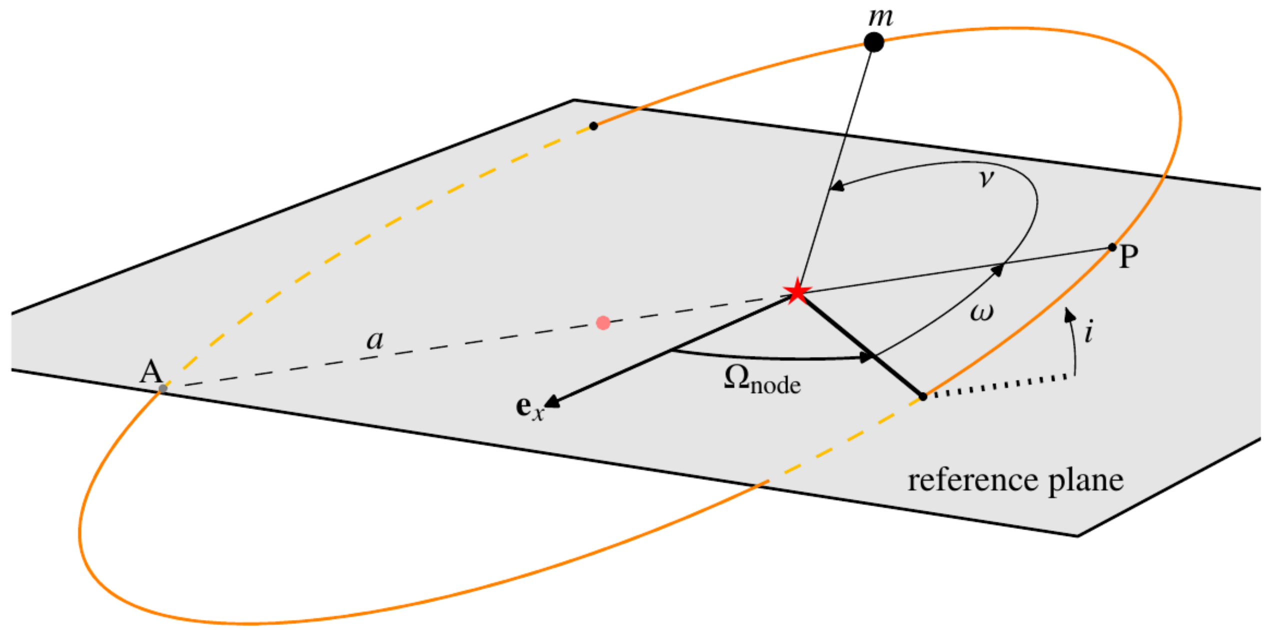
Kepler orbit



orbital elements:

a	semi-major axis
e	eccentricity
$\mathbf{e}_x$	unit vector (coordinate frame)
Ω_{node}	longitude ascending node
ω	argument of periapsis
ν	true anomaly
i	inclination

Kepler orbit

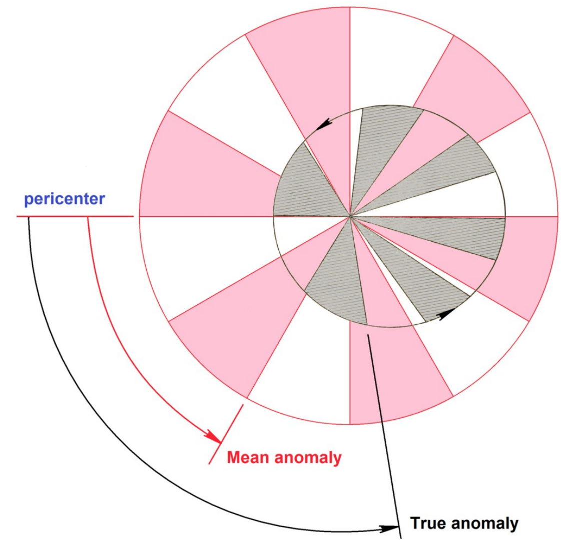


longitude of periapsis:

$$\varpi = \omega + \Omega_{node}$$

mean longitude:

