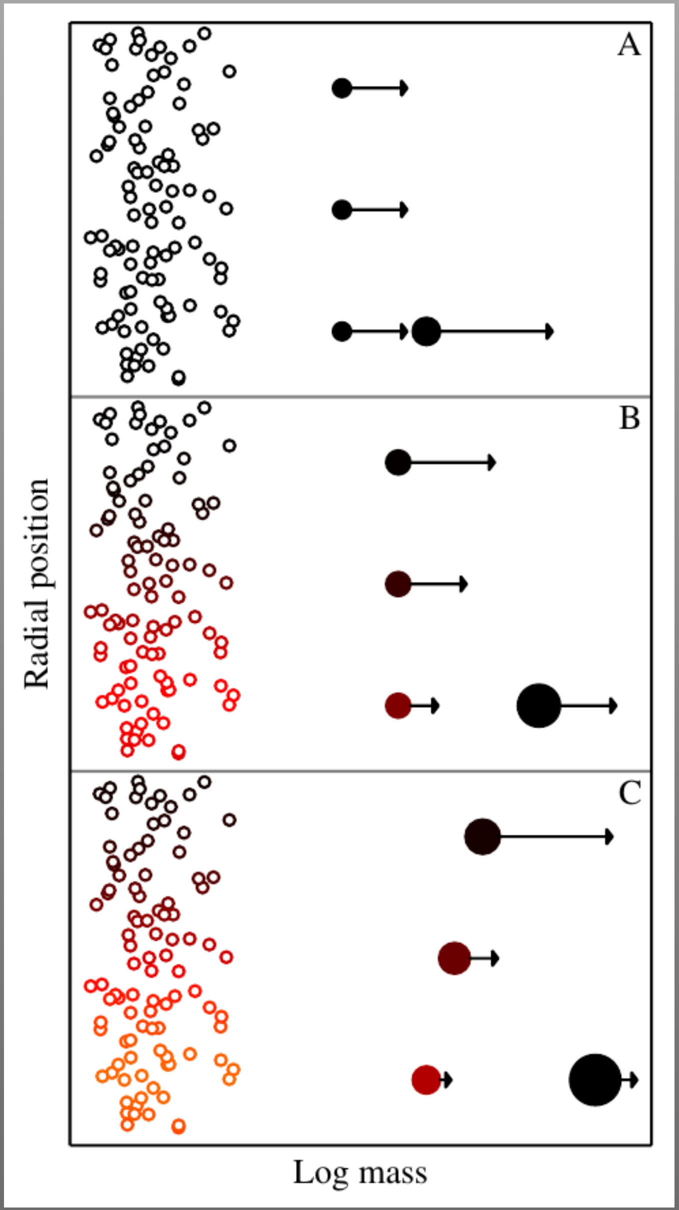
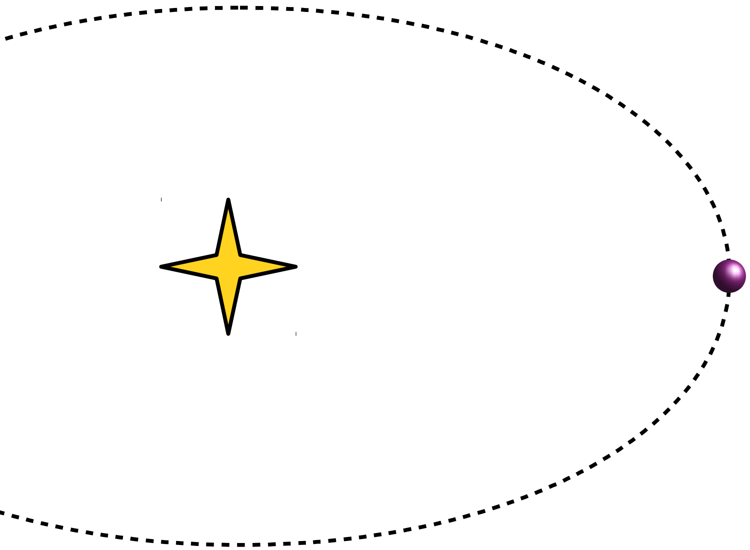
L11: Runaway & Oligarchic growth



Large and small



During planet formation
Consider a large body interacting with small body/particle, e.g.:

– massive vs light planetesimal
– embryo vs planetesimal
– embryo vs particle

Large body is on a circular (*e*=0), Keplerian orbit b/c of:

– dynamical friction
– tidal damping

Condition becomes invalid when:

– resonances
– removal gas → orbit crossing

Velocity regimes

Dispersion-dominated regime
Relative velocity (v_∞) determined by eccentric motion of planetesimal

