Hecke algebras and harmonic analysis

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August 2006, ICM Madrid
Outline

1. Introduction
   - Origin and Motivation

2. Harmonic analysis
   - Plancherel formula for p-adic reductive groups
   - The Plancherel formula for affine Hecke algebras

3. Noncommutative geometry
   - K-theory and index functions
Hecke algebras were introduced by Shimura (1950’s) in modern abstract setting, modelled after the famous work of Hecke (1930’s) on Euler products for automorphic forms.
Hecke algebras have turned out to be widely applicable. Besides in number theory Hecke algebras arise in various areas of representation theory, in combinatorics, in the theory of knots, quantum groups, integrable models, orthogonal polynomials. This represents the work of many mathematicians with a variety of backgrounds (see the references in the proceedings article...).
In this talk I will discuss the natural application of Hecke algebras in the **harmonic analysis** of \( p \)-adic reductive groups.

This places affine Hecke algebras in an analytic context. As we will see, this gives rise to new problems, but also it sheds new light on old solutions.

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What we are to discuss here is relevant to various other applications of Hecke algebras as mentioned above, or even in a more general context.
The main problem of harmonic analysis on a locally compact, unimodular, type I group $G$ (e.g. a $p$-adic reductive group):

$$\delta = \int_{\pi \in \hat{G}} \chi_{\pi} \, d\mu(\pi)$$

$\delta$ is the delta distribution at $e \in G$.
$\mu$ is called the Plancherel measure of $G$. It is a positive Radon measure on $\hat{G}$, unique up to normalization of the Haar measure on $G$. 
The Plancherel formula presents a hierarchy of very challenging partial problems.

- Describe the unitary dual $\hat{G}$ of $G$, or at least the support of the Plancherel measure $\text{Supp}(\mu) \subset \hat{G}$.
- Determine the irreducible characters of $G$.
- Compute the Plancherel measure $\mu$. 
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- For finite reductive groups this was exploited very successfully (Howlett, Lehrer, Lusztig,...).
- In this talk we want to study this method to $p$-adic groups.
- For a reductive $p$-adic group $G$ the main underlying idea for this to work is the Bernstein decomposition theorem of the category of smooth representations of $G$. 
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Splitting the Plancherel formula in blocks

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- For a reductive $p$-adic group $G$ the main underlying idea for this to work is the Bernstein decomposition theorem of the category of smooth representations of $G$. 
Due to the work of many (e.g. Borel, Bushnell, Kutzko, Morris, Lusztig, Prasad, Moy,...) the Bernstein blocks are often known to be equivalent to the representation category of certain explicit affine Hecke algebras, the so-called affine Iwahori-Matsumoto Hecke algebras or simply affine Hecke algebras.
In this talk $W$ will be an extended affine Weyl group. $W$ acts discretely by isometries on a Euclidean space $V$, and

- $W$ acts transitively on the set of alcoves (closed chambers) of an affine root hyperplane arrangement in $V$.
- We choose a fundamental alcove $C$. The number of separating affine hyperplanes between $C$ and $w(C)$ is called the length $l(w)$ of $w \in W$.
- $W$ contains the set $S$ of reflections in the walls of $C$. $S \subset W$ is called the set of affine simple reflections.
- We put $W^a = \langle S \rangle \rtimes W$, and $\Omega \subset W$ for the subgroup of length 0 elements. Then $W = W^a \rtimes \Omega$. 
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The Plancherel formula for affine Hecke algebras
$W$ contains a lattice of translations $X \subset V$ as a normal subgroup.

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Definition of the affine Hecke algebra

The affine Hecke algebra $\mathcal{H}$ is the complex, unital associative algebra with basis $N_w$ ($w \in W$) and relations:

- If $u, v \in W$ and $l(uv) = l(u) + l(v)$ then $N_u N_v = N_{uv}$.
- For all affine simple reflections $s \in S$:
  \[(N_s - q(s)^{1/2})(N_s + q(s)^{-1/2}) = 0\]
  for certain positive real numbers $q(s)$ ($s \in S$) such that $q(s) = q(s')$ if $s \sim s'$ in $W$.

- The affine Hecke algebra $\mathcal{H}(W, q)$ is a deformation of the group algebra $\mathbb{C}[W]$ of the affine Weyl group, in the positive parameters $q(s)$. 

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for certain positive real numbers $q(s)$ ($s \in S$) such that $q(s) = q(s')$ if $s \sim s'$ in $W$.

