Update, Probability, Knowledge and Belief

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Abstract

The paper compares two kinds of models for logics of knowledge and belief, neighbourhood models and epistemic weight models. We give sound and complete calculi for both, and we show that our calculus for neighbourhood models is sound but not complete for epistemic weight models. Epistemic weight models combine knowledge and probability by using epistemic accessibility relations and weights to define subjective probabilities. Our Probability Comparison Calculus for this class of models is a further simplification of the calculus that was presented in AIML 2014.

Keywords: Probability, epistemic modal logic, dynamic epistemic logic, probabilistic update, Bayesian learning.

1 Probability and Information

A Bayesian learner is an agent who uses new information to update a subjective probability distribution that somehow captures what she knows or believes about the world. In a multi-agent setting, various learners could receive differents pieces of information, and if multi-agent logics of knowledge of belief are extended with update procedures that implement the processing of new information, then this also gives a perspective on learning in a multi-agent setting. So it is natural to combine probability theory and the update perspective on knowledge and belief from dynamic modal logic.

Indeed, there exists already a modest tradition in combining DEL (Dynamic Epistemic Logic) and probability theory. The first combinations are in Kooi's thesis [23], in Van Benthem's [3], and in the combined effort of Van Benthem,

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Gerbrandy and Kooi in [5]; [10] gives an overview. Inspiration for this goes back to work of Fagin and Halpern in the 1990s [13]. A simplified combined system of DEL and probability theory is presented in [12]. Further simplication of the base logic is provided in [9]. The present paper extends work that was presented at AIML 2014 in [12].

A natural notion of belief that turns up in a setting of probabilistic dynamic epistemic logic is betting belief (or: Bayesian belief) in ϕ: $P(\phi) > P(\neg\phi)$. Van Eijck & Renne [11] give a logical calculus for this, combined with an S5 operator for knowledge, that we will review and extend below. This is in fact an extension of a calculus proposed by Burgess in [7]. A variation on this is threshold belief in ϕ: $P(\phi) > t$, for some specific t with $\frac{1}{2} \leq t < 1$. This is also known as Lockean belief. John Locke suggests in his work that a person's belief in a proposition ϕ is somehow connected to that person's confidence in ϕ. This confidence should then be connected in turn to the evidence that the person has for ϕ. If it makes sense to talk about degree of belief at all, then subjective probability is one way of making this precise [16]. A logic with KD45 belief and an explicit belief comparison operator is presented in [21]. See [28] for an overview of the extensive literature on belief comparison operators.

In this paper we will use a neighbourhood semantics ([8, Ch. 8] and [19]) for 'belief as willingness to bet'. Related to neighbourhood models for belief are the evidence models proposed in [4].

Attempts to develop a qualitative notion of probability date back to De Finetti [14,15]. He proposed the following requirements for a binary relation $\succeq$ on a (finite and non-empty) set W:

$$\begin{aligned}
\text{nonnegativity } & A \succeq \emptyset \\
\text{nontriviality } & \emptyset \not\succeq W \\
\text{totality } & A \succeq B \text{ or } B \succeq A \\
\text{transitivity } & \text{if } A \succeq B \text{ and } B \succeq C \text{ then } A \succeq C \\
\text{quasi-additivity } & \text{if } (A \cup B) \cap C = \emptyset \text{ then } A \succeq B \text{ iff } A \cup C \succeq B \cup C
\end{aligned}$$

A probability measure on W is a function $\mu : \mathcal{P}(W) \to \mathbb{R}$ satisfying $\mu(\emptyset) = 0$, $\mu(W) = 1$ and $\mu(A \cup B) = \mu(A) + \mu(B)$ for $A, B \subseteq W$ with $A \cap B = \emptyset$ (additivity). The conjecture that the five requirements completely determine a probability measure on W was refuted in [24].

Theorem 1.1 *There is a relation satisfying De Finetti's axioms that does not agree with any probability measure [24].*

Proof. Consider $W = \{p, q, r, s, t\}$ with a weight map $\nu : W \to \mathbb{N}$ given by $\nu(p) = 4, \nu(q) = 1, \nu(r) = 3, \nu(s) = 2, \nu(t) = 6$. Extend ν to subsets of W by putting $\nu(A) = \sum_{a \in A} \nu(a)$. Let the relation $\succeq_\nu$ on W be given by $A \succeq_\nu B$ iff $\nu(A) \geq \nu(B)$. Next, writing pq for $\{p, q\}$, define $\succeq$ as

$$\succeq \;:=\; \succeq_\nu - \{(st, pqr)\}.$$

This yields (writing $A \approx B$ for $A \succeq B \wedge B \succeq A$ and $A \succ B$ for $A \succeq B \wedge B \not\succeq A$): $p \approx qr, rs \approx pq, qt \approx pr, pqr \succ st$. Check that this relation satisfies the De

Fenetti axioms: Transitivity still holds, because pqr and st are the only two sets that have ν-weights adding up to 8, so there can be no set A different from pqr and st with $st \succeq A$ and $A \succeq pqr$. Quasi-additivity still holds since the only C with $C \cap pqrst = \emptyset$ is $C = \emptyset$. The other three properties are obvious. But the relation does not agree with any probability measure μ. For it follows from $\mu(p) = \mu(qr)$, $\mu(rs) = \mu(pq)$, $\mu(qt) = \mu(pr)$ that $\mu(st) = \mu(pqr)$. Thus, μ cannot agree with $\{p, q, r\} \succ \{s, t\}$. □

Scott [29] gave an algebraic reformulation of the new axioms proposed in [24]. Formulated in terms of subsets of a universe W, a pair of k-length sequences of sets $(A_1, \ldots, A_k)$ and $(B_1, \ldots, B_k)$ is *balanced* if for each $w \in W$ it holds that $|\{i \mid w \in A_i\}| = |\{i \mid w \in B_i\}|$. The Scott axiom for $\succeq$ for length k says:

k-cancellation if $(A_1, \ldots, A_k, X)$ and $(B_1, \ldots, B_k, Y)$ are balanced, and $A_i \succeq B_i$ for each i with $1 \leq i \leq k$, then $Y \succeq X$.

It is not hard to see that if a relation $\succeq$ is representable by a probability measure, then $\succeq$ must satisfy cancellation for any k. Scott showed that any $\succeq$ relation satisfying nonnegativity, nontriviality, totality and cancellation for any $k \in \mathbb{N}$ determines a probability measure, thus replacing the set of conditions proposed by De Finetti by a complete set.

Notice that the example in the proof of Theorem 1.1 does not satisfy 3-cancellation, for the pair (p, rs, qt, pqr) and (qr, pq, pr, st) is balanced and satisfies $p \succeq qr$, $rs \succeq pq$, $qt \succeq pr$, but $st \succeq pqr$ does not hold.

Segerberg [30] showed how the balancedness requirements could be translated into modal logic (be it at formidable coding cost), and Segerberg [30] and Gärdenfors [17] proposed axiomatisations for a logic with $\succeq$, using an infinite number of modal schemes (one for each sequence length k) to cover the balancedness conditions identified by Scott. This approach was later adopted in Lenzen [26] for a logic of conviction (German: *Überzeugung*) and belief where an agent's conviction of ϕ is identified with assigning probability 1 to ϕ, and belief in ϕ with assignming a probability greater than $\frac{1}{2}$. Later, Herzig also incorporated action, in [20]. A difference between Lenzen's approach and ours are that Lenzen's conviction does not imply truth. Another difference is that we will not use the Scott axioms. The main difference between the approach of Herzig and ours is that he relates betting belief to a KD45 modality, while we relate betting belief to S5 knowledge.

2 Epistemic Neighbourhood Models

We now introduce epistemic neighbourhood models, where belief is represented as truth in a neighbourhood.

Definition 2.1 [Epistemic Neighbourhood Models] An **Epistemic Neighbourhood Model** $\mathcal{M}$ is a tuple $(W, \sim, N, V)$ where

- W is a non-empty set of worlds.

