Hints to selected problems

(c) Chris Ormel; see lecture notes part II for the exercises

Hand in HW-problems by Mar. 15 13:00 to Jacob Arcangeli

Exercise 1.13
Use conservation of angular momentum and energy. Shown in class.

Exercise 1.14
This is a straightforward but rather math-heavy problem. The goal is to find the precise deflection angle of a gravitational scattering (hyperbolic encounter).

(a) Note that derivatives (primes) are with respect to $\theta$, not to time!

(b) Somewhat mathy, but straightforward.

(c) The boundary conditions that you should apply are those at the beginning where $v_r = -v_\infty$ and $\theta = \pi$.

(d) Please do (e) first. This is a rather algebraically-heavy exercise that you perhaps want to do on a rainy Sunday afternoon. Substitute $\sin \theta = \sqrt{1 - \cos^2 \theta}$, isolate the square-root, and solve the quadratic equation. The answer is:

$$\theta = \pm \arccos \frac{b_{90}}{b_{90} + R}$$

(1)

As important than deriving this expression is to inspect that the limiting cases ($b_{90} \ll R$) and $b_{90} \gg R$ make sense.

(e) Look at the solution for a hint.

Exercise 1.15 (HW) [4pt]
For this exercise, order-of-magnitude expressions suffice! Discard numerical constant and use $\sim$ instead of equal signs, ie. $M \sim \rho R^3$, $\pi \sim 1$, etc. Note that question (c) has been split in two and that we ask you to produce some plots.

(a) Here you should derive the given $de/dt$ expression due to visous stirring. Consider the following:

– the dispersion-dominated regime holds: the approach velocities (here $\Delta v$, in class also referred to as $v_\infty$) are determined by the eccentric motion of the planetesimal.

– inclinations are similar to eccentricities, $i \approx e/2$.

Substitute $GM_* = a^3 \Omega^2$ at some point.
(b) Back to aerodynamics. Because we are considering planetesimals, you can safely assume that $C_D$ is order unity and that the drag force is quadratic in velocity: $F_{\text{drag}} = -\frac{1}{2} C_D \rho_{\text{gas}} (\Delta v)^2$ with $\Delta v$ now the relative motion with respect to the gas. You can assume that $\Delta v$ is again due to the eccentric motion of the planetesimal and ignore the sub-Keplerian motion of the gas; in other words, you can assume that $\epsilon v_K \gg \eta v_K$ (which is not always the case).

(c) Constant means here that $\Theta$ is independent of the mass (or radius) of the embryo, assuming constant internal density. The Safronov factor will depend on the location in the disk and on $F_{\text{aero}}$ (properties of the planetesimals). Also, the question should read 10 Earth-mass planets (not size!).

(d) In this question you are asked to make some figures. First, choose your favorite “realistic” disk surface density profile, $\Sigma_{\text{gas}}(r)$, $\Sigma_{\text{solids}}(r)$, $T(r)$ where $r$ is the disk orbital radius. From the disk profile you can further calculate $\rho_{\text{gas}}$ and $h_{\text{gas}}$ at any point $r$. You can assume that planetesimals represent the bulk of the solid mass (rather than the embryos). Finally, choose a planetesimal size $s_P$ somewhere between 1 and 1,000 km. Other constants (internal densities, $b$) are up to you. Just mention the parameters that you have adopted.

Now that we have an expression for the Safronov number $\Theta$ from (c), you can calculate the growth timescale $t_{\text{growth}}$, which is a function of embryo mass, disk orbital radius and the planetesimal (aerodynamical) properties. This is of course the growth timescale without focusing divided by the Safronov factor:

$$t_{\text{growth}} \sim \frac{R H_{\text{Hill}}}{\Sigma \Omega} \frac{1}{\Theta}.$$

With your favorite plotting tool, plot as function of disk orbital radius:

- $t_{\text{growth}}$ as function of disk orbital radius $r$ for an embryo mass of 10 Earth masses.
- The embryo mass $M_p$ that corresponds to a growth time of 10 Myr.

Please return clear plots with logarithmic axes, large-enough fonts and label them correctly!

Exercise 1.16 (HW) [2pt]

Note that in this exercise "PA" means pebble accretion, while "Saf" refers to the previously-derived planetesimal accretion.