$$\lambda = \varpi + M$$



$\mathbf{e}_x$	unit vector (coordinate frame)
Ω_{node}	longitude ascending node
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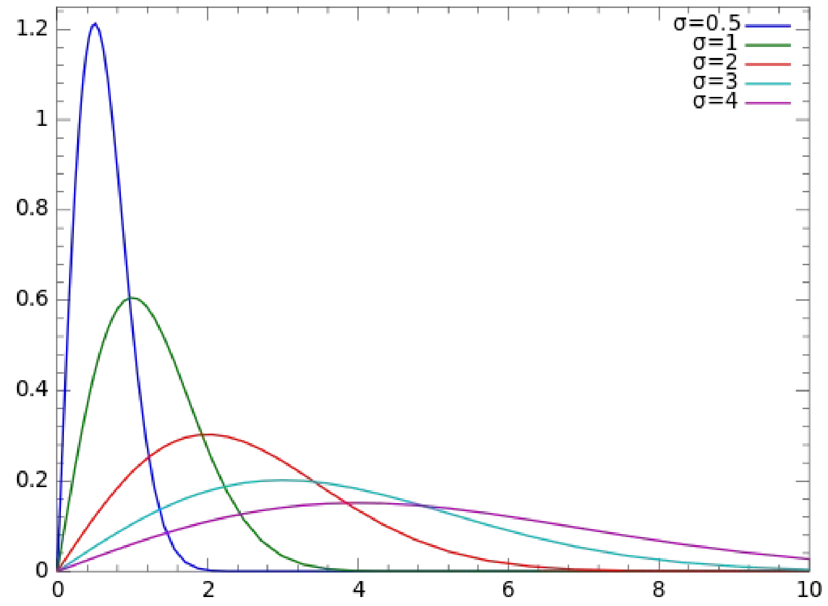
Distributions

Swarms of bodies (planetesimals)
Provided that there are many mutual dynamical interactions, they follow distributions:

Rayleigh distributions
inclination, eccentricity

Uniform distribution
mean anomaly, argument of periapsis, longitude of ascending node, etc.



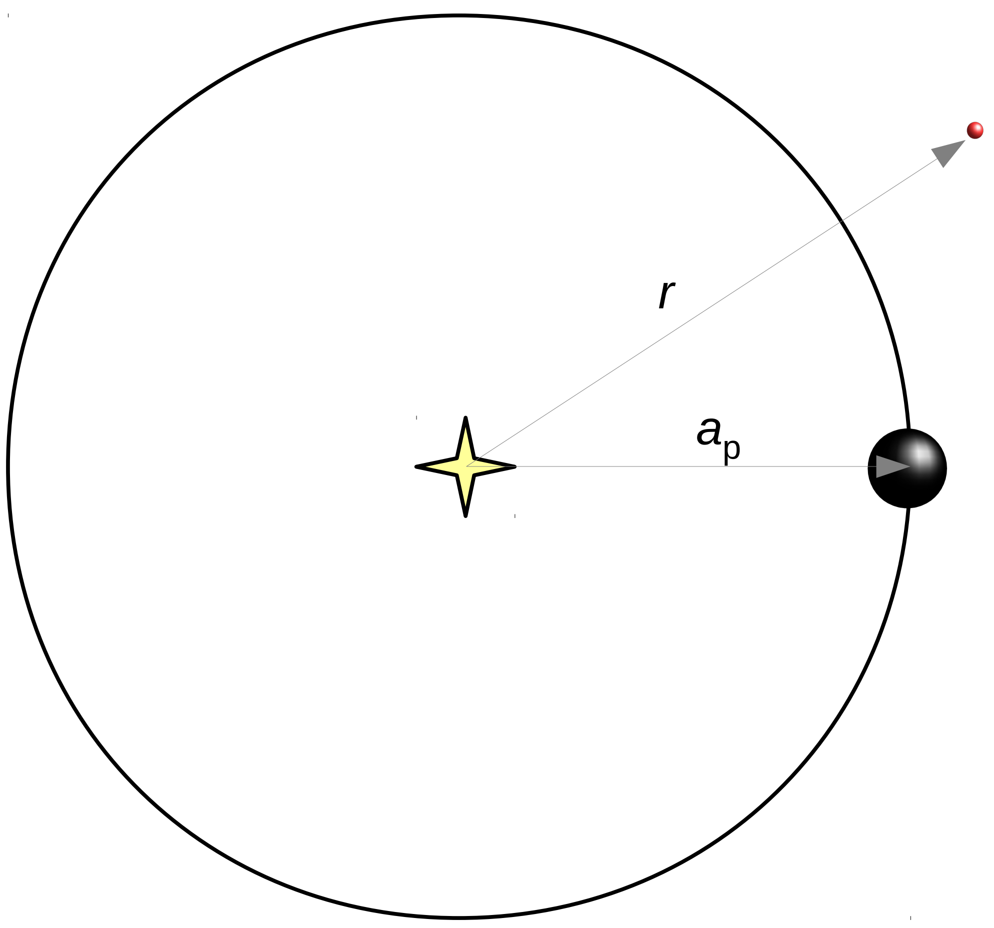
Rayleigh distribution
By Krishnavedala - Own work, CC0, https://commons.wikimedia.org/w/index.php?curid=25067844

