Intuitionism

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Intuitionism is one of the main points of view in the philosophy of mathematics, nowadays usually set opposite *formalism* and *Platonism*. As such intuitionism is best seen as a particular manner of implementing the idea of *constructivism* in mathematics, a manner due to the Dutch mathematician Brouwer and his pupil Heyting. Constructivism is the point of view that mathematical objects exist only in so far they have been constructed and that proofs derive their validity from constructions; more in particular, existential assertions should be backed up by *effective* constructions of objects. Mathematical truths are rather seen as being created than discovered. Intuitionism fits into *idealistic* trends in philosophy: the mathematical objects constructed are to be thought of as idealized objects created by an idealized mathematician (IM), sometimes called *the creating* or *the creative subject*. Often in its point of view intuitionism skirts the edges of *solipsism* when the idealized mathematician and the proponent of intuitionism seem to fuse.

Much more than formalism and Platonism, intuitionism is in principle *normative*. Formalism and Platonism may propose a foundation for existing mathematics, a reduction to logic (or set theory) in the case of Platonism, or a consistency proof in the case of formalism. Intuitionism in its stricter form leads to a reconstruction of mathematics: mathematics as it is, is in most cases not acceptable from an intuitionistic point of view and it should be attempted to rebuild it according to principles that are constructively acceptable. Typically it is not acceptable to prove $\exists x\, \phi(x)$ (for some x, $\phi(x)$ holds) by deriving a contradiction from the assumption that $\forall x \neg \phi(x)$ (for each x, $\phi(x)$ does not hold): *reasoning by contradiction*. Such a proof does not create the object that is supposed to exist.

Actually, in practice the intuitionistic point of view hasn't lead to a large scale and continuous rebuilding of mathematics. For what has been done in this respect, see e.g. [1]. In fact, there is less of this kind of work going on now even than before. On the other hand, one might say that intuitionism describes a particular portion of mathematics, the constructive part, and that it has been described very adequately by now what the meaning of that constructive part is. This is connected with the fact that the intuitionistic point of view has been very fruitful in *metamathematics*, the construction and study of systems in which parts of mathematics are formalized. After Heyting this has been pursued by Kleene, Kreisel and Troelstra (see for this, and an extensive treatment of most other subjects discussed here, and many other ones [13]). Heyting's [7] will always remain a quickly readable but deep introduction to the intuitionistic ideas. In theoretical computer science many of the formal systems that are of foundational importance are formulated on the basis of intuitionistic logic.

L.E.J. Brouwer first defended his constructivist ideas in his dissertation of 1907 ([4]). There were predecessors defending constructivist positions. Mathematicians like Kronecker, Poincaré, Borel. Kronecker and Borel were prompted by the increasingly abstract character of concepts and proofs in the mathematics of the end of the 19th century, and Poincaré couldn't accept the formalist or Platonist ideas proposed by Frege, Russell and Hilbert. In particular, Poincaré maintained in opposition to the formalists and Platinists that mathematical induction (over the natural numbers) cannot be reduced to a more primitive idea. However, from the start Brouwer was more radical, consistent and encompassing than his predecessors. The most distinctive features of intuitionism are:

1. The use of a distinctive logic: *intuitionistic logic*. (Ordinary logic is then called *classical* logic.)

2. Its construction of the *continuum*, the totality of the real numbers, by means of *choice sequences*.

Intuitionistic logic was introduced and axiomatized by A. Heyting, Brouwer's main follower. The use of intuitionistic logic has most often been accepted by other proponents of constructive methods, but the construction of the continuum much less so. The particular construction of the continuum by means of choice sequences involves principles that contradict classical mathematics. Constructivists of other persuasion like the school of Bishop often satisfy themselves in trying to constructively prove theorems that have been proved in a classical manner, and shrink back from actually contradicting ordinary mathematics.

We shall first discuss in this article intuitionistic logic, then spend some time on intuitionistic (natural) number theory and analysis. We then treat the notion of realizability, after which we return to intuitionistic logic in connection with some theories formalized in it. We end up with a discussion of a recently developed game for intuitionistic propositional logic [9].

Intuitionistic logic. We will indicate the formal system of intuitionistic propositional logic by **IPC** and intuitionistic predicate logic by **IQC**; the corresponding classical systems will be named **CPC** and **CQC**. Formally the best way to characterize intuitionistic logic is by a *natural deduction system* à la Gentzen. (For an extensive treatment of natural deduction and sequent systems, see [14].) In fact, natural deduction is more natural for intuitionistic logic than for classical logic. A natural deduction system has *introduction* rules and *elimination* rules for the logical connectives $\wedge$ (and), $\vee$ (or) and $\to$ (if ..., then) and *quantifiers* $\forall$ (for all) and $\exists$ (for at least one). The rules for $\wedge$, $\vee$ and $\to$ are:

- $I\wedge$: From ϕ and ψ conclude $\phi \wedge \psi$,

- $E\wedge$: From $\phi \wedge \psi$ conclude ϕ and conclude ψ,
- $E\rightarrow$: From ϕ and $\phi \rightarrow \psi$ conclude ψ,
- $I\rightarrow$: If one has a derivation of ψ from premise ϕ, then one may conclude to $\phi \rightarrow \psi$ (simultaneously dropping assumption ϕ),
- $I\vee$: From ϕ conclude to $\phi \vee \psi$, and from ψ conclude to $\phi \vee \psi$,
- $E\vee$: If one has a derivation of χ from premise ϕ and a derivation of χ from premise ψ, then one is allowed to conclude χ from premise $\phi \vee \psi$ (simultaneously dropping assumptions ϕ and ψ),
- $I\forall$: If one has a derivation of $\phi(x)$ in which x is not free in any premise, then one may conclude $\forall x\phi(x)$,
- $E\forall$: If one has a derivation of $\forall x\phi(x)$, then on may conclude $\phi(t)$ for any term t,
- $I\exists$: From $\phi(t)$ for any term t one may conclude $\exists x\phi(x)$,
- $E\exists$: If one has a derivation of ψ from $\phi(x)$ in which x is not free in in ψ itself or in any premise other than $\phi(x)$, then one may conclude ψ from premise $\exists x\phi(x)$, dropping the assumption $\phi(x)$ simultaneously.

One usually takes negation $\neg$ (not) of a formula ϕ to be defined as ϕ implying a contradiction ($\perp$). One adds then the *ex falso sequitur quodlibet rule* that

- anything can be derived from $\perp$.

If one wants to get classical propositional or predicate logic one adds the rule that

- if $\perp$ is derived from $\neg\phi$, then one can conclude to ϕ, simultaneously dropping the assumption $\neg\phi$.

Note that this is not a straightforward introduction or elimination rule as the other rules.

The natural deduction rules are strongly connected with the so-called BHK-interpretation (named after Brouwer, Heyting and Kolmogorov) of the connectives and quantifiers. This interpretation gives a very clear foundation of intuitionistically acceptable principles and makes intuitionistic logic one of the very few non-classical logics in which reasoning is clear, unambiguous and all encompassing but nevertheless very different from reasoning in classical logic.

In classical logic the meaning of the *connectives*, i.e. the meaning of complex statements involving the connectives, is given by supplying the *truth conditions* for complex statements that involve the informal meaning of the same connectives. For example:

- $\phi \wedge \psi$ is true if and only if ϕ is true *and* ψ is true,

- $\phi \vee \psi$ is true if and only if ϕ is true *or* ψ is true,
- $\neg\phi$ is true iff ϕ is *not* true

The BHK-interpretation of intuitionistic logic is based on the notion of *proof* instead of truth. (N.B! *Not* formal proof, or derivation, as in natural deduction or Hilbert type axiomatic systems, but intuitive (informal) proof, i.e. convincing mathematical argument.) The meaning of the connectives and quantifiers is then just as in classical logic explained by the informal meaning of their intuitive counterparts:

- A proof of $\phi \wedge \psi$ consists of a proof of ϕ *and* a proof of ψ plus the conclusion $\phi \wedge \psi$,
- A proof of $\phi \vee \psi$ consists of a proof of ϕ *or* a proof of ψ plus a conclusion $\phi \vee \psi$,
- A proof of $\phi \rightarrow \psi$ consists of a *method of converting* any proof of ϕ into a proof of ψ,
- *No* proof of $\perp$ exists,
- A proof of $\exists x\, \phi(x)$ consists of a name d of *an* object constructed in the intended domain of discourse plus a proof of $\phi(d)$ and the conclusion $\exists x\, \phi(x)$,
- A proof of $\forall x\, \phi(x)$ consists of a method that *for any* object d constructed in the intended domain of discourse produces a proof of $\phi(d)$.

For negations this then means that a proof of $\neg\, \phi$ is a method of converting any supposed proof of ϕ into a proof of a contradiction. That $\perp \rightarrow \phi$ has a proof for any ϕ is based on the intuitive counterpart of the ex falso principle. This may seem somewhat less natural then the other ideas, and Kolmogorov did not include it in his proposed rules.

Together with the fact that statements containing negations seem less contentful constructively this has lead Griss to consider doing completely without negation. Since however it is often possible to prove such more negative statements without being able to prove more positive counterparts this is not very attractive. Moreover, one can do without the formal introduction of $\perp$ in natural mathematical systems, because a statement like $1=0$ can be seen to satisfy the desired properties of $\perp$ without making any ex falso like assumptions. More precisely, not only statements for which this is obvious like $3=2$, but all statements in those intuitionistic theories are derivable from $1=0$ without the use of the rules concerning $\perp$. If one nevertheless objects to the ex falso rule, one can use the logic that arises without it, called *minimal logic*.