- $\sim$ is a function that assigns to every agent $i \in Ag$ an equivalence relation $\sim_i$ on W. We use $[w]_i$ for the $\sim_i$ class of w, i.e., for the set $\{v \in W \mid w \sim_i v\}$.
- N is a function that assigns to every agent $i \in Ag$ and world $w \in W$ a collection $N_i(w)$ of sets of worlds—each such set called a *neighbourhood* of w—subject to the following conditions.
  (c) $\forall X \in N_i(w) : X \subseteq [w]_i$.
  (n) $[w]_i \in N_i(w)$.
  (a) $\forall v \in [w]_i : N_i(v) = N_i(w)$.
  (m) $\forall X \subseteq Y \subseteq [w]_i :$ if $X \in N_i(w)$, then $Y \in N_i(w)$.
  (d) $\forall X \in N_i(w), [w]_i - X \notin N_i(w)$.
  (sc) $\forall X, Y \subseteq [w]_i$: if $[w]_i - X \notin N_i(w)$ and $X \subsetneq Y$, then $Y \in N_i(w)$.
- V is a valuation function that assigns to every $w \in W$ a subset of *Prop*.

Epistemic neighbourhood models are a variation on the well-known neighbourhood models from modal logic. The difference is that they include an epistemic component $\sim_i$ for each agent i. Since $[w]_i$ is the set of worlds agent i knows to be possible at w, each $X \in N_i(w)$ represents a proposition that the agent believes at w. Notice that it follows from these semantics that knowledge implies belief.

Property (c) This ensures that what is known is also believed.

Property (n) This ensures that what is logically true is believed.

Property (a) If X is believed, then it is known that X is believed.

Property (m) Belief is monotonic: if an agent believes X, then she believes all propositions $Y \supseteq X$ that logically follow from X.

Property (d) If i believes a proposition then i does not also believe the complement of that proposition.

Property (sc) This is a form of "strong commitment": if the agent does not believe the complement $\overline{X}$, then she must believe any strictly weaker Y implied by X.

If follows from (n) and (d) that $\emptyset \notin N_i(w)$. Indeed, by (n) $[w]_i \in N_i(w)$, hence by (d), $\emptyset = [w]_i - [w]_i \notin N_i(w)$. It follows from (d) and (m) that the intersection of two neighbourhoods of w is non-empty. Indeed, suppose $X, Y \in N_i(w)$. Then $[w]_i - X \notin N_i(w)$, by (d). Therefore, $Y \not\subseteq [w]_i - X$, by (m). Thus, $X \cap Y \neq \emptyset$. Incidentally, (m) follows from (d) and (sc). For let $X \in N_i(w)$. Then $[w]_i - X \notin N_i(w)$ by (d). Let $[w]_i \supseteq Y \supseteq X$. If $Y = X$ then $Y \in N_i(w)$ by what is given about X. If $Y \supsetneq X$ then $Y \in N_i(w)$ by (sc).

Definition 2.2 [ED Language] Let p range over a set of basic propositions P and i over a finite set of agents A.

$$\phi ::= \top \mid p \mid \neg\phi \mid (\phi \wedge \phi) \mid K_i\phi \mid B_i\phi.$$

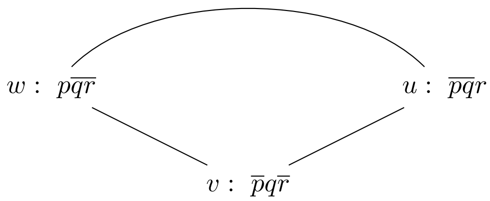
We will employ the usual abbreviations for $\perp, \vee, \rightarrow$ and $\leftrightarrow$, and we use $\check{K}_i\phi$ for $\neg K_i \neg\phi$ and $\check{B}_i\phi$ for $\neg B_i \neg\phi$. This makes $\check{K}_i, \check{B}_i$ behave as duals (diamonds) to the boxes K_i, B_i.

Definition 2.3 [Truth in Neighbourhood Models] Key clauses are

$$\begin{aligned}
\mathcal{M}, w \models K_i\phi \ \text{ iff } & \text{ for all } v \in [w]_i : \mathcal{M}, v \models \phi. \\
\mathcal{M}, w \models B_i\phi \ \text{ iff } & \text{ for some } X \in N_i(w) \\
& \quad \text{it holds that } X = \{v \in [w]_i \mid \mathcal{M}, v \models \phi\}.
\end{aligned}$$

Our first example illustrates that neighbourhood belief is not closed under conjunction.

Example 2.4



$$N(w) = N(v) = N(u) = \{\{w, v\}, \{v, u\}, \{w, u\}, \{w, v, u\}\}$$

- In all worlds, $K(p \vee q \vee r)$ is true.
- In all worlds $B\neg p$, $B\neg q$, $B\neg r$ are true.
- In all worlds $B(\neg p \wedge \neg q)$, $B(\neg p \wedge \neg r)$, $B(\neg q \wedge \neg r)$ are false.

Example 2.4 illustrates that the lottery puzzle [25] is “solved” in neighbourhood models for belief by non-closure of belief under conjunction. A calculus for ED logic is given in Figure 1.

Theorem 2.5 *The schema $K_i\phi \to B_i\phi$ is derivable in the ED calculus.*

Proof. Write $\vdash \phi$ for derivability in the ED calculus. By propositional logic, $\vdash K_i\phi \to K_i(\top \to \phi)$. From this and (M), by propositional reasoning: $\vdash K_i\phi \to (B_i\top \to B_i\phi)$. By propositional reasoning: $\vdash B_i\top \to (K_i\phi \to B_i\phi)$. From this and (N), by MP: $\vdash K_i\phi \to B_i\phi$. □

Henceforth we use $K_i\phi \to B_i\phi$ as an extra axiom scheme, and call it (KB).

Theorem 2.6 *The rule*

$$\frac{B_i\phi \qquad \phi \to \psi}{B_i\psi}$$

is derivable in the calculus of epistemic-doxastic neighbourhood logic.

Proof. Here is the derivation:

$$\dfrac{B_i\phi \qquad \dfrac{\dfrac{\phi \to \psi}{K_i(\phi \to \psi)}\,\text{Nec-K} \qquad K_i(\phi \to \psi) \to B_i\phi \to B_i\psi \ \ \text{M}}{B_i\phi \to B_i\psi}\,\text{MP}}{B_i\psi}\,\text{MP}$$

□

Theorem 2.7 (Soundness) *The calculus for ED logic is sound for epistemic neighbourhood models.*

Axioms

(Taut)	All instances of propositional tautologies
(Dist-K)	$K_i(\phi \to \psi) \to K_i\phi \to K_i\psi$
(T)	$K_i\phi \to \phi$
(PI-K)	$K_i\phi \to K_iK_i\phi$
(NI-K)	$\neg K_i\phi \to K_i\neg K_i\phi$
(N)	$B_i\top$.
(PI-KB)	$B_i\phi \to K_iB_i\phi$
(NI-KB)	$\neg B_i\phi \to K_i\neg B_i\phi$
(M)	$K_i(\phi \to \psi) \to B_i\phi \to B_i\psi$
(D)	$B_i\phi \to \check{B}_i\phi$.
(SC)	$\check{B}_i\phi \wedge \check{K}_i(\neg\phi \wedge \psi) \to B_i(\phi \vee \psi)$

Rules

$$\frac{\phi \to \psi \quad \phi}{\psi}\ \text{(MP)} \qquad \frac{\phi}{K_i\phi}\ \text{(Nec-K)}$$

Fig. 1. ED Calculus

Proof. The soundness of Dist-K follows from the fact that the K_i are modal operators. The soundness of T, PI-K and NI-K follows from the fact that the K_i are interpreted as equivalences. The soundness of N follows from property (n). The soundness of PI-KB follows from property (a). The soundness of NI-KB also follows from property (a). The soundness of M follows from property (m). The soundness of D follows from property (d). The soundness of SC follows from property (sc). The two rules preserve soundness, so all theorems of the calculus are sound. □

The following theorem was stated and proved for a weaker notion of neighbourhood model and a weaker calculus, in the unpublished [11].

Theorem 2.8 *Every consistent formula ϕ determines a canonical epistemic neighbourhood model $\mathcal{M}_\phi$.*

Proof. Suppose ϕ is consistent, i.e., $\nvdash \neg\phi$. We construct a canonical epistemic neighbourhood model for ϕ.

