The Disjunction Property according to Kreisel and Putnam

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This paper is dedicated to my longtime friend and colleague Roel de Vrijer on the occasion of his sixtieth birthday. With its subject I have tried to go a little in his direction by taking a very syntactic subject.

The work is part of a project in progress in cooperation with Rosalie Iemhoff and Nick Vaporis. It concerns the disjunction property in intermediate logics, i.e. logics extending the intuitionistic propositional logic **IPC**, and in particular the method to prove such statements due to [8]. It is well-known that **IPC** has the *disjunction property*:

$$\text{If } \vdash_{\mathbf{IPC}} A \vee B, \text{ then } \vdash_{\mathbf{IPC}} A \text{ or } \vdash_{\mathbf{IPC}} B.$$

Łukasiewicz [10] conjectured this property to be characteristic for **IPC** in the sense that no stronger logic has this property. In 1957, G. Kreisel and H. Putnam [8] disproved this conjecture by showing hat the logic, now called **KP**, obtained by adding the scheme

$$(\neg A \to B \vee C) \to (\neg A \to B) \vee (\neg A \to C)$$

to **IPC** has the *disjunction property*: If $\vdash_{\mathbf{KP}} A \vee B$, then $\vdash_{\mathbf{KP}} A$ or $\vdash_{\mathbf{KP}} B$.

Their method has not been much used since and it gives stronger results than the methods in use nowadays. Their result has a connection to my thesis as well. My adviser S.C. Kleene asked me to show that a stronger disjunction property does characterize **IPC** and I successfully executed that project ([5, 6]).

Noteworthy is that at the time the disjunction property was not advertised so much as a positive property (which of course has some connection to the BHK-interpretation of intuitionistic logic), but as a refutation rule. Łukasiewicz had hoped to obtain a very compact and neat system of refutation rules for intuitionistic logic in this manner. The title of the Kreisel-Putnam paper: "Eine Unableitbarkeitsbeweismethode für den Intuitionistischen Aussagenkalkül", clearly shows how one felt about this.

Of course, they could not use Kripke models for their result because these did not exist, Saul Kripke was 16 years old. And even though H.B. Curry writes to E.W. Beth on January 24, 1957 (see [7]),

> I have recently been in communication with a young man in Omaha, Nebraska, named Saul Kripke.... This young man is a mere boy of 16 years; yet he has read and mastered my Notre Dame lectures and writes me letters which would do credit to many a professional logician.

Kripke's first abstract on modal logic [9] only appeared in 1959. Not surprisingly, their method was the application of cutfree proofs but in fact it does not matter so much since the basic result they use, and after which they proceed purely syntactically, can be proved in various ways, also using Kripke models. This basic result is in its simple form:

If $\vdash (A \to B) \to C \vee D$, then
$\vdash (A \to B) \to C$ or
$\vdash (A \to B) \to D$ or
$\vdash (A \to B) \to A$,

This is a generalization of *Harrop's rule* [2]:

If $\vdash \neg A \to B \vee C$, then $\vdash \neg A \to B$ or $\vdash \neg A \to C$.

Equivalently (modulo the disjunction property), Harrop's rule can be stated as a so-called *admissible rule*:

If $\vdash \neg A \to B \vee C$, then $\vdash (\neg A \to B) \vee (\neg A \to C)$,
abbreviated as $\neg A \to B \vee C/(\neg A \to B) \vee (\neg A \to C)$.

The basic result in its extended form is:

If $\vdash (A_1 \to B_1) \wedge \cdots \wedge (A_k \to B_k) \to C \vee D$, then
$\vdash (A_1 \to B_1) \wedge \cdots \wedge (A_k \to B_k) \to C$ or
$\vdash (A_1 \to B_1) \wedge \cdots \wedge (A_k \to B_k) \to D$ or
$\vdash (A_1 \to B_1) \wedge \cdots \wedge (A_k \to B_k) \to A_i$ for some $i \leq k$.