The intuitionistic meaning of a disjunction is only superficially close to the classical meaning. To prove a disjunction one has to be able to prove one of its members. This makes it immediately clear that there is no general support for $\phi \vee \neg \phi$: there is no way to invariably guarantee a proof of ϕ or a proof of $\neg \phi$. However, many of the laws of classical logic remain valid under the BHK-interpretation. Various *decision methods* for **IPC** are known, but it is often easy to decide intuitively:

- A disjunction is hard to prove: for example, of the four directions of the two *de Morgan laws* only $\neg(\phi \wedge \psi) \rightarrow \neg\phi \vee \neg\psi$ is not valid, other examples of such invalid formulas are
  - $\phi \vee \neg\phi$ (the law of the *excluded middle*)
  - $(\phi \rightarrow \psi) \rightarrow \neg\phi \vee \psi$
  - $(\phi \rightarrow \psi \vee \chi) \rightarrow (\phi \rightarrow \psi) \vee (\phi \rightarrow \chi)$
  - $((\phi \rightarrow \psi) \rightarrow \psi) \rightarrow (\phi \vee \psi)$
- An existential statement is hard to prove: for example, of the four directions of the classically valid interactions between negations and quantifiers only $\neg \forall x\, \phi \rightarrow \exists x \neg\, \phi$ is not valid,
- statements directly based on the two-valuednes of truth values are not valid, e.g. $\neg\neg\, \phi \rightarrow \phi$ or $((\phi \rightarrow \psi) \rightarrow \phi) \rightarrow \phi$ (*Peirce's law*), and *contraposition* in the form $(\neg\psi \rightarrow \neg\phi) \rightarrow \phi \rightarrow \psi)$,
- On the other hand, many basic laws naturally remain valid, commutativity and associativity of conjunction and disjunction, both distributivity laws, and
  - $(\phi \rightarrow \psi \wedge \chi) \leftrightarrow (\phi \rightarrow \psi) \wedge (\phi \rightarrow \chi)$,
  - $(\phi \rightarrow \chi) \wedge (\psi \rightarrow \chi) \leftrightarrow ((\phi \vee \psi) \rightarrow \chi))$,
  - $(\phi \rightarrow (\psi \rightarrow \chi)) \leftrightarrow (\phi \wedge \psi) \rightarrow \chi$.
  - $(\phi \vee \psi) \wedge \neg\phi \rightarrow \psi)$ (needs *ex falso*!),
  - $(\phi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\phi \rightarrow \chi))$,
  - $(\phi \rightarrow \psi) \rightarrow (\neg\psi \rightarrow \neg\phi)$ (the *converse* form of contraposition),
  - $\phi \rightarrow \neg\neg\phi$,
  - $\neg\neg\neg\phi \leftrightarrow \neg\phi$ (triple negations are not needed).

Slightly less obvious is that *double negation shift* is valid for $\wedge$ and $\rightarrow$ but not for $\forall$, at least in one direction. Valid are:

- $\neg\neg(\phi \wedge \psi) \leftrightarrow \neg\neg\phi \wedge \neg\neg\psi$,
- $\neg\neg(\phi \rightarrow \psi) \leftrightarrow \neg\neg\phi \rightarrow \neg\neg\psi$,
- $\neg\neg\forall x \phi(x) \rightarrow \forall x \neg\neg\phi(x)$ (but not its converse).

The BHK-interpretation was independently given by Kolmogorov and Heyting, with Kolmogorov's formulation in terms of the solution of problems rather than in terms of executing proofs. Of course, both extracted the idea from Brouwer's work. In any case, it is clear from the above that, if a logical schema is (formally) provable in **IPC** (say, by natural deduction), then any instance of the scheme will have an informal proof following the BHK-interpretation.

Clearly, in the most direct sense intuitionistic logic is weaker than classical logic. However, from a different point of view the opposite is true. By Gödel's so-called *negative translation* classical logic can be translated into intuitionistic logic. To translate a classical statement one puts $\neg\neg$ in front of all atomic formulas and then replaces each subformula of the form $\phi \vee \psi$ by $\neg(\neg\phi \wedge \neg\psi)$ and each subformula of the form $\exists x\,\phi(x)$ by $\neg\forall x\,\neg\phi(x)$ in a recursive manner. The formula obtained is provable in intuitionistic logic exactly when the original one is provable in classical logic. Some examples are:

- $p \vee \neg p$ becomes in translation $\neg(\neg\neg p \wedge \neg\neg\neg p)$,
- $(\neg q \to \neg p) \to (p \to q)$ becomes $(\neg\neg\neg q \to \neg\neg\neg p) \to (\neg\neg p \to \neg\neg q)$,
- $\neg\forall x\,Ax \to \exists x \neg\,Ax$ becomes $\neg\forall x \neg\neg Ax \to \neg\forall x \neg\neg Ax$

Thus, one may say that intuitionistic logic accepts classical reasoning in a particular form and is therefore richer than classical logic.

Kripke models. A semantics for intuitionistic logic along the lines of the well-known possible worlds models developed by Kripke for modal logic has been extremely useful to obtain very many results about intuitionistic logic, even though in itself it is not faithful to the BHK-interpetation. Actually, this semantics was developed immediately to a high extent by Kripke himself. He proved completeness for both **IPC** and **IQC** with respect to his models, and the finite model property and thus decidability for **IPC** (see [8]). He employed semantic tableaux for this since the Henkin-type completeness proofs for modal logics stem from a later day.