Let Φ be the closure of ϕ, that is, Φ is the minimal set such that (i) $\phi \in \Phi$, $\top \in \Phi$, (ii) Φ is closed under taking subformulas (i.e., if χ is a subformula of a formula ψ in Φ, then χ is in Φ), (iii) if $\psi \in \Phi$, and ψ is not a negation, then $\neg\psi \in \Phi$.

We define the canonical model $\mathcal{M}_\phi = (W, \sim, N, V)$. W is the set of all maximal consistent subsets of Φ. W is non-empty because ϕ is supposed to be consistent. Valuations are defined as follows: $V(\mathbf{w}) = \mathit{Prop} \cap \mathbf{w}$.

Since each $\mathbf{w} \in W$ is a set of formulas, we can talk about what is derivable from this set: $\mathbf{w} \vdash \psi$ means that ψ is derivable in the calculus from $\mathbf{w}$.

Relations are defined as follows:

$$\mathbf{w} \sim_i \mathbf{u} \text{ iff for all } \psi \in \Phi : \mathbf{w} \vdash K_i\psi \text{ iff } \mathbf{u} \vdash K_i\psi \text{ and } \mathbf{w} \vdash K_iB_i\psi \text{ iff } \mathbf{u} \vdash K_iB_i\psi.$$

Clearly, all $\sim_i$ are equivalence relations.

Next, we define N_i by means of:

$$N_i(\mathbf{w}) := \{\{\mathbf{v} \in [\mathbf{w}]_i \mid \psi \in \mathbf{v}\} \mid \psi \in \Phi, \mathbf{w} \vdash B_i\psi\}.$$

We check that the properties of neighbourhood functions hold. (c) holds by definition of N_i. (n) holds by the fact that $\mathbf{w} \vdash B_i\top$ (by (N)), and $\{\mathbf{v} \in [\mathbf{w}]_i \mid \top \in \mathbf{v}\} = [\mathbf{w}]_i$.

To show that (a) holds assume $\mathbf{w} \sim_i \mathbf{u}$. Let $X \in N_i(\mathbf{w})$. Then for some $\psi \in \Phi$, $X = \{\mathbf{v} \in [\mathbf{w}]_i \mid \psi \in \mathbf{v}\}$ and $\mathbf{w} \vdash B_i\psi$. Therefore, by (KB), $\mathbf{w} \vdash K_iB_i\psi$. By the definition of $\sim_i$ it follows that $\mathbf{u} \vdash K_iB_i\psi$, and therefore by (T), $\mathbf{u} \vdash B_i\psi$, and we have that $X \in N_i(\mathbf{u})$.

To see that (m) holds, let $\psi \in \Phi$ and assume $\{\mathbf{v} \in [\mathbf{w}]_i \mid \psi \in \mathbf{v}\} \in N_i(\mathbf{w})$. Then $\mathbf{w} \vdash B_i\psi$. Let $\psi' \in \Phi$ and assume $\vdash \psi \to \psi'$. Then by Theorem 2.6, $\mathbf{w} \vdash B_i\psi'$. Therefore $\{\mathbf{v} \in [\mathbf{w}]_i \mid \psi' \in \mathbf{v}\} \in N_i(\mathbf{w})$.

To see that (d) holds, assume for some $\psi \in \Phi$, $\{\mathbf{v} \in [\mathbf{w}]_i \mid \psi \in \mathbf{v}\} \in N_i(\mathbf{w})$. Then $\mathbf{w} \vdash B_i\psi$. By (D), $\mathbf{w} \vdash \neg B_i\neg\psi$. Thus, $\{\mathbf{v} \in [\mathbf{w}]_i \mid \neg\psi \in \mathbf{v}\} \notin N_i(\mathbf{w})$.

Finally, we show that (sc) holds. Let $\psi \in \Phi$ and let $X = \{\mathbf{v} \in [\mathbf{w}]_i \mid \psi \in \mathbf{v}\}$. Then $[\mathbf{w}]_i - X = \{\mathbf{v} \in [\mathbf{w}]_i \mid \neg\psi \in \mathbf{v}\}$. Assume $[\mathbf{w}]_i - X \notin N_i(\mathbf{w})$. Then $\mathbf{w} \vdash \neg B_i\neg\psi$. If $X = [\mathbf{w}]_i$ there is nothing to prove. So assume there is some $\mathbf{v} \in [\mathbf{w}]_i - X$. Then there is $\psi' \in \Phi$ with $\psi' \in \mathbf{v}$ and for all $\mathbf{u} \in X : \psi' \notin \mathbf{u}$. Then $\mathbf{v} \vdash (\psi' \wedge \neg\psi)$, and therefore by (T), $\mathbf{v} \vdash \check{K}(\psi' \wedge \neg\psi)$. Thus by (SC), $\mathbf{v} \vdash B_i(\psi \vee \psi')$, and by (KB), $\mathbf{v} \vdash K_iB_i(\psi \vee \psi')$. From this, by the definition of $\sim_i$, $\mathbf{w} \vdash K_iB_i(\psi \vee \psi')$, and by (T), $\mathbf{w} \vdash B_i(\psi \vee \psi')$. It follows that $X \cup \{\mathbf{v}\} \in N_i(\mathbf{w})$. So we have indeed defined an epistemic neighbourhood model. $\Box$

Lemma 2.9 (Truth Lemma) *For all formulas $\psi \in \Phi$, we have $\mathcal{M}_\phi, \mathbf{w} \models \psi$ iff $\psi \in \mathbf{w}$.*

Proof. Induction on ψ. The cases of $\top$, p and the Boolean combinations are straightforward. For the case of $K_i\chi$. We switch for convenience to the dual, and prove $\mathcal{M}_\phi, \mathbf{w} \models \check{K}_i\chi$ iff $\check{K}_i\chi \in \mathbf{w}$.

From left to right: $\mathcal{M}_\phi, \mathbf{w} \models \check{K}_i\chi$ iff there is a $\mathbf{v} \in [\mathbf{w}]_i$ with $\mathcal{M}_\phi, \mathbf{v} \models \chi$ iff there is a $\mathbf{v} \in [\mathbf{w}]_i$ with $\chi \in \mathbf{v}$ by induction hypothesis only if $\check{K}_i\chi \in \mathbf{v}$ by (T) and maximal consistency of $\mathbf{v}$ iff $\check{K}_i\chi \in \mathbf{w}$ by definition of $[\mathbf{w}]_i$ and maximal consistency of $\mathbf{w}$.

From right to left: Suppose $\check{K}_i\chi \in \mathbf{w}$. We have to show $\mathcal{M}_\phi, \mathbf{w} \models \check{K}_i\chi$. For that we have to construct $\mathbf{v} \in [\mathbf{w}]_i$ with $\chi \in \mathbf{v}$. So let

$$\mathbf{v}^- := \{\chi\} \cup \{K_i\psi \in \Phi \mid K_i\psi \in \mathbf{w}\} \cup \{\neg K_i\psi \in \Phi \mid \neg K_i\psi \in \mathbf{w}\}.$$

Then $\mathbf{v}^-$ is consistent. For suppose not. Then there are

$$K_i\psi_1, \ldots, K_i\psi_n, \neg K_i\sigma_1, \ldots, \neg K_i\sigma_m \in \mathbf{v}^-$$

with

$$\vdash K_i\psi_1 \to \cdots \to K_i\psi_n \to \neg K_i\sigma_1 \to \cdots \to \neg K_i\sigma_m \to \neg\chi.$$

With the Nec rule:

$$\vdash K_i(K_i\psi_1 \to \cdots \to K_i\psi_n \to \neg K_i\sigma_1 \to \cdots \to \neg K_i\sigma_m \to \neg\chi).$$

From this, with $n + m$ applications of (Dist-K) and propositional reasoning:

$$\vdash K_iK_i\psi_1 \to \ldots \to K_iK_i\psi_n \to K_i\neg K_i\sigma_1 \to \cdots \to K_i\neg K_i\sigma_m \to K_i\neg\chi.$$

By construction of $\mathbf{v}^-$, $K_i\psi_1, \ldots, K_n\psi_n, \neg K_i\sigma_1, \ldots, \neg K_i\sigma_m$ are in $\mathbf{w}$. Therefore, by (PI-K) and (NI-K),

$$K_iK_i\psi_1, \ldots, K_iK_i\psi_n, K_i\neg K_i\sigma_1, \ldots, K_i\neg K_i\sigma_m$$

follow from $\mathbf{w}$. But this means $K_i\neg\chi \in \mathbf{w}$, by maximal consistency of $\mathbf{w}$, and contradiction with $\check{K}_i\chi \in \mathbf{w}$. It follows that $\mathbf{v}^-$ is consistent. Therefore an extension $\mathbf{v}$ that is Φ maximal consistent exists. By construction, $\mathbf{v} \in [\mathbf{w}]_i$, and $\chi \in \mathbf{v}$. Therefore, by the induction hypothesis, $\mathcal{M}, \mathbf{v} \models \chi$, and it follows that $\mathcal{M}, \mathbf{w} \models \check{K}_i\chi$.