- The affine Hecke algebra $\mathcal{H}(W, q)$ is a deformation of the group algebra $\mathbb{C}[W]$ of the affine Weyl group, in the positive parameters $q(s)$. 
Theorem (Bernstein, Zelevinski, Lusztig)

Let $T$ be the complex algebraic torus with character lattice $X$. There exists an injective algebra homomorphism $\theta : \mathbb{C}[T] \rightarrow \mathcal{H}$ such that $\mathcal{H}$ is generated by the commutative subalgebra $\mathcal{A} = \theta(\mathbb{C}[T]) \subset \mathcal{H}$ and by the elements $N_s$ with $s \in S_0$ (the simple reflections in $W_0$). The cross relations:

$$N_s.a - a^s.N_s = (q(s)^{1/2} - q(s)^{-1/2}) \frac{a - a^s}{1 - \theta_{-\alpha}}$$

Corollary: the center of $\mathcal{H}$

The center $\mathcal{Z}$ of $\mathcal{H}$ is equal to $\mathcal{Z} = \mathcal{A}^{W_0} = \mathbb{C}[T]^{W_0}$. 
\( \mathcal{H} \) is a \( \ast \) algebra with distinguished tracial state.

The anti-linear anti-involution \( \ast \)

We define an anti-linear anti-involution on \( \mathcal{H} \) by \( N_w^\ast := N_{w^{-1}} \).

The distinguished trace \( \tau \) on \( \mathcal{H} \)

We define \( \tau(N_w) = \delta_{w,e} \).

Positivity: \( \tau \) is positive, hence a tracial state

The basis \( N_w \) is orthonormal with respect to the Hermitian form \( (x, y) = \tau(x^\ast y) \) on \( \mathcal{H} \). In particular, \( \tau \) is a positive tracial state on the \( \ast \)-algebra \( (\mathcal{H}, \ast) \).
Abstract Plancherel formula (Matsumoto)

There exists a compact topological space $\hat{\mathcal{H}}$ of irreducible $\ast$-representations of $\mathcal{H}$, and a unique positive measure $\mu_{Pl}$ on $\hat{\mathcal{H}}$ such that

$$\tau = \int_{\pi \in \hat{\mathcal{H}}} \chi_\pi d\mu_{Pl}(\pi)$$

There is a finite-to-one map $z : \hat{\mathcal{H}} \to W_0 \setminus T$. 
The Plancherel formula for affine Hecke algebras

Eisenstein functionals

Weyl denominator and q-Weyl denominator

We define

\[ D(t) = \prod_{\alpha \in R_0, +} (1 - \alpha(t)^{-1}) \]

\[ D_q(t) = \prod_{\alpha \in R_0, +} (1 - q_{\alpha}^{-1} \alpha(t)^{-1}) \]

Eisenstein functionals on \( \mathcal{H} \)

There exists a unique holomorphic function \( E : T \rightarrow \mathcal{H}^* \) (\( \mathcal{H}^* \) is the linear dual of \( \mathcal{H} \)) such that \( E_t(1) = D(t) \) and such that for all \( t \in T \):

\[ E_t(\theta_x h) = E_t(h\theta_x) = x(t)E_t(h) \]
The Plancherel formula for affine Hecke algebras

**Integral formula for the trace $\tau$ of $\mathcal{H}$**

**Contour integral for the trace**

Write $T = T_v T_u$ for the polar decomposition of $T$ in a vector group $T_v \cong \mathbb{R}^n$ and a compact torus $T_u \cong (S^1)^n$. Then

$$
\tau(h) = q(w_0)^{-1} \int_{t \in pT_u} \frac{E_t(h)}{D(t)} \frac{D(t)D(t^{-1})}{D_q(t)D_q(t^{-1})} dt
$$

where $p \in T_v$ is deep in the negative chamber of $T_v$. 
The poles of the kernel

We would like to refine the contour integral to a spectral decomposition of $\tau$, or at least the diagonalization of the center in the Hilbert space completion $L^2(\mathcal{H})$ of $\mathcal{H}$. This is a computation of residue distributions.

Residual cosets

A residual coset $L \subset T$ is a coset of a (complex) subtorus of $T$ along which the pole order $o_L$ of

$$\frac{D(t)D(t^{-1})}{D_q(t)D_q(t^{-1})}$$

is at least equal to $\text{codim}(L)$. 
The Plancherel formula for affine Hecke algebras

Existence and Uniqueness of Residues

**Tempered real form of a residual coset**

The **tempered part** $L^t$ of a residual coset $L$ is the unique compact form of $L$ such that its projection $c_L$ to $T_v$ has minimal distance to $e \in T_v$. We call $c_L$ the **center** of $L$.