As always one has a set of worlds and a valuation on them. One can imagine uRv between the worlds u and v to mean that v is a possible later state of knowledge as seen from u. It is natural then, contrary to the usual models of modal logic, that, once a formula is true it stays true, i.e. if ϕ is true in u and uRv, then ϕ is true in v (this is called *persistency*).
The rules for *satisfaction* of the formulas are:

1. $w \models \phi \wedge \psi$ iff $w \models \phi$ and $w \models \psi$,
2. $w \models \phi \vee \psi$ iff $w \models \phi$ or $w \models \psi$,
3. $w \models \phi \to \psi$ iff, for all w' such that wRw', if $w' \models \phi$, then $w' \models \psi$,
4. $w \not\models \bot$.

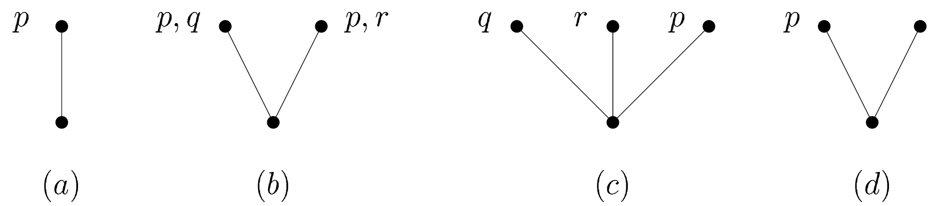

Figure 1: Counter-models for the propositional formulas

It is helpful to note that $w \models \neg\neg\phi$ iff, for each w' such that wRw', there exists w'' with $w'Rw''$ and $w'' \models \phi$. For finite models this simplifies to $w \models \neg\neg\phi$ iff for all maximal nodes w' above w, $w' \models \phi$.

Usually, Kripke models will be *rooted* models, they have a least node (often w_0), a *root*. For the predicate calculus each node w of a model is equipped with a domain D_w in such a way that, if wRw', then $D_w \subseteq D_{w'}$. Persistency comes in this case down to the fact that D_w is a submodel of $D_{w'}$ in the normal sense of the word. The clauses for the quantifiers are (adding names for the elements of the domain to the language):

1. $w \models \exists x\phi(x)$ iff, for some $d \in D_w$, $w \models \phi(d)$.
2. $w \models \forall x\phi(x)$ iff, for each w' with wRw' and all $d \in D_{w'}$, $w' \models \phi(d)$.

One of the earliest theorems proved about intuitionistic logic is Glivenko's theorem which states that $\vdash_{\mathbf{CPC}} \phi$ iff $\vdash_{\mathbf{IPC}} \neg\neg\phi$. The reader will be able to prove this for himself, either by means of finite Kripke models, or by induction on the length of a proof in a natural deduction or other proof system. The result implies for example that $\vdash_{\mathbf{IPC}} \neg\neg(p \vee \neg p)$. This does not extend to the predicate calculus or arithmetic. As we will see, $\nvdash_{\mathbf{IPC}} \neg\neg\forall x(Ax \vee \neg Ax)$.

The following models invalidate respectively $p \vee \neg p$, $\neg\neg p \to p$ (both Figure 1a), $(\neg\neg p \to p) \to p \vee \neg p)$ (Figure 1d), $(p \to q \vee r) \to (p \to q) \vee (p \to r)$ (Figure 1b), $(\neg p \to q \vee r) \to (\neg p \to q) \vee (\neg p \to r)$ (Figure 1c), $\neg\neg\forall x(Ax \vee \neg Ax)$ (Figure 2a, constant domain $\mathbb{N}$), $\forall x(A \vee Bx) \to A \vee \forall x Bx$ (Figure 2b).

Arithmetic. Classical arithmetic of the natural numbers is formalized in **PA** by the so-called *Peano axioms* (the idea of which is originally due to Dedekind). These axioms

- $x + 1 \neq 0$,
- $x + 1 = y + 1 \to x = y$,
- $x + 0 = x$,
- $x + (y + 1) = (x + y) + 1$,
- $x\,.\,0 = 0$,

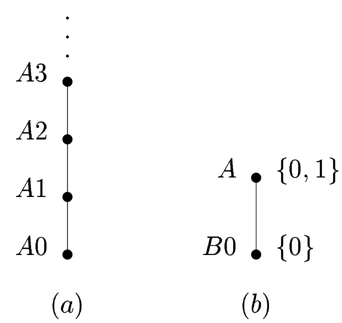

Figure 2: Counter-models for the predicate formulas

- $x\,.\,(y+1)=x\,.\,y+x$,

and the *induction* scheme

- For each $\phi(x)$, $\phi(0)\wedge\forall x(\phi(x)\rightarrow\phi(x+1))\rightarrow\forall x\phi(x)$.

together with using intuitionistic logic instead of classical logic can also be used to axiomatize the intuitionistic version **HA** of natural number theory, *Heyting's arithmetic*. Of course an intuitionist does not simply accept these axioms face value but checks their (intuitive) provability from the basic idea of what natural numbers are (Brouwer in his inaugural address: "... This intuition of two-oneness, the basal intuition of mathematics, creates not only the numbers one and two, but also all finite ordinal numbers, inasmuch as one of the elements of the two-oneness maybe thought of as a new two-oneness, which process may be repeated indefinitely ...").

Worth while noting is that the scheme

- For each $\phi(x)$, $\exists x\phi(x)\rightarrow\exists x(\phi(x)\wedge\forall y<x\,\neg\phi(y))$

is classically but not intuitionistically equivalent to the induction scheme. (Here $y<x$ is defined as $\exists z(y+(z+1)=x)$.)