This rule (which can of course be given as an infinite sequence of admissible rules for **IPC**) is nowadays often called *Visser's rule*, VR for short. Of course, at that time it was still many years before A. Visser even began his logic studies, but he gave an extended version of the rule in provability logic of which R. Iemhoff has made extensive use ([3, 4]). Among other things she proved (roughly stated) that all admissible rules for **IPC** can be derived form these.

The disjunction property will now be proved for some logics in the following manner. If the logic L is axiomatized by the scheme A one uses the fact that $\vdash_L C \vee D$ iff $\vdash_{\mathbf{IPC}} A_1 \wedge \ldots A_k \to C \vee D$ for some substitution instances $A_1 \wedge \ldots A_k$ of A. One proves, by induction on k, that for such sequences of substitution instances indeed $\vdash_{\mathbf{IPC}} A_1 \wedge \ldots A_k \to C \vee D$ iff $\vdash_{\mathbf{IPC}} A_1 \wedge \ldots A_k \to C$ or $\vdash_{\mathbf{IPC}} A_1 \wedge \ldots A_k \to D$.

If one succeeds in doing this one gets a very strong result. If $\vdash_L C \vee D$, then not only $\vdash_L C$ or $\vdash_L D$, but C or D is provable using beyond **IPC** only the same instances of L's axioms that were used to prove $C \vee D$. Among other things one has proved the disjunction property not only for the logic itself but also for any logic axiomatized by substitution instances of the axioms of the logic in question. We will apply this later in some examples.

1 The proof for KP

Teaching an intuitionistic logic class this spring I rediscovered essentially the proof that Kreisel and Putnam gave[1]. Let us introduce some notation first. For any formulas A, B, C we write $\mathrm{KP(A, B, C))}$ for

$$(\neg A \to B \vee C) \to (\neg A \to B) \vee (\neg A \to C)$$

We leave the A, B, C off and write KP if it is clear what we mean and we write $\mathrm{KP_1}$ for $\mathrm{KP(A_1, B_1, C_1)}$ etc. We will write $\mathrm{KP}^{(n)}$ for $\mathrm{KP_1} \wedge \cdots \wedge \mathrm{KP_n}$. Also, we will write $\vdash$ without further ado if we mean $\vdash_{\mathbf{IPC}}$. We prove

Theorem 1.1 *If* $\vdash \mathrm{KP(A_1, B_1, C_1)} \wedge \cdots \wedge \mathrm{KP(A_n, B_n, C_n)} \to \mathrm{X} \vee \mathrm{Y}$*, then* $\vdash \mathrm{KP(A_1, B_1, C_1)} \wedge \cdots \wedge \mathrm{KP(A_n, B_n, C_n)} \to \mathrm{X}$ *or* $\vdash \mathrm{KP(A_1, B_1, C_1)} \wedge \cdots \wedge \mathrm{KP(A_n, B_n, C_n)} \to \mathrm{Y}$*.*

Proof. We prove this by induction on n. The characteristic point of the proof is that to be able to execute the induction step we prove something stronger:

- If $\vdash \mathrm{KP_1} \wedge \cdots \wedge \mathrm{KP_n} \wedge \neg \mathrm{D_1} \wedge \cdots \wedge \neg \mathrm{D_k} \to \mathrm{X} \vee \mathrm{Y}$, then $\vdash \mathrm{KP_1} \wedge \cdots \wedge \mathrm{KP_n} \wedge \neg \mathrm{D_1} \wedge \cdots \wedge \neg \mathrm{D_k} \to \mathrm{X}$ or $\vdash \mathrm{KP_1} \wedge \cdots \wedge \mathrm{KP_n} \wedge \neg \mathrm{D_1} \wedge \cdots \wedge \neg \mathrm{D_k} \to \mathrm{Y}$

Actually, in the above, we can assume k to be 1, since conjunctions of negations are equivalent to negations in **IPC**.

BASIS. $n = 0$. This is Harrop's rule.

INDUCTION STEP. Assume the claim for n, and assume $\vdash \mathrm{KP_1} \wedge \cdots \wedge \mathrm{KP_{n+1}} \wedge \neg \mathrm{D} \to \mathrm{X} \vee \mathrm{Y}$. By Visser's rule there are three possibilities. The first two give the result immediately. So, we can assume w.l.o.g.:

- $\vdash \mathrm{KP_1} \wedge \cdots \wedge \mathrm{KP_{n+1}} \wedge \neg \mathrm{D} \to (\neg \mathrm{A_{n+1}} \to \mathrm{B_{n+1}} \vee \mathrm{C_{n+1}})$. Since
- $\vdash \neg A_{n+1} \to \mathrm{KP_{n+1}}$ we get
- $\vdash \mathrm{KP_1} \wedge \cdots \wedge \mathrm{KP_n} \wedge \neg \mathrm{D} \wedge \neg \mathrm{A_{n+1}} \to \mathrm{B_{n+1}} \vee \mathrm{C_{n+1}}$. Applying the induction hypothesis we get
- $\vdash \mathrm{KP_1} \wedge \cdots \wedge \mathrm{KP_n} \wedge \neg \mathrm{D} \wedge \neg \mathrm{A_{n+1}} \to \mathrm{B_{n+1}}$ or
- $\vdash \mathrm{KP_1} \wedge \cdots \wedge \mathrm{KP_n} \wedge \neg \mathrm{D} \wedge \neg \mathrm{A_{n+1}} \to \mathrm{C_{n+1}}$. In both cases
- $\vdash \mathrm{KP_1} \wedge \cdots \wedge \mathrm{KP_n} \wedge \neg \mathrm{D} \to \mathrm{KP_{n+1}}$ follows.

We can now apply the induction hypothesis to get the desired result. $\dashv$

The result immediately applies to some weaker logics which have been only under investigation more recently.

- $\mathbf{ND_k} = \mathbf{IPC} + (\neg A \to \neg C_1 \vee \cdots \vee \neg C_k) \to (\neg A \to \neg C_1) \vee \cdots \vee (\neg A \to \neg C_k)$
- **ND** is the union of all $\mathbf{ND_k}$.

ND is not finitely axiomatizable. Note that if one defines $\mathbf{KP_2}$ in the same manner as $\mathbf{ND_2}$, $\mathbf{KP_2}$ derives **KP**.

Corollary 1.2 **ND** *and* $\mathbf{ND_k}$*, for each* k*, have the disjunction property.*

[1] I have to thank the students in my class for pushing me in the right, syntactic, direction.

2 The proof for Scott's logic

Kreisel and Putnam mention in their paper that D. Scott has shown that with a similar method the logic

$$\mathbf{Sc} = \mathbf{IPC} + ((\neg\neg A \to A) \to A \vee \neg A) \to \neg A \vee \neg\neg A,$$

since then called Scott's logic, can be proved to have the disjunction property. Let us now prove the disjunction property of Scott's logic by the method. We first give the proof for the basic case to get in the right mood. We write $\mathrm{Sc}(A)$ (or Sc for short if the A is clear) for $((\neg\neg A \to A) \to A \vee \neg A) \to \neg A \vee \neg\neg A$ and Ant for the antecedent of Sc etc.

Lemma 2.1 *If* $\vdash \mathrm{Sc} \to \mathrm{X} \vee \mathrm{Y}$*, then* $\vdash \mathrm{Sc} \to \mathrm{X}$ *or* $\vdash \mathrm{Sc} \to \mathrm{Y}$.

Proof. Assume $\vdash \mathrm{Sc} \to \mathrm{X} \vee \mathrm{Y}$. To prove is: $\vdash \mathrm{Sc} \to \mathrm{X}$ or $\vdash \mathrm{Sc} \to \mathrm{Y}$.

Then, the Visser rule gives us three possibilities of which the first two immediately give the desired result. So, we assume that the third possibility applies.

- $\vdash \mathrm{Sc} \to \mathrm{Ant}$, i.e.
- $\vdash \mathrm{Sc} \to ((\neg\neg \mathrm{A} \to \mathrm{A}) \to \mathrm{A} \vee \neg \mathrm{A})$. It is obvious that
- $\vdash (\neg\neg A \to A) \to \mathrm{Sc}$, so we have
- $\vdash (\neg\neg A \to A) \to A \vee \neg A$. By applying the Visser rule, we obtain
- $\vdash (\neg\neg A \to A) \to A$ or $\vdash (\neg\neg A \to \neg A) \to \neg A$ or $\vdash (\neg\neg A \to A) \to \neg\neg A$.

This gives us $\vdash \neg A$ or $\vdash \neg\neg A$. In both cases we have $\vdash \mathrm{Sc}$, and hence $\vdash X \vee Y$. By the simple disjunction property for **IPC** the desired result now follows. $\dashv$

For the induction step we first prove a lemma. We also introduce the notation $\mathrm{Sc}^{(\mathrm{n})}$ for $\mathrm{Sc}_1 \wedge \cdots \wedge \mathrm{Sc}_\mathrm{n}$.

Lemma 2.2 *If* $\vdash \mathrm{Sc}^{(\mathrm{n})} \wedge (\neg\neg \mathrm{B}_1 \to \mathrm{B}_1) \wedge \ldots (\neg\neg \mathrm{B}_\mathrm{k} \to \mathrm{B}_\mathrm{k}) \to \mathrm{C} \vee \neg \mathrm{C}$*, then* $\vdash \neg C$ *or* $\vdash \neg\neg C$*, or* $\vdash \neg\neg B_i$ *for some* $i \leq k$*, or* $\vdash \mathrm{Sc}_\mathrm{j}$ *for some* $j \leq n$.

Proof. By induction on n.

BASIS $n = 0$. Asssume $\vdash (\neg\neg B_1 \to B_1) \wedge \ldots (\neg\neg B_k \to B_k) \to C \vee \neg C$. Then, by VR, C,$\neg C$ or some $\neg\neg B_i$ will be provably implied by a classical tautology. So, respectively, $\vdash \neg\neg C$, $\vdash \neg C$, $\vdash \neg\neg B_i$ follows.

INDUCTION STEP Assume $\vdash \mathrm{Sc}^{(\mathrm{n}+1)} \wedge (\neg\neg \mathrm{B}_1 \to \mathrm{B}_1) \wedge \ldots (\neg\neg \mathrm{B}_\mathrm{k} \to \mathrm{B}_\mathrm{k}) \to \mathrm{C} \vee \neg \mathrm{C}$. Apply the Visser rule. We run through the relevant possibilities.

- $\vdash \mathrm{Sc}^{(\mathrm{n}+1)} \wedge (\neg\neg \mathrm{B}_1 \to \mathrm{B}_1) \wedge \ldots (\neg\neg \mathrm{B}_\mathrm{k} \to \mathrm{B}_\mathrm{k}) \to \mathrm{C}$. Since all the antecedents are classical tautologies we get $\vdash \neg\neg C$.
- $\vdash \mathrm{Sc}^{(\mathrm{n}+1)} \wedge (\neg\neg \mathrm{B}_1 \to \mathrm{B}_1) \wedge \ldots (\neg\neg \mathrm{B}_\mathrm{k} \to \mathrm{B}_\mathrm{k}) \to \neg \mathrm{C}$. As above we get $\vdash \neg C$.

- $\vdash \mathrm{Sc^{(n+1)}} \wedge (\neg\neg \mathrm{B_1} \to \mathrm{B_1}) \wedge \ldots (\neg\neg \mathrm{B_k} \to \mathrm{B_k}) \to \neg\neg \mathrm{B_i}$ for some i. Again we get $\vdash \neg\neg B_i$.
- $\vdash \mathrm{Sc^{(n)}} \wedge (\neg\neg \mathrm{B_1} \to \mathrm{B_1}) \wedge \ldots (\neg\neg \mathrm{B_k} \to \mathrm{B_k}) \wedge (\neg\neg \mathrm{A} \to \mathrm{A}) \to \mathrm{A} \vee \neg \mathrm{A}$ (w.l.o.g.). Then the induction hypothesis applies, and in the cases $\neg A_{n+1}$ and $\vdash \neg\neg A_{n+1}$ obtained from its application $\vdash \mathrm{Sc_{n+1}}$ follows. The other cases are obvious.

$\dashv$

Theorem 2.3 *If* $\vdash \mathrm{Sc^{(n)}} \to \mathrm{X} \vee \mathrm{Y}$, *then* $\vdash \mathrm{Sc^{(n)}} \to \mathrm{X}$ *or* $\vdash \mathrm{Sc^{(n)}} \to \mathrm{Y}$.

Proof. By induction on n. The basis is given by Lemma 2.1. Assume the result for n, and assume

$$\vdash \mathrm{Sc^{(n+1)}} \to \mathrm{X} \vee \mathrm{Y}.$$

Using the Visser rule, and reasoning as before we get w.l.o.g. the relevant possibility

- $\vdash \mathrm{Sc^{(n+1)}} \to ((\neg\neg \mathrm{A_{n+1}} \to \mathrm{A_{n+1}}) \to \mathrm{A_{n+1}} \vee \neg \mathrm{A_{n+1}})$ and from that
- $\vdash \mathrm{Sc^{(n)}} \to ((\neg\neg \mathrm{A_{n+1}} \to \mathrm{A_{n+1}}) \to \mathrm{A_{n+1}} \vee \neg \mathrm{A_{n+1}})$.

Now apply Lemma 2.2. In all cases we obtain $\mathrm{Sc_j}$ for some j. We can then apply the induction hypothesis. $\dashv$

Let us call **GR** the logic that is axiomatized by $\mathrm{Sc}(\neg B \vee \neg C)$ after Gene Rose, who proved that this formula is always realizable [12], and thereby started the still continuing quest for the realizability logic, the logic of all formulas that are always realizable. It is of course an immediate corollary that the logic **GR** has the disjunction property. This applies as well to the logic axiomatized by the infinitely many axiom schemata $\mathrm{Sc}(\neg B_1 \vee \cdots \vee \neg B_n)$ for any n.

3 Some Thoughts

Although we have no proof of this it seems out of the question that the method can be used to prove the disjunction property for any intermediate logic that has the property. Another important point is that what one proves here is not a property of the logic but of its axiomatization. One axiomatization of a logic may have the property and another one might not. A trivial example is easy to find. For example add to Scott's logic the axiom $\mathrm{Sc(A, B, C)} \vee \mathrm{D}$. One still has Scott's logic. Obviously by means of the new axiom one can prove now $\vdash_{\mathbf{Sc}} \mathrm{Sc(A, B, C)} \vee \mathrm{p}$. And it is equally obvious that one cannot hope to prove $\mathrm{Sc(A, B, C)}$ or p in **IPC** from this axiom alone.

A much nicer way of showing that it is not the logic that has the Kreisel-Putnam property but the axiomatization would be by giving two serious axiomatizations of a logic, one of which has the property, but the other one does not. We tried one obvious candidate but the attempt failed. We will show this failure.

If one has an axiomatization of a logic by a scheme of the form $Z \to X \vee Y$, then an equivalent axiomatization is given by $Z \wedge (X \to C) \wedge (Y \to C) \to C$. (From left to right this is just a logical consequence of **IPC**, from right to left just substitute $X \vee Y$ for C.) If we apply this idea to Scott's logic, we obtain the new axiomatization:

$$\mathrm{Sc}'(\mathrm{A}) \;=\; ((\neg\neg\mathrm{A} \to \mathrm{A}) \to \mathrm{A} \vee \neg\mathrm{A}) \wedge (\neg\mathrm{A} \to \mathrm{C}) \wedge (\neg\neg\mathrm{A} \to \mathrm{C}) \to \mathrm{C}.$$

Somewhat unexpectedly, the Kreisel-Putnam method does apply. Let us do the basic case in detail.