(a) 

(b) 

(c) You should give an expression for $R/R_{\text{Hill}}$ in which either $R$ (the radius of the embryo) or $M$ (its mass) does not appear. Fill in appropriate numbers and give the dependence of $R/R_{\text{Hill}}$ as function of orbital distance $r$.

(d) For planetesimals, the zero-eccentricity limit always implies the shear-dominated regime. This maximizes the focusing factor. Express $\Theta_{\text{Saf}}$ in terms of $R_{\text{Hill}}/R$. Then compare.

Exercise 1.17 (HW-Bonus) [1pt]

Here we have again assumed the shear-dominated regime for pebble accretion, where the impact parameter $b_{\text{col}}$ is large. We also assumed that the pebbles have settled to the midplane, which renders the system 2D. Hence, the form of $M$. 

2
Exercise 1.18

(a) You should of course use that $P = c_s^2 \rho$ where $c_s$ is constant.

(b)

(c) We can just divide the above expression by $4\pi R_c^2/3$:

$$
\frac{M_{\text{adm}}}{M_c} = \frac{3 R_b \rho_{\text{neb}}}{R_c \rho_c} \exp \left[ \frac{R_b}{R_c} \right] = \frac{3 \rho_{\text{neb}} \exp[x]}{\rho_c x}
$$

where $x = R_b/R_c$. Of course $\rho_{\text{neb}}/\rho_c \ll 1$ but this gets compensated by already moderate $x$.

Exercise 1.19

In this adiabatic case case $c_s$ and the temperature are no longer constant. Therefore, $R_b$ is defined in terms of the nebular parameters: $R_b = GM_p/c_s^2$.

Exercise 1.20 (HW) [3pt]

(a) 

(b) The $M$ that appears in $T(r)$ and $\rho(r)$ is the total mass, $M_{\text{tot}}$.

(c) 

(d) We already discussed this partly in class. Also give a physical explanation for the dependence of the critical core mass on the parameter $W$.

Exercise 1.21 (HW-Bonus) [2pt]

Unfortunately, we won’t be able to discuss orbit crossing in our Friday lecture. This exercise is therefore bonus. I have also re-arranged this exercise. Take care. You can assume $\Delta = 10$ as was discussed in the text. Furthermore take an internal density of 5 g cm$^{-3}$ for the embryos. Assume that resonances are not applicable; planets positions are not correlated. Take a purely probabilistic approach. Naturally, an order-of-magnitude approach suffices.

(a) In the answer $r$ denotes the "radius", but it would in fact be more precise to replace it with the semi-major axis $a$. The key to the solution is to calculate the fraction of the orbit where a collision becomes possible. Also: convert the derived probabilities into a collision time $t_{\text{coll}}$ for both cases. Express your answers symbolically in terms of the orbital period $t_K$, $R_{12}$, $i$, and $a$.

(b) Plug in numbers for both the 2D and 3D cases. For the inclinations you can assume $i \sim e$, although this is incorrect (see below).

(c) Accounting for gravitational focusing, how would this change the 2D and 3D timescales, derived above?

(d) Additional question: Why is the assumption $i \sim e$ invalid for this case? Remark that initially (after the gas of the disk has dissipated) the embryos start out in well-separated, non-crossing orbits and can be found in approximately the same orbital plane.
Exercise 2.1

(a) HW-BONUS. [1pt] Errata: the numerical constant in front of equation (2.3) \( \frac{2}{3} \) is correct! In contrast to what was previously mentioned. Note that \( \Delta a \) cannot be negative; hence \( \Delta a = |a_1 - a_0| \). Hint: do a Taylor-approximation on \( \Omega(a) \) around the semi-major axis of one of the bodies (say, the one of \( a = a_0 \)):

\[
\Omega_k(a) \approx \Omega_k(a_0) + \left( \frac{d\Omega}{da} \right)_{a_0} (a - a_0) \quad (4)
\]

This will tell you the relative angular speed at which the two bodies separate. (The rest should really be straightforward...)

(b) There should be no \( a \) in eq. (2.4b). Because \( e \ll 1, M \approx \nu \) (but not quite).