$v_\infty = e v_K$

Shear-dominated regime
v_∞ determined by Keplerian shear

$v_\infty = (3/2) b \Omega_K$

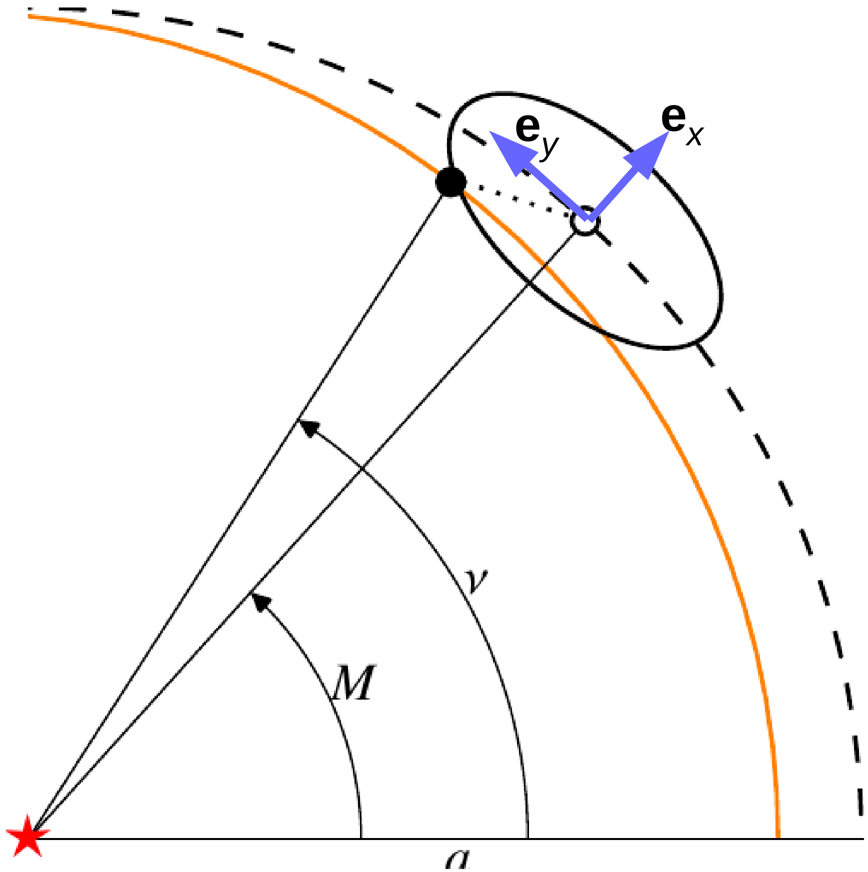
Headwind regime
v_∞ determined by sub-Keplerian headwind gas

$v_\infty = \eta v_K$

Planetesimal Accretion
gas drag damps eccentricity on long timescales ($\tau_{stop} >> 1$)

Pebble Accretion
gas drag acts *during* encounter (t_{stop} small)

Guiding center approximations



GC-approximation

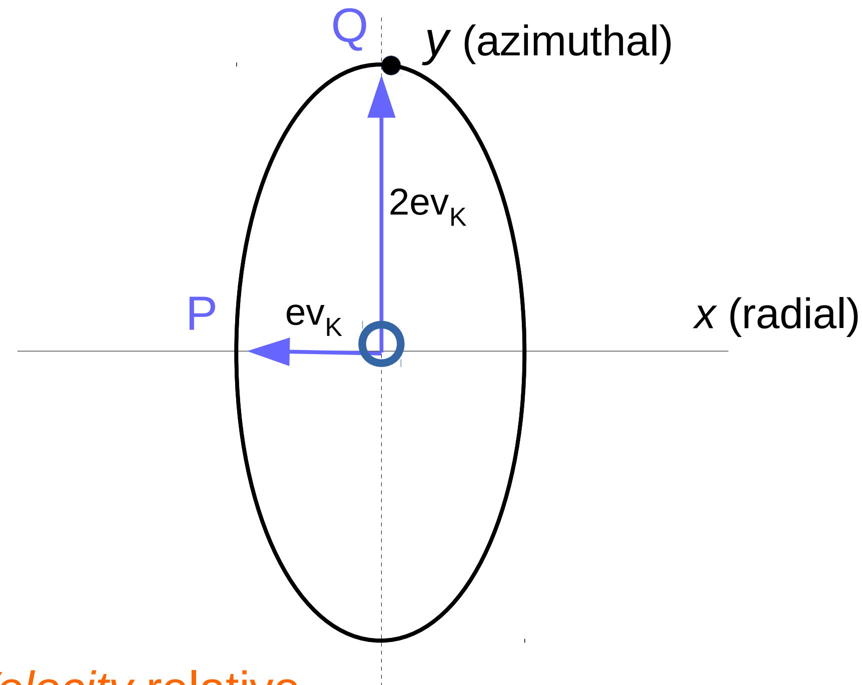
$$r = a - e\cos M$$
$$\nu = M + 2e\sin M$$

$$x = r\cos(\nu - M) - a \approx e\,a\cos M$$
$$y = r\sin(\nu - M) \approx 2\,e\,a\sin M$$