**Theorem (with Heckman)**

Let $h \in \mathcal{H}$. There exist **unique** distributions $\mathcal{X}_L^h$ on $c_L T_u$ supported on $L^t \subset c_L T_u$ such that for all $a \in \mathcal{A} = \mathbb{C}[T]$,

$$
\tau(ha) = \sum_{L \text{ residual}} \mathcal{X}_L^h(a|_{c_L T_u})
$$
The Plancherel formula for affine Hecke algebras
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From distributions to positive measures

**Positivity Theorem**

The restriction of $\mathcal{X}^1 = \sum_L \mathcal{X}_L^1$ to $\mathcal{Z} = A^{W_0} = \mathbb{C}[T]^{W_0}$ (the center of $\mathcal{H}$) defines a positive, $W_0$-invariant measure $\nu = \mathcal{X}^1|_\mathcal{Z}$ on $T$, supported in the union $S = \cup_L L^t$ of the tempered residual cosets.

**Continuity Theorem**

For any $h \in \mathcal{H}$ we have: $\mathcal{X}^h = \sum_L \mathcal{X}_L^h$ is defines a complex measure which is absolutely continuous with respect to $\nu$.

**Support Theorem (no cancellation of residues)**

$$\text{Supp}(\nu) = S = \cup_L L^t$$
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Implications for Root Systems

**Bound on Pole Order (to appear)**

For any coset $L$ of a subtorus of $T$ the pole order $o_L$ along $L$ is at most equal to $\text{codim}(L)$. Explicitly:

\[
o_L := \left| \{ \alpha \in R_0 \mid \alpha\big|_L = q_{\alpha^\vee} \} \right| - \left| \{ \alpha \in R_0 \mid \alpha\big|_L = 1 \} \right| \leq \text{codim}(L)
\]

**Parabolic induction of residual cosets**

Let $L$ be a residual coset, and let $R_L \subset R_0$ be the subset of roots which are constant on $L$. Then $\text{rank}(R_L) = \text{codim}(L)$ and there exists a $R_L$-residual point (i.e. a zero dimensional residual coset) $r_L \in T_L$ such that $L = r_LT_L$. 
\(W_0\)-orbits of residual points are hermitian (to appear)

Let \( r \in T \) be a residual point. Then \( r^{-1} \in W_0 r \).

No embedded tempered residual cosets (to appear)

If \( L, M \) are residual cosets and \( L \neq M \) then \( M^t \not\subset L^t \)
(equivalently, the spectral measure \( \nu_L \) on \( L^t \) is smooth).
The Plancherel formula for affine Hecke algebras

Diagonalization of the center \( \mathcal{Z} \) in \( L^2(\mathcal{H}) \)

**Spectral decomposition for \( \mathcal{H} \) as \( \mathcal{Z} \)-module**

We have

\[
\tau(h) = \int_{t \in S \subset T} \chi_t(h) d\nu(t)
\]

where:

- \( t \rightarrow \chi_t(h) \) is a \( W_0 \)-invariant function on \( S \).
- For all \( t \in S \), \( \chi_t \) is a positive tracial state with central character \( W_0 t \).
- \( \chi_t \) is a positive linear combination of (finitely many) irreducible tempered characters of \( \mathcal{H} \) with central character \( W_0 t \).
- On each \( L^t \), \( \nu \) is a smooth measure, given by an explicit product formula.
Unknowns: the set of discrete series at residual points

**Discrete series characters**

- For each $W_0$-orbit $W_0 r \in T$ of residual points let $\Delta_{W_0 r}$ denote the (finite, nonempty) set of equivalence classes of irreducible discrete series representations $(\delta, V_\delta)$ (with $\chi_\delta \in L^2(\mathcal{H})$), and $d_\delta \in \mathbb{R}_+$ such that

  $$\chi_r = \sum_{\delta \in \Delta_{W_0 r}} d_\delta \chi_\delta$$

- If we put $q(s) = q^{f_s}$ for fixed $f_s \in \mathbb{R}$ (for $s \in S$) then the numbers $d_\delta > 0$ are independent of $q > 1$ (we say: $d_\delta$ is independent of scaling isomorphisms).
The Plancherel formula for affine Hecke algebras
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Equal parameter case

In the equal parameter case with \( X = P \) (the weight lattice) there is a natural bijection between the set of \( W_0 \)-orbits of residual points and \( \mathcal{L}G \)-conjugacy classes of pairs \((s, u) \in \mathcal{L}G \times \mathcal{L}G\) with \( s \in \mathcal{L}G \) semisimple such that the centralizer \( C_{\mathcal{L}G}(s) \) is semisimple, and \( u \in C_{\mathcal{L}G}(s) \) distinguished unipotent.