Gödels' negative translation is applicable to **HA**/**PA**. Of course, also Gödel's incompleteness theorem applies to **HA**: there exists a ϕ such that neither $\vdash_{\mathbf{HA}}\phi$, nor $\vdash_{\mathbf{HA}}\neg\phi$, and this ϕ can be taken to have the form $\forall x\psi(x)$ for some $\psi(x)$ such that, for each n, $\vdash_{\mathbf{HA}}\psi(\bar{n})$. (Here $\bar{n}$ stands for $1+\ldots+1$ with n ones, a term with the value n.)

Free choice sequences. A great difficulty in setting up constructive versions of mathematics is the continuum. It is not difficult to reason about individual real numbers via for example *Cauchy sequences*, but one loses that way the intuition of the totality of all real numbers which does seem to be a primary intuition. Brouwer based the continuum on the idea of choice sequences. For example, a choice sequence α of natural numbers is viewed as an ever unfinished, ongoing process of choosing natural number values $\alpha(0), \alpha(1), \alpha(2), \cdots$ by the ideal mathematician IM. At any stage of IM's activity only finitely many values have been determined by IM, plus possibly some restrictions on future choices. This straightforwardly leads to the idea that a function f giving values to all choice sequences can do so only by having the value $f(\alpha)$ for any particular choice sequence α determined by a finite initial segment $\alpha(0), \ldots, \alpha(m)$ of that choice sequence, in the sense that all choice sequences β starting with the same initial segment $\alpha(0), \ldots, \alpha(m)$ have to get the same value under the function: $f(\beta) = f(\alpha)$. This idea will lead us to Brouwer's theorem that every real function on a bounded closed interval is necessarily uniformly continuous. Of course, this is in clear contradiction with classical mathematics.

Before we get to a characteristic example of a less severe distinction between classical and intuitionistic mathematics, the *intermediate value theorem*, let us discuss the fact that counterexamples to classical theorems in logic or mathematics can be given as *weak* counterexamples or *strong* counterexamples. A weak counterexample to a statement just shows that one cannot hope to prove that statement, a strong counterexample really derives a contradiction from the general application of the statement. For example, to give a weak counterexample to $p \vee \neg p$ it suffices to give a statement ϕ that has not been proved or refuted, especially a statement of a kind that can always be reproduced if the original problems is solved after all. A strong counterexample to $\phi \vee \neg \phi$ cannot consist of proving $\neg (\phi \vee \neg \phi)$ for some particular ϕ, since $\neg (\phi \vee \neg \phi)$ is even in intuitionistic logic contradictory (it is directly equivalent to $\neg \phi \wedge \neg \neg \phi$). But a predicate $\phi(x)$ in intuitionistic analysis can be found such that $\neg \forall x\, (\phi(x) \vee \neg \phi(x))$ can be proved, which can reasonably be called a strong counterexample.

For weak counterexamples Brouwer often used the decimal expansion of π. For example consider the number $a = 0, a_0 a_1 a_2 \ldots$ for which the decimal expansion[1] defined as follows:

As long as no sequence 1234567890 has occurred in the decimal expansion of π, a_n is defined to be 3. If a sequence 1234567890 has occcurred in the decimal expansion of π starting at some m with $m \leqslant n$, then, if the first such m is even a_n is 0 for all $n \geqslant m$, if it is odd, $a_m = 4$ and $a_n = 0$ for all $n > m$. As long as the problem has not been solved whether such a sequence exists it is not known whether $a < \frac{1}{3}$ or $a = \frac{1}{3}$ or $a > \frac{1}{3}$. That this is time bound is shown by the fact that in the meantime this particular problem has been solved, m does exist and is even, so $a < \frac{1}{3}$ [2]. But that does not matter, such problems can, of course, be multiplied endlessly, and (even though we don't take the trouble to change our example) this shows that it is hopeless to try to prove that, for any a, $a < \frac{1}{3} \vee a = \frac{1}{3} \vee a > \frac{1}{3}$. Note that, also a cannot be shown to be rational, because for that, p and q should be given such that $a = \frac{p}{q}$, which

[1] To make arguments easier to follow, we discuss these problems regarding real numbers with arguments pertaining to their decimal expansions. This was not Brouwer's habit, he even showed with a weak counteraxmple that not all real numbers have a decimal expansion (how to start the decimal expansion of a if one does not know whether it is smaller than, equal to, or greater than 0?).

clearly cannot be done without solving the problem. On the other hand, obviously, $\neg\neg(a < \frac{1}{3} \vee a = \frac{1}{3} \vee a > \frac{1}{3})$ does hold, a is not not rational. In any case, weak counterexamples are not mathematical theorems, but they do show which statements one should not try to prove. Later on, Brouwer used unsolved problems to provide weak and strong counterexamples in a stronger way by making the decimal expansion of a dependent on the creating subjects' insight whether he had solved a particular unsolved problem at the moment of the construction of the decimal in question. Attempts to formalize these so-called *creative subject arguments* have lead to great controversy and sometimes paradoxical consequences. For a reconstruction more congenial to Brouwer's ideas that avoids such problematical consequences, see [10].