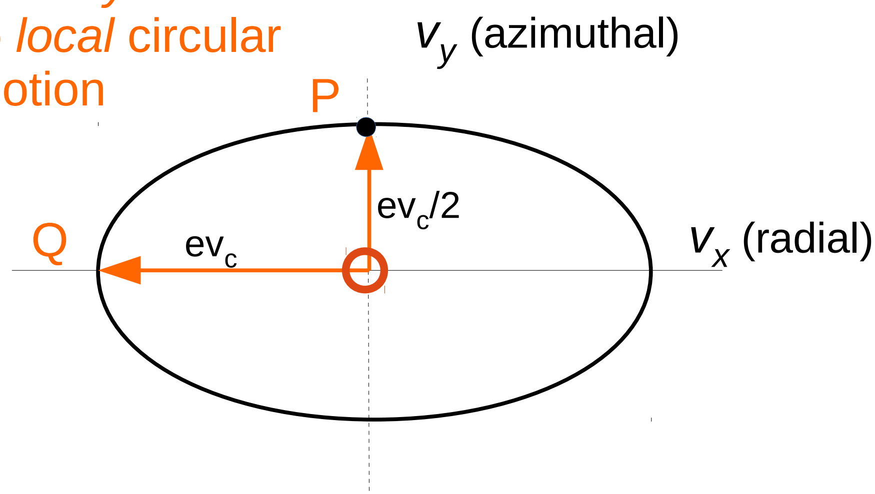
Use:

$$\cos(x) \approx 1 + O(x^2)$$
$$\sin(x) \approx x + O(x^3)$$

Guiding center approximations



Velocity relative to local circular motion



GC-approximation

$$r = a - ea\cos M$$

$$\nu = M + 2e\sin M$$

velocity w.r.t. GC-motion

$$v_r \approx e\, v_K \sin M$$

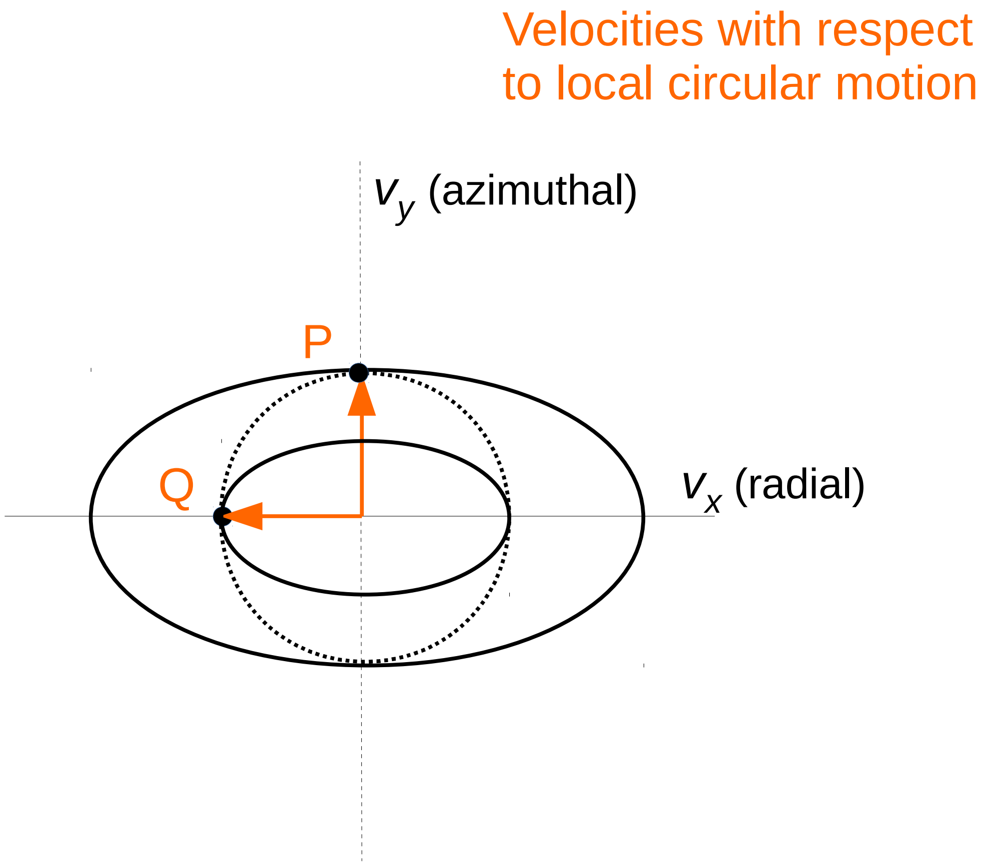
$$v_\theta \approx v_K (1 + e\cos M)$$

velocity w.r.t. *local* circular velocity (v_c)