Theorem (Kazhdan-Lusztig)

Let \( W_0r \) correspond to the orbit of \((s, u)\).
Then \( \Delta_{W_0r} \) is in bijection with the set of geometric irreducible characters of \( C_{\mathcal{L}G}(s, u)/ZC_{\mathcal{L}G}(s, u)^0 \).
The Plancherel formula for affine Hecke algebras

Formal dimension product formula

The Plancherel measure of $\delta \in \Delta_{W_0 r}$ equals:

$$\mu_{Pl}(\{\delta\}) = d_\delta \nu(\{W_0 r\}) = d_\delta \lambda_{W_0 r} q(w_0) \frac{\prod'_{\alpha \in R_0} (\alpha(r) - 1)}{\prod'_{\alpha \in R_0} (q_{\alpha^\vee} \alpha(r) - 1)}$$

where $\lambda_{W_0 r} \in \mathbb{Q}$ and where $\prod'$ denotes the product in which we omit the factors which are equal to 0.
The Plancherel formula for affine Hecke algebras

**Application of product formula: formation of $L$-packets**

The product formula of $\nu$ combined with the scale invariance of the $d_\delta$ were used to determine the unipotent discrete series $L$-packets for the exceptional $p$-adic Chevalley groups of adjoint type (Heckman-O. (based on conjectures which have been solved now), Reeder). Lusztig has determined the $L$-packets in general for $p$-adic Chevalley groups of adjoint type by other methods.
The Plancherel formula for affine Hecke algebras

**Groupoid of induction data**

- $\mathcal{H}$ determines a **compact orbifold groupoid** $\mathcal{W}_\Xi$ whose set of objects $\Xi$ consists of parabolic induction data $\xi = (R_P, \delta, t)$ with $R_P \subset R_0$ a standard parabolic subsystem, with $\delta \in \Delta(\mathcal{H}_P)$ an irreducible discrete series character for $\mathcal{H}_P = \mathcal{H}(X_P, R_P, Y_P, R_P^\vee)$, and with $t \in T_P$.

- The set of arrows $\mathcal{W}_{\xi, \eta}$ from $\xi = (P, \delta, t) \in \Xi$ to $\eta = (Q, \delta', t') \in \Xi$ in the groupoid $\mathcal{W}_\Xi$ are given by twisting of $\xi$ with Weyl group elements $w$ (and by elements a certain finite abelian group) such that $w(\xi) = \eta$ (in particular, $w(P) = Q$).

- There is a natural finite surjective continuous map

  $$|\Xi| : = \mathcal{W}\setminus \Xi \rightarrow W_0\setminus S \subset W_0\setminus T$$

  $$(P, \delta, t) \rightarrow W_0(r_\delta t)$$
The Plancherel formula for affine Hecke algebras

The Fourier transform

**Plancherel formula and Fourier transform**

There exists a smooth projective unitary representation $\pi$ of the groupoid $\mathcal{W}_\Xi$ in $\text{Mod}_{\text{unitary}}^{\text{fd}}(\mathcal{H})$, and a unique measure $\mu_{Pl}$ such that the Fourier transform

$$\mathcal{F} : L^2(\mathcal{H}) \rightarrow L^2(\Xi, \text{End}(\pi), \mu_{Pl})^\mathcal{W}$$

$$h \rightarrow \{\xi \rightarrow \pi(\xi)(h)\}$$

is an isometric isomorphism of $\mathcal{H} \times \mathcal{H}$-bimodules.
Fourier isomorphism on The Schwartz algebra

Theorem/Definition Schwartz algebra

We define \( S = \{ s = \sum_{w \in W} c_w N_w \mid \forall n \in \mathbb{N} : p_n(s) := \sup_{w \in W} \{|c_w| l(w)^n\} < \infty \} \). This a Fréchet algebra with respect to the seminorms \( p_n \).

Theorem (Delorme-O.): Support of Plancherel measure

The support of \( \mu_{Pl} \) is the set \( \hat{S} \) of irreducible tempered representations of \( \mathcal{H} \), i.e. those irreps which extend continuously to \( S \).
The Schwartz algebra

**Theorem (Delorme-O.)**

The Fourier isomorphism restricts to an isomorphism

\[ \mathcal{F}_S : S \rightarrow C^\infty(\Xi, \text{End}(\pi))^W \]  

(1)

**Corollary: The center \( \mathcal{Z}_S \subset S \)**

The center \( \mathcal{Z}_S = C^\infty(\Xi)^W \) of the algebra \( S \) is the algebra of smooth functions on the orbifold \( \mathcal{W}_\Xi \).

**Corollary**

For every irreducible tempered representation \( \rho \) there is a unique orbit \( \mathcal{W}_\xi \) so that \( \rho \) is equivalent to a summand of \( \pi(\xi) \). We call \( \mathcal{W}_\xi \) the tempered central character \( z_S(\rho) \) of \( \rho \).
Support of Plancherel measure

Fundamental problem: The support of $\mu_{Pl}$

Determine (parametrize) $\text{Supp}(\mu_{Pl}) \subset \hat{\mathcal{H}}$. 
Reduction to discrete series

Let $\Delta = \bigcup_P (P, \Delta(\mathcal{H}_P))$ ($P$ standard parabolic subset), with $\Delta(\mathcal{H}_P)$ the set of irreducible discrete series of $\mathcal{H}_P$. Then $\mathcal{W}$ acts on this set by twisting, and this defines a finite groupoid $\mathcal{W}_\Delta$.