This leaves the case of $B_i\chi$. $\mathcal{M}_\phi, \mathbf{w} \models B_i\chi$ iff (truth definition for $B_i\chi$) $\{\mathbf{v} \in [\mathbf{w}]_i \mid \mathcal{M}_\phi, \mathbf{v} \models \chi\} \in N_i(\mathbf{w})$ iff (induction hypothesis) $\{\mathbf{v} \in [\mathbf{w}]_i \mid \chi \in \mathbf{v}\} \in N_i(\mathbf{w})$ iff ($B_i\chi$ and χ are in Φ) $\{\mathbf{v} \in [\mathbf{w}]_i \mid \chi \in \mathbf{v}\} \in N_i(\mathbf{w})$ iff (definition of N_i, plus the fact that $B_i\chi \in \Phi$) $B_i\chi \in \mathbf{w}$. □

Theorem 2.10 (Completeness of ED Logic) *The calculus of epistemic-doxastic neighbourhood logic is complete for epistemic neighbourhood models:*

$$\textit{If } \models \phi \textit{ then } \vdash \phi.$$

Proof. Let $\nvdash \phi$. Then $\neg\phi$ is consistent, and one can find a maximal consistent set $\mathbf{w}$ in the closure of $\neg\phi$ with $\neg\phi \in \mathbf{w}$. By the Truth Lemma, $\mathcal{M}_{\neg\phi}, \mathbf{w} \models \neg\phi$, i.e., $\mathcal{M}_{\neg\phi}, \mathbf{w} \not\models \phi$. Therefore, $\not\models \phi$. □

3 Epistemic Weight Models and Incompleteness

An alternative semantics for the language of epistemic doxastic logic can be given with respect to *Epistemic Weight Models*. As it turns out, the calculus given above is *sound* but *incomplete* for this alternative semantics.

Definition 3.1 [Epistemic Weight Models] An **epistemic weight model** for agents I and basic propositions P is a tuple $\mathcal{M} = (W, R, L, V)$ where

- W is a non-empty countable set of worlds,
- R assigns to every agent $i \in I$ an equivalence relation $\sim_i$ on W,
- L assigns to every $i \in I$ a function $\mathbb{L}_i$ from W to $\mathbb{Q}^+$ (the positive rationals), subject to the following boundedness condition (*).

$$\forall i \in I \forall w \in W \sum_{u \in [w]_i} \mathbb{L}_i(u) < \infty. \tag{*}$$

where $[w]_i$ is the cell of w in the partition induced by $\sim_i$.

- V assigns to every $w \in W$ a subset of P,

Use $\mathbb{L}_i(X)$ for $\sum_{x \in X} \mathbb{L}_i(x)$.

Definition 3.2 [Truth of ED formulas in Epistemic Weight Models] Key clauses are:

$$\mathcal{M}, w \models K_i\phi \text{ iff } \quad \text{for all } v \in [w]_i : \mathcal{M}, v \models \phi.$$

$$\mathcal{M}, w \models B_i\phi \text{ iff}$$

$$\mathbb{L}_i(\{v \in [w]_i \mid \mathcal{M}, v \models \phi\}) > \mathbb{L}_i(\{v \in [w]_i \mid \mathcal{M}, v \models \neg\phi\}).$$

If we interpret ED formulas in epistemic weight models, we get:

Theorem 3.3 *The ED calculus is sound for epistemic weight models.*

Proof. The axioms are sound, and the rules preserve soundness. □

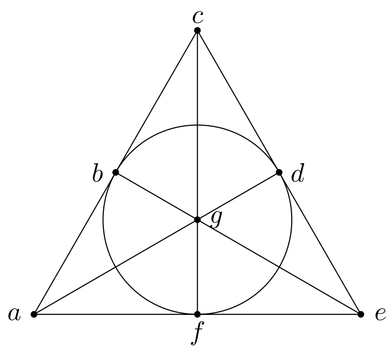
Definition 3.4 [Agreement] Let $\mathcal{M} = (W, R, N, V)$ be a neighbourhood model and let L be a weight function for $\mathcal{M}$. Then L *agrees with* $\mathcal{M}$ if it holds for all agents i and all $w \in W$ that

$$X \in N_i(w) \text{ iff } \mathbb{L}_i(X) > \mathbb{L}_i([w]_i - X).$$

The following theorem shows that the ED calculus is incomplete for epistemic weight models. A version of this theorem for a weaker notion of neighbourhood model was proved in the unpublished [11], as an adaptation of example 2 from [31, pp. 344-345].

Theorem 3.5 *There exists an epistemic neighbourhood model $\mathcal{M}$ that has no agreeing weight function.*

Proof. Consider the Fano plane from finite geometry (see, e.g., [27, Chapter 9]).



Every pair of distinct points determines a line, every line has exactly three points on it. Let $Prop := \{a, b, c, d, e, f, g\}$. Assume a single agent 0. Define $\mathcal{X}$ as the set of lines in the Fano plane (notation: xyz for $\{x, y, z\}$):

$$\mathcal{X} := \{abc, cde, afe, agd, cgf, egb, bdf\}.$$

No complement of a line contains a line (check this in the figure). Moreover, if one extends the complement of a line with another point, the result will contain a line. This is because any element of a line lies on two points of the complement (check this in the figure). Thus, the members of $\mathcal{X}'$ are the maximal sets that do not contain a line:

$$\begin{aligned}\mathcal{X}' &:= \{\overline{abc}, \overline{cde}, \overline{afe}, \overline{agd}, \overline{cgf}, \overline{egb}, \overline{bdf}\}\\ &= \{defg, abfg, bcdg, bcef, abde, acdf, aceg\}.\end{aligned}$$

Now define the neighbourhoods $\mathcal{Y}$ as the sets that contain (the points of) at least one line:

$$\mathcal{Y} := \{Y \mid \exists X \in \mathcal{X} : X \subseteq Y \subseteq W\}.$$

Let $\mathcal{M} := (W, R, N, V)$ be defined by $W := \mathit{Prop}$, $R_0 = W \times W$, $V(w) = \{w\}$, and for all $w \in W$, $N_0(w) = \mathcal{Y}$. So the neighbourhoods are the sets that contain (the points of) at least one line from the Fano plane.

Check that $\mathcal{X}' \cap \mathcal{Y} = \emptyset$. This shows that condition (d) holds. Condition (sc) holds because adding a point to any member of $\mathcal{X}'$ yields a neighbourhood. The other conditions for neighbourhood models are also easily checked. So $\mathcal{M}$ is a neighbourhood model.

Toward a contradiction, suppose there exists a weight function L that agrees with $\mathcal{M}$. Since each letter $p \in W$ occurs in exactly three of the seven members of $\mathcal{X}$, we have:

$$\sum_{X \in \mathcal{X}} \mathbb{L}_0(X) = \sum_{p \in W} 3 \cdot \mathbb{L}_0(\{p\}).$$

Since each letter $p \in W$ occurs in exactly four of the seven members of $\mathcal{X}'$, we have:

$$\sum_{X \in \mathcal{X}'} \mathbb{L}_0(X) = \sum_{p \in W} 4 \cdot \mathbb{L}_0(\{p\}).$$

On the other hand, from the fact that $\mathbb{L}_0(X) > \mathbb{L}_o(\overline{X})$ for all members X of $\mathcal{X}$ we get:

$$\sum_{X \in \mathcal{X}} \mathbb{L}_0(X) > \sum_{X \in \mathcal{X}} \mathbb{L}_0(\overline{X}) = \sum_{X \in \mathcal{X}'} \mathbb{L}_0(X).$$

Contradiction. So no such $\mathbb{L}_0$ exists. □

4 Epistemic Weight Models and Completeness

We will now rephrase the system of simplified probabilistic epistemic logic of [12], using an alternative syntax, following [9]. We call the result *Epistemic Comparison Logic*. In the next definition, we use $\oplus$ as a list-forming operation for formulas, so $\phi_1 \oplus \cdots \oplus \phi_n$ should be thought of as the n-element formula list $(\phi_1, \dots, \phi_n)$. We will use Φ to range over formula lists, and $\phi \oplus \Phi$ for the extension of the formula list Φ at the front with the formula ϕ.

