Unifiability in extensions of **K4**

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Abstract

We extend and generalize the work on unifiability of [8]. We give a semantic characterization for unifiability and non-unifiability in the extensions of **K4**. We apply this in particular to extensions of **KD4**, **GL** and **K4.3** to obtain a syntactic characterization and give a concrete decision procedure for unifiability for those logics. For that purpose we use universal models.

Keywords: Unification, unifier, provability logic, closed formula, universal model

1 Introduction

The research of unification for logical systems was originally motivated by automatic deduction tools and its starting point was the existence of a most general unifier for any unifiable formula in Boolean logic. Much later, Ghilardi [4] proved that there are no most general unifiers in **IPC** but there is a finite set of maximal general unifiers. Ghilardi [5] extended this result to various modal logics including **GL**. These results provide a connection between unification and admissibility of inference rules.

The research of admissible rules was stimulated by a question asking whether admissibility of rules in **IPC** is decidable. The problem was investigated mainly by Rybakov and he proved the decidability of the admissible rules for **IPC** in 1975 and then extended this result to a large class of modal logics [7].

A logical implication of Ghilardi's finitary results for unification is the fact that an algorithm computing the finitely many maximal unifiers of a formula yields a new solution to Friedman's problem of recognizing the admissibility of inference rules in **IPC**.

The paper [8] directed itself to unifiability, i.e., the pure existence of unifiers instead of to the form or the properties of unifiers. A uniform syntactic characterization was given for unifiability of formulas in **KD4** and its extensions:

Theorem 1 [8] For any modal logic λ extending **KD4** and any modal formula α, α is not unifiable in λ iff $\vdash_\lambda \Box\alpha \wedge \alpha \rightarrow \bigvee_{p \in Var(\alpha)} (\Diamond p \wedge \Diamond\neg p)$.

This includes of course many of the best known modal logics like **S4** and **S5**. For **IPC** the question of unifiability is uninteresting since the answer is the same as for classical logic **CPC**: all and only consistent formulas are unifiable. Characterization of non-unifiability of formulas in a logic brings with it a characterization of its so-called passive admissible rules (see [8]).

In this paper we prove similar results for **GL** and its extensions as well as for **K4.3**. The situation is more complicated than for **KD4**, not all extensions of **GL** behave in the same way. Our method consists of first providing a semantic characterization of unifiability for **K4** and its extensions. This characterization is based on the use of universal models (see for more details on their construction than are given here, e.g. [1] or [3]). For the logics studied we provide a concrete decision prodecure for unifiability and thereby for the passive admissible rules.

2 Preliminaries

Definition 1 *The* language of modal propositional logic *consists of the propositional variables: p, q, r,..., connectives: $\vee, \wedge, \rightarrow, \leftrightarrow, \neg, \top, \bot$ and a unary modal operator $\Box$.*
The modal logic **K** *is axiomatized by the following schemes:*

- *All propositional tautologies in the modal language,*
- $\Box(\alpha \rightarrow \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta)$.

The modal logic **K4** *is axiomatized by adding the scheme 4 to* **K***:*

- *4:* $\Box\alpha \rightarrow \Box\Box\alpha$.

The modal logic **K4.3** *is axiomatized by adding the scheme 3 to* **K4***:*

- *3:* $\Box(\boxdot\alpha \rightarrow \beta) \vee \Box(\boxdot\beta \rightarrow \alpha)$ *where* $\boxdot\alpha = \alpha \wedge \Box\alpha$.

The modal logic **S4** *is axiomatized by adding the scheme T to* **K4***:*

- *T:* $\Box\alpha \rightarrow \alpha$.

The modal logic **KD4** *is axiomatized by adding the scheme D to* **K4***:*

- *D:* $\Box\bot \rightarrow \bot$, *i.e.,* $\neg\Box\bot$.

The modal logic **GL** *is axiomatized by adding the scheme L to* **K4** (*or equivalently to* **K**)*:*

- *L:* $\Box(\Box\alpha \to \alpha) \to \Box\alpha$.

Inference rules for these logics are modus ponens $\frac{\alpha, \alpha\to\beta}{\beta}$ *and* necessitation $\frac{\alpha}{\Box\alpha}$.

The scheme L plays an essential role in **GL** where $\Box\phi$ is read as "it is provable that ϕ". It is named after Löb, who proved L as a theorem of the provability logic of **PA** (see [6]).

Definition 2

1. *A* Kripke frame *for* **K** *is a pair* $< W, R >$ *with* W *a nonempty set of so-called* worlds *or* nodes*, and* R *a binary relation, the so-called* accessibility relation.
2. *A* Kripke frame *for* **K4** *is a pair* $< W, R >$ *with* R *transitive.*
3. *A* Kripke frame *for* **K4.3** *is a pair* $< W, R >$ *with* R *transitive, upwards linear.*
4. *A* Kripke frame *for* **GL** *is a pair* $< W, R >$ *with* R *a transitive relation such that the converse of* R *is well-founded* (*there is no infinite sequence* $x_0 R x_1 R x_2 R \ldots$). (*This excludes cycles and loops, and in the finite case comes down to irreflexivity.*)

Definition 3

1. *A* Kripke model *for* **K (K4, K4.3, GL)** *is a triple* $< W, R, \Vdash >$ *with* $< W, R >$ *a Kripke frame for* **K(K4, K4.3, GL)** *together with a satisfaction relation* $\Vdash$ *between worlds and propositional variables. We usually write* $w \Vdash p$ *for* $\mathfrak{M}, w \Vdash p$*, etc. The relation* $\Vdash$ *is extended to a relation between worlds and all formulas by the stipulations* $w \Vdash \neg\alpha$ *iff* $w \nVdash \alpha$, $w \Vdash \alpha \wedge \beta$ *iff* $w \Vdash \alpha$ *and* $w \Vdash \beta$*, and similarly for the other connectives,* $w \Vdash \Box\alpha$ *iff for all* w' *such that* wRw', $w' \Vdash \alpha$.
2. *If* $\mathfrak{M} =< W, R, \Vdash>$*, and* $\mathfrak{M}, w \Vdash \alpha$ *for each* $w \in W$*, and we write* $\mathfrak{M} \Vdash \alpha$ *and we say that* α *is* valid in $\mathfrak{M}$.