- If we would be interested in $\text{Supp}(\mu_{Pl})$ only up to $\mu_{Pl}$ 0-sets then it is enough to determine $\mathcal{W}_\Delta$, because this determines $\mathcal{W}_\Xi$.

- **Problem A’**: Determine the finite groupoid $\mathcal{W}_\Delta$.

- To describe $\text{Supp}(\mu_{Pl})$ precisely we will need slightly more information: Choose a realization $(V_\delta, \delta)$ for each class $(P, [\delta])$ in $\Delta$, and an intertwining isomorphism $\tilde{\delta}_w : V_\delta \to V_{\delta'}$ for each twisting isomorphism $w : \mathcal{H}_P \to \mathcal{H}_Q$. This yields a projective unitary representation $\tilde{\delta}$ of $\mathcal{W}_\Delta$, unique up to natural isomorphisms. Let $\eta_\Delta \in H^2(\mathcal{W}_\Delta, \mathbb{C}^\times)$ be its cohomology class.

- **Problem A**: Determine $\mathcal{W}_\Delta$ and $\eta_\Delta$. 


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- **Problem A:** Determine $\mathcal{W}_\Delta$ and $\eta_\Delta$. 
Reduction to discrete series

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- If we would be interested in $\text{Supp}(\mu_{Pl})$ only up to $\mu_{Pl}$ 0-sets then it is enough to determine $\mathcal{W}_\Delta$, because this determines $\mathcal{W}_\Xi$.

- Problem $A'$: Determine the finite groupoid $\mathcal{W}_\Delta$.

- To describe $\text{Supp}(\mu_{Pl})$ precisely we will need slightly more information: Choose a realization $(V_\delta, \delta)$ for each class $(P, [\delta])$ in $\Delta$, and an intertwining isomorphism $\tilde{\delta}_w : V_\delta \to V_{\delta'}$ for each twisting isomorphism $w : \mathcal{H}_P \to \mathcal{H}_Q$. This yields a projective unitary representation $\tilde{\delta}$ of $\mathcal{W}_\Delta$, unique up to natural isomorphisms. Let $\eta_\Delta \in H^2(\mathcal{W}_\Delta, \mathbb{C}^\times)$ be its cohomology class.

- Problem $A$: Determine $\mathcal{W}_\Delta$ and $\eta_\Delta$. 
Explicit answers to $A$ are known in special cases, in particular in the case of equal parameters and $X = P$, by the work of Kazhdan and Lusztig.

Lusztig has solved $A$ (in principal at least) for more general cases, by geometric methods.
Definition: Analytic R-groups (Delorme-O.)

For each orbit $\xi \in \Xi$ we define (up to isomorphism) an analytic R-group $\mathcal{R}_\xi \subset \mathcal{W}_{\xi,\xi}$ depending on the density of $\mu_{Pl}$ at $\xi$. In addition we define a 2-cocycle $\eta_\xi$ for $\mathcal{R}_\xi$ in terms of $\eta_\Delta$ (recall Problem A).

Knapp-Stein type Theorem (Delorme-O.)

The module $\pi(\xi)$ is a unitary $\mathcal{S} - \mathbb{C}[\mathcal{R}_\xi, \eta_\xi]$-bimodule which defines a Morita equivalence

$$E_\xi : \text{Mod}(\mathbb{C}[\mathcal{R}_\xi, \eta_\xi]) \to \text{Mod}_{\mathcal{W}_{\xi,\text{unitary}}} (\mathcal{S}).$$

Reduction to discrete series

The above Theorem reduces the problem to describe $\text{Supp}(\mu_{Pl})$ completely to Problem A.
The Plancherel formula for affine Hecke algebras

Fundamental problem: Computation of $\mu_P$

The problem to compute $\mu_P$ reduces to

**Computation constants $d_\delta(H_P)$**

**Problem B**: For $(P, \delta) \in \Delta$, compute the positive real constants $d_\delta(H_P)$.

Answers to B are known for exceptional root systems in the equal par. case (M. Reeder). In the general equal parameter case there is a conjecture for their value; in general not even that.

**Conjecture**

The $d_\delta$ are rational numbers.
The remaining part of this talk

In the remaining part of this talk I want to discuss conjectures in noncommutative geometry which might help to solve problems A and B. This approach is different from the algebro-geometric approach but based on deformations in the parameters $q(s)$. We expect this to be especially useful (effective) in unequal label cases.