Let us now move to using a weak counterexample to show that one cannot hope to prove the so-called *intermediate value theorem.* A continuous function f that has value -1 at 0 and value 1 at 1 reaches the value 0 for some value between 0 and 1 according to classical mathematics. This does not hold in the constructive case: a function f that moves linearly from value -1 at 0 to value $a - \frac{1}{3}$ at $\frac{1}{3}$, stays at value $a - \frac{1}{3}$ until $\frac{2}{3}$ and then moves linearly to 1 cannot be said to reach the value 0 at a particular place if one does not know whether $a > \frac{1}{3}$, $a = \frac{1}{3}$ or $a < \frac{1}{3}$. Since there is no method to settle the latter problem in general, one cannot determine a value x where $f(x) = 0$. (See Figure 3.)

Constructivitsts of the Russian school did not accept the intuitionistic construction of the continuum, but neither did they shrink from results contradicting classical mathematics. They obtained such results in a different manner however, by assuming that effective constructions are *recursive* constructions, and thus in particular when one restricts functions to effective functions that all functions are recursive functions. Thus, in opposition to the situation in classical mathematics, accepting the so-called *Church-Turing thesis* that all effective functions are recursive does influence the validity of mathematical results directly.

Let us remark finally that, no matter what ones standpoint is, the resulting formalized intuitionistic analysis has a more complicated relationship to classical analysis than the one between **HA** and **PA**, the negative translation does no longer apply.

Realizability. Kleene used recursive functions in a different manner than the Russian constructivists. Starting in the 1940's he attempted to give a faithful interpretation of intuitionistic logic and (formalized) mathematics by means of recursive functions. To understand this, we need to know two basic facts. The first is that there is a recursive way of coding pairs of natural numbers by a single one, j is a bijection from $\mathbb{N}^2$ to $\mathbb{N}$: $j(m,n)$ codes the pair (m,n) as a single natural number. Decoding is done by the functions $(\,)_0$ and $(\,)_1$: if $j(m.n) = p$, then $(p)_0 = m$ and $(p)_1 = n$. The second insight is that all recursive functions, or easier to think about, all the Turing machines that calculate them can be coded by natural numbers as well. If e codes a Turing machine, then $\{e\}$ is the function that is calculated by it, i.e. for each natural number n, $\{e\}(n)$ has a certain value if on input n the Turing machine coded by e delivers that value.

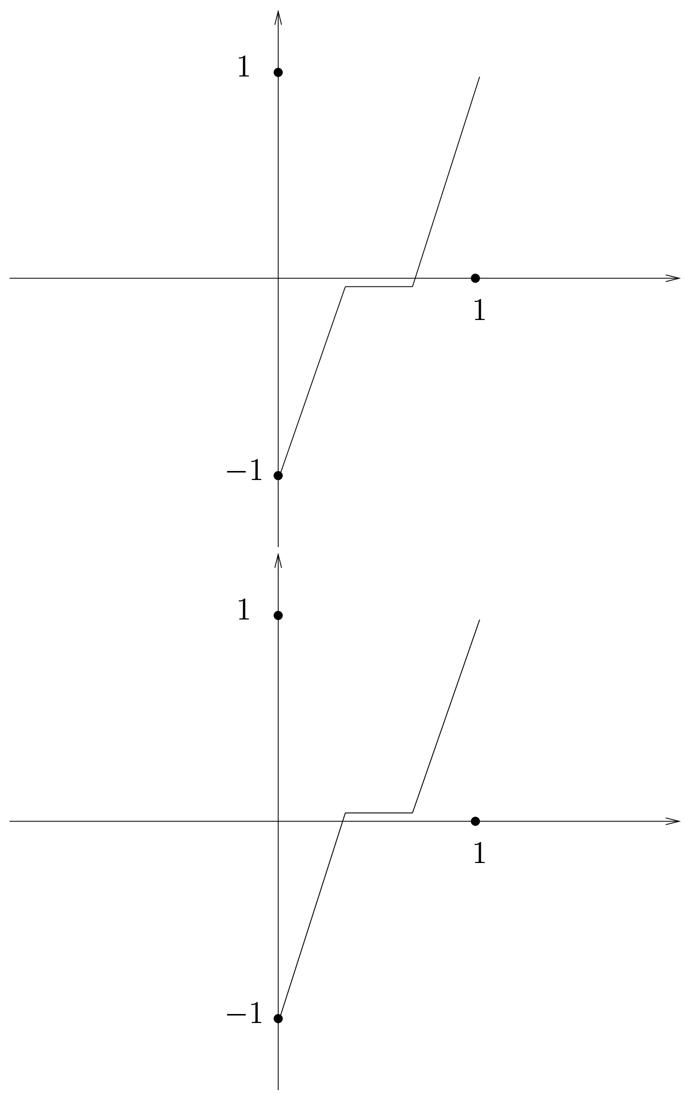

Figure 3: Counter-example to the intermediate value theorem

Now Kleene defines how a natural number realizes an arithmetic statement (in the language of **HA**):

- Any $n \in \mathbb{N}$ realizes an atomic sentence iff the statement is true,
- n realizes $\phi \wedge \psi$ iff $(n)_0$ realizes ϕ and $(n)_1$ realizes ψ,
- n realizes $\phi \vee \psi$ iff $(n)_0 = 0$ and $(n)_1$ realizes ϕ, or $(n)_0 = 1$ and $(n)_1$ realizes ψ,
- n realizes $\phi \to \psi$ iff, for any $m \in \mathbb{N}$ that realizes ϕ, $\{n\}(m)$ has a value that realizes ψ,
- n realizes $\forall x \phi(x)$ iff, for each $m \in \mathbb{N}$, $\{n\}(m)$ has a value that realizes $\phi(\overline{m})$,
- n realizes $\exists x \phi(x)$ iff, $(n)_1$ realizes $\phi(\overline{(n)_0})$.