CR3BP

CR3BP
Circular, restricted three-body problem:

– secondary on circular orbit
– tertiary a test particle (massless)

One constant of motion
J: Jacobi energy



properties/applications Jacobi energy

rotating frame:

$$J = \frac{1}{2}\dot{\boldsymbol{r}}^2 + \Phi - \frac{1}{2}(\boldsymbol{\omega} \times \boldsymbol{r})^2$$

inertial frame (Exc. 2.2a):

$$J = E - \boldsymbol{\omega} \cdot \boldsymbol{l} = E - n_p l_z$$

interpretation: energy *E* and A.M. l_z are exchanged, while *J* is conserved!

properties/applications Jacobi energy

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In orbital elements (Exc 2.2b):

$$J = -\frac{Gm_\star}{2a} - n_p \sqrt{Gm_\star(1-e^2)a}\cos i$$

a.k.a. Tisserand relation; written $a = a_p + b$ we can approximate (Exc. 2.2c)

$$J \approx \frac{Gm_\star}{a_p}\left(-\frac{3}{8}\frac{b^2}{a_p^2} + \frac{e^2+i^2}{2}\right)$$

A change in *e* (or *i*) results in a change in *b* and vice-versa!

Zero velocity curves

CR3BP concepts

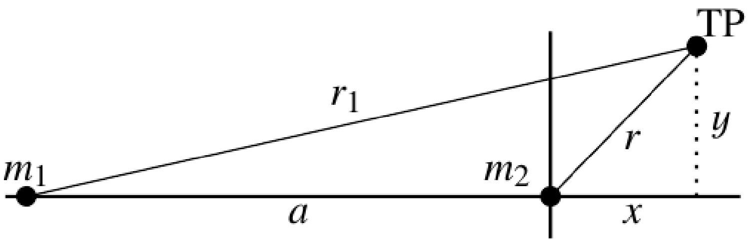
J: Jacobi energy (integral of motion)

Φ_{eff}: effective potential (includes centrifugal term)

zero-velocity curves: constant Φ_{eff}

Hill approximation: local frame (*x*,*y*) centered around planet

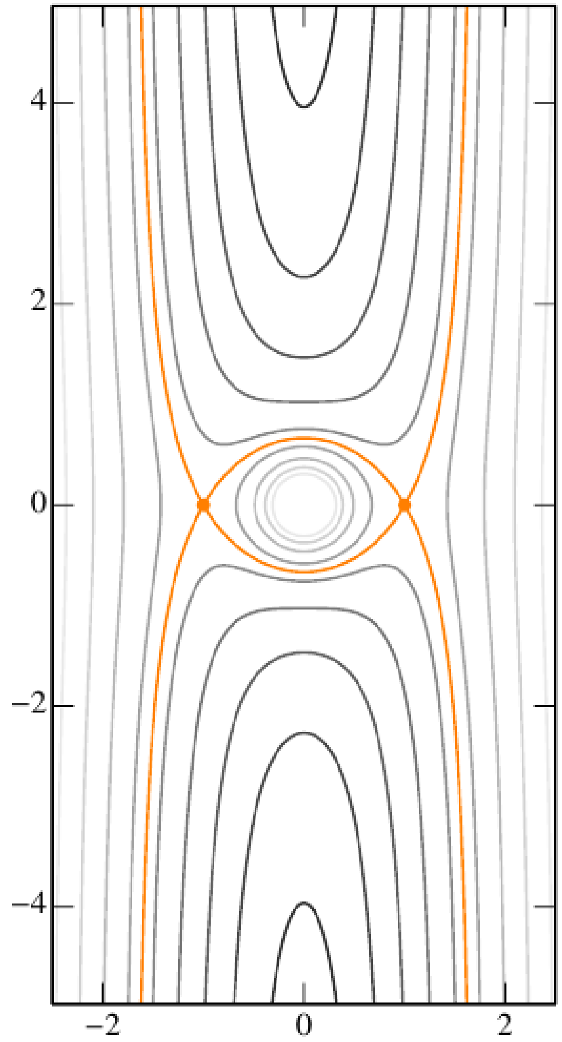
– neglects curvature
– approximates Φ_{eff}



