Glivenko and the negative translation for **IPC**

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Theorem 1. (Glivenko, Theorem 5 of course notes)
$\vdash_{\mathbf{CPC}} \varphi$ iff $\vdash_{\mathbf{IPC}} \neg\neg\varphi$
Proof. The direction from right to left is trivial, so we only give the proof from left to right. First a syntactic proof using the Hilbert type system.

For a proof one has to show $\neg\neg\varphi$ for all **CPC**-axioms φ, and prove that modus ponens preserves this property: If $\vdash_{\mathbf{IPC}} \neg\neg\varphi$ and $\vdash_{\mathbf{IPC}} \neg\neg(\varphi \to \psi)$, then $\vdash_{\mathbf{IPC}} \neg\neg\psi$.

Of course $\neg\neg\varphi$ for **IPC**-axioms follows from the **IPC**-theorem $\varphi \to \neg\neg\varphi$. That leaves of the axioms only $\neg\neg\varphi \to \varphi$. And $\neg\neg(\neg\neg\varphi \to \varphi)$ follows from useful equivalences 12 and 7. That modus ponens preserves the property follows from useful equivalence 12 (ue12).

This finishes the proof, but we will also give a semantic proof. That proof assumes the completeness of **IPC** with regard to finite models. Assume $\vdash_{\mathbf{CPC}} \varphi$. Then, for any endpoint w of any finite model $w \models \varphi$, because in an endpoint the valuations behave as in classical logic. So, for any v in any finite model, for all endpoints w with vRw, $w \models \varphi$; so $v \models \neg\neg\varphi$. Therefore, $\vdash_{\mathbf{IPC}} \neg\neg\varphi$. $\dashv$

Let φ^n be defined for **IPC** as in the course notes.

Lemma 2. *For each φ, $\varphi^n \sim \neg\neg\varphi^n \sim \neg\neg\varphi$.*
Proof. By induction on the length of φ. We give one example, the step for $\to$. Assume the statement holds for φ and ψ. Then $\varphi^n \to \psi^n \sim \neg\neg\varphi^n \to \neg\neg\psi^n \sim \neg\neg(\varphi^n \to \psi^n)$ (by IH1, ue12) and also $\neg\neg\varphi^n \to \neg\neg\psi^n \sim \neg\neg\varphi \to \neg\neg\psi \sim \neg\neg(\varphi \to \psi)$ (by IH2, ue12). $\dashv$

The second part of this lemma fails for **IQC**. And therefore the next proof as well.

Theorem 3. (Theorem 30 of course notes) $\vdash_{\mathbf{CPC}} \varphi$ iff $\vdash_{\mathbf{IPC}} \varphi^n$
Proof. Immediate from Theorem 1 and Lemma 2. $\dashv$

Theorem 4. *(Extended version of Glivenko)*

$$\psi_1, \ldots, \psi_n \vdash_{\mathbf{CPC}} \varphi \quad \textit{iff} \quad \neg\neg\psi_1, \ldots, \neg\neg\psi_n \vdash_{\mathbf{IPC}} \neg\neg\varphi$$

The proof will be for the exercises. It is worthwhile to note though that if one wants to prove the Glivenko Theorem through natural deduction one has no choice but to prove the extended version.