One cannot say that realizability is a faithful interpretation of intuitionism, as Kleene later realized very well. For example, it turns out that at least from the classical point of view there exist in **IPC** unprovable formulas all of whose arithmetic instances are realizible. But realizability has been an enormously successful concept that has multiplied into countless variants. One important fact Kleene was immediately able to produce by means of realizability is that, if **HA** proves a statement of the form $\forall x \exists y \phi(x, y)$, then ϕ is satisfied by a recursive function $\{e\}$, and even, for each $n \in \mathbb{N}$, **HA** proves $\phi(\overline{n}, \overline{\{e\}(n)})$.

Intuitionistic logic in intuitionistic formal systems. Intuitionistic logic, in the form of propositional logic or predicate logic satisfies the so-called *disjunction property*: if $\phi \vee \psi$ is derivable, then ϕ is derivable or ψ. This is very characteristic for intuitionistic logic: for classical logic $p \vee \neg p$ is an immediate counterexample to this assertion. The property also transfers to the usual formal systems for arithmetic and analysis. Of course, this is in harmony with the intuitionistic philosophy. If $\phi \vee \psi$ is formally provable, then if things are right it is informallly provable as well. But then, according to the BHK-interpretation, ϕ or ψ should be provable informally as well. It would at least be nice if the formal system were complete enough to provide this formal proof, and in the usual case it does. For existential statements something similar holds, an *existence property*, if $\exists x\, \phi(x)$ is derivable in Heyting's arithmetic, then $\phi(\bar{n})$ is derivable for some $\bar{n}$. Statements of the form $\forall y \exists x\, \phi(y, x)$ express the existence of functions, and, for example for Heyting's arithmetic, the existence property then transforms in: if such a statements is derivable, then also some instantiation of it as a recursive function as was stated above already. In classical Peano arithmetic such properties only hold for particularly simple, e.g. quantifier-free, ϕ. In fact, with regard to the latter statements, classical and intuitionistic arithmetic are of the same strength.

Some formal systems may be decidable (e.g. some theories of order) and then one will have classical logic in most cases. However, in Heyting's arithmetic one has de Jongh's *arithmetic completeness theorem* stating that its logic is exactly the intuitionistic one: if a formula is not derivable in intuitionistic logic an arithmetic substitution instance can be found that is not derivable in Heyting's arithmetic (see e.g. [6], [11]). For the particular case of $p \vee \neg p$ this is easy to see, it follows immediately from Gödel's incompleteness theorem and the disjunction property: by Gödel a sentence ϕ exists which **HA** can neither prove nor refute, by the disjunction property **HA** will then not be able to prove $\phi \vee \neg\phi$ either.

Mezhirov's game for IPC. We like to end up with something that has recently been developed: a game that is sound and complete for intuitionistic propositional logic announced in [9]. In this last section we will also give full details of the mathematical proofs.

The games played are *ϕ-games* with ϕ being a formula of the propositional calculus. The game has two players P (*proponent*) and O (*opponent*). The playing field is the set of subformulas of ϕ. A *move* of a player is *marking* a formula that has not been marked before. Only O is allowed to mark atoms. The first move is made by P, and consists of marking ϕ. Players do not move in turn; whose move it is is determined by the *state of the game*. The player who has to move in a state where no move is available *loses*. The state of the game is determined by the markings and by a classical valuation Val that is developed along with the markings. The rules for this valuation are at each stage

- for atoms that $Val(p) = 1$ iff p is marked,
- for complex formulas $\psi \circ \chi$ that, if $\psi \circ \chi$ is unmarked, $Val(\psi \circ \chi) = 0$, and if $\psi \circ \chi$ is marked, $Val(\psi \circ \chi) = Val(\psi) \circ_B Val(\chi)$ where $\circ_B$ is the Boolean function associated with $\circ$.

If a player has marked a formula that gets the valuation 0, then that is considered to be a *fault* by that player. If P has a fault and O doesn't then P moves, in all other cases (i.e. if O has fault and P does or doesn't, or if neither player has a fault) O moves. The completeness theorem can be stated as follows.