Lemma 3.1 *If* $\vdash \mathrm{Sc}' \to \mathrm{X} \vee \mathrm{Y}$*, then* $\vdash \mathrm{Sc}' \to \mathrm{X}$ *or* $\vdash \mathrm{Sc}' \to \mathrm{Y}$.

Proof. Assuming $\vdash \mathrm{Sc}' \to \mathrm{X} \vee \mathrm{Y}$, as usual we can disregard the first two alternatives given by VR and concentrate on the third one:

- $\vdash \mathrm{Sc}' \to \mathrm{Ant}$. We just need:
- $\vdash \mathrm{Sc}' \to ((\neg\neg\mathrm{A} \to \mathrm{A}) \to \mathrm{A} \vee \neg\mathrm{A})$. As usual, but in a slightly different manner,
- $\vdash (\neg\neg A \to A) \to A \vee \neg A$ follows. Again as usual this gives three possibilities. The first one is
- $\vdash (\neg\neg A \to A) \to A$. Then $\vdash \neg\neg A$ and thus, since $\neg\neg A \to C$ is one of the antecedents, $\vdash \mathrm{Sc}'$. This is of course sufficient; we can apply the disjunction property of **IPC** itself to $\vdash X \vee Y$.

The other two cases $\vdash (\neg\neg A \to A) \to \neg A$ and $\vdash (\neg\neg A \to A) \to \neg\neg A$ are similar. ⊣

The lemma that provides the induction step is essentially the same as in the proof for Sc itself and is proved by induction in a similar way.

Lemma 3.2 *If* $\vdash \mathrm{Sc}'^{(\mathrm{n})} \wedge (\neg\neg\mathrm{B}_1 \to \mathrm{B}_1) \wedge \cdots \wedge (\neg\neg\mathrm{B}_\mathrm{k} \to \mathrm{B}_\mathrm{k}) \to \mathrm{C} \vee \neg\mathrm{C}$*, then* $\vdash \neg C$ *or* $\vdash \neg\neg C$*, or* $\vdash \neg\neg B_i$ *for some* $i \leq k$*, or* $\vdash \mathrm{Sc}'_\mathrm{j}$ *for some* $j \leq n$.

As stated in the beginning of this section, it seems extremely likely that there are intermediate logics for which no axiomatization with the right property can be found. To make this into a theorem could be hard. It seems that the need is for a semantic or other characterization of such axiom systems.

4 Other Logics

Can the method be applied to other logics? Two obvious candidates (actually sequences of candidates) have resisted attempts so far. Whether the method really does not apply only time will tell. One sequence of systems is formed by the the elements of the Rieger-Nishimura lattice higher than Scott's formula, the first one of which is $(((\neg\neg A \to A) \to A \vee \neg A) \to \neg A \vee \neg\neg A) \to \neg\neg A \vee (\neg\neg A \to A)$. The other is the sequence $\mathbf{T_n}$ of the Gabbay-deJongh logics [1], starting with the logic $\mathbf{T_2}$ that characterizes the finite frames with only splittings of 2 or less (no node has more than 2 immediate successors). Its axiomatization is:

$$((A \to B \vee C) \to B \vee C) \wedge ((B \to A \vee C) \to A \vee C) \wedge ((C \to A \vee B) \to A \vee B) \to A \vee B \vee C.$$

In both cases the result applies up to two axioms but the induction step has evaded us up to now. Note that the result for one axiom always has the form of a rule admissible for **IPC**. In the case of **KP** this is Harrop's rule:

$$\neg A \to B \vee C / (\neg A \to B) \vee (\neg A \to C)$$

In the case of Scott's logic this is the rule:

$$(\neg\neg A \to A) \to A \vee \neg A / \neg A \vee \neg\neg A.$$

And in the case of $\mathbf{T_2}$ it is the curious rule:

$$(A \to B \vee C) \to B \vee C, (B \to A \vee C) \to A \vee C, (C \to A \vee B) \to A \vee B \;/\; A \vee B \vee C.$$

However, there is a sequence of logics $\mathbf{B_n}$, which is somewhat close to $\mathbf{T_n}$, and has been studied by H. Ono ([11]), which is susceptible to the Kreisel-Putnam method. I will show this for the first one of these logics with the disjunction property, $\mathbf{B_3}$:

$$(\neg A \leftrightarrow B \vee C) \wedge (\neg B \leftrightarrow A \vee C) \wedge (\neg C \leftrightarrow A \vee B) \to A \vee B \vee C.$$

Our notation for this formula will be $\mathrm{O(A, B, C)}$ with its obvious variants. The proof resembles the one for **KP**.