$$v_r \approx e\, v_c \sin M$$

$$v_\theta \approx v_c \left(1 + \frac{e}{2}\cos M\right)$$

Viscous stirring



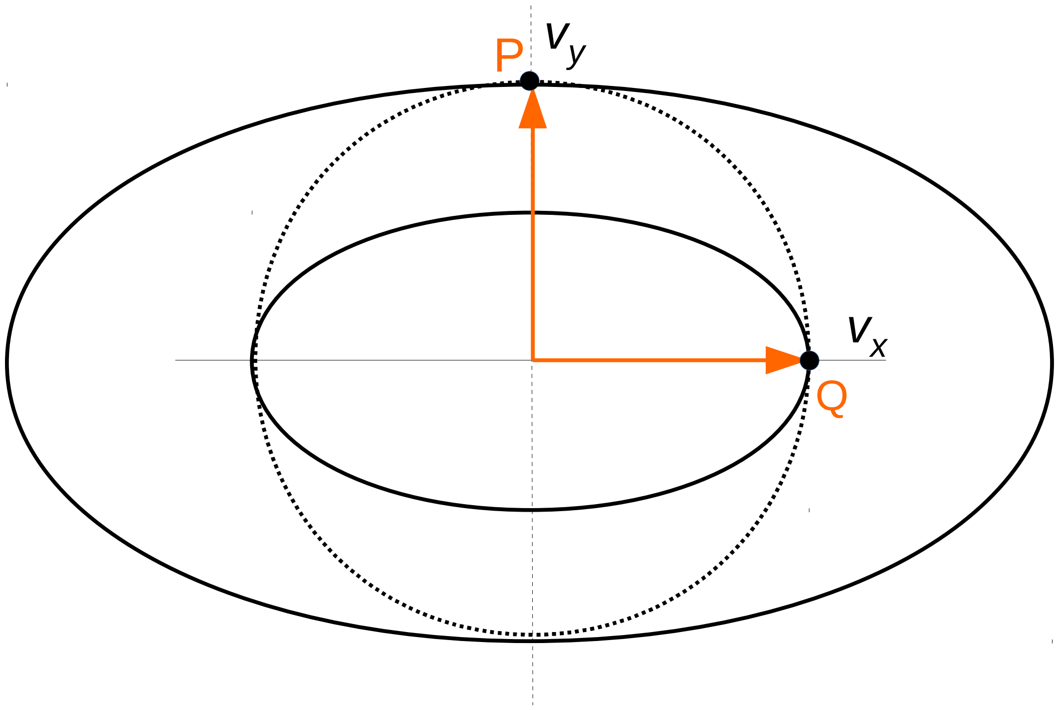
Scatterings

A close encounter randomizes the phase (argument of periapsis), while preserving the magnitude of relative velocity vector (δv)

Imagine an encounter with a more massive body moving on a circular orbit

A scattering from P→Q (or A→Q) decreases the eccentricity by a factor 2

Viscous stirring



Viscous stirring

Conversely, a scattering from Q→P doubles the eccentricity

The averaged effect (of P→Q and Q→P) is positive:

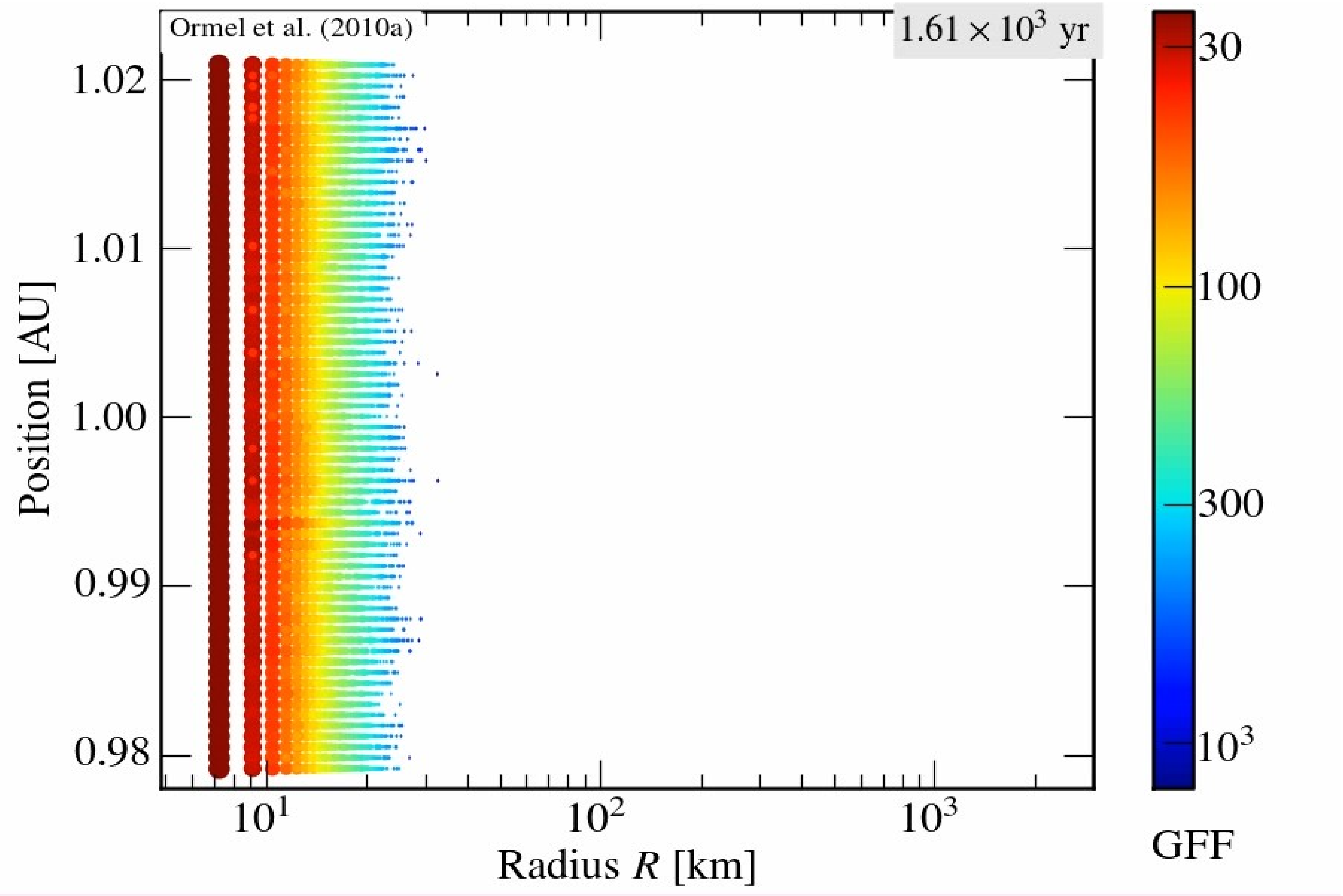
$\langle \Delta e \rangle = (+e - e/2)/2 = e/4$

This net growth in *e* is referred to as *viscous stirring*

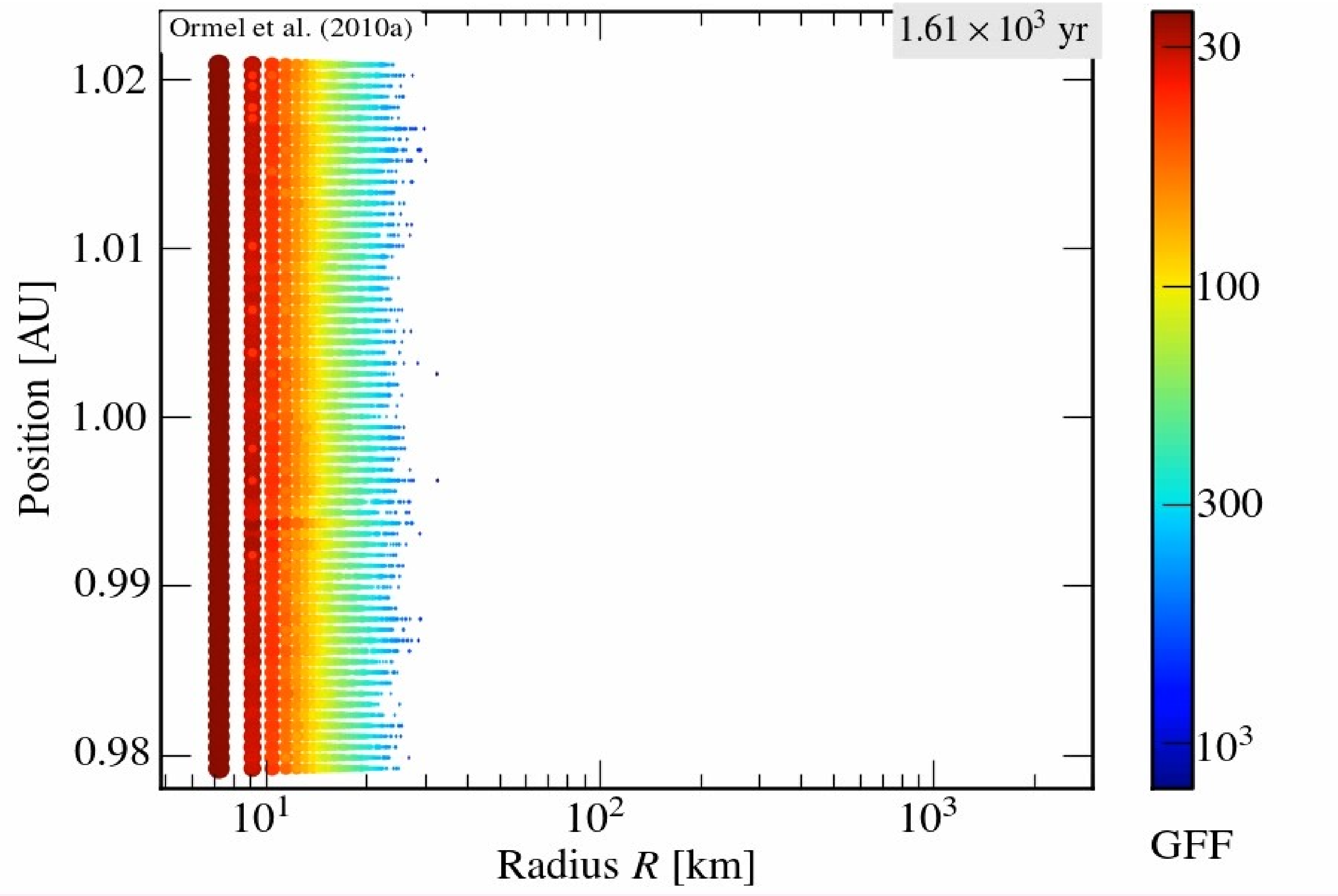
As *e* increases the GFF factor Θ decreases.