Definition 4.1 [EC Language]

$$\phi ::= \top \mid p \mid \neg\phi \mid \phi \wedge \phi \mid \Phi \leq_i \Phi \qquad \Phi ::= \phi \mid \phi \oplus \Phi$$

Abbreviations: As usual for $\bot, \vee, \rightarrow, \leftrightarrow$. $\Phi <_i \Psi$ for $\Phi \leq_i \Psi \wedge \neg\Psi \leq_i \Phi$. $\Phi =_i \Psi$ for $\Phi \leq_i \Psi \wedge \Psi \leq_i \Phi$. $B_i\phi$ for $(\neg\phi) <_i \phi$, $\check{B}_i\phi$ for $(\neg\phi) \leq_i \phi$ ("Belief as willingness to bet"), $K_i\phi$ for $\top \leq_i \phi$, $\check{K}_i\phi$ for $\bot <_i \phi$ ("Knowledge as certainty").

Definition 4.2 [Truth for EC Logic] Let $\mathcal{M} = (W, R, L, V)$ be an epistemic weight model, let $w \in W$.

$$
\begin{array}{lll}
[\![\phi]\!]_{\mathcal{M}} & := & \{w \in W \mid \mathcal{M}, w \models \phi\} \\
[\![\phi]\!]_{\mathcal{M}}^{w,i} & := & [\![\phi]\!]_{\mathcal{M}} \cap [w]_i \\
\mathbb{L}_{w,i}\phi & := & \mathbb{L}_i([\![\phi]\!]_{\mathcal{M}}^{w,i}) \\
\mathcal{M}, w \models \top & & \text{always} \\
\mathcal{M}, w \models \neg\phi & \text{iff} & \text{not } \mathcal{M}, w \models \phi \\
\mathcal{M}, w \models \phi_1 \wedge \phi_2 & \text{iff} & \mathcal{M}, w \models \phi_1 \text{ and } \mathcal{M}, w \models \phi_2 \\
\mathcal{M}, w \models \Phi \leq_i \Psi & \text{iff} & \sum_{\phi\in\Phi} \mathbb{L}_{w,i}\phi \leq \sum_{\psi\in\Psi} \mathbb{L}_{w,i}\psi
\end{array}
$$

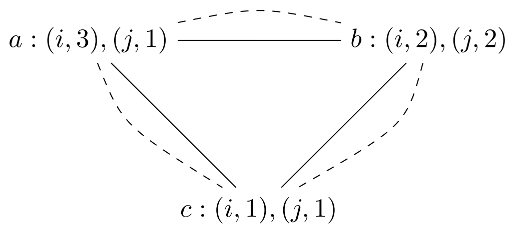
Note that in $\sum_{\phi\in\Phi}$, we sum over *occurrences* of ϕ in the list Φ.

Weight function and epistemic accessibility relation together determine probability:

$$P_{w,i}^{\mathcal{M}}\phi := \frac{\mathbb{L}_{w,i}\phi}{\mathbb{L}_{w,i}\top} \left(= \frac{\mathbb{L}_i([\![\phi]\!]_{\mathcal{M}} \cap [w]_i)}{\mathbb{L}_i([w]_i)} \right)$$

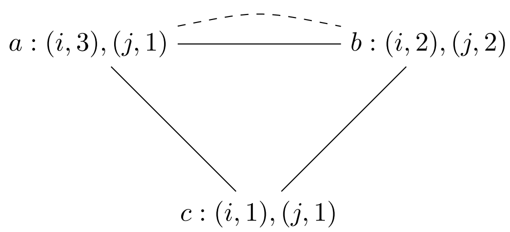
In a slogan: "Probabilities are weights normalized for epistemic partition cells."

Example 4.3 Two bankers i, j consider buying stocks in three firms a, b, c that are involved in a takeover bid. There are three possible outcomes: a for "a wins", b for " b wins", and c for "c wins." i takes the winning chances to be $3 : 2 : 1$, j takes them to be $1 : 2 : 1$. i: solid lines, j: dashed lines.



We see that i is willing to bet $1 : 1$ on a, while j is willing to bet $3 : 1$ against a. It follows that in this model i and j have an opportunity to gamble, for, to put it in Bayesian jargon, they do not have a common prior.

Suppose j has foreknowledge about what firm c will do.



Taut	instances of propositional tautologies
ProbT	$(\top \leq_i \phi) \to \phi$
ProbImpl	$\top \leq_i (\phi \to \psi) \to (\phi \leq_i \psi)$
PropPos	$(\Phi \leq_i \Psi) \to \top \leq_i (\Phi \leq_i \Psi)$
PropNeg	$(\Phi >_i \Psi) \to \top \leq_i (\Phi >_i \Psi)$
PropAdd	$(\phi \wedge \psi) \oplus (\phi \wedge \neg\psi) =_i \phi$
Tran	$(\Phi \leq_i \Psi) \wedge (\Psi \leq_i \Xi) \to (\Phi \leq_i \Xi)$
Tot	$(\Phi \leq_i \Psi) \vee (\Psi \leq_i \Phi)$
ComL	$(\Phi_1 \oplus \Phi_2 \leq_i \Psi) \leftrightarrow (\Phi_2 \oplus \Phi_1 \leq_i \Psi)$
ComR	$(\Phi \leq_i \Psi_1 \oplus \Psi_2) \leftrightarrow (\Phi \leq_i \Psi_2 \oplus \Psi_1)$
Add	$(\Phi_1 \leq_i \Psi_1) \wedge (\Phi_2 \leq_i \Psi_2) \to (\Phi_1 \oplus \Phi_2 \leq_i \Psi_1 \oplus \Psi_2)$
Succ	$(\Phi \oplus \top \leq_i \Psi \oplus \top) \to (\Phi \leq_i \Psi)$
MP	From $\vdash \phi$ and $\vdash \phi \to \psi$ derive $\vdash \psi$
NEC	From $\vdash \phi$ derive $\vdash 1 \leq_i \phi$

Fig. 2. EC Calculus

The probabilities assigned by i remain as before. The probabilities assigned by j have changed, as follows. In worlds a and b, j assigns probability $\frac{1}{3}$ to a and $\frac{2}{3}$ to b. In world c, j is sure of c.

We may suppose that this new model results from j being informed about the truth value of c, while i is aware that j received this information, but without i getting the information herself. So i is aware that j's subjective probabilities have changed, and it would be unwise for i to put her beliefs to the betting test. For although i cannot distinguish the three situations, she knows that j can distinguish the c situation from the other two. Willingness of j to bet against a at any odds can be interpreted by i as an indication that c is true, thus forging an intimate link between action and information update.

The probability comparison calculus is given in Figure 2.

Definition 4.4 [EC Derivability] $\Gamma \vdash \phi$ holds if either $\phi \in \Gamma$, or ϕ is an axiom, or ϕ follows by means of the rules of the calculus from axioms or members of Γ, while taking care that application of NEC only is allowed when the set of premisses Γ is empty.

Let $\vdash_{EC}$ denote derivability in the probability comparison calculus. The deduction theorem holds for this calculus:

Theorem 4.5 $\Gamma \cup \{\phi\} \vdash \psi$ *iff* $\Gamma \vdash \phi \to \psi$.

Proof. The proof for this is folklore, but see [18] for an explanation of why the restriction on the use of NEC is crucial for this. □

The deduction theorem will help us to state a number of useful derivable principles.

(i) From ProbImpl, with propositional reasoning:

$$\vdash \top \leq_i (\phi \leftrightarrow \psi) \rightarrow (\phi =_i \psi).$$

(ii) Consider the following instance of ProbT: $\vdash (\top \leq_i \bot) \rightarrow \bot$. By propositional logic, $\vdash \neg(\top \leq_i \bot)$, i.e., $\vdash \bot <_i \top$.