Henceforth we restrict attention to transitive frames.

Definition 4

1. *A* root *is a node* w *such that* wRw' *for all* $w \neq w'$ *in the frame.*
2. *The* depth *or* level m *of a node* w *is the maximal number* m *for which there are nodes* $w = w_0 R \ldots R w_{m-1}$ *such that* $w_i R w_{i+1}$ *and* $\neg(w_{i+1} R w_i)$ *for every* $i < m - 1$*. If the maximum does not exist, the depth is* infinite.
3. *The* depth *of a model is the maximum of the depth of its nodes.*

Note that the depth of an *end point* (a node without successors) is 1. If there are cycles in the model the definition should be adapted so that a whole cycle (or all the nodes in it) should get the same depth, but we will not need to discuss models with cycles.

Definition 5 *Let* $< W, R >$ *be a frame.* $A \subseteq W$ *is called an* antichain *if* $|A| > 1$ *and for each* $w, v \in A$, $w \neq v$ *implies* $\neg(wRv)$ *and* $\neg(vRw)$. *We say that a set* $A \subseteq W$ totally covers *a point* v $(v \prec A)$ *if* A *is the set of all immediate successors of* v. *In case* A *consists of a single element* w, *we write* $v \prec w$.

3 Admissibility, Closed Formulas and 0-Universal Models

In this section first we show connections between unifiability and admissibility and then give a semantic characterization for unifiability of formulas in the extensions of **K4** using 0-universal models of these extensions. These models are really useful only in case the logics do have the finite model property for closed formulas.

Definition 6 *A formula* $\alpha(p_1, ..., p_n)$ *is* unifiable *in a logic* λ *iff there is a tuple of formulas* $\delta_1, ..., \delta_n$ *such that* $\vdash_\lambda \alpha(\delta_1, ..., \delta_n)$. *The formulas* $\delta_1, ..., \delta_n$ *are called* unifiers *for the formula* α.

Definition 7 *A rule* $\varphi_1, ..., \varphi_k/\psi$ *is* admissible *in logic* λ *if for each substitution* σ *of formulas* $\theta_1, ..., \theta_m$ *for the atoms in* $\varphi_1, ..., \varphi_k$ *such that* $\sigma(\varphi_1), ..., \sigma(\varphi_m)$ *are theorems of* λ, $\sigma(\psi)$ *is a theorem as well. In other words, unifiers for premises are unifiers for the conclusion as well.*

In **CPC** there are no nontrivial admissible rules; if a rule is admissible, then the conclusion is derivable from the premises. In **IPC** there are well-known admissible non-derivable rules. The best known is called Harrop's rule:

$$\neg\varphi \to \psi \vee \chi/(\neg\varphi \to \psi) \vee (\neg\varphi \to \chi).$$

In modal logic a well-known admissible non-derivable rule is $\Box\varphi/\varphi$ in **K** or **K4**.

If α is non-unifiable, α/β is admissible for trivial reasons. Then the rule α/β is called a *passive rule*. Clearly, if p does not occur in α, then α/p is admissible iff α is non-unifiable. Therefore decidability of unifiability and of the passive admissible rules of a logic is equivalent.

In **CPC** and **IPC** a formula is unifiable iff it is consistent (satisfiable). This is not true for modal logic. In general, $\Diamond p \wedge \Diamond\neg p$ will be consistent but not unifiable.

Definition 8 *A formula is called a* closed *formula if it is built up from the formulas* $\top$, $\bot$ *by Boolean connectives and* $\Box$.

The following lemma was obvious in [8], even if it was not stated as such.

Lemma 1 *If a formula* $\alpha(p_1, ..., p_n)$ *is* unifiable *in a logic* λ*, then it has a sequence of closed unifiers* $\delta_1, ..., \delta_n$*.*

Proof. Just substitute $\perp$ for all the propositional variables in a sequence of unifiers for α. $\dashv$

An immediate corollary is:

Corollary 1

1. *If* λ_1 *and* λ_2 *prove the same closed formulas, then the sets of unifiable formulas of* λ_1 *and* λ_2 *are the same.*
2. *For each* λ *the set of its unifiable formulas is uniquely determined by its closed fragment.*

This means that to determine the set of unifiable formulas of extensions of **K4** it is sufficient to determine the set of unifiers of logics extending **K4** by closed formulas only. For the study of such fragments so-called 0-universal models are very useful. We will now introduce them. As is the case for n-universal models in general, 0-universal models can be seen as the part of the 0-*canonical model* (constructed using closed formulas only) consisting of its nodes of finite depth (see [1]).

Definition 9 *The* 0-Universal model $\mathcal{U}_{\mathbf{K4}}(0)$ *of* **K4** *is constructed as follows: It contains two maximal elements, a reflexive and an irreflexive element. Under any finite anti-chain* A *in* $\mathcal{U}_{\mathbf{K4}}(0)$ *we put a new reflexive element that is covered by* A*, and a new irreflexive element that is covered by* A*. Under each irreflexive element* w *we put a reflexive* v_1 *such that* $v_1 \prec w$ *and an irreflexive* v_2 *such that* $v_2 \prec w$*.* $\mathcal{U}_{\mathbf{K4}}(0)$ *is the result of iterating this procedure.*

An extensive discussion of universal models is given in [1] or [3]. Note that a 0-universal model is a frame because there is no valuation. In this case there is no distinction between universal model and universal frame. The for us most important facts about such a universal model are stated in the next theorem.