Viscous stirring provides negative feedback to the growth of runaway bodies!

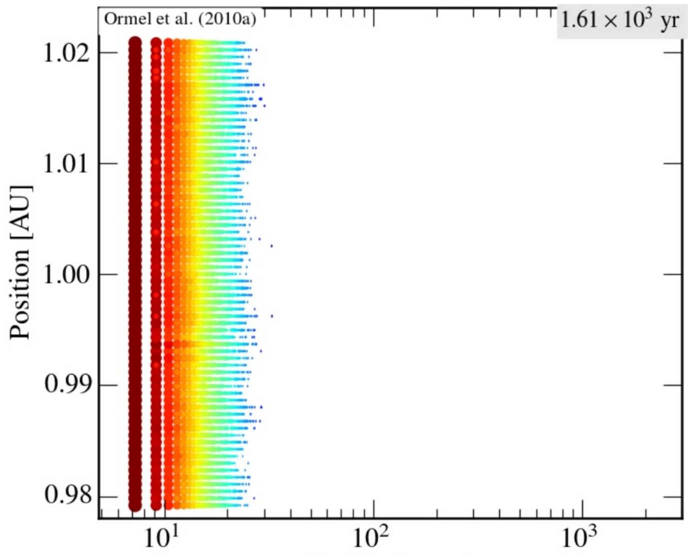
Runway growth → Oligarchy



Runway growth → Oligarchy



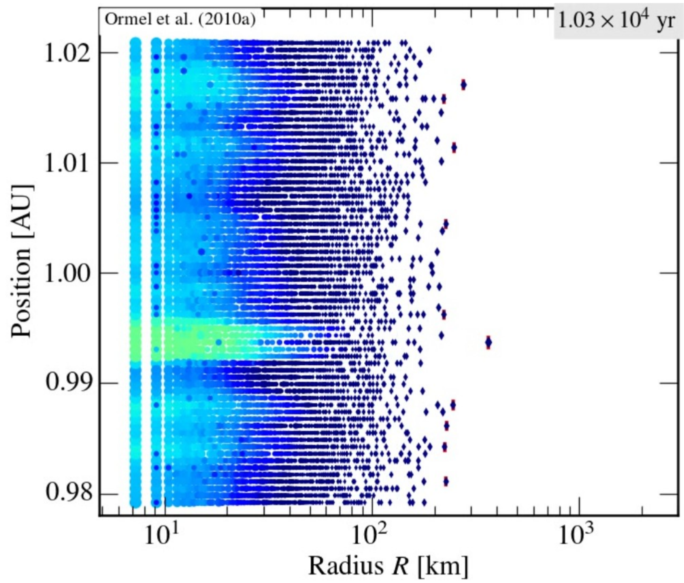
Stages runaway growth → oligarchy



Initial
(color = GFF wrt largest body in simulation)

Larger bodies are kept cool by *dynamical friction* (energy equipartition)