We will assume (for simplicity) from here on that the root datum is semisimple, i.e. that $\mathbb{Q}R_0 = \mathbb{Q} \otimes X$. 
The Euler-Poincaré map

Theorem (O., Reeder) (unpublished)

Every $\mathcal{H}$-module of finite length has homological dimension $\leq \dim \mathbb{C}(T)$. In fact we have defined an explicit functor $\mathcal{F} : \text{Mod}^{\text{fl}}(\mathcal{H}) \to K_b(\mathcal{H})$ (the category of bounded complexes of projective $\mathcal{H}$-modules), together with a projective resolution $\mathcal{F}_\bullet(V) \to V \to 0$.

Change of base ring

Let $K_0(\mathcal{H})$ be Grothendieck group of the finitely generated projective $\mathcal{H}$ modules, and let $G(S)$ be the Grothendieck group of the $S$-modules of finite length. We have natural homomorphisms $\beta : K_0(\mathcal{H}) \to K_0(S)$ (change of base ring) and $\rho : G(S) \to G(\mathcal{H})$ (forget tempered).
We denote by

**Euler-Poincaré map**

We define an anti-linear Euler-Poincaré map $\varepsilon : G^C(S) \to K_0^C(S)$ such that $\varepsilon(V) = \beta(\sum_i (-1)^i [F_i(\rho(V))])$ for all $V \in \text{Mod}^{\text{fl}}(S)$.

**Definition**

Let $G_i^C(S)$ denote the subspace of $G^C(S)$ generated by the modules induced from proper parabolic subquotient algebras $\mathcal{H}_P$. We define $\text{Ell}^{\text{temp}} := G^C(S)/G_i^C(S)$, a $\mathcal{Z}_S$-module.
Theorem (after Arthur)

The equivalence $E_\xi$ induces a linear isomorphism $G(E_\xi) : G^c(\mathcal{R}_\xi, \eta_\xi) \rightarrow G^c(S)_W\xi$. Via $G(E_\xi)$ the specialization $Ell_{W\xi}^{\text{temp}} = Ell_{W\xi}^{\text{temp}} \otimes \mathcal{Z}_S \mathcal{C}_W\xi$ corresponds to the space $Ell(\mathcal{R}_\xi, \eta_\xi)$ of $\eta$-twisted class functions of $\mathcal{R}_\xi$ which have support on the set of elliptic conjugacy classes of $\mathcal{R}_\xi$, i.e. the elements $r \in \mathcal{R}_\xi$ for which $\xi$ is an isolated fixed point.

Corollary

The space $Ell_{W}^{\text{temp}}$ is a finite dimensional semisimple $\mathcal{Z}_S = C^\infty(\Xi)_W$-module.
The index pairing

Definition of the index pairing

There is a natural “index pairing” $[\cdot, \cdot] : K_0(S) \times G(S) \to \mathbb{Z}$ defined as follows: given an idempotent $p \in M_n(\mathbb{C}) \otimes S$ and a $(V, \pi) \in \text{Mod}^{\text{fl}}(S)$ we set $[p, [\pi]] := \text{rank}((\text{Id} \otimes \pi)(p))$.

The discrete part $K_{0}^{\text{discr}}(S) \subset K_{0}^{\mathbb{C}}(S)$ of $K_0(S)$

We define the support of a class $[\alpha] \in K_{0}^{\mathbb{C}}(S)$ to be the subset $\{ \mathcal{W}_\xi \in \mathcal{W} \setminus \Xi | \exists \pi \in G_{\mathcal{W}_\xi}(S) : [\alpha, [\pi]] \neq 0 \}$. We define $K_{0}^{\text{discr}}(S) \subset K_{0}^{\mathbb{C}}(S)$ to be the subspace of classes whose support has dimension 0.
The index pairing

Theorem

The anti-linear map $\epsilon$ factors through $\epsilon : \Ell_{\text{temp}} \to K_0^{\text{discr}}(S)$.

Hermitian pairing on $\Ell_{\text{temp}}$

We define a Hermitian pairing $\langle \cdot, \cdot \rangle_{\text{el}}$ on $\Ell_{\text{temp}}$ by

$\langle \chi, \psi \rangle := [\epsilon(\chi), \psi]$. 
The index pairing

Conjecture A ("Arthur type formula")

Let $\phi, \psi \in \text{Ell}(\mathcal{K}_\xi, \eta_\xi)$ be elliptic twisted characters of $\mathcal{K}_\xi$. Then

$$\langle G(E_\xi)(\phi), G(E_\xi)(\psi) \rangle_{el} = |\mathcal{K}_\xi|^{-1} \sum_{r \in \mathcal{K}_\xi} \det(1 - r) \overline{\phi(r)} \psi(r)$$

Elliptic representations with distinct tempered central character are orthogonal.