Theorem 2

1. *Each finite Kripke frame for* **K4** *can be mapped p-morphically onto a generated submodel of* $\mathcal{U}_{\mathbf{K4}}(0)$ *in a unique manner.*
2. *For each closed formula* α, $\mathbf{K4} \vdash \alpha$ *iff* $\mathcal{U}_{\mathbf{K4}}(0) \Vdash \alpha$*.*
3. *For each node* w *of* $\mathcal{U}_{\mathbf{K4}}(0)$ *there exists a (closed) formula* φ_w *such that* $v \Vdash \varphi_w$ *iff* $v = w$*.*

circl=[circle, fill=black!100,thick,inner sep=0pt,minimum size=2mm] circu=[circle, draw=black!100,thick,inner sep=0pt,minimum size=2mm] [circl] (1) at (0,0) ; [circu] (2) at (1,0) ; [circu] (3) at (2,0) ; [circl] (4) at (3,0) ; [circl] (5) at (4,0) ; [circu] (6) at (5,0) ; [circl] (7) at (6,0) ; [circu] (8) at (7,0) ; [circl] (9) at (8,0) ; [circu] (10) at (9,0); [circl] (11) at (1.5,1.5) ; [circu] (12) at (3.5,1.5) ; [circl] (13) at (5.5,1.5) ; [circu] (14) at (7.5,1.5) ; [circl] (15) at (2.5,3) ; [circu] (16) at (6.5,3);
[-] (1) – (11); [-] (2) – (11); [-] (2) – (12); [-] (3) – (11); [-] (3) – (12); [-] (3) – (13); [-] (4) – (11); [-] (4) – (12); [-] (4) – (13); [-] (4) – (14); [-] (5) – (11); [-] (5) – (13); [-] (6) – (11); [-] (6) – (16); [-] (7) – (11); [-] (7) – (12); [-] (7) – (16); [-] (8) – (11); [-] (8) – (14); [-] (9) – (11); [-] (9) – (12); [-] (9) – (14); [-] (10) – (11); [-] (10) – (13); [-] (10) – (14); [-] (11) – (15); [-] (12) – (15); [-] (13) – (15); [-] (14) – (15); [-] (13) – (16); [-] (14) – (16);

Figure 1: 0-universal model of **K4**

Proof. See [3]. ⊣

In figure 1 irreflexive nodes are indicated by a dot, reflexive points by a small circle. The two nodes of level one and the four nodes of level two have all been given, but of the nodes of level three only ones have been drawn that are connected to the leftmost node of level two (and only half of those, either the reflexive ones, or the irreflexive ones). No nodes of higher levels have been drawn.

The important fact that we do not need cycles in $\mathcal{U}_{\mathbf{K4}}(0)$ is connected to clause (1) in the above theorem: any cycle in a **K4**-frame can p-morphically be replaced by a reflexive node. Clause (1) of Theorem 2 then shows that we do not want/need to introduce such cycles in $\mathcal{U}_{\mathbf{K4}}(0)$; the same holds for single predecessors of single reflexive nodes.

Let us say that a logic λ has the 0-*fmp property* if λ has the finite model property with respect to closed formulas. For all we know each extension of **K4** by closed formulas (or even any formulas) may have this property, but no proof is known to us.

Definition 10

1. *For a* 0*-fmp logic* λ *extending* **K4** *the* 0-universal model *and* frame $\mathcal{U}_\lambda(0)$ *is the restriction of the* 0-universal model $\mathcal{U}_{\mathbf{K4}}(0)$ *to those nodes* w *for which the upward closed set generated by* w *is a* λ*-frame.*
2. *A subset* $A \subseteq \mathcal{U}_\lambda(0)$ *is called* definable *or* admissible *in* $\mathcal{U}_\lambda(0)$ *iff there exists a (closed) formula* α *such that* $A = \{x | x \in \mathcal{U}_\lambda(0), x \Vdash \alpha\}$*. A valuation* v *on* $\mathcal{U}_\lambda(0)$ *is called* admissible *iff, for any propositional variable* p_i *from the domain of* v*,* $v(p_i)$ *is admissible.*
3. *The restriction to the elements of depth* n *or less of the* 0*-universal model* $\mathcal{U}_\lambda(0)$ *is written* $(\mathcal{U}_\lambda(0))_n$*.*

Theorem 3 *There is a* 1-1 *correspondence between* 0*-fmp extensions of* **K4** *by closed formulas and upsets in* $\mathcal{U}_{\mathbf{K4}}(0)$.

Proof. Straightforward. ⊣

Corollary 2 *There are uncountably many* 0*-fmp extensions of* **K4** *by closed formulas.*

Proof. It is sufficent to find an infinite antichain in $\mathcal{U}_{\mathbf{K4}}(0)$. One can find such a sequence for example in the reflexive elements in Figure 3, since $\mathcal{U}_{\mathbf{K4.3}}(0)$ can of course be embedded in $\mathcal{U}_{\mathbf{K4}}(0)$. ⊣

For 0-fmp λ extending **K4** a theorem analogous to Theorem 2 applies.

Theorem 4 *For each* 0*-fmp extension of* **K4**,

1. *Each finite Kripke frame for* λ *can be mapped p-morphically onto a generated submodel of* $\mathcal{U}_\lambda(0)$ *in a unique manner.*
2. *For each closed formula* α, $\lambda \vdash \alpha$ *iff* $\mathcal{U}_\lambda(0) \Vdash \alpha$.
3. *For each node* w *of* $\mathcal{U}_\lambda(0)$ *there exists a formula* φ_w *such that* $v \Vdash \varphi_w$ *iff* $v = w$.

Proof. See [3]. ⊣

It is also obvious that

Theorem 5 *Let* λ *be a* 0*-fmp logic extending* **K4**, *and* $\gamma_1, ..., \gamma_n$ *be closed formulas. Then, for any* $\alpha(p_1, \ldots, p_n)$, $\vdash_\lambda \alpha(\gamma_1, ..., \gamma_n)$ *iff* $\mathcal{U}_\lambda(0) \Vdash \alpha(\gamma_1, ..., \gamma_n)$.

Proof. Just note that $\alpha(\gamma_1, ..., \gamma_n)$ is closed if $\gamma_1, ..., \gamma_n$ are and apply Theorem 4(2). ⊣

We can now formulate the following general theorem.

Theorem 6 *For each* 0*-fmp* λ *extending* **K4** *and each* $\alpha(p_1, ..., p_n)$, α *is unifiable in* λ *iff there exists an admissible valuation* v *on the* 0*-universal frame* $\mathcal{U}_\lambda(0)$ *such that* $\mathcal{U}_\lambda(0) \Vdash_v \alpha(p_1, ..., p_n)$.

Proof. ($\Rightarrow$): If $\alpha(p_1, ..., p_n)$ is unifiable then there are closed formulas $\gamma_1, ..., \gamma_n$ such that $\vdash_\lambda \alpha(\gamma_1, ..., \gamma_n)$. So, $\mathcal{U}_\lambda(0) \Vdash \alpha(\gamma_1, ..., \gamma_n)$, by Theorem 5. Take $v(p_i) = v(\gamma_i)$ then $\mathcal{U}_\lambda(0) \Vdash \alpha(p_1, ..., p_n)$.
($\Leftarrow$): Suppose there is an admissible valuation v on $\mathcal{U}_\lambda(0)$. Since v is admissible $v(p_i) = v(\gamma_i)$ for some closed γ_i, for each i. So $\mathcal{U}_\lambda(0) \Vdash \alpha(\gamma_1, ..., \gamma_n)$ and hence $\vdash_\lambda \alpha(\gamma_1, ..., \gamma_n)$ by Theorem 5. Therefore α is unifiable. ⊣

Of course one may see this thorem as a reformulation of the fact that $\alpha(p_1, ..., p_n)$ is unifiable in λ iff $\alpha(p_1, ..., p_n)$ is satisfiable in the free 0-generated algebra of λ. This fact is not directly useful for us.

This theorem by itself does in general not lead to a concrete decision procedure for unifiability in a logic. But if one succeeds in exhibiting an effective procedure that provides for each formula α an n such that the existence of an admissible valuation on $\mathcal{U}_\lambda(0)$ is guaranteed by the existence of such a valuation on $(\mathcal{U}_\lambda(0))_n$, then decidability follows. Of course, this decidability was known by the decidability of the admissible rules for these logics, but the decision procedure is much more concrete. We have succeeded in the calculation of such an n for the logics **K4.3** and **GL** but not for **K4** itself. These decision procedures have much lower computational complexity than the generic algorithms for admissibility: the semantic criteria for unifiability in **GL** or **KD4** are NP, whereas nonadmissibility in these logics was just known to be NEXP-complete.

4 Semantic results on Unifiability in KD4 and GL and their extensions, and in K4.3

In this section we give semantic results for the unifiability and non-unifiability of a formula in various logics. We start with **KD4**.

Theorem 7 *The* 0*-universal model* $\mathcal{U}_{\mathbf{KD4}}(0)$ *of* **KD4** *and all extensions of* **KD4** *consists of a single reflexive point.*
Proof. Obvious. ⊣

To obtain results for **GL** and its extensions, we use 0-universal models as planned. In addition, to obtain non-unifiability results, we consider α-soundness of **GL**-models and validity of boxed subformulas of formulas in these models.

Lemma 2 [2]. *Let* w *be node in a* **GL***-model.* $w \Vdash \Box^n\bot$ *iff* $depth(w) \leq n$.
Proof. By induction on n. ⊣

The following is the *normal form theorem* for closed formulas in **GL**.

Theorem 8 [2]. *Any closed formula* α *in* **GL** *is equivalent to a Boolean combination of some* $\Box^n\bot$.
Proof. See [2]. ⊣

Corollary 3 [2]. *For each closed formula* α *of* **GL** *there exists a finite or cofinite subset* F_α *of* $\mathbb{N}$ *such that for each node* w *of finite depth,* $w \Vdash \alpha$ *iff* $depth(\alpha) \in F_\alpha$.

Theorem 9 *The* 0*-universal model* $\mathcal{U}_{\mathbf{GL}}(0)$ *of* **GL** *consists of the set of irreflexive worlds* $\{w_i \mid i \in \mathbb{N}\backslash\{0\}\}$ *where* $w_i R w_j$ *iff* $j < i$.
Proof. Obvious. ⊣

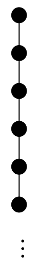

Figure 2: 0-universal model of **GL**

By Theorem 6 we then have immediately:

Theorem 10 *For each $\alpha(p_1, ..., p_n)$, α is unifiable in* **GL** *iff there exists an admissible valuation v on $\mathcal{U}_{\mathbf{GL}}(0)$ such that $\mathcal{U}_{\mathbf{GL}}(0) \Vdash_v \alpha(p_1, ..., p_n)$.*

We will now show how we can restrict this universal model to an upper part of it that is sufficient for our purposes.

Definition 11 *A Kripke model K is* α-sound *if K is rooted and in its root w, $\Vdash \Box\beta \to \beta$ holds for each subformula $\Box\beta$ of α.*

The following lemma is a slight generalization (to models containing reflexive nodes) of a lemma in [9].

Lemma 3 *Let K be α-sound, and let K' be defined by adding a new root u below K with its satisfaction relation identical to the one at w for all atoms. Then $u \Vdash \beta$ iff $w \Vdash \beta$, for all subformulas β of α.*

Proof. Let K be α-sound, and K' be defined by adding a new root u below K with the forcing identical to w for all the atoms. We prove by induction on the length of α that for all subformulas β of α that $u \Vdash \beta$ iff $w \Vdash \beta$. This is trivial for atoms and Boolean combinations.

Let $\beta = \Box\delta$ and the theorem hold for the formula δ. If $u \Vdash \Box\delta$ then $w \Vdash \Box\delta$ since uRw and R is transitive. If $w \Vdash \Box\delta$ then, not only for all v such that wRv, $v \Vdash \delta$, but also, by the α-soundness of K, $w \Vdash \delta$. By the induction hypothesis, $u \Vdash \delta$ as well. But then, irregardless of whether u is reflexive or irreflexive, for all v such that uRv, $v \Vdash \delta$, i.e., $u \Vdash \Box\delta$. Therefore, for every subformula β of α, $u \Vdash \beta$ iff $w \Vdash \beta$. $\dashv$

Theorem 11 *Let m be the number of subformulas of the form $\Box\beta$ in α plus one. Then, for each $\alpha(p_1, ..., p_n)$, α is unifiable in* **GL** *iff there exists a valuation v on $(\mathcal{U}_{\mathbf{GL}}(0))_m$ such that $(\mathcal{U}_{\mathbf{GL}}(0))_m \Vdash_v \alpha(p_1, ..., p_n)$.*

Proof.
(⇒): Follows from Theorem 6.
(⇐): Assume v is a valuation on $(\mathcal{U}_{\mathbf{GL}}(0))_m$ such that $(\mathcal{U}_{\mathbf{GL}}(0))_m \Vdash_v \alpha(p_1, ..., p_n)$. $(\mathcal{U}_{\mathbf{GL}}(0))_m$ is simply a chain of depth m. By the pigeonhole principle there is a $k < m$ such that the set of subformulas $\Box\beta$ of α that are forced at w of depth k and u of depth $k+1$ are the same because going up the number of such formulas can only increase or stay equal. Let K_k^* be the submodel of $(\mathcal{U}_{\mathbf{GL}}(0))_m$ generated by w. For each subformula $\Box\beta$ of α, $w \Vdash \Box\beta \rightarrow \beta$ holds because, if $w \Vdash \Box\beta$, then $u \Vdash \Box\beta$ and hence $w \Vdash \beta$. Therefore K_k^* is α-sound. Moreover, K_k^* is a model of $\alpha \wedge \Box\alpha$. By Lemma 3 we can conclude that by adding a new root w' to K_k^* with the same valuation as w we obtain a model K' that again satisfies $\alpha \wedge \Box\alpha$. Continuing by similarly adding w'' to obtain K'', w''' to obtain K''', etc. we get an infinite linear model for $\alpha \wedge \Box\alpha$. The special property of this model is that the valuation of p_i, is constant from depth k downwards for $1 \leq i \leq l$. That is because we kept the valuation constant each time we added a new root.

This means that p_i is equivalent to a closed formula γ_i on this model for each i, $1 \leq i \leq l$. The infinite linear frame of the model is of course nothing but $\mathcal{U}_{\mathbf{GL}}(0)$. The valuation v is determined by the formulas γ_i and is therefore admissible. Since $v(\alpha) = 1$ everywhere on the model, by Theorem 6, α is unifiable in **GL**. ⊣

Now consider a logic λ extending **GL**. To determine the set of unifiers of extensions of **GL** it is, by Corollary 1, sufficient to determine the set of unifiers of logics extending **GL** by closed formulas only. It is well-known that extensions of **GL** are 0-fmp (see e.g. [3]). But we have a more precise description of the extensions by closed formulas only.

Theorem 12

1. *The closed fragments of extensions of* **GL** *are the closed fragment of* **GL** *itself, and the logics axiomatized by* $\Box^n\bot$ *for some* $n > 0$ *over the closed fragment of* **GL**.

2. *An extension* λ *of* **GL** *has the same closed fragment as* **GL** *iff, for no* n, $\lambda \vdash \Box^n\bot$.

Proof. See [3]. ⊣

This enables us to extend the characterization of the unifiable formulas for **GL** to its extensions.

Definition 12 *For a logic* λ *extending* **GL** *the* 0-universal model *and* frame $\mathcal{U}_\lambda(0)$ *consists of the set of irreflexive worlds* $\{w_i \,|\, i \in \mathbb{N}\backslash\{0\}\}$ *where* $w_i R w_j$ *iff* $j < i$, *if for no* n, $\Box^n\bot$ *is provable in* λ, *and of the set of irreflexive worlds* $\{w_1, \ldots, w_n\}$ *ordered in the same way if* n *is the smallest number for which* $\Box^n\bot$ *is provable in* λ.

Theorem 13 *Let λ be a logic extending* **GL**. *The formula α is unifiable in λ iff, for some valuation, α is valid in $(\mathcal{U}_\lambda(0))_n$, where n is the number of $\Box$-subformulas of α plus 1.*
Proof. Proof is the same as the proof of Theorem 11. ⊣

Now consider the logic **K4.3**.

Definition 13 *The 0-universal model of* **K4.3** *is constructed as follows: The set of worlds consists of a set of irreflexive worlds $\{w_i \mid i \in \mathbb{N}\backslash\{0\}\}$ and a set of reflexive worlds $\{\bar{w}_i \mid i \in \mathbb{N}\backslash\{0\}\}$ where*

$w_i R w_j$ *iff* $j < i$,

$\bar{w}_i R w_j$ *iff* $j < i$,

$\bar{w}_i R \bar{w}_j$ *iff* $i = j$,

not $w_i R \bar{w}_j$.

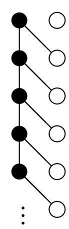

Figure 3: 0-universal model of **K4.3**

Theorem 14 *The formula α is unifiable in* **K4.3** *iff, for some valuation, α is valid in $(\mathcal{U}_{\mathbf{K4.3}}(0))_n$ where n is the number of $\Box$-subformulas of α plus 1.*
Proof. The proof is done similarly to proof of Theorem 11. Just note that $\mathcal{U}_{\mathbf{K4.3}}(0)$ is upwards linear, and apply the pigeonhole principle to the chain of irreflexive elements in $(\mathcal{U}_{\mathbf{K4.3}}(0))_{n+1}$. ⊣

The theorem of course applies to the extensions of **K4.3** as well. But note that because the set of reflexive elements in $\mathcal{U}_{\mathbf{K4.3}}(0)$ is an infinite antichain there are uncountably many of such extensions among which many undecidable ones.