Large bodies growth faster than smaller ones

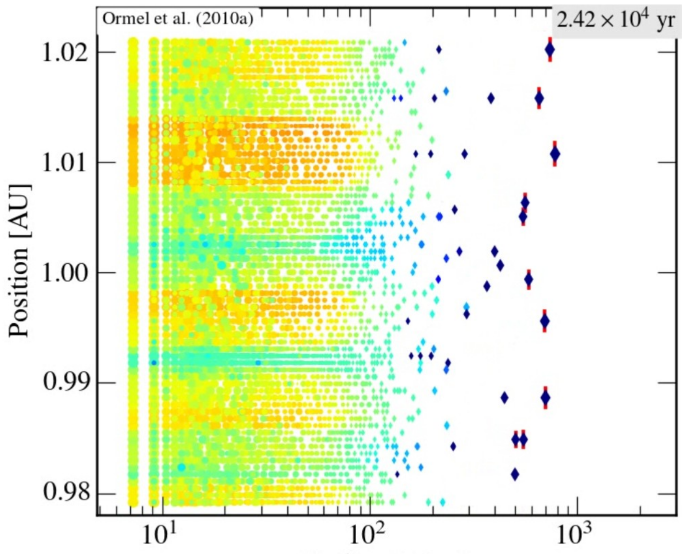


Runaway growth peaks
GFF are huge and growth is very rapid!

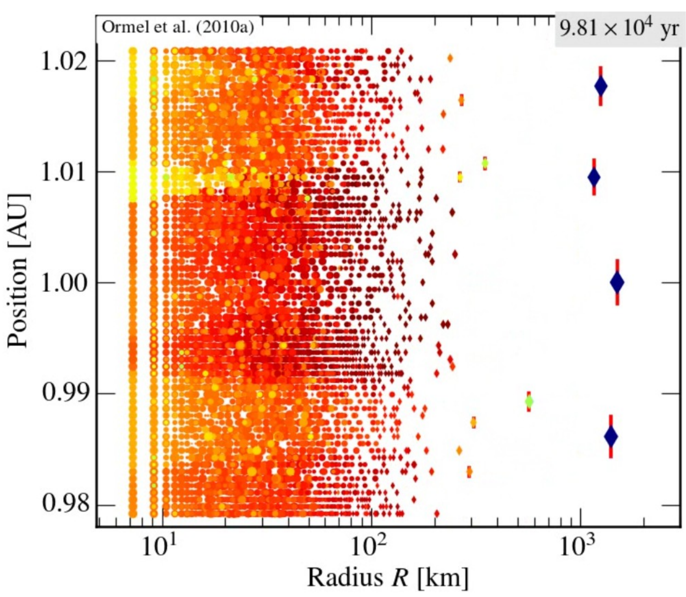
Yet 1 embryo has already excited the bodies in its feeding zone

It's growth stalls relative to bodies in neighboring zones, identifying the transition to oligarchic growth

Stages runaway growth → oligarchy



Oligarchic growth
– embryos in neighboring zones converge on each other in terms of mass
– embryos in same zones separate



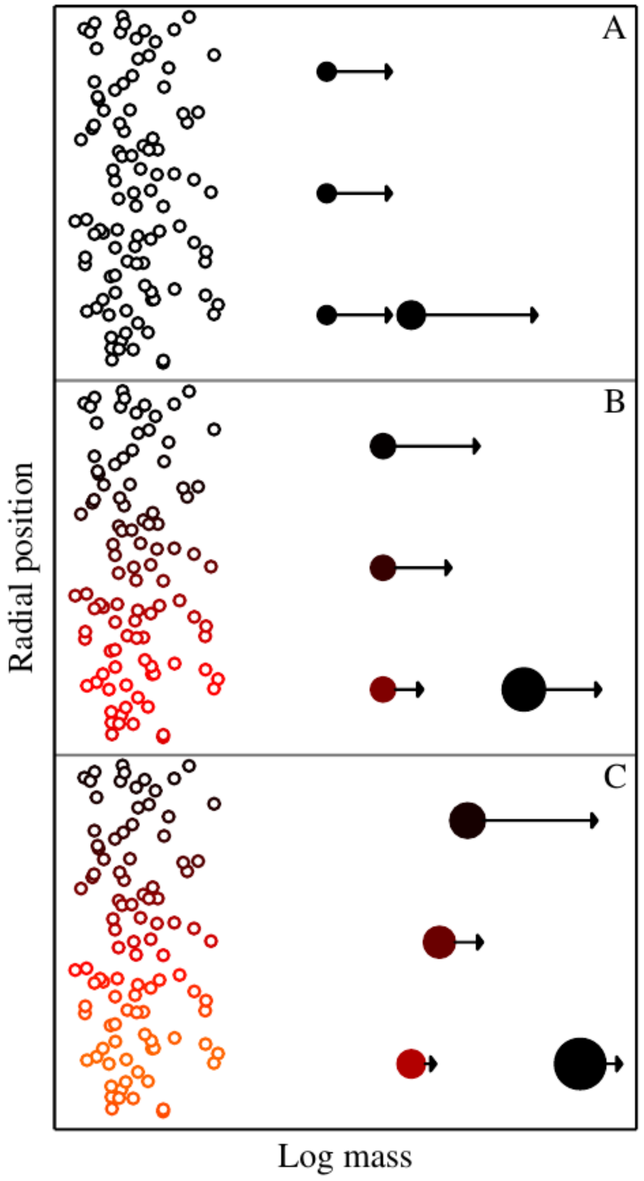
Oligarchic growth
Embryos merge once their "dynamical spacing" (=distance in terms of Hill radii) decreases.

