Intuitionisitic Logic

Spring 2012

Homework 3

(due Monday, 12th March)

1. Prove that the canonical model $\mathfrak{M}_{\mathbf{KC}_{prop}}$ of $\mathbf{KC}_{prop} := \mathbf{IPC} + \neg p \vee \neg\neg p$ for propositional variables p only, has a largest element above each world, and simultaneously that $\mathbf{KC}_{prop}$ is equal to $\mathbf{KC}$, in the following manner:

   (a) Take a node Γ of the canonical model of $\mathbf{K}C_{prop}$. Show that Γ contains either $\neg p_i$ or $\neg\neg p_i$ for each i. [1 pt]

   (b) Assume that Γ contains $\neg p_i$ for $i \in A$, $\neg\neg p_j$ for $j \in B$, where $A \cup B = \mathbb{N}$. Show that then $\Gamma \cup \{\neg p_i \mid i \in A\} \cup \{p_j \mid j \in B\}$ is consistent. [1 pt]

   (c) Show that there is a unique Δ in $\mathfrak{M}_{\mathbf{KC}_{prop}}$ containing $\{\neg p_i \mid i \in A\} \cup \{p_j \mid j \in B\}$ (you may use the fact that $\mathbf{CPC}_{prop} := \mathbf{IPC} + p \vee \neg p$ for propositional variables p only, is equal to $\mathbf{CPC}$), and that this Δ is the largest element above Γ. [2 pts]

   (d) Conclude that any such Γ satisfies all of $\mathbf{KC}$ and therefore $\mathbf{KC}_{prop}$ is equal to $\mathbf{KC}$. [2 pts]

2. Let $\mathfrak{F}$ be a frame with a non-empty domain attached to each node.

   (a) Assume the domain is the same for each node (constant domains). Show that $\forall x(\varphi \vee \psi(x)) \to \varphi \vee \forall x\psi(x)$ (x not free in φ) is valid on this frame. [2 pts]

   (b) Assume wRu with $D_w \subset D_u$ (the domain of w is strictly contained in the domain of u). Give a valuation to p and an interpretation of A so that, in the resulting model on $\mathfrak{F}$, $\forall x(p \vee A(x)) \to p \vee \forall x A(x)$ is not valid. [2 pts]

3. Let $\mathfrak{F}$ be a frame.

   (a) Assume for each $w \in W$ there exists an endpoint u with wRu. Show that $\forall x\neg\neg\varphi(x) \to \neg\neg\forall x\varphi(x)$ is valid on this frame. [2 pts]

   (b)* Assume there exists a sequence $w_1R^+w_2R^+ \ldots$ (xR^+y standing for xRy and $x \neq y$) in W such that for each $w \in W$ there is an n such that wRw_n. Give domains for all worlds in the model and an interpretation for A such that, in the resulting model on $\mathfrak{F}$, $\forall x\neg\neg A(x) \to \neg\neg\forall x A(x)$ is not valid. [2 pts]

4. Prove that $\vdash_{\mathbf{IPC}} \varphi$ implies $\vdash_{\mathbf{S4}} \varphi^{\Box}$ in the following manner:

   Assume $\mathfrak{M}$ on $\mathfrak{F}$ is an **S4**-countermodel to $\varphi^{\Box}$. Take the frame $\mathfrak{G}$ that is obtained from $\mathfrak{F}$ by replacing each cluster (collection of nodes that are pairwise accessible from each other) by a single node (try to define this exactly.) There is an obvious function from $\mathfrak{F}$ onto $\mathfrak{G}$. Show that it is a **frame**-p-morphism. Define a valuation on $\mathfrak{G}$ in such a way that the resulting model $\mathfrak{N}$ is an **IPC**-model. Show, by induction on the length of $\psi(p_1, \ldots, p_n)$ that, for each $w \in W$, $\mathfrak{M}, w \models \psi^{\Box}$ (as an **S4**-model) iff $\mathfrak{N}, f(w) \models \psi$ (as an **IPC**-model). Finally conclude that $\mathfrak{N}$ (as an **IPC**-model) falsifies φ. [7 pts]