Hecke algebras and harmonic analysis

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Abstract. Iwahori–Hecke algebras are ubiquitous. One encounters these algebras in subjects as diverse as harmonic analysis, equivariant K-theory, orthogonal polynomials, quantum groups, knot theory, algebraic combinatorics, and integrable models in statistical physics. In this exposition we will mostly concentrate on the analytic aspects of affine Hecke algebras and study them from the perspective of operator algebras. We will discuss the Plancherel theorem for these type of algebras, and based on a conjectural invariance property of their (operator algebraic) K-theory, study the structure of the tempered dual.

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1. Introduction

Let G be a group and let $K \subset G$ be an almost normal subgroup of G, i.e. a subgroup whose double cosets are finite unions of one-sided cosets. The Hecke algebra of the pair (G, K) is the convolution algebra of $\mathbb{Z}$-valued functions with finite support on the double coset space $K\backslash G/K$. More generally, given a K-module (M, σ) (over some commutative, unital ring R) one considers the convolution algebra $H_R(G, K, \sigma)$ of $\mathrm{End}_R(M^\vee)$-valued $(K, \sigma^\vee)$-spherical functions which are supported on finitely many double cosets of K. Hecke algebras were introduced in this abstract setting by Shimura in the 1950s, following the original work of Hecke on certain linear operators acting in a space of modular forms. The study of representations of Hecke algebras in spaces of modular forms is of basic importance for the study of modular forms.

Later it became apparent that this concept is also fundamental for understanding the representation theory of finite reductive groups, and this seems also true for p-adic reductive groups. Let k be a non-archimedean local field and let G be the group of k-points of a connected reductive group defined over k, equipped with the locally compact, totally disconnected Hausdorff topology it inherits from k. Any compact open subgroup $K \subset G$ is almost normal, hence in this situation we have a large supply of Hecke algebras of the form $H_{\mathbb{C}}(G, K, \sigma)$ where (V_σ, σ) is a complex finite dimensional smooth representation of K.

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The fundamental and beautiful result which is the underpinning of the application of these Hecke algebras to the representation theory of G is Bernstein's Decomposition Theorem [11]. It states that the category of smooth representations of G has a canonical decomposition as a product of blocks $\mathfrak{R}_{\mathfrak{s}}$ which are parametrized by the components $\mathfrak{s}$ of the Bernstein variety Σ of "supercuspidal pairs" (L, ρ) modulo G-conjugacy, where $L \subset G$ is a Levi-subgroup, and where ρ is an irreducible supercuspidal representation of L, i.e. a representation whose matrix coefficients are compactly supported modulo the center of L. