Intuitionistic Logic
Spring 2012

Homework 6
(due Monday, 23rd April)

1. Prove in **HA**:
   (a) $x \cdot (y + z) = x \cdot y + x \cdot z$ [2 pts]
   (b) $\forall x \forall y (x = y \vee x \neq y)$ [3 pts]

   You may use that Robinson's axiom $x = 0 \vee \exists y(x = y + 1)$ is provable in **HA**, shown in class.

2. (a) Show that **HA** has the existence property: if $\mathbf{HA} \vdash \exists x \varphi(x)$, then $\mathbf{HA} \vdash \varphi(\overline{n})$ for some n. [2 pts]
   (b) Show that, if $\vdash_{\mathbf{HA}} \neg\varphi \to \exists x \psi(x)$, then $\vdash_{\mathbf{HA}} \neg\varphi \to \psi(\bar{n})$ for some n (here φ has no free variables.) [1 pt]
   (c) Add a predicate $A(x)$ to the language of **HA** with the axiom $A(0) \wedge \forall x(A(x) \to A(x+1))$, but do **not** add induction for formulas containing A. Show the disjunction property for this system. [2 pts]
   (d) Add a predicate $B(x)$ to the language of **HA** with the axiom $\exists x B(x)$. Does this system have the disjunction property? [2 pts]

3. Spell out the clause for $x \,\underline{\mathtt{rn}}\, (A \vee B)$ on the basis of the definition

$$A \vee B := \exists x(x = 0 \to A) \wedge (x \neq 0 \to B)$$

   and show the equivalence with the notion of realizability for formulas with disjunction as a primitive and extra realizability clause

$$x \,\underline{\mathtt{rn}}\, (A \vee B) := (\mathbf{p}_0 x = 0 \wedge \mathbf{p}_1 x \,\underline{\mathtt{rn}}\, A) \vee (\mathbf{p}_0 x \neq 0 \wedge \mathbf{p}_1 x \,\underline{\mathtt{rn}}\, B)$$

   Show how to construct ϕ_A, ψ_A such that

$$\vdash x \,\underline{\mathtt{rn}}\, A \to \phi_A(x) \,\underline{\mathtt{rn}}'A$$

$$\vdash x \,\underline{\mathtt{rn}}'A \to \psi_A(x) \,\underline{\mathtt{rn}}\, A$$

   where $\underline{\mathtt{rn}}'$ is realizability with disjunction treated as a primitive. [3pts]

4. Complete the proof of the soundness theorem for $\underline{\mathtt{rn}}$ and $\underline{\mathtt{rnt}}$. N.B. In the system of axioms and rules for E-logic in the handbookarticle, the axiom Ex for free variables has been inadvertently omitted. [3pts]