Constructive and nonconstructive proofs

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1 Example of constructive and nonconstructive proofs

Since this the only subject all of you know I have chosen a very simple proof from modal logic

Theorem 1.1. *$\Box p \to p$ characterizes the reflexive frames*

- We want to prove that $\Box p \to p$ characterizes the reflexive frames, i.e. If $\mathfrak{F} = (W, R)$ is a reflexive frame, then $\mathfrak{F} \vDash \Box p \to p$, and, if $\mathfrak{F} \vDash \Box p \to p$, then $\mathfrak{F}$ is a reflexive frame. The first part is easy, we just do the second.

- First proof (nonconstructive).

- Assume that $\mathfrak{F}$ is not reflexive. Then there exists a node $w \in W$ such that not $w\,R\,w$. Now define $V(p) = \{u \in W \,|\, w\,R\,u\}$. Then $w \vDash \Box p$, but $w \nvDash p$ because not $w\,R\,w$. So, $\mathfrak{F} \nvDash \Box p \to p$.

- This is not a constructive proof. Starting with "Assume that $\mathfrak{F}$ is not reflexive" is already a wrong move, because, even if we would be able to derive $\mathfrak{F} \nvDash \Box p \to p$, we still would not have proved the theorem, since contraposition is not acceptable in this form. But, also the conclusion "Then there exists a node $w \in W$ such that not $w\,R\,w$." is not warranted. How would we find w? W may be infinite, R may be undecidable.

- Second proof (constructive).

- Assume that $\mathfrak{F} \vDash \Box p \to p$. Take an arbitrary $w \in W$. Define $V(p) = \{u \in W \,|\, w\,R\,u\}$. Clearly, $w \Vdash \Box p$. Since, $\mathfrak{F} \vDash \Box p \to p$, $w \vDash p$ follows. This means that $w \in V(p)$. By the definition of $V(p)$ this implies that $w\,R\,w$. Since $w \in W$ was arbitrary, $\mathfrak{F}$ is a reflexive.

- This is a proper constructive proof.

- One has to take care. One cannot, for example, require V to be a function with $V(p, u) = 1$ or $V(p, u) = 0$ for each u: '$V(p, u) = 1$ iff $w\,R\,u$' is not a proper definition of a function if R is not decidable. A proper definition of a function tells you how to evaluate it for its arguments. Decidability of R can certainly not be assumed. In the canonical model for example, R will not be decidable. Note, by the way, that decidability is expressible in the language: $\forall x, y(x\,R\,y \vee \neg x\,R\,y)$. This is very different from the situation in classical systems.

Non-constructive proof revisited:

- The nonconstructive proof does have a constructive side.
- It is proved that:
- If a frame has a non-reflexive node, then there exists a valuation that makes $\Box p \to p$ false at that node.
- What has to be proved has the form $\forall x\, Ax \to \forall y\, By$.
- To prove $\neg\forall y\, By \to \neg\forall x\, Ax$ is not sufficient, it is weaker than what one has to prove.
- To prove $\exists x \neg\, By \to \exists x \neg\, Ax$, though stronger than the previous, is still insufficient, but it is qua strength incomparable to the statement to be proved.
- Statements in classical systems may have many non-equivalent forms in constructive systems.

2 The BHK-interpretation

- Brouwer-Heyting-Kolmogorov Interpretation of connectives and quantifiers. Comparable to Tarski's truth definition for classical logic.. Natural deduction closely related to BHK.
- Interpretation by means of proofs (nonformal, nonsyntactical objects, mind constructions),
- A proof of $\varphi \wedge \psi$ consists of proof of φ plus proof of ψ (plus conclusion),
- A proof of $\varphi \vee \psi$ consists of proof of φ or of proof of ψ (plus conclusion),
- A proof of $\varphi \to \psi$ consists of method that applied to any conceivable proof of φ will deliver proof of ψ,
- Nothing is a proof of $\perp$,
- Proof of $\neg\varphi$ is method that given any proof of φ gives proof of $\perp$,
- A proof of $\exists x\, \varphi(x)$ consists of object d from domain plus proof of $\varphi(d)$ (plus conclusion),
- A proof of $\forall x\, \varphi(x)$ consists of method that applied to any element d of domain will deliver proof of $\varphi(d)$,

Is adherence to the BHK interpretation the only thing that matters? Certainly not, according to both Brouwer and Heyting. If one would say yes, one would get pretty close to formalism. Any set of axioms formulated in a first order language would do. For all practical purposes the only difference then with classical logical systems would be that the reasoning would proceed without the excluded middle ($\varphi \vee \neg\varphi$).

Proofs are only one type of construction. The objects one is talking about are constructions as well. If one has axioms these should somehow be clear from the kind of thing the objects are. According to Heyting the axioms are proved to hold. Such proofs can of course not be based on the content of the BHK-interpretation alone.

In Introduction to Intuitionism Heyting actually proves for example the axioms for **HA** including the induction axiom for natural numbers:

The theorem of complete induction admits a proof of the same kind. Suppose $E(x)$ is a predicate of natural numbers such that, for every natural number n, $E(n)$ implies $E(n')$, where n' is the successor of n. Let p be any natural number. Running over $1 \to p$ we know that the property E, which is true for 1, will be preserved at every step in the construction of p; therefore $E(p)$ holds.

I really see this as the only coherent viewpoint. If one wants to maintain that truth is provability (in a non-formal sense, of course), then one should prove everything one considers to be true.

With Brouwer this is not so easy to determine. Brouwer stayed inside mathematics, and considered these matters internally. For him the induction axiom was so obvious that no discussion was warranted. He does say that these statements are experienced as true.

To clarify this in a related matter. Brouwer would never say that it is possible to create systems that contradict classical mathematics. He simply said that he proves classical mathematics to be contradictory. He "refutes" the principle of the excluded third.

In any case, the axioms are defended in a way very reminiscent of the defense of the axioms of set theory by Platonistic mathematicians/philosophers. Also there one experiences the truth of certain "axioms" and provides a coherent defense of the truth of the whole collection of axioms.