$$J = \frac{1}{2}\dot{r}^2 + \Phi_{\rm eff}$$

$$\Phi_{\rm eff} = -\frac{3}{2}n_2^2 x^2 + \frac{1}{2}n_2^2 z^2 - \frac{Gm_2}{r}$$

zero-velocity curves
These are not orbits!



Hill's approximation (Exc. 2.3)

EOM in Hill's approximation

$$\ddot{x} = -\frac{Gm_p}{r^3}x + 2n_p v_y + 3n_p^2 x$$
$$\ddot{y} = -\frac{Gm_p}{r^3}y - 2n_p v_x$$

Equilibrium point at $(x,y) = (R_{\rm Hill}, 0)$
Hill radius $R_{\rm Hill}$:

$$R_{\rm Hill} = a_p \left(\frac{m_p}{3m_\star}\right)^{1/3}$$

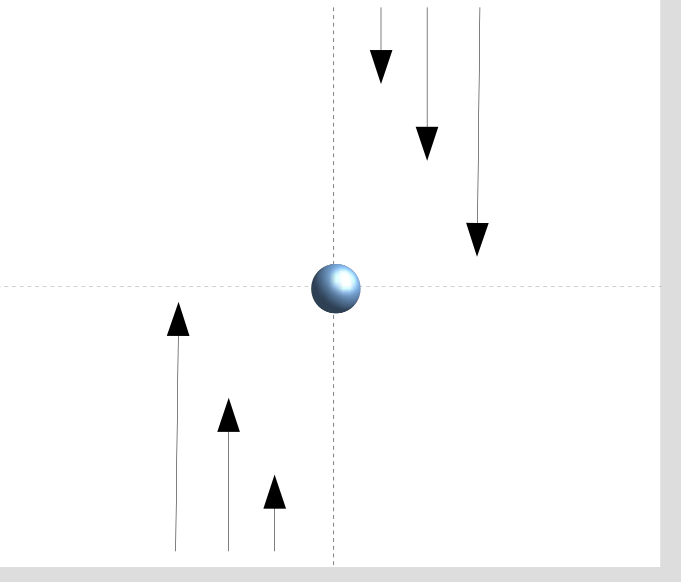
EOM in **Hill units**:

$$\ddot{x} = -\frac{3x}{r^3} + 2v_y + 3x \qquad \ddot{y} = -\frac{3y}{r^3} + 2v_x$$

The unperturbed solution
(for 0-eccentricy & far from the planet)

$$v_x = 0 \qquad v_y = -\frac{3}{2}x n_p$$

which is known as the *shearing sheet*



Encounters

close, distant encounters
There are 3 types of interactions:

1. Horseshoe orbits
2. Close (Hill-penetrating) encounters
3. Distant encounters

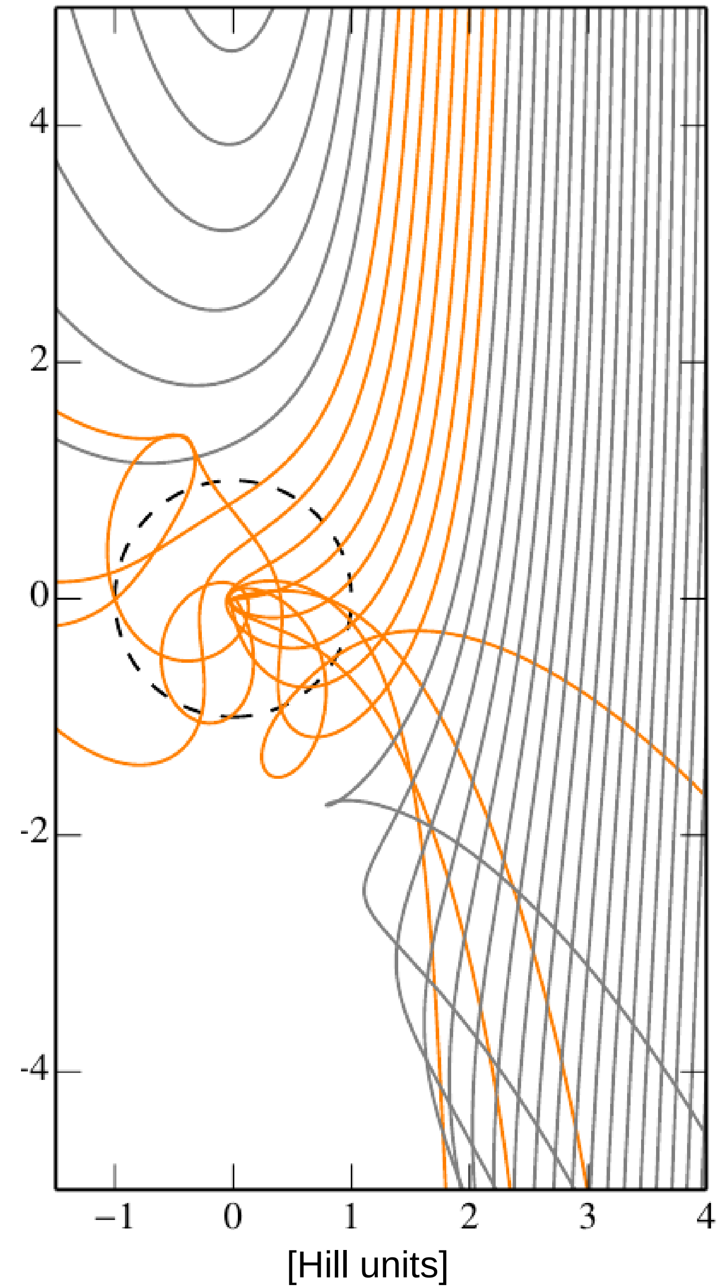
→ encounters for $e = 0$:
approach velocity is $v_{\rm rel} = 3n_{\rm p}x/2$

dispersion- and shear-dominated regimes:

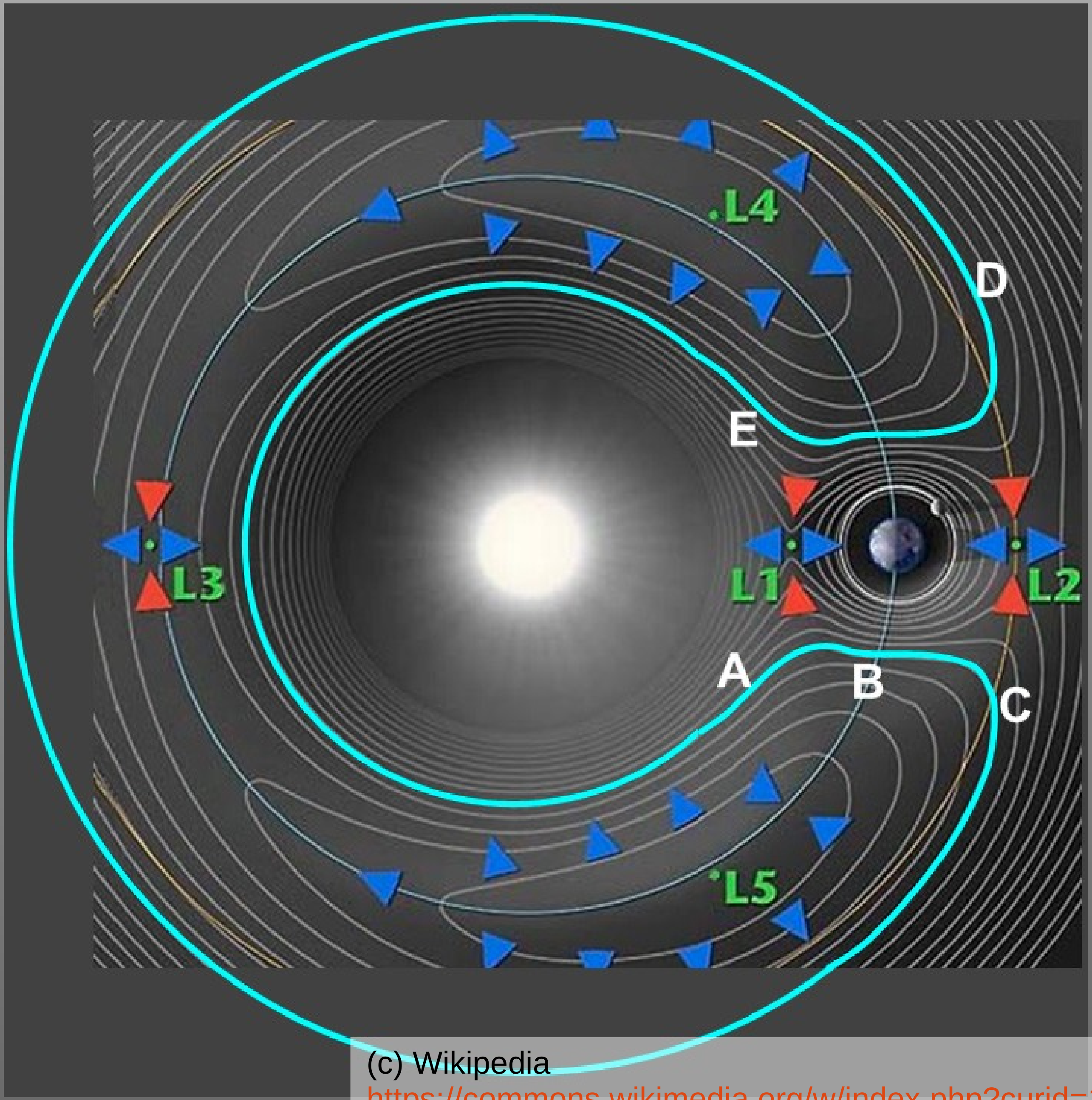
1. d.d.: $v_{\rm rel}$ is set by eccentricity: $v_{\rm rel} \sim ev_K$
2. s.d.: $v_{\rm rel}$ is set by shear: $v_{\rm rel} \sim n_{\rm p}R_{\rm Hill}$

or:

e >~ $R_{\rm Hill}/a$: d.d.-regime
e <~ $R_{\rm Hill}/a$: s.d.-regime

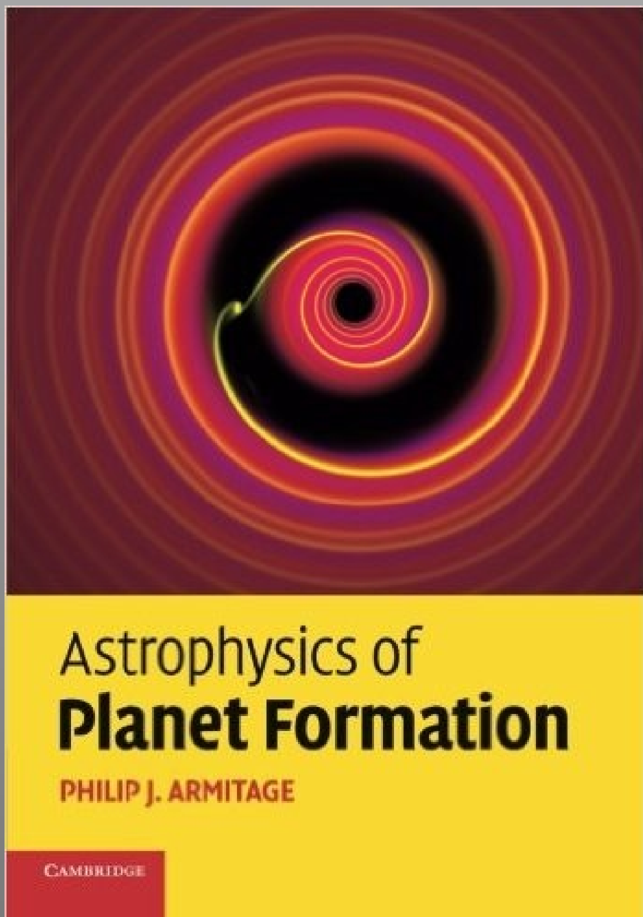


Horseshoe orbits (global frame)



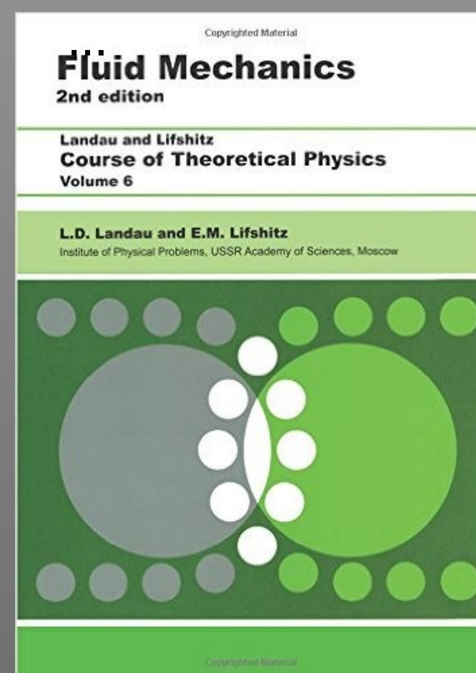
(c) Wikipedia
https://commons.wikimedia.org/w/index.php?curid=15823454

Reading material



Best overall guide to planet formation

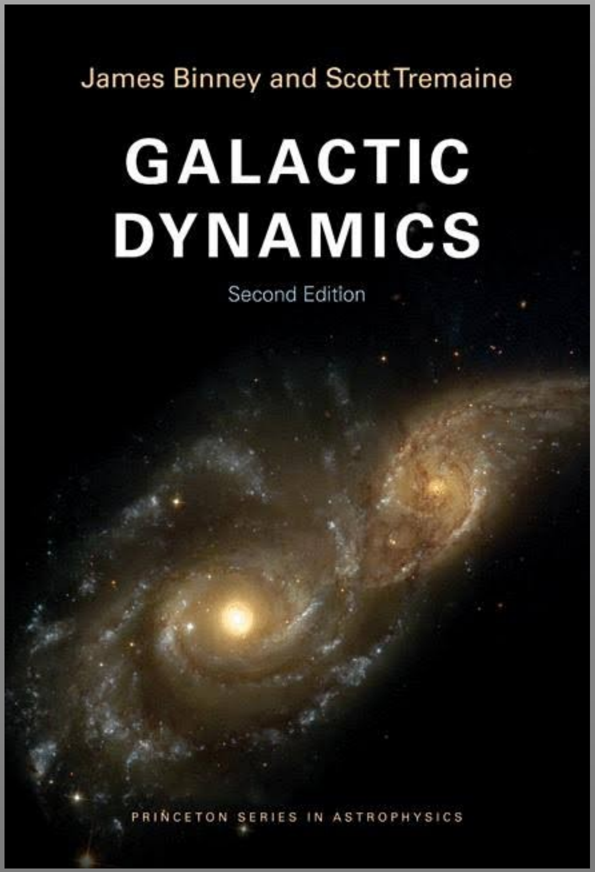
See also
http://arxiv.org/abs/1509.06382



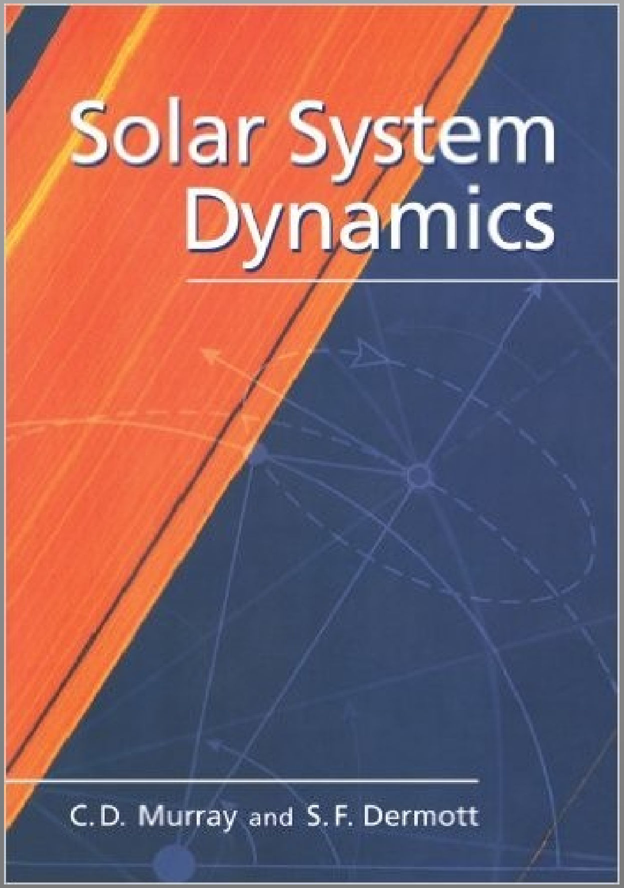
Fluid dynamics:
Turbulence, flows, hydro numbers

(hard to read)

Reading material



Gas & stellar dynamics
Gravitational interactions, Toomre-Q, epicycle approx, etc.



2-body, 3-body problem (Ch. 2, 3)

Exercise 2.1

Exercise 2.1 Guiding center:

(a) Consider two bodies in Kepler orbits separated by Δa in semimajor axis where $\Delta a \ll a$ and a is the semimajor axis of one of the bodies. Show that the *synodical period*, which is the time between successive conjunctions (close encounters), is

$$P_{\rm syn} = \frac{2P}{3}\left(\frac{a}{\Delta a}\right) \qquad (2.3)$$

where P is the orbital period corresponding to a.

(b) Show that for $e \ll 1$ the equations of motions (Eq. [2.1]) can be approximated:

$$r - a \simeq -ae\cos(M) \qquad (2.4a)$$

$$\nu - M \simeq 2ae\sin(M) \qquad (2.4b)$$

which is the *guiding center approximation*. The Keplerian motion is approximated by a superposition of a circle and an ellipse.

Bonus HW
Synodical period

Exercise 2.2

Exercise 2.2 Jacobi integral:

(a) Converting Equation (2.6) back to the inertial frame, show that:

$$J = E - \boldsymbol{\omega} \cdot \boldsymbol{l} = E - n_p l_z \tag{2.7}$$

where E and $\boldsymbol{l}$ are the energy and angular momentum measured in the inertial frame. Hence, in the CR3BP interactions will exchange E and $\boldsymbol{l}$, while J stays constant.

(b) Express J in orbital elements:

$$J = -\frac{Gm_\star}{2a} - n_p \sqrt{Gm_\star(1-e^2)a} \cos i \tag{2.8}$$

where n_p is the mean motion of the secondary and the other symbols refer to the test particle. Written in the form of Equation (2.8) (or analogous) the Jacobi integral is called the *Tisserand relation*.

(c) Let $a = a_p + b$ with a_p the semimajor axis corresponding to n_p and consider the limits where $b/a_0 \ll 1$, $i \ll 1$ and $e \ll 1$. Show that in that case:

$$J \approx \frac{Gm_\star}{a_p} \left(-\frac{3}{8} \frac{b^2}{a_p^2} + \frac{e^2 + i^2}{2} \right) \tag{2.9}$$

where we have discarded a constant term from J.

$$J = \frac{1}{2}\dot{\boldsymbol{r}}^2 + \Phi - \frac{1}{2}(\boldsymbol{\omega} \times \boldsymbol{r})^2$$

(a) Find relation for velocity in local frame and inertial frame

(b) Insert orbital elements

(c) Taylor-expands in terms of $b/a_p << 1$

Exercise 2.3 (HW)

Exercise 2.3 Hill's equations:

(a) Show that the equations of motion in Hill's approximation are:

$$\ddot{x} = -\frac{Gm_p}{r^3}x + 2n_p v_y + 3n_p^2 x \quad (2.13a)$$

$$\ddot{y} = -\frac{Gm_p}{r^3}y - 2n_p v_x \quad (2.13b)$$

where $r^2 = x^2 + y^2$ if we restrict the motion to the orbital plane.

(b) Show that zero eccentricity particles at distances far from the secondary obey $v_y = -\frac{3}{2}n_p x$ and $v_x = 0$. This (local) approximation of the Keplerian flow is known as the *shearing sheet*.

(c) Equilibrium points are points where $\ddot{\boldsymbol{r}} = \dot{\boldsymbol{r}} = 0$. Show that these *Lagrange points* are located at $(x, y) = (\pm R_{\rm Hill}, 0)$ where $R_{\rm Hill}$ is the *Hill radius*:

$$R_{\rm Hill} = a_p \left(\frac{m_p}{3m_\star}\right)^{1/3} \quad (2.14)$$

(d) Are these stable or unstable equilibrium points?

(e) What is the Jacobi constant at the Lagrange point (J_L)? And what is the Jacobi constant far from the perturber (J_∞), assuming $e = 0$. What is the half-width $x_{\rm hs}$ of the corresponding horseshoe orbit?

(e) You can assume dr/dt = 0 at the Lagrange point