5 Syntactic Results on Unifiability

In this section we give syntactic results for unifiability and non-unifiability of a formula for the logics considered section 4.

Theorem 15 [8] For any modal logic λ extending **KD4** and any modal formula α, α is not unifiable in λ iff $\vdash_\lambda \Box\alpha \wedge \alpha \to \bigvee_{p \in Var(\alpha)} (\Diamond p \wedge \Diamond\neg p)$.

Proof
($\Leftarrow$): Assume α is unifiable in λ extending **KD4**. By Theorem 6 there exists a valuation on $\mathcal{U}_\lambda(0)$ validating α and hence also $\Box\alpha$. By Theorem 7, $\mathcal{U}_\lambda(0)$ is a single reflexive node. On that node $\bigvee_{p \in Var(\alpha)} (\Diamond p \wedge \Diamond\neg p)$ will always be falsified. Hence, $\Box\alpha \wedge \alpha \to \bigvee_{p \in Var(\alpha)} (\Diamond p \wedge \Diamond\neg p)$ is not provable in λ.
($\Rightarrow$): Assume $\Box\alpha \wedge \alpha \to \bigvee_{p \in Var(\alpha)} (\Diamond p \wedge \Diamond\neg p)$ is not provable in λ. Then there exists a **KD4**-model $\mathfrak{M}$ with a node w verifying $\Box\alpha \wedge \alpha$ and falsifying $\bigvee_{p \in Var(\alpha)} (\Diamond p \wedge \Diamond\neg p)$. Thus, all nodes accessible from w (including possibly w itself), verify the same atoms. Consider a successor u of w (guaranteed to exist by the axiom D) and the submodel $\mathfrak{M}_u$ generated by u. Since each node has a successor in this model, and each node satisfies the same atoms, a p-morphism from $\mathfrak{M}_u$ onto a model on a single reflexive node exists. But this is a model on $\mathcal{U}_\lambda(0)$ and it still validates $\Box\alpha \wedge \alpha$. So, again applying Theorem 6, α is unifiable in λ. $\dashv$

We now move to discuss **GL**.

Definition 14 *For $n > 0$, D_n denotes the formula $\Box^n\bot \wedge \neg\Box^{n-1}\bot$ for some n, where $\Box^0\bot \equiv \bot$.*

Note that $F_{D_n} = \{n\}$ (see Corr. 3).

Theorem 16 *For each formula $\alpha(p_1, ..., p_l)$, α is not unifiable in* **GL** *iff $\alpha \wedge \Box\alpha \to (D_n \to \bigvee_{p_i \in Var(\alpha)} \bigvee_{k<n} [\Diamond(D_k \wedge p_i) \wedge \Diamond(D_k \wedge \neg p_i)])$ is provable in* **GL** *for some $n > 0$.*

In the proof we will see that the number of $\Box$-subformulas of α plus 1 is a bound on the n, thereby again providing a concrete decision procedure for non-unifiability in **GL**.

Proof.
($\Leftarrow$): Assume α is unifiable and the formula

$$\alpha \wedge \Box\alpha \to (D_n \to \bigvee_{p_i \in Var(\alpha)} \bigvee_{k<n} [\Diamond(D_k \wedge p_i) \wedge \Diamond(D_k \wedge \neg p_i)])$$

is provable in **GL**. We have to obtain a contradiction. By the fact that α is unifiable there is a substitution g of unifiers in place of the variables of α such that $g(\alpha) \in$ **GL** (and hence $g(\Box\alpha) \in$ **GL**):

$$g(\alpha \wedge \Box\alpha \to (D_n \to \bigvee \lim 1its_{p_i \in Var(\alpha)} \bigvee_{k<n} [\Diamond(D_k \wedge p_i) \wedge \Diamond(D_k \wedge \neg p_i)])) \in \mathbf{GL}.$$

Take a linear frame of depth n. Its root w_n validates $g(\alpha)$, $g(\Box\alpha)$ and D_n and hence, for some p_i,

$$w_n \Vdash \bigvee_{k<n} [\Diamond(D_k \wedge g(p_i)) \wedge \Diamond(D_k \wedge \neg g(p_i))].$$

At some depth below n there should be two nodes of that depth satisfying the contradictory formulas $g(p_i)$ and $\neg g(p_i)$. This is impossible on a linear frame.
($\Rightarrow$): Assume $\alpha \wedge \Box\alpha \to (D_n \to \bigvee_{p_i \in Var(\alpha)} \bigvee_{k<n} [\Diamond(D_k \wedge p_i) \wedge \Diamond(D_k \wedge \neg p_i)]) \notin \mathbf{GL}$ for all n. We have to show that α is unifiable.

Since the formula $\alpha \wedge \Box\alpha \to (D_n \to \bigvee_{p_i \in Var(\alpha)} \bigvee_{k<n} [\Diamond(D_k \wedge p_i) \wedge \Diamond(D_k \wedge \neg p_i)])$ is not provable in **GL** there is, for each n, a **GL**-model $\mathfrak{M}_n$ of depth n that invalidates this formula in its root w_n, i.e.,

$$w_n \Vdash \alpha \wedge \Box\alpha \wedge D_n,\ w_n \nVdash \bigvee_{k<n} [\Diamond(D_k \wedge p_i) \wedge \Diamond(D_k \wedge \neg p_i)].$$