(iii) $\vdash \Phi =_i \Phi$ by Tot.

(iv) From PropAdd, $\vdash (\phi \wedge \bot) \oplus (\phi \wedge \top) =_i \phi$. By propositional reasoning, $\vdash \bot \oplus \phi =_i \phi$.

(v) $\vdash \bot \oplus \Phi =_i \Phi$ by an easy induction using the previous two items.

(vi) Plugging in $\top$ as a special case in (3), we get $\vdash \bot \oplus \top =_i \top$.

(vii) Plugging in $\bot$ as a special case in (3), we get $\vdash \bot \oplus \bot =_i \bot$.

(viii) From $\vdash \phi \rightarrow \psi$, derive $\vdash \top \leq_i (\phi \rightarrow \psi)$, by NEC. Combine this with $\vdash \top \leq_i (\phi \rightarrow \psi) \rightarrow (\phi \leq_i \psi)$ (ProbImpl) to get $\vdash \phi \leq_i \psi$. Thus we have derived the rule PR:

$$\frac{\vdash \phi \rightarrow \psi}{\vdash \phi \leq_i \psi}$$

(ix) Assume $\vdash \phi \leftrightarrow \psi$. Then also $\vdash \phi \rightarrow \psi$ and $\vdash \psi \rightarrow \phi$. From the former, with PR (previous item), $\vdash \phi \leq_i \psi$, and from the latter $\vdash \psi \leq_i \phi$. Combining these, we get $\vdash \phi =_i \psi$. This gives the derived inference rule:

$$\frac{\vdash \phi \leftrightarrow \psi}{\vdash \phi =_i \psi}$$

(x) From PropAdd, $\vdash (\top \wedge \phi) \oplus (\top \wedge \neg\phi) =_i \top$, so by propositional reasoning, $\vdash \phi \oplus \neg\phi =_i \top$.

(xi) Assume $\vdash \phi =_i \top$. Then $\vdash \neg\phi \oplus \phi =_i \neg\phi \oplus \top$, and hence $\vdash \neg\phi \oplus \top =_i \top$, by the previous item. Since $\vdash \bot \oplus \top =_i \top$ we get $\vdash \neg\phi \oplus \top =_i \bot \oplus \top$, and therefore, by Succ, $\vdash \neg\phi =_i \bot$. By the deduction theorem, we have derived $\vdash \phi =_i \top \rightarrow \neg\phi =_i \bot$.

(xii) In a similar way we can derive: $\vdash \phi =_i \bot \rightarrow \neg\phi =_i \top$.

(xiii) Assume $\vdash \top \leq_i \phi$ and $\vdash \top \leq_i \phi \rightarrow \psi$. From this: $\vdash \phi =_i \top$ and $\vdash \phi \rightarrow \psi =_i \top$. From $\vdash \phi =_i \top$, $\vdash \neg\phi \wedge \neg\psi =_i \bot$ and from $\vdash \phi \rightarrow \psi =_i \top$ we get that $\vdash \phi \wedge \neg\psi =_i \bot$. Since $\vdash \neg\psi =_i (\phi \wedge \neg\psi) \oplus (\neg\phi \wedge \neg\psi)$, by PropAdd, we derive that $\vdash \neg\psi =_i \bot$, and it follows that $\vdash \psi =_i \top$, hence $\vdash \top \leq_i \psi$. By the deduction theorem, we have derived the distribution principle for certainty: $\vdash (\top \leq_i \phi \wedge \top \leq_i (\phi \rightarrow \psi) \rightarrow \top \leq_i \psi$.

(xiv) Using the previous item, we get:

$$\dfrac{\dfrac{\vdash \phi \rightarrow \psi}{\vdash \top \leq_i (\phi \rightarrow \psi)}\;\text{NEC}}{\vdash \top \leq_i \phi \rightarrow \top \leq_i \psi}\;\text{Distribution}$$

(xv) From Tot, by the definitions of $<_i$, $>_i$ and $=_i$:

$$\vdash \Phi <_i \Psi \vee \Phi =_i \Psi \vee \Phi >_i \Psi.$$

(xvi) By Tran and the definition of $<_i$:
$\vdash \Phi <_i \Psi \wedge \Psi \leq_i \Xi \rightarrow \Phi <_i \Xi$,
$\vdash \Phi <_i \Psi \wedge \Psi \leq_i \Xi \rightarrow \Phi <_i \Xi$,
$\vdash \Phi <_i \Psi \wedge \Psi <_i \Xi \rightarrow \Phi <_i \Xi$.

(xvii) By Add, and the definition of $=_i$:

$$\vdash \Phi_1 =_i \Phi_2 \wedge \Psi_1 =_i \Psi_2 \rightarrow \Phi_1 \oplus \Psi_1 =_i \Phi_2 \oplus \Psi_2.$$

(xviii) Assume $\vdash \phi \leq_i \psi$. Then by Add, $\vdash \neg\phi \oplus \neg\psi \oplus \phi \leq_i \neg\phi \oplus \neg\psi \oplus \psi$. By rearranging and applying (x), $\vdash \neg\psi \oplus 1 \leq_i \neg\phi \oplus 1$. By Succ, $\vdash \neg\psi \leq_i \neg\phi$. By the deduction theorem we have shown $\vdash \phi \leq_i \psi \rightarrow \neg\psi \leq_i \neg\phi$.

Given the abbreviations for $K_i\phi$ and $B_i\phi$ given after Definition 4.1, we see that we can take the ED language of Definition 2.2 to be a fragment of the EC language of Definition 4.1. It then turns out that everything that is derived in the ED calculus is also derived in the EC calculus.

Theorem 4.6 *For all ϕ in the ED language and Γ of ED-formula sets: if $\Gamma \vdash_{ED} \phi$ then $\Gamma \vdash_{EC} \phi$.*

Proof. We show that the (translations of the) axioms and rules of the ED calculus are all derivable in the EC calculus.

K_i necessitation. This is NEC.

Dist-K $K_i(\phi \rightarrow \psi) \rightarrow K_i\phi \rightarrow K_i\psi$. This was proved in (xiii) above.

T: Follows from PropT.

PI-K: Follows from ProbPos

NI-K: Follows from ProbNeg

N: This is $\perp <_i \top$. Proved in (ii) above.

PI-KB: Follows from ProbNeg.

NI-KB: Follows from ProbPos.

M: Assume $\vdash \top \leq_i (\phi \rightarrow \psi)$ and $\vdash \neg\phi <_i \phi$. We have to show that $\vdash \neg\psi <_i \psi$. From $\vdash \top \leq_i (\phi \rightarrow \psi)$ get $\vdash \phi \leq_i \psi$ by ProbImpl. From $\vdash \phi \leq_i \psi$ we get $\vdash \neg\psi \leq_i \neg\phi$ by (xviii) above. From $\vdash \neg\phi <_i \phi$ and $\vdash \phi \leq_i \psi$, by Trans, $\vdash \neg\phi < \psi$. From this and $\vdash \neg\psi \leq_i \neg\phi$ by Trans $\vdash \neg\psi < \psi$.

D: From $\vdash \neg\phi <_i \phi$ by Tot $\vdash \neg\phi \leq_i \phi$.

SC: Assume $\vdash (\neg\phi <_i \phi)$ and $\vdash \perp <_i (\neg\phi \wedge \psi)$. We have to show $\vdash (\neg\phi \wedge \neg\psi) <_i (\phi \vee \psi)$. For this we can use the equivalence of $\phi \vee \psi$ and $(\phi \wedge \neg\psi) \vee (\phi \wedge \psi) \vee (\neg\phi \wedge \psi)$. □

Theorem 4.7 *Every consistent EC formula ϕ determines a canonical epistemic weight model $\mathcal{M}_\phi$.*

Proof. Suppose ϕ is consistent, i.e., $\nvdash \neg\phi$. We construct a canonical epistemic weight model for ϕ.

Let $\mathbf{\Phi}$ be the set of all subformulas of ϕ, closed under single negations. The subformulas of $\Psi \leq_i \Xi$ are all subformulas of formulas ψ that occur as $\oplus$ terms in Ψ or Ξ, plus the results $\Psi' \leq_i \Xi'$ of leaving out $\oplus$ terms in Ψ or Ξ while taking care that Ψ' and Ξ' are not empty.

