Homework 6, due Tuesday 1 April, 12:00

1. Prove that $\vdash_{\mathbf{IPC}} \varphi$ implies $\vdash_{\mathbf{S4}} \varphi^{\Box}$ in the following manner:

   Assume $\mathfrak{M}$ on $\mathfrak{F}$ is an **S4**-countermodel to $\varphi^{\Box}$. Take the frame $\mathfrak{G}$ that is obtained from $\mathfrak{F}$ by replacing each cluster (collection of nodes that are pairwise accessible from each other) by a single node. (Try to define this exactly.) There is an obvious function from $\mathfrak{F}$ onto $\mathfrak{G}$. Show that it is a p-morphism. Define an appropriate valuation on $\mathfrak{G}$ in such a way that the resulting model $\mathfrak{N}$ is an **IPC**-model. Show, by induction on the length of $\psi(p_1, \ldots, p_n)$ that, for each $w \in W$, $\mathfrak{M}, w \Vdash \psi^{\Box}$ (as an **S4**-model) iff $\mathfrak{N}, f(w) \models \psi$ (as an **IPC**-model). Finally conclude that $\mathfrak{N}$ (as an **IPC**-model) falsifies φ. [6 pts]

2. (a) Show that **HA** has the existence property: if **HA**$\vdash \exists x \varphi(x)$, then **HA**$\vdash \varphi(\overline{n})$ for some n. [2 pts]

   (b) Add a predicate $A(x)$ to the language of **HA** with the axiom $A(0) \wedge \forall x(A(x) \to A(x+1))$, but do **not** add induction for formulas containing A. Show the disjunction property for this system. [2 pts]

   (c) Add a predicate $B(x)$ to the language of **HA** with the axiom $\exists x B(x)$. Does this system have the disjunction property? [2 pts]

3. (a) Let φ be a propositional formula not containing $\vee$.

   Let $\mathfrak{M}$ be a model and w a node in $\mathfrak{M}$ such that w has proper successors (i.e., there is at least one v in $\mathfrak{M}$ with wRv and $w \neq v$). Suppose

   - φ holds in all proper successors of w (i.e., for all v with $wRv, w \neq v$, we have $\mathfrak{M}, v \models \varphi$)
   - for all propositional variables p, we have that p is true in w iff p is true in all proper successors of w (i.e., $w \in V(p)$ iff $\forall v\ (wRv, w \neq v \Rightarrow v \in V(p))$.

     (In other words, the valuation in w is maximal for propositional variables considering persistency).

   Show that φ is true in w. [4 pts]

   (b) Show on the basis of the above that, if φ is a propositional formula not containing $\vee$ and $\vdash_{\mathbf{IPC}} \varphi \to \psi \vee \chi$, then $\vdash_{\mathbf{IPC}} \varphi \to \psi$ or $\vdash_{\mathbf{IPC}} \varphi \to \chi$. [2 pts]