Premaster Examination: Logic in Action

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Question 1. Rephrase the following independent statements in natural language (English, Dutch, etc; you may use mathematical terms):

1. $A \in 2^B$ or $A \in \mathcal{P}(B)$
   Answer: set A is an element of a power set of B; set A is a subset of the set B

2. $A = \varnothing$
   Answer: set A is an empty set; set A has no elements

3. $\forall x, f$
   Answer: f holds for all x

Question 2. Rephrase the following independent statements with quantifiers ($\forall$, $\exists$, etc; do not use $\cap$, $\cup$, $\setminus$):

1. $A \subseteq B \setminus C$
   Answer: $\forall x \in A, x \in B \wedge x \notin C$, or $\forall x, x \in A \Rightarrow x \in B \wedge x \notin C$

2. $|A \cup B| = 0$
   Answer: $\nexists x : x \in A \vee x \in B$

3. $A \cap B \in 2^C$
   Answer: $\forall x : x \in A \wedge x \in B \Rightarrow x \in C$

Question 3. Rephrase the following independent statements with relations on sets ($\subseteq$, $\setminus$, $\cup$, $\cap$, etc; do not use any quantifiers):

1. $\forall x \notin A, x \in B$
   Answer: $A = \overline{B}$, or $I \setminus A = B$, or $I \setminus B = A$ (where I is the universal set)

2. $\exists x \notin A, x \in B$
   Answer: $B \setminus A \neq \varnothing$

3. $\exists A \in 2^B, \exists x \in A$
   Answer: $B \neq \varnothing$, or $|B| > 0$

Question 4. Are there more valid potential XML documents or more valid XSLT potential programs? Prove it.
Answer: The number is the same: since both kinds can be represented by files on a computer but can reach arbitrary lengths, there are countably infinite number of them, or $|\mathcal{L}| = \aleph_0$, where $\mathcal{L}$ is the language.

Question 5. Consider the following grammar:

$$F \quad ::= \quad \diamond \mid \ddagger \tag{1}$$
$$E \quad ::= \quad \Psi F \mid F \asymp F \mid F \equiv F \mid \bowtie \tag{2}$$

Recognise correct E terms according to the grammar and draw parse trees of correct ones:

1. $\asymp\diamond\bowtie$
   Answer: Nonsense.

2. $\ddagger\equiv\diamond$
   Answer: Correct.

3. $\diamond\asymp\Psi$
   Answer: Nonsense.

4. $\diamond\asymp\diamond\equiv\ddagger$
   Answer: Incorrect: the infix production rules of E are not recursive.

5. $\Psi\diamond$
   Answer: Correct.

6. $\Psi\bowtie$
   Answer: Incorrect: the first alternative of E does not take us back to F.

Question 6. Consider the following *less than* relation on natural numbers:

$$< \; = \{(n, m) \mid n \in \mathbb{N}, m \in \mathbb{N}, n < m\}$$

Is $<$ reflexive? Irreflexive? Symmetric? Asymmetric? Antisymmetric? Transitive?
Answer: Not reflexive because $(1,1) \notin <$. Irreflexive because $\forall n \in \mathbb{N}, (n,n) \notin <$. Not symmetric because $(2,1) \notin <$ while $(1,2) \in <$. Asymmetric because $\forall n \in \mathbb{N}, \forall m \in \mathbb{N}, ((n,m) \in < \Rightarrow (m,n) \notin <$. Antisymmetric because asymmetric. Transitive because $\forall n \in \mathbb{N}, \forall m \in \mathbb{N}, \forall s \in \mathbb{N}, ((n,m) \in < \wedge (m,s) \in < \Rightarrow (n,s) \in <)$.

Question 7. Define the transitive closure and the reflexive transitive closure of the relation from the previous question.

Answer: Any (preferably formal) definition of the following: $<^+ = <$; $<^* = \leqslant$;

Question 8. Inductively prove that the formula holds for all $n \in \mathbb{N}$:

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} = 2 - \frac{1}{2^n}$$

Answer: Try the base case of $n = 1$. Then assume that it holds for a given n and express the formula for $n + 1$ through it.

Question 9. Consider this sequence of x_k:

$$x_1 = \varnothing; \quad x_{n+1} = 2^{x_n} \cap x_n$$

How many elements does the set of all the elements of this sequence have?
Answer: One, because all x_k are the same.