Remark

The pairing on Ell($\mathcal{K}_\xi, \eta_\xi$) is positive definite (since the support of the function $\det(1 - r)$ is exactly equal to the set of elliptic conjugacy classes of $\mathcal{K}_\xi$).
This conjecture is the analogue of well known results for $p$-adic groups.

Kazhdan conjectured that the elliptic pairing of elliptic characters of a $p$-adic reductive group $G$ can be rewritten as integral over the set of regular semisimple elliptic conjugacy classes of $G$ with respect to a canonical measure $d\mu$.

The Kazdan conjecture was proved by Bezrukavnikov, and independently by Schneider-Stuhler.

Arthur derived the pairing in terms of analytic $R$-groups from this integral over the elliptic classes, using the local trace formula.

For the Hecke algebra $\mathcal{H}$ the intermediate step, the integral over the regular elliptic conjugacy classes make no sense. Also these results are based on the theory of orbital integrals, which is not available for Hecke algebras.
Corollary

The Hermitian pairing $\langle \cdot, \cdot \rangle_{el}$ on $\text{Ell}^{\text{temp}}$ is positive definite.

Corollary

$\epsilon : \text{Ell}^{\text{temp}} \rightarrow K_0^{\text{discr}}(S)$ is an anti-linear isomorphism.

Corollary

The irreducible discrete series representations $(U, \delta) \in \Delta(\mathcal{H})$ give rise to an orthonormal set of vectors $\{[U]\}$ in $\text{Ell}^{\text{temp}}$.

Index function

Let $(U, \delta) \in \text{Mod}^{\text{fl}}(S)$. There exists an index function $f_U \in \mathcal{H}$ for $U$ such that for any virtual tempered character $\chi \in G^C(S)$ we have $\langle \chi_U, \chi \rangle_{el} = \chi(f_U)$. 
If \((U, \delta) \in \Delta_{W_0r}\) then

\[
\mu_{Pl}(\{\delta\}) = d_\delta \nu(\{W_0r\})
\]

\[
= d_\delta \lambda_{W_0r} q(w_0) \frac{\prod'_{\alpha \in R_0} (\alpha(r) - 1)}{\prod'_{\alpha \in R_0} (q_{\alpha \vee}(r) - 1)}
\]

\[
= \tau(f_U)
\]

\[
= \sum (-1)^{\dim(f)} \sum_{\sigma \in \text{Irr}(\mathcal{H}_f \rtimes \Omega_f)} [U |_{(\mathcal{H}_f \rtimes \Omega_f) \otimes \epsilon_f : \sigma}] d_\sigma(q)
\]

where \(f\) runs over a complete set of representatives of the \(\Omega\)-orbits of faces of the alcove \(C\), and where \(d_\sigma(q) \in \mathbb{Q}(q_s^{1/2})\) denotes the formal dimension of \(\sigma\) in the finite dimensional Hilbert algebra \(\mathcal{H}_f \rtimes \Omega_f\) whose trace is the restriction of the trace \(\tau\) of \(\mathcal{H}\).
Conjecture A for $\langle \cdot, \cdot \rangle_{el}$ applies to problem B.

By Euler-Poincaré identity, we see that $d_\delta \in \mathbb{Q}_+$. 
Towards problem A: Finding the support of $\mu_{P_l}$

- Assume that $q(s) = q^{f_s}$ for all $s \in S$ and fixed $f_s \in \mathbb{R}$. Given a tempered irreducible representation $(V, \pi)$ of $\mathcal{H}$, there exists a unique continuous family $(V, \pi(q))$ (with $q > 1$) of tempered irreducible representations of $\mathcal{H}(q)$ such that the unitary part of the central character of $\pi(q)$ is independent of $q$. The limit $\pi(1)$ exists and is a tempered representation (not semisimple, in general) of $\mathcal{H}(1) = \mathbb{C}[W]$.

- This defines a linear map $\text{ev}_{q \to 1} : G(S(q)) \to G(S(W))$, preserving all “K-types” (Tits’ deformation lemma) and sending $G_I(S(q))$ to $G_I(S(W))$.

Consequence of Conjecture A

The induced map $\text{ev}_{q \to 1, el}$ is isometry for $\langle \cdot, \cdot \rangle_{el}$, and thus injective on $\text{Ell}_{\text{temp}}^\mu(q)$. 
Towards problem A: Describing the support of $\mu_{PL}$

Maarten Solleveld has defined a homomorphism $\alpha(q): K_0(S(W)) \rightarrow K_0(S(q))$, using the continuous family of Fréchet algebras $S(q)$ and the fact that the Fréchet algebras $S(q)$ and $S(q')$ are isomorphic for any $q, q' > 1$.