Theorem 1. *$\vdash_{\mathbf{IPC}} \phi$ iff P has a winning strategy in the ϕ-game.*

We will first prove

Theorem 2. *If $\nvdash_{\mathbf{IPC}} \phi$, then O has a winning strategy in the ϕ-game.*

Proof. We write the sequences of formulas marked by O and P respectively as **O** and **P**. O keeps in mind a *minimal* counter-model for ϕ, i.e., in the root w_0, ϕ is not satisfied, but in all other nodes of the model ϕ is satisfied. The strategy of O is as follows. As long as P does not choose formulas false in nodes higher up in the model O just picks formulas that are true in w_0. As soon as P does choose a formula ψ that is falsified at a higher up in the model, O keeps in mind the submodel generated by a maximal node w that falsifies ψ. O keeps repeating the same tactic with respect to the node where the game has lead the players. It is sufficient to prove the following:

Claim. If there are no formulas left for O to choose when following this strategy, i.e. all formulas that are true in the w that is fixed in O's mind have been marked, then it is P's move.

This is sufficient because it means that in such a situation P can only move onwards in the model, or, in case w is a maximal node, P loses.

Proof of Claim. We write $|\theta|_w$ for the truth value of θ in w. As we will see it is sufficient to show that, if the situation in the game is as in the assumptions of the claim, then $|\theta|_w = Val(\theta)$ for all θ. We prove, by induction on θ, $|\theta|_w = 1 \iff Val(\theta) = 1$.

- If θ is atomic, then O has marked all the atoms that are forced in w and no other, so those have become true and no other.
- Induction step $\Rightarrow$: Assume $|\theta \circ \psi|_w = 1$. Then $\theta \circ \psi$ is marked, because otherwise O could do so, contrary to assumption. We have $|\theta|_w \circ_B |\psi|_w = 1$. By IH, $Val(\theta) \circ_B Val(\psi) = 1$, so $Val(\theta \circ \psi) = 1$.
- Induction step $\Leftarrow$:
- $Val(\theta \wedge \psi) = 1 \Rightarrow Val(\theta) = 1$ and $Val(\psi) = 1 \Rightarrow_{IH}$ $|\theta|_w = 1$ and $|\psi|_w = 1 \Rightarrow |\theta \wedge \psi|_w = 1$.
- $\vee$ is same as $\wedge$.
- $Val(\theta \to \psi) = 1 \Rightarrow Val(\theta) = 0$ or $Val(\psi) = 1$, and thus by IH, $|\theta|_w = 0$ or $|\psi|_w = 1$. Also, $\theta \to \psi$ is marked, hence in **O** or **P**. In the first case $|\theta \to \psi|_w = 1$ immediate, in the second, $|\theta \to \psi|_s = 1$ for all $s > w$ (P has marked no formulas false higher up, otherwise O would have shifted attention another node) and hence $|\theta|_s = 0$ or $|\psi|_s = 1$ for all $s > w$. Indeed, $|\theta \to \psi|_w = 1$.

We are now faced with the fact that O has only chosen formulas true in the world in O's mind and those stay true higher up in the model. So, $Val(\theta) = 1$ for all $\theta \in \mathbf{O}$. On the other hand, P has at least one fault, the formula ξ chosen by P that landed the game in w in the first place: $Val(\xi) = 0$. Indeed, it is P's move.□

We now turn to the second half:

Theorem 3. *If $\vdash_{\mathbf{IPC}} \phi$, then P has a winning strategy in the ϕ-game.*

Proof. P's strategy is to choose only formulas that are provable from **O**. Note that P's first forced choice of ϕ is in line with this strategy. For this case it is sufficient to prove the following claim.

Claim If all formulas that are provable from **O** are marked, then it is O's move.

This is sufficient because it means that in such a situation O can only mark a completely new formula, and when there are no such formulas left loses.

Proof of Claim. Create a model in the following manner. Assume $\chi_1, \ldots, \chi_k$ are the formulas unprovable from **O** and hence the unmarked ones. By the completeness of **IPC** there are k models making **O** true and falsifying respectively $\chi_1, \ldots, \chi_k$ in their respective roots. Adjoin to these models a new root r verifying exactly the **O**-atoms (this obeys persistency). As in the other direction we will prove: $|\theta|_r = Val(\theta)$ for all θ, or, equivalently, $|\theta|_r = 1 \Longleftrightarrow Val(\theta) = 1$.

- Atoms are forced in r iff marked by O and then have Val 1, otherwise 0.
- Induction step $\Rightarrow$: Assume $|\theta \circ \psi|_r = 1$. Then $\theta \circ \psi$ is marked, because if it wasn't it would be one of the χ_i, falsifying persistency. We can reason on as in the other direction.
- Induction step $\Leftarrow$:
- $\vee$ and $\wedge$ as in the other direction.
- $Val(\theta \to \psi) = 1 \Rightarrow Val(\theta) = 0$ or $Val(\psi) = 1$, and thus by IH, $|\theta|_r = 0$ or $|\psi|_r = 1$. Also, $\theta \to \psi$ is marked, hence in **O** or **P**, and so $|\theta \to \psi|_s = 1$ and hence $|\theta|_s = 0$ or $|\psi|_s = 1$ for all $s > r$. Indeed, $|\theta \to \psi|_r = 1$.

We are now faced with the fact that P has only marked formulas provable from **O** and those will remain provable from **O**. So, $Val(\theta) = 1$ for all $\theta \in \mathbf{P}$. So, P has no fault. By the rules of the game it is O's move. □

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