Therefore, $\mathfrak{M}_n$ has depth n. Let us take the case that n is the number of $\Box$-subformulas of α plus 1. Because all nodes at each depth $k < n$ have the same valuation we can apply a p-morphism onto a linear model of depth n by mapping all nodes of depth $k < n$ to one single node of depth k with that valuation. So w.l.o.g. we can assume $\mathfrak{M}_n$ to be linear. Also $\alpha \wedge \Box\alpha$ is forced everywhere in this model $\mathfrak{M}_n$. Of course, $\mathfrak{M}_n$ is a model on $(\mathcal{U}_{\mathbf{GL}}(0))_n$ so that, by Theorem 11, the theorem follows. $\dashv$

Theorem 17 *If λ is an extension of* **GL***, then*

1. *if, for no n, $\lambda \vdash \Box^n\bot$, then α is not unifiable in λ iff $\alpha \wedge \Box\alpha \to (D_n \to \bigvee_{p_i \in Var(\alpha)} \bigvee_{k<n} [\Diamond(D_k \wedge p_i) \wedge \Diamond(D_k \wedge \neg p_i)])$ is provable in λ for some $n > 0$.*
2. *if m is the smallest number for which $\lambda \vdash \Box^m\bot$, then α is not unifiable in λ iff $\alpha \wedge \Box\alpha \to (D_m \to \bigvee_{p_i \in Var(\alpha)} \bigvee_{k<m} [\Diamond(D_k \wedge p_i) \wedge \Diamond(D_k \wedge \neg p_i)])$ is provable in λ for some $n \leq m$.*

Proof. (1) ($\Leftarrow$) : Let $\alpha \wedge \Box\alpha \to (D_n \to \bigvee_{p_i \in Var(\alpha)} \bigvee_{k<n} [\Diamond(D_k \wedge p_i) \wedge \Diamond(D_k \wedge \neg p_i)]) \notin \lambda$ and $\lambda \vdash \Box^n\bot$, for no n. We have to show that α is unifiable.
In this case λ has the same closed fragment as **GL**, by Theorem 12. Since then λ and **GL** have the same finite linear models, the proof is given as for Theorem 16.

($\Rightarrow$) : Let $\lambda \vdash \Box^n\bot$, for no n. Assume α is unifable and the formula $\alpha \wedge \Box\alpha \to (D_n \to \bigvee_{p_i \in Var(\alpha)} \bigvee_{k<n} [\Diamond(D_k \wedge p_i) \wedge \Diamond(D_k \wedge \neg p_i)])$ is provable in λ. Since λ and **GL** have the same closed fragment and the same finite linear models the proof is given as for Theorem 16.

(2)($\Leftarrow$) : Let $\alpha \wedge \Box\alpha \to (D_n \to \bigvee_{p_i \in Var(\alpha)} \bigvee_{k<m} [\Diamond(D_k \wedge p_i) \wedge \Diamond(D_k \wedge \neg p_i)]) \notin \lambda$, where m is the smallest number for which $\lambda \vdash \Box^m\bot$. We have to show that α is

unifiable. By the same method as before we get a model for $\alpha \wedge \Box\alpha$ on a linear model of m elements. This is, in this case, the 0-universal model for λ.

$(\Rightarrow)$: Let m be the smallest number for which $\lambda \vdash \Box^m \bot$. Assume α is unifable and the formula $\alpha \wedge \Box\alpha \rightarrow (D_n \rightarrow \bigvee_{p_i \in Var(\alpha)} \bigvee_{k<n} [\Diamond(D_k \wedge p_i) \wedge \Diamond(D_k \wedge \neg p_i)])$ is provable in λ for some $n \leq m$. Using a linear frame of depth n we obtain a contradiction as in the proof of Theorem 16. ⊣

Though true, this theorem is somewhat misleading in that the logic λ may have only upward linear models (e.g. if λ is **GL3**). Then clearly,

$$\vdash_\lambda \neg \bigvee_{p_i \in Var(\alpha)} \bigvee_{k<n} [\Diamond(D_k \wedge p_i) \wedge \Diamond(D_k \wedge \neg p_i)],$$

so the condition reduces to $\vdash_\lambda \alpha \wedge \Box\alpha \rightarrow \neg D_n$ for some n.

For **K4.3** upward linearity is of course in force as well. Nevertheless, the syntactic conditions are rather complicated. Let us name the formulas guaranteed to exist for the reflexive worlds $\bar{w}_i$ by Theorem 4(3), $\bar{D}_i$. Then, if m is the number of $\Box$-subformulas of α, $\nvdash \alpha \wedge \Box\alpha \rightarrow \neg\bar{D}_i$ for each $i \leq m$ is not sufficient to guarantee a model for α on $(\mathcal{U}_{K4.3}(0))_m$ because the valuations on the different counter-models with reflexive nodes as roots which can be obtained may not be the same on the irreflexive nodes. Let us first determine when $w \Vdash \bar{D}_i$ holds an a **K4.3**-model (or frame, $\bar{D}_i$ is closed).

An arbitrary **K4.3**-frame $F = \ < W, R >$ is a linear order of a number of irreflexive elements and clusters (possibly a single reflexive element). For the purpose here we split F into two subframes, an initial segment $F^1 = \ < W^1, R >$ and a tail of irreflexive elements $F^2 = \ < W^2, R >$ in such a way that F^1 is serial, i.e., for each $u \in W^1$, $v \in W^2$, $u\,R\,v$, and ., for each $u \in W^1$, there exists $v \in W^1$ with $u\,R\,v$. Either W^1 or W^2 may be empty. In some irrelevant cases the splitting may not be uniquely determined. Of course, **K4.3**-models can be split similarly. Using that terminology we can state:

Lemma 4 *In a* **K4.3**-*model* M, $w \Vdash \bar{D}_i$ *iff a splitting of* F *can be made such that* F^2 *is a chain of exactly* $i-1$ *elements, and* $w \in W^1$.
Proof. $(\Leftarrow)$: Assume F^2 is a chain of exactly $i-1$ elements, and $w \in W^1$. It is easy to see that the function f from M to the 0-universal model of **K4.3** that that maps all of W^1 onto $\bar{w}_i$ and W^2 onto $\{w_1, \ldots, w_{i-1}\}$ in the obvious manner is a p-morphism. Thus, $\bar{D}_i$ is true everywhere in M^1.
$(\Rightarrow)$: Assume in a **K4.3**-model M, $w \Vdash \bar{D}_i$. From the fact that $\bar{D}_i$ implies $\Diamond D_{i-1}$, $\neg\Diamond D_i$ and $\neg\Diamond D_j$ for any j, it is clear that in any splitting F^2 will have to contain exactly i elements, and $w \in W^1$. ⊣

What we have to do is to check whether there is a model for $\alpha \wedge \Box\alpha$ on each subframe of $(\mathcal{U}_{\mathbf{K4.3}}(0))_m$ generated by one of its reflexive elements such that all of these models have the same valuation on the irreflexive elements they share. Let v be a valuation on $(\mathcal{U}_{\mathbf{K4.3}}(0))_m$, i.e., v is defined for $p_1, \ldots, p_n$ for each w_i

($1 \leq i \leq m$, we write v_i) and each $\bar{w}_i, 1 \leq i \leq m$ (we write $\bar{v}_i$). Let us define $p_j^{v_i}$ to be p_j if $v_i(p_j) = 1$ and $\neg p_j$ if $v_i(p_j) = 0$, similarly for $p_j^{\bar{v}_i}$. Finally, let us write θ_v^i for

$$p_1^{\bar{v}_i} \wedge \cdots \wedge p_n^{\bar{v}_i} \wedge \bigwedge_{j=1}^{i-1} \Diamond(D_j \wedge p_1^{v_j} \wedge \cdots \wedge p_n^{v_j}).$$

Then truth of θ_v^k in $\bar{w}_k$ aims to express that the valuation v holds in the node itself and in the irreflexive nodes above $\bar{w}_k$. We can then state the following theorem.

Theorem 18 *Let m be the number of $\Box$-subformulas of $\alpha(p_1, ..., p_n)$ plus 1. Then α is unifiable in* **K4.3** *iff there exists a valuation $v = v_1, \ldots, v_m, \bar{v}_1, \ldots, \bar{v}_m$ such that, for all $i \leq m$, $\alpha \wedge \Box\alpha \wedge \bar{D}_i \to \Diamond(\bar{D}_i \wedge \neg\theta_v^i)$ is not provable in* **K4.3**.

Proof. ($\Rightarrow$): Assume α is unifiable in **K4.3**. Then, by Theorem 6, there exists a valuation v on $(\mathcal{U}_{K4.3}(0))_m$ validating α and hence $\Box\alpha$, where m is the number of $\Box$-subformulas of α plus 1. Using this valuation as v the formulas θ_v^i are defined. In each $\bar{w}_i$, $\bar{D}_i$ is satisfied as well as θ_v^i, and since the only node accessible from $\bar{w}_i$ that satisfies $\bar{D}_i$ is $\bar{w}_i$ itself, $\Box(\bar{D}_i \to \theta_v^i)$ as well. Therefore, for each $i \leq m$, $\alpha \wedge \Box\alpha \wedge \bar{D}_i \to \Diamond(\bar{D}_i \wedge \neg\theta_v^i)$ is not provable in **K4.3**.
($\Leftarrow$): Assume v is such that none of the formulas $\alpha \wedge \Box\alpha \wedge \bar{D}_i \to \Diamond(\bar{D}_i \wedge \neg\theta_v^i)$ is provable in **K4.3** for $i \leq m$. We want to show that α is unifiable. Since these formulas are not provable, there exists, for each $i \leq m$, a **K4.3** model M_i with a node w satisfying $\alpha \wedge \Box\alpha \wedge \bar{D}_i \wedge \Box(\bar{D}_i \to \theta_v^i)$. Of course, we can assume that M_i is generated by w. By Lemma 4, the set of worlds of M_i splits into W_i^1 and W_i^2 with all elements of W_i^1 satisfying $\bar{D}_i$ and the elements of W_i^2 forming a chain of $i-1$ elements. Since $w \Vdash \Box(\bar{D}_i \to \theta_v^i)$ all elements of W_i^1 have the same valuation. Thus, a p-morphism of M_i onto a model N_i on the subframe of $(\mathcal{U}_{\mathbf{K4.3}}(0))_m$ generated by $\bar{w}_i$ exists. Because of the fact that in each of these models N_i the irreflexive nodes of a particular depth have the same valuation determined by v, all the N_i are identical on the irreflexive elements they share, so they can be glued together to one model on the frame $(\mathcal{U}_{\mathbf{K4.3}}(0))_m$. The resulting model is then a model on $(\mathcal{U}_{\mathbf{K4.3}}(0))_m$ satisfying $\alpha \wedge \Box\alpha$ everywhere so that, by Theorem 14, α is unifable. $\dashv$

6 Passive Admissible Rules

In [8] it was shown (Proposition 3.3) that it is an immediate consequence of the Theorem 15, that the rules $r_n := (\Diamond p_1 \wedge \Diamond\neg p_1) \vee ... \vee (\Diamond p_n \wedge \Diamond\neg p_n)/q$ form a basis for all passive rules for any logic λ extending **KD4**. (And that in fact this can be improved to show that this basis can be reduced to a finite one consisting of r_1 by itself). In the same way we can now formulate a corresponding theorem for **GL**.

Theorem 19 *For any logic* λ *extending* **GL** *the following rules* $g_{n,l}$ *form a basis for all passive rules:* $g_{n,l} := (D_n \to \bigvee_{i \leq l} \bigvee_{k<n} [\Diamond(D_k \wedge p_i) \wedge \Diamond(D_k \wedge \neg p_i)])/q$.

Obviously for extensions λ proving some $\Box^n\bot$ this can be reduced to the formulas $g_{m,l}$ for $m < n$. For **GL** itself it is clear that no finite basis can be sufficient. We will not spell out the similar result for **K4.3** which can be derived.

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