Hints to selected problems

(c) Chris Ormel; **see lecture notes part II for the exercises**

Hand in HW-problems by Mar. 15 13:00 to Jacob Arcangeli

Exercise 1.13

Use conservation of angular momentum and energy. Shown in class.

Exercise 1.14

This is a straightforward but rather math-heavy problem. The goal is to find the precise deflection angle of a gravitational scattering (hyperbolic encounter).

(a) Note that derivatives (primes) are with respect to θ, not to time!

(b) Somewhat mathy, but straightforward.

(c) The boundary conditions that you should apply are those at the beginning where $v_r = -v_\infty$ and $\theta = \pi$.

(d) **Please do (e) first**. This is a rather algebraically-heavy exercise that you perhaps want to do on a rainy Sunday afternoon. Substitute $\sin\theta = \sqrt{1-\cos\theta^2}$, isolate the square-root, and solve the quadratic equation. The answer is:

$$\theta = \pm \arccos \frac{b_{90}}{b_{90}+R} \tag{1}$$

As important than deriving this expression is to inspect that the limiting cases ($b_{90} \ll R$) and $b_{90} \gg R$ make sense.

(e) Look at the solution for a hint.

Exercise 1.15 (HW) [4pt]

For this exercise, order-of-magnitude expressions suffice! Discard numerical constant and use $\sim$ instead of equal signs, *ie.* $M \sim \rho_\bullet R^3$, $\pi \sim 1$, etc. Note that question (c) has been split in two and that we ask you to produce some plots.

(a) Here you should derive the given de/dt expression due to visous stirring. Consider the following:

- the dispersion-dominated regime holds: the approach velocities (here Δv, in class also referred to as v_∞) are determined by the eccentric motion of the planetesimal.
- inclinations are similar to eccentricities, $i \simeq e/2$.

Substitute $GM_\star = a^3\Omega^2$ at some point.

(b) Back to aerodynamics. Because we are considering planetesimals, you can safely assume that C_D is order unity and that the drag force is quadratic in velocity: $F_{\mathrm{drag}} = -\frac{1}{2}C_D A\rho_{\mathrm{gas}}(\Delta v)^2$ with Δv now the relative motion with respect to the gas. You can assume that Δv is again due to the eccentricic motion of the planetesimal and ignore the sub-Keplerian motion of the gas; in other words, you can assume that $ev_K \gg \eta v_K$ (which is not always the case).

(c) Constant means here that Θ is independent of the mass (or radius) of the embryo, assuming constant internal density. The Safronov factor will depend on the location in the disk and on $\mathcal{F}_{\mathrm{aero}}$ (properties of the planetesimals). Also, the question should read **10 Earth-mass** planets (not size!).

(d) **In this question you are asked to make some figures.** First, choose your favorite "realistic" disk surface density profile, $\Sigma_{\mathrm{gas}}(r)$, $\Sigma_{\mathrm{solids}}(r)$, $T(r)$ where r is the disk orbital radius. From the disk profile you can further calculate ρ_{gas} and h_{gas} at any point r. You can assume that planetesimals represent the bulk of the solid mass (rather than the embryos). Finally, choose a planetesimal size s_P somewhere between 1 and 1,000 km. Other constants (internal densities, $\tilde{b}$) are up to you. Just mention the parameters that you have adopted.

Now that we have an expression for the Safronov number Θ from (c), you can calculate the growth timescale t_{growth}, which is a function of embryo mass, disk orbital radius and the planetesimal (aerodynamical) properties. This is of course the growth timescale without focusing divided by the Safronov factor:

$$t_{\mathrm{growth}} \sim \frac{R\rho_\bullet}{\Sigma\Omega}\frac{1}{\Theta} \tag{2}$$

With your favorite plotting tool, plot as function of disk orbital radius:

- t_{growth} as function of disk orbital radius r for an embryo mass of 10 Earth masses.
- The embryo mass M_p that corresponds to a growth time of 10 Myr.

Please return clear plots with logarithmic axes, large-enough fonts and label them correctly!

