Intuitionistic Logic

Spring 2012

Homework 7

(due Monday, 7 May)

For the following exercises you can use, if $\alpha(x)$ expresses a primitive recursive predicate, then $\vdash_{\text{HA}} \forall x(\alpha(x) \vee \neg\alpha(x))$. You can use the fact that for each n, $\vdash_{\mathbf{HA}} \alpha(\bar{n})$ or $\vdash_{\mathbf{HA}} \neg\alpha(\bar{n})$ holds. You can also assume that **HA** proves only true formulas, and you can use that it is not decidable whether a primitive recursive predicate is empty or not.

1. Show that, if $\alpha(x)$ expresses a primitive recursive predicate and $\vdash_{\text{HA}} \neg\neg\exists x\alpha(x)$, then $\vdash_{\text{HA}} \exists x\alpha(x)$. Hint: This is easy. You are allowed to reason very nonconstructively metamathematically. [2 pts]
2. Consider the sentence $\beta = \exists x((x = 0 \wedge \gamma) \vee ((x = 1 \wedge \neg\gamma))$ where γ is a gödelsentence of **HA**. Show that $\nvdash_{\text{HA}} \beta$ but $\vdash_{\text{HA}} \neg\neg\beta$. [2 pts]
3. Show that, if $\alpha(x)$ expresses a primitive recursive predicate, then $\vdash_{\text{HA}} \neg\neg\forall x\alpha(x) \to \forall x\alpha(x)$. [2 pts]
4. Let $\text{Prf}(x, y)$ be the primitive recursive proof predicate of **HA**, γ the gödelsentence of **HA** and $\ulcorner\gamma\urcorner$ the numeral corresponding to the gödelnumber of γ. Show that $\vdash_{\text{HA}} \forall x(\neg\text{Prf}(\ulcorner\gamma\urcorner, x) \vee \exists y\,\text{Prf}(\ulcorner\gamma\urcorner, y))$ but $\nvdash_{\text{HA}} \forall x\neg\text{Prf}(\ulcorner\gamma\urcorner, x) \vee \exists y\,\text{Prf}(\ulcorner\gamma\urcorner, y)$. [2 pts]
5. Use an exercise in the previous homework to prove that not for all $\alpha(x)$ that express primitive recursive formulas, $\vdash_{\mathbf{HA}} \neg\neg\exists x\alpha(x) \to \exists x\alpha(x)$ (Markov's principle). [3 pts]