Consequence of Conjecture A

We have $\alpha^C(q)(K_0^{\text{discr}}(S_W)) \subset K_0^{\text{discr}}(S(q))$, and $\epsilon_q^{-1} \circ \alpha^C(q) \circ \epsilon_{q=1}$ is the adjoint of $\text{ev}_{q \rightarrow 1, e_l}$. In particular, this restriction of $\alpha^C(q)$ to $K_0^{\text{discr}}(S_W)$ is onto $K_0^{\text{discr}}(S(q))$.
Conjecture B: The invariance of the groups $K_0(S(q))$

The linear map $\alpha^C(q)$ is an isomorphism.

For equal parameters this follows from general results of Maarten Solleveld combined with a Theorem of Baum and Nistor for the periodic cyclic homology $HP_*(\mathcal{H})$.

- Solleveld proved that the Chern character yields an isomorphism $ch : K_*^C(S) \xrightarrow{\sim} HP_*(S)$.
- Solleveld also proved $HP_*(S) \simeq HP_*(\mathcal{H})$.
- Baum and Nistor proved that $HP_*(\mathcal{H}) \simeq HP_*(\mathbb{C}[W])$ ($\mathcal{H}$ with equal parameters). With Solleveld’s results this implies the conjecture for this case.

- For Hecke algebras of reductive $p$-adic groups this conjecture is stated by Baum, Connes and Higson (1994) in relation to the Baum-Connes conjecture.
Conjecture A and Conjecture B imply

**Conjecture C**

\[ \text{ev}_{q \rightarrow 1, e^I} : \text{Ell}_{temp}^\bullet(S(q)) \rightarrow \text{Ell}_{temp}^\bullet(S(W)) \]  \hspace{1cm} (2)

is an isomorphism.

This was proved by Reeder for equal parameters and \( X = P \). It was shown in greater generality by Waldspurger (I think).

**Refinement of Conjecture C**

The family \( q \rightarrow \text{Ell}_{temp}^\bullet(S(q)) \) of semisimple \( \mathcal{Z} \approx \mathbb{C}[T]^{W_0} \)-modules is **continuous**.
\[ q(s_1) > q(s_0) > 1 \]
$q(s_1) = q(s_0) > 1$
\[ q(s_1) = q(s_0) = 1 \]
Same events for \( \text{Ell}^{\text{temp}} \)
K-theory and index functions

\[ q(s_1) > q(s_0) > 1 \]
K-theory and index functions

$q(s_1) = q(s_0) > 1$
\[ q(s_1) = q(s_0) = 1 \]
Towards **problem A**: Finding the support of $\mu_P$.

**Reduction to generic case**

Assuming conjecture C (refined) we can reduce the classification problem for $\text{Supp}(\mu_P)$ in principle to the generic case. This is a considerably simpler problem for the non simply laced cases.
Example: The 3-parameter case $C_{n}^{\text{aff}}$

- We assume $X = \mathbb{Z}^n$, $R_0 = \{ \pm e_i, \pm e_j \}$) The orbits of residual points for generic parameters $q = (q^1, q^2, q^3)$ are parametrized by ordered pairs $(\lambda, \mu)$ of partitions with $|\lambda| + |\mu| = n$ (easy).

- On the other hand, the elliptic conjugacy classes of the $R$-groups for $q = (1, 1, 1)$ (the group algebra of the affine Weyl group) are also parameterized by pairs $(\kappa, \nu)$ of $|\kappa| + |\nu| = n$ (easy).

- But we know that each residual orbit carries at least one irreducible discrete series representation!

- Hence **Conjecture C** implies: in the generic case, each orbit of residual points carries one irreducible discrete series representation.
Example: The 3-parameter case $C_{n}^{aff}$

- For non generic $q^{0}$: we can still easily parameterize the orbits $W_{0}r$ of residual points. But what about $\Delta_{W_{0}r}$?
- By Conjecture C (refined): the set $\Delta_{W_{0}r}$ is in bijection with the set of generic residual orbits which coincide with $W_{0}r$ when we specialize at $q = q^{0}$.
- This determines the action of $\mathcal{V}$ on $\Delta(q = q^{0})$.
- One easily proves that always $\eta_{\Delta} = 1$.
- Using the theory of the analytic $R$-group we can now reconstruct the tempered dual.
It was verified by Klaas Slooten that this gives rise to the familiar combinatorial description of the Kazhdan-Lusztig parameters in terms of Lusztig’s family symbols when $q_1 = q_2 = q_3 = q$ (which is highly nongeneric of course).

For other “special values” of $q$, Slooten gave a similar combinatorial description based on the generalized symbols of Lusztig and Shoji (in the real central character case).