1. Find the largest bisimulation on the following model, and also give the corresponding partition.

Answer: the largest bisimulation is the bisimulation \( Z \) given by the following list of pairs:
\[
\{(0, 0), (1, 1), (1, 2), (1, 5), (2, 1), (2, 2), (2, 5), (5, 1), (5, 2), (5, 5), \\
(3, 3), (3, 4), (3, 6), (4, 3), (4, 4), (4, 6), (6, 3), (6, 4), (6, 6)\}
\]

The partition that corresponds with this equivalence relation:
\[
\{\{0\}, \{1, 2, 5\}, \{3, 4, 6\}\}.
\]

2. Give the bisimulation minimal version of the model from the previous exercise, and show the stages in which this bisimulation minimal model gets computed by the partition refinement algorithm.

Answer: the bisimulation minimal model looks like this:
This is computed by the partition refinement algorithm in two steps. Initial blocks are \{0, 1, 2, 5\} and \{3, 4, 6\}, for these are the blocks of worlds with the same valuation. Next, the first block gets split, for 1, 2, 5 have a red arrow to the other block, while 0 has not. The other block does not get split in this round. This gives partition \{\{0\}, \{1, 2, 5\}, \{3, 4, 6\}\}. In the next round no further splitting takes place, so this partition is a fixpoint.

3. Find the largest bisimulation on the following model, and also give the corresponding partition:

Answer: The largest bisimulation is:

\{(0, 0), (0, 1), (0, 2), (0, 5), (1, 0), (1, 1), (1, 2), (1, 5), (2, 0), (2, 1), (2, 2), (2, 5),\}
The corresponding partition:

\[
\{\{0, 1, 2, 5\}, \{3, 4, 6, 7\}\}.
\]

4. Give the bisimulation minimal version of the model from the previous exercise, and show the stages in which this bisimulation minimal model gets computed by the partition refinement algorithm.

Answer: The bisimulation minimal model looks like this:

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0

\(\rightarrow\)

1
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This is found by the partition refinement algorithm in two steps.

First, all the worlds are put together in a single block (for they all have the same valuation). Next, the block is split because worlds 0, 1, 2, and 5 have a black arrow pointing to the block, but worlds 3, 4, 6 and 7 have not. In the next round, no further splitting takes place, so the partition \(\{\{0, 1, 2, 5\}, \{3, 4, 6, 7\}\}\) is a fixpoint.

5. The generated submodel of a Kripke model \(M, s\) (where \(s\) is a designated state, called the root) is the model \(M', s\) that results from restricting the state set of \(M\) to

\[
\{t \mid (s, t) \in \left(\bigcup_{b \in B} R_b\right)^*\},
\]

where \(\{R_b \mid b \in B\}\) are the accessibility relations of \(M\).

Show that \(M, s \leftrightarrow M', s\).

(Recall from the lectures that \(\leftrightarrow\) denotes modal equivalence.)

Answer: We will show that the relation \(Z\) that links those \(s\) in \(M\) that belong to the set of worlds of \(M'\) to the corresponding \(s\) in \(M'\) is a bisimulation. The result then follows from Theorem 1 in the Lecture Notes on Bisimulation.

The Invariance property follows at once from the fact that the states linked by \(Z\) are the same.
For the Zig property, assume \( Z : M, w \leftrightarrow M', w \) and assume \( w \overset{a}{\rightarrow} w' \) in \( M \). Then \((s, w) \in (\bigcup_{b \in B} R_b)\), so \((s, w') \in (\bigcup_{b \in B} R_b)^* \). Thus, \( s' \) is in the state set of \( M' \), and \( w \overset{a}{\rightarrow} w' \) in \( M' \), and \( Z \) links \( w' \) in \( M \) to \( w' \) in \( M' \).

The Zag property follows immediately from the way \( M' \) is defined.

6. Prove that if \( Z \) and \( Z' \) are bisimulations on a model \( M \), then \( Z \cup Z' \) is also a bisimulation on \( M \).

Answer: We show that \( Z \cup Z' \) satisfies the invariance, zig and zag conditions.

Assume \( s(Z \cup Z')t \).

Invariance: Either \( sZt \) or \( sZ't \). In both cases \( s \) and \( t \) have the same valuation, either by the invariance condition of \( Z \) or by that of \( Z' \).

Zig: suppose \( s \overset{a}{\rightarrow} s' \). Case 1. \( sZt \). Then by the zig condition of \( Z \) there is a \( t' \) with \( t \overset{a}{\rightarrow} t' \) and \( sZ't \). But then also \( s'(Z \cup Z')t' \), and the zig condition for \( Z \cup Z' \) is satisfied. Case 2. \( sZ't \). Then by the zig condition of \( Z' \) there is a \( t' \) with \( t \overset{a}{\rightarrow} t' \) and \( s'Z't' \). But then also \( s'(Z \cup Z')t' \), and the zig condition for \( Z \cup Z' \) is satisfied.

Zag: similar reasoning.

7. Prove that every bisimulation \( Z \) on a model is contained in a largest bisimulation.

Answer. Let \( \{Z_k \mid k \in K\} \) be the set of all bisimulations on a model \( M \). Then it follows from the previous exercise that \( Z = \bigcup_{k \in K} Z_k \) is a bisimulation on \( M \). By definition, \( Z \) is the largest bisimulation on \( M \).

8. Does it hold that if \( Z \) and \( Z' \) are bisimulations on a model \( M \), then \( Z \cap Z' \) is a bisimulation on \( M \)? Give a proof or a counterexample.

Answer: no, this does not hold. Consider the following model.

The dotted lines picture the relation \( Z \), the dashed lines the relation \( Z' \) (reflexive arrows are not shown). Surely, these are bisimulations. But the only pairs in the intersection of \( Z \) and \( Z' \) are \((0, 0)(0, 1), (1, 1)\), and \( \{(0, 0)(0, 1), (1, 1)\} \) is not a bisimulation for it does not satisfy the zig and zag requirements.