We define the canonical model $\mathcal{M}_\phi = (W, R, L, V)$. W is the set of all maximal consistent subsets of $\mathbf{\Phi}$. W is non-empty because ϕ is supposed to be consistent.

Valuations are defined as follows: $V(\mathbf{w}) = \mathit{Prop} \cap \mathbf{w}$.

Let $\text{sat}(\mathbf{w}) = \{\psi \in \mathbf{\Phi} \mid \mathbf{w} \vdash \psi\}$, that is, $\text{sat}(\mathbf{w})$ is the set of $\mathbf{\Phi}$-formulas that are provable from $\mathbf{w}$.

Notice that it follows by the soundness of the probability comparison calculus that all members of $\text{sat}(\mathbf{w})$ are true in $\mathbf{w}$.

Relations $\sim_i$ are defined as follows: $\mathbf{w} \sim_i \mathbf{u}$ iff $\text{sat}(\mathbf{w})$ and $\text{sat}(\mathbf{u})$ contain the same i-comparison formulas. Clearly, all $\sim_i$ are equivalence relations.

Now it remains to define L. Consider an agent i and an equivalence class $[\mathbf{w}]_i$ in the canonical model $\mathcal{M}_\phi$. All worlds $\mathbf{u}$ of $[\mathbf{w}]_i$ contain the same i-comparison formulas.

We show how to transform all these i-comparison formulas in a system of linear inequalities that is consistent.

For all $\mathbf{u} \in W$, we write $\phi_{\mathbf{u}}$ for the conjunction of all formulas in $\mathbf{u}$. We have:

- $\vdash \phi_{\mathbf{u}} \to \neg\phi_{\mathbf{v}}$ if $\mathbf{u} \neq \mathbf{v}$ by propositional logic.

Given any formula ψ of $\mathbf{\Phi}$, we have

- $\vdash \psi \leftrightarrow \bigvee_{\{\mathbf{u} \in W \mid \psi \in \mathbf{u}\}} \phi_{\mathbf{u}}$ by propositional logic.

Since the $\phi_{\mathbf{u}}$ are all mutually inconsistent we can prove in the calculus:

- $\vdash \psi =_i \bigoplus\{\phi_{\mathbf{u}} \mid \mathbf{u} \in W \text{ and } \psi \in \mathbf{u}\}$.

Now, when we i-compare $\phi_{\mathbf{u}}$ to $\perp$ in $\mathbf{w}$, we should obtain $>_i$ iff $\mathbf{u} \in [\mathbf{w}]_i$. Let us prove this fact.

- If $\mathbf{u} \in [\mathbf{w}]_i$, we have:
  (i) $\vdash \phi_{\mathbf{u}} \to \perp <_i \phi_{\mathbf{u}}$ by (ProbT) and (PropAdd).
  (ii) $\perp <_i \phi_{\mathbf{u}} \in \text{sat}(\mathbf{u})$;
  (iii) $\perp <_i \phi_{\mathbf{u}} \in \text{sat}(\mathbf{w})$ because $\mathbf{u} \in [\mathbf{w}]_i$.
  Therefore, $\perp <_i \phi_{\mathbf{u}}$ follows from $\mathbf{w}$.
- Suppose $\mathbf{u} \notin [\mathbf{w}]_i$. Then $\mathbf{u}$ and $\mathbf{w}$ differ by at least one i-comparison formula $\Psi \leq_i \Xi \in \mathbf{\Phi}$. Without loss of generality, assume $\Psi \leq_i \Xi \in \mathbf{w}$ and $\Psi \leq_i \Xi \notin \mathbf{u}$. Then we have:
  (i) $\vdash \phi_{\mathbf{w}} \to \Psi \leq_i \Xi$ by propositional logic;
  (ii) $\vdash \Psi \leq_i \Xi \to \neg\phi_{\mathbf{u}}$ by propositional logic;
  (iii) $\vdash \Psi \leq_i \Xi \to \top \leq_i (\Psi \leq_i \Xi)$ by axiom (PropT);
  (iv) $\vdash \phi_{\mathbf{w}} \to \top \leq_i (\Psi \leq_i \Xi)$ by propositional logic;
  (v) $\vdash \phi_{\mathbf{w}} \to \top \leq_i (\neg\phi_{\mathbf{u}})$ by ii and iv.
  (vi) $\vdash \phi_{\mathbf{w}} \to (\phi_{\mathbf{u}} \leq_i \perp)$ from the above by PropAdd and propositional reasoning.
  Therefore $\phi_{\mathbf{u}} \leq_i \perp$ follows from $\mathbf{w}$.

Thus, ψ has the same i-weight as $\{\phi_{\mathbf{u}} \mid \mathbf{u} \in [\mathbf{w}]_i \text{ and } \psi \in \mathbf{u}\}$. We can prove in the calculus:

- $\vdash \psi =_i \bigoplus\{\phi_{\mathbf{u}} \mid \mathbf{u} \in [\mathbf{w}]_i \text{ and } \psi \in \mathbf{u}\}$.

Now let $\Psi \leq_i \Xi$ be any i-comparison formula of $\mathbf{w}$. Then we can replace any $\oplus$ term ψ occurring in either Ψ or Ξ by a list of terms $\bigoplus\{\phi_{\mathbf{u}} \mid \mathbf{u} \in [\mathbf{w}]_i$ and $\psi \in \mathbf{u}\}$ with the same i-weight. Let the result of this be $\Psi' \leq_i \Xi'$. Regrouping the $\oplus$ terms in Ψ' and Ξ', using the abbreviation $n\chi$ for $\underbrace{\chi \oplus \cdots \oplus \chi}_{n \text{ times}}$, 0 for $\bot \oplus \cdots \oplus \bot$, m for $\underbrace{\top \oplus \cdots \oplus \top}_{m \text{ times}}$, and replacing $\oplus$ by $+$ and $\leq_i$ by $\leq$ gives a linear inequality

$$a_1\phi_{\mathbf{u}_1} + \cdots + a_n\phi_{\mathbf{u}_n} + k \leq b_1\phi_{\mathbf{v}_1} + \cdots + b_m\phi_{\mathbf{v}_m} + l$$

where a_i, b_j, k, l are non-negative integers, and the $\phi_{\mathbf{u}}$ and $\phi_{\mathbf{v}}$ figure as variables. Applying this recipe to each i-comparison formula in $\mathbf{w}$, we get a system of linear inequalities made up of i-inequalities in $\mathbf{w}$.

The set sat($\mathbf{w}$) is consistent so the above system, which is a rephrasing of inequations that are in sat($\mathbf{w}$), is also consistent and therefore satisfiable [13, Theorem 2.2]. Let $(x_u^*)_{\mathbf{u} \in [\mathbf{w}]_i}$ be a solution, and define $\mathbb{L}_i(\mathbf{u}) = x_u^*$. □

Lemma 4.8 (Truth Lemma) *Let ϕ be a consistent EC formula and $\mathbf{\Phi}$ the set of all its subformulas closed under single negations. Then for all formulas $\psi \in \mathbf{\Phi}$, we have $\mathcal{M}_\phi, \mathbf{w} \models \psi$ iff $\psi \in \mathbf{w}$.*

Proof. Induction on ψ. □

Theorem 4.9 (Completeness of Epistemic Comparison Logic) *The calculus of epistemic comparison logic given in Figure 2 is complete for epistemic weight models:*

$$\textit{If } \models \phi \textit{ then } \vdash_{EC} \phi.$$

Proof. Let $\nvdash \phi$. Then $\neg\phi$ is consistent, and one can find a maximal consistent set $\mathbf{w}$ in the closure of $\neg\phi$ with $\neg\phi \in \mathbf{w}$. By the Truth Lemma, $\mathcal{M}_{\neg\phi}, \mathbf{w} \models \neg\phi$, i.e., $\mathcal{M}_{\neg\phi}, \mathbf{w} \not\models \phi$. Therefore, $\not\models \phi$. □