Theorem 4.1 *If* $\vdash \mathrm{O^{(n)}} \to \mathrm{X} \vee \mathrm{Y}$*, then* $\vdash \mathrm{O^{(n)}} \to \mathrm{X}$ *or* $\vdash \mathrm{O^{(n)}} \to \mathrm{Y}$.

Proof. We prove again, by induction on n, that if $\vdash \mathrm{O^{(n)}} \wedge \neg \mathrm{D} \to \mathrm{X} \vee \mathrm{Y}$, then $\vdash \mathrm{O^{(n)}} \wedge \neg \mathrm{D} \to \mathrm{X}$ or $\vdash \mathrm{O^{(n)}} \wedge \neg \mathrm{D} \to \mathrm{Y}$.

BASIS $n = 0$. This is Harrop's rule.

INDUCTION STEP Assume the result holds for n and

- $\vdash \mathrm{O^{(n+1)}} \wedge \neg \mathrm{D} \to \mathrm{X} \vee \mathrm{Y}$. Applying VR
  and ignoring the first two cases gives
- $\vdash \mathrm{O^{(n+1)}} \wedge \neg \mathrm{D} \to \mathrm{Ant_{n+1}}$, which again implies
- $\vdash \mathrm{O^{(n)}} \wedge \neg \mathrm{D} \to \mathrm{Ant_{n+1}}$. In particular
- $\vdash \mathrm{O^{(n)}} \wedge \neg \mathrm{D} \wedge \neg \mathrm{A_{n+1}} \to \mathrm{B_{n+1}} \vee \mathrm{C_{n+1}}$. By induction hypothesis, w.l.o.g.
- $\vdash \mathrm{O^{(n)}} \wedge \neg \mathrm{D} \to (\neg \mathrm{A_{n+1}} \to \mathrm{B_{n+1}})$. Also
- $\vdash \mathrm{O^{(n)}} \wedge \neg \mathrm{D} \to (\mathrm{B_{n+1}} \to \neg \mathrm{C_{n+1}})$. So,
- $\vdash \mathrm{O^{(n)}} \wedge \neg \mathrm{D} \to (\neg \mathrm{A_{n+1}} \to \neg \mathrm{C_{n+1}})$. But, we have also
- $\vdash \mathrm{O^{(n)}} \wedge \neg \mathrm{D} \to (\neg\neg \mathrm{A_{n+1}} \to \neg \mathrm{C_{n+1}})$ from

- $\vdash \mathrm{O}^{(n)} \wedge \neg \mathrm{D} \to (\mathrm{C}_{n+1} \to \neg \mathrm{A}_{n+1})$. So,
- $\vdash \mathrm{O}^{(n)} \wedge \neg \mathrm{D} \to (\neg \mathrm{A} \vee \neg\neg \mathrm{A}_{n+1} \to \neg \mathrm{C}_{n+1})$ from which
- $\vdash \mathrm{O}^{(n)} \wedge \neg \mathrm{D} \to \neg \mathrm{C}_{n+1}$ follows. This gives us
- $\vdash \mathrm{O}^{(n)} \wedge \neg \mathrm{D} \to \mathrm{A}_{n+1} \vee \mathrm{B}_{n+1}$ and finally
- $\vdash \mathrm{O}^{(n)} \wedge \neg \mathrm{D} \to \mathrm{O}_{n+1}$. So,
- $\vdash \mathrm{O}^{(n)} \wedge \neg \mathrm{D} \to \mathrm{X} \vee \mathrm{Y}$,

and we can apply the induction hypothesis once more. $\dashv$

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