Exercise 1.16 (HW) [2pt]

Note that in this exercise "PA" means pebble accretion, while "Saf" refers to the previously-derived planetesimal accretion.

(a)

(b)

(c) You should give an expression for R/R_{Hill} in which either R (the radius of the embryo) or M (its mass) does *not* appear. Fill in approprate numbers and give the dependence of R/R_{Hill} as function of orbital distance r.

(d) For planetesimals, the zero-eccentricity limit always implies the shear-dominated regime. This maximizes the focusing factor. Express Θ_{Saf} in terms of R_{Hill}/R. Then compare.

Exercise 1.17 (HW-Bonus) [1pt]

Here we have again assumed the shear-dominated regime for pebble accretion, where the impact parameter b_{col} is large. We also assumed that the pebbles have settled to the midplane, which renders the system 2D. Hence, the form of $\dot{M}$.

Exercise 1.18

(a) You should of course use that $P = c_s^2 \rho$ where c_s is constant.

(b)

(c) We can just divide the above expression by $4\pi\rho_c R_c^3/3$:

$$\frac{M_{\rm atm}}{M_c} = \frac{3R_c\rho_{\rm neb}}{R_b\rho_c} \exp\left[\frac{R_b}{R_c}\right] = \frac{3\rho_{\rm neb}}{\rho_c}\frac{\exp[x]}{x} \tag{3}$$

where $x = R_b/R_c$. Of course $\rho_{\rm neb}/\rho_c \ll 1$ but this gets compensated by already moderate x.

Exercise 1.19

In this adiabatic case case c_s and the temperature are no longer constant. Therefore, R_b is defined in terms of the nebular parameters: $R_b = GM_p/c_{\rm neb}^2$.

Exercise 1.20 (HW) [3pt]

(a)

(b) The M that appears in $T(r)$ and $\rho(r)$ is the total mass, $M_{\rm tot}$.

(c)

(d) We already discussed this partly in class. Also give a physical explanation for the dependence of the critical core mass on the parameter W.

Exercise 1.21 (HW-Bonus) [2pt]

Unfortunately, we won't be able to discuss orbit crossing in our Friday lecture. This exercise is therefore bonus. I have also re-arranged this exercise. Take care. You can assume $\Delta = 10$ as was discussed in the text. Furthermore take an internal density of 5 g cm^{-3} for the embryos. Assume that resonances are not applicable; planets positions are not correlated. Take a purely probabilistic approach. Naturally, an order-of-magnitude approach suffices.

(a) In the answer r denotes the "radius", but it would in fact be more precise to replace it with the semi-major axis a. The key to the solution is to calculate the fraction of the orbit where a collision becomes possible. **Also:** convert the derived probabilities into a collision time $t_{\rm coll}$ for both cases. Express your answers symbolically in terms of the orbital period t_K, R_{12} i and a.

(b) Plug in numbers for both the 2D and 3D cases. For the inclinations you can assume $i \sim e$, although this is incorrect (see below).

(c) Accounting for gravitational focusing, how would this change the 2D **and** 3D timescales, derived above?

(d) **Additional question: Why is the assumption $i \sim e$ invalid for this case?** Remark that initially (after the gas of the disk has dissapated) the embryos start out in well-separated, non-crossing orbits and can be found in approximately the same orbital plane.

Exercise 2.1

(a) **HW-BONUS. [1pt] Errata: the numerical constant in front of equation (2.3) [$\frac{2}{3}$] is correct! In contrast to what was previously mentioned.** Note that Δa cannot be negative; hence $\Delta a = |a_1 - a_0|$. Hint: do a Taylor-approximation on $\Omega(a)$ around the semi-major axis of one of the bodies (say, the one of $a = a_0$):

$$\Omega_K(a) \approx \Omega_K(a_0) + \left(\frac{d\Omega}{da}\right)_{a_0} (a - a_0) \tag{4}$$

This will tell you the relative angular speed at which the two bodies separate. (The rest should really be straightforward...)

(b) **There should be no a in eq. (2.4b)**. Because $e \ll 1$, $M \approx \nu$ (but not quite).