From Epistemic Probability Models to Epistemic Neighbourhood Models: If $\mathcal{M} = (W, R, L, V)$ is an epistemic weight model, then $\mathcal{M}^\bullet$ is the tuple (W, R, N, V) given by replacing the weight function by a function N, where N is defined as follows, for $i \in Ag$, $w \in W$.

$$N_i(w) = \{X \subseteq [w]_i \mid \mathbb{L}_i(X) > \mathbb{L}_i([w]_i - X)\}.$$

Theorem 4.10 *For any epistemic weight model $\mathcal{M}$ it holds that $\mathcal{M}^\bullet$ is a neighbourhood model.*

Proof. Check that all neighbourhood conditions hold for $\mathcal{M}^\bullet$. In particular, (sc) holds, because for every agent every world has strictly positive weight. □

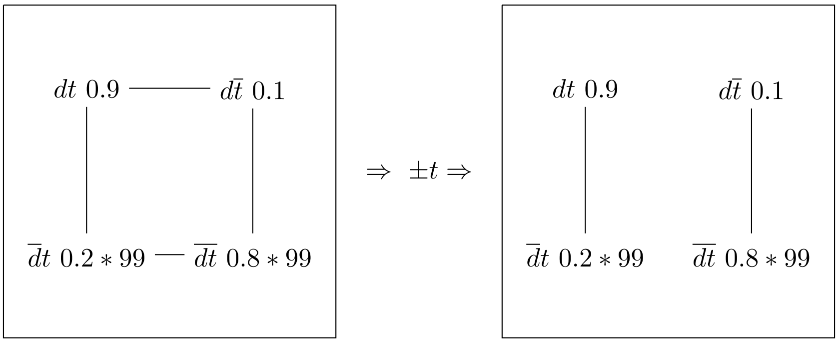
Finally, it follows from the fact that the interpretation of $B_i\phi$ formulas only uses neighbourhoods that it makes no difference whether we interpret ED formulas in epistemic weight models or in their corresponding neighbourhood models.

Theorem 4.11 *For all ED formulas ϕ, for all epistemic probability models $\mathcal{M}$, for all worlds w of $\mathcal{M}$: $\mathcal{M}^{\bullet}, w \models \phi$ iff $\mathcal{M}, w \models \phi$.*

5 Updates

You are from a population with a statistical chance of 1 in 100 of having disease D. The initial screening test for this has a false positive rate of 0.2 and a false negative rate of 0.1. You tested positive (T). Should you believe you have disease D? We can model this with public announcement update.

Example 5.1 [Disease and Test]



Extend the EC language with an operator $[\pm\phi]$, for publicly announcing the *value* of ϕ. This operator is defined as a map on epistemic weight models that maps $\mathcal{M}$ to $\mathcal{M}^{\pm\phi}$, where $\mathcal{M}^{\pm\phi}$ is given by the next definition.

Definition 5.2 If $\mathcal{M} = (W, \sim, L, V)$ is an epistemic weight model and ϕ is a formula of the EC language, then $\mathcal{M}^{\pm\phi} = (W^{\pm\phi}, \sim^{\pm\phi}, L^{\pm\phi}, V^{\pm\phi})$ is given by:

- $W^{\pm\phi} = W$,
- $\sim_i^{\pm\phi} = \{(w, v) \in W^2 \mid w \sim_i v$ and $\mathcal{M}, w \models \phi$ iff $\mathcal{M}, v \models \phi\}$.
- $L^{\pm\phi} = L$,
- $V^{\pm\phi} = V$.

Intuitively, what the operation $[\pm\phi]$ does is cut the i-accessibility links between ϕ and $\neg\phi$ worlds, for all agents i. Everything else remains the same.

The model shows that after the update with t the probability of d equals $\frac{0.9}{0.9+0.2*99} = \frac{9}{207} = \frac{1}{23}$, and after the update with $\neg t$ the probability of d equals $\frac{0.1}{0.1+0.8*88} = \frac{1}{704}$. A huge difference, but with both outcomes it is vastly more probable that you do not have the disease than that you have it.

Compare this with applying Bayes' Rule:

Example 5.3 [Applying Bayes' Rule]

$$P(D|T) = \frac{P(T|D)P(D)}{P(T)} = \frac{P(T|D)P(D)}{P(T|D)P(D) + P(T|\neg D)P(\neg D)}$$

Filling in $P(T|D) = 0.9, P(D) = 0.01, P(\neg D) = 0.99, P(T|\neg D) = 0.2$ gives $P(D|T) = \frac{1}{23}$.

Public announcement update of an epistemic weight model and application of Bayes' rule give the same result, in the precise sense that the probability in the input model gives the prior and the probability in the updated model the posterior.

Public announcement can also be defined for epistemic neighbourhood models, which gives us a kind of poor man's Bayesian update. To interpret $[\pm\phi]$ in epistemic neighbourhood models, define:

Definition 5.4 If $\mathcal{M} = (W, \sim, N, V)$ is an epistemic neighbourhood model and ϕ is a formula of the ED language, then $\mathcal{M}^{\pm\phi} = (W^{\pm\phi}, \sim^{\pm\phi}, N^{\pm\phi}, V^{\pm\phi})$ is given by:

- $W^{\pm\phi} = W$,
- $\sim_i^{\pm\phi} = \{(w, v) \in W^2 \mid w \sim_i v$ and $\mathcal{M}, w \models \phi$ iff $\mathcal{M}, v \models \phi\}$.
- $N_i^{\pm\phi}(w) = \{X \cap [w]_i^{\pm\phi} \mid X \in N_i(w)\}$,
- $V^{\pm\phi} = V$.

Thus, in $\mathcal{M}^{\pm\phi}$ the epistemic accessibilities for all agents are adjusted by cutting the links between ϕ and $\neg\phi$ worlds, and all neighbourhoods are restricted to the appropriate new epistemic accessibility cells. It is not hard to check that the definition is correct: if $\mathcal{M}$ is an epistemic neighbourhood model, then $\mathcal{M}^{\pm\phi}$ is an epistemic neighbourhood model. Axiomatizing the logic of poor man's Bayesian belief (the logic of ED plus the $[\pm\phi]$ operation on epistemic neighbourhood models) is future work.

6 Conclusion and Further Work

We have compared two classes of models for knowledge and belief, one based on neighbourhood semantics, one based on adding weights to worlds in S5 Kripke models, and we have given sound and complete calculi for both model classes. We have also shown that the calculus for neighbourhood models is sound but incomplete for weight models. One can view this as a point in favour of weight models, but one might just as well say that neighbourhood models allow us to make distinctions between shades of belief that are lost in weight models.

In [5], update models for probabilistic epistemic logic are built from sets of formulas that are mutually exclusive. In the present set-up, it is possible to stay a bit closer to the original update model from [2]. A weighted update model is like a weighted epistemic model, but with the valuation function replaced by a function that assigns preconditions and actions (substitutions) to events. The actions take care of factual changes: changes in the values of proposition letters. Update is a product operation, as in [2]. The new i-weight for (w, e) is computed as the product the weights of w and of e. Update products satisfy the requirements for epistemic weight models. Example 5.1 is a special case of this definition. A full blown probabilistic logic of communication and change, in the spirit of [6], is given in [1]. It is future work to give a similar axiomatic treatment to extend the EC calculus.

Alternatively, one might wish to carve out a 'natural' subspace of the set of all possible operations on epistemic weight models, by focussing on a limited set of operations, such as $[\pm\phi_Q]$, for the operation that reveals the value of ϕ to a subset Q of the set of all agents, and $[(\phi, N, M)_Q]$ (with N, M strictly positive natural numbers), for the operation that multiplies the i-weights of the ϕ worlds by N/M (and that of the other worlds by $1 - N/M$) for agents $i \in Q$, $[\sigma_Q]$, for the operation that applies the factual change operation σ, and makes the result visible to agents in Q, while other agents confuse this with 'no change', and finally, $[(p := ?, N, M)]$ for nondetermined change, an update action that makes p true with probability N/M. Each of these operators presents an axiomatisation challenge of its own. Next, one would like to find an illuminating description of the class of updates that the combination of $[\pm\phi_Q]$, $[(\phi, N, M)_Q]$, $[\sigma_Q]$ and $[(p := ?, N, M)]$ allows.

In future, we also would like to extend our treatment to capture the important distinction made in [22] between **risk** and **uncertainty**.

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