Homework 4

1. In the following, assume that x is not a free variable of ψ. Which of the following statements are intuitionistically valid? (If yes, give a proof, if not, give a countermodel).

   (a) $(\exists x\varphi(x) \to \psi) \to \forall x(\varphi(x) \to \psi)$

   (b) $\forall x(\varphi(x) \to \psi) \to (\exists x\varphi(x) \to \psi)$

   (c) $(\forall x\varphi(x) \to \psi) \to \exists x(\varphi(x) \to \psi)$

   (d) $\exists x(\varphi(x) \to \psi) \to (\forall x\varphi(x) \to \psi)$ [4 pts]

2. (a) Show that the following is valid: If $\vdash_{\mathbf{IPC}} (\varphi \to \psi) \to (\chi \vee \theta)$, then $\vdash_{\mathbf{IPC}} (\varphi \to \psi) \to \chi$ or $\vdash_{\mathbf{IPC}} (\varphi \to \psi) \to \theta$ or $\vdash_{\mathbf{IPC}} (\varphi \to \psi) \to \varphi$. [4 pts]

   (b) Give an example such that the first two alernatives of (a) do not apply, but the last one does. [2 pts]

   (c)* Let α be $(\neg\phi \to \psi \vee \chi) \to (\neg\phi \to \psi) \vee (\neg\phi \to \chi)$.
   Show that, if $\vdash_{IPC} \alpha \to \beta \vee \gamma$, then
   $\vdash_{IPC} \alpha \to \beta$ or $\vdash_{IPC} \alpha \to \gamma$.[2 pts]

3. (a) Let φ contain only $\wedge, \vee$ and $\to$ but no $\neg$ and no $\bot$. Let $\mathfrak{M}$ be any Kripke-model (for the language of φ). Extend the model $\mathfrak{M}$ to $\mathfrak{M}^+$ by adding one more node x at the top above all the nodes of $\mathfrak{M}$, and making all the propositional variables of φ true in x.

   Show that, for all the nodes w in $\mathfrak{M}$ we have:

   $$\mathfrak{M}, w \models \varphi \text{ iff } \mathfrak{M}^+, w \models \varphi$$

   (satisfaction in the old and new model is the same for φ). [4 pts]

   (b) Let φ contain only $\wedge, \vee$ and $\to$ but no $\neg$ and no $\bot$. Show that $\vdash_{\mathbf{IPC}} \varphi$ iff $\vdash_{\mathbf{KC}} \varphi$. (You may use what is claimed about completeness of **KC** in class.) [2 pts]