GFF decrease, growth slows down

Towards oligarchy

Towards oligarchy
– diverging (runaway) growth w/i same zone
– converging (normal) growth for different zones

During oligarchy
embryos feast on planetesimals, but also merge; feeding zone stays several R_{Hill}.



Exercise 1.15 (HW)

Exercise 1.15 oligarchy: Assume that the separation between embryos is a couple of mutual Hill radii, $\Delta a \sim \tilde{b} R_{\rm Hill}$. Then, it may be argued that the (effective) number density of embryos seen by a planetesimal is $n_M \simeq [(2\pi a) \times (\tilde{b} R_{\rm Hill}) \times (2ia)]^{-1}$.

(a) Argue that viscous stirring increases the eccentricity of the planetesimals at a rate:

$$\left.\frac{de}{dt}\right)_{\rm vs} \sim (n_M \sigma_{90} \Delta v) \Delta e \sim \frac{q_p^{5/3} \Omega_K}{e^3 \tilde{b}} \qquad (1.47)$$

where $q_p = M/m_\star$. Gas friction, on the other hand, damps the eccentricity of the planetesimals. For planetesimals, the gas drag law depends quadratically on velocity/eccentricity. Let us therefore define:

$$\left.\frac{de}{dt}\right)_{\rm gas} \equiv -e^2 \Omega_K \mathcal{F}_{\rm aero} \qquad (1.48)$$

where the dimensionless $\mathcal{F}_{\rm aero}$ depends on the aerodynamical properties of gas and planetesimals.

(b) Give the expression for $\mathcal{F}_{\rm aero}$. Balancing viscous stirring and gas damping gives therefore an *equilibrium eccentricity* of $e \sim q_p^{1/3} / (\tilde{b} \mathcal{F}_{\rm aero})^{1/5}$.

(c) Show that for these eccentricities, the Safronov numbers are constant. How far out in the disk can 10 Earth-size planets form within 10 Myr?

Equate:
– increase de/dt by viscous stirring
– decrease de/dt by gas drag

↓

Equilibrium eccentricity

↓

Safronov focusing factor Θ

↓

Growth timescale $t_{\rm growth}$

Exercise 1.14

Exercise 1.14 gravitational scattering: Calculate the trajectory of a gravitational scattering. Consider polar coordinates (r, θ) with $\theta = 0$ the *direction* of the unperturbed velocity and $r(\theta = -\pi) = \infty$ initially (see Figure 1.15). Let primes denote derivatives towards θ;

(a) show that energy conservation implies $v_r v_r' + v_\theta v_\theta' + GMr'/r^2 = 0$ and that conservation of angular momentum gives $v_\theta = bv_\infty/r$ with b the impact parameter and v_∞ the initial velocity.

(b) Show that $v_r = bv_\infty r'/r^2$ and retrieve the following ODE for $r(\theta)$:

$$rr'' - 2(r')^2 - r^2 + r^3 \frac{b_{90}}{b^2} = 0 \tag{1.39}$$

where $b_{90} \equiv GM/v_\infty^2 (= \Theta R)$. This equation can be simplified by substituting $u = 1/r$:

$$u'' + u = \frac{b_{90}}{b^2}. \tag{1.40}$$

which has the solution $u = A\cos\theta + B\sin\theta + b_{90}/b^2$ where A and B are integration constants.

(c) Determine these to find:

$$r(\theta) = \frac{b}{b_{90}(1+\cos\theta)/b - \sin(\theta)}. \tag{1.41}$$

(d) Find the angle θ corresponding to the collisional focusing impact parameter of Equation (1.38), *i.e.* the location where the particle impacts the big body.

(e) Finally, derive the scattering angle – the direction the test particle is heading to after the scattering:

$$\theta_{\text{scat}} = \arcsin\left(\frac{2bb_{90}}{b^2 + b_{90}^2}\right) \tag{1.42}$$

and explain the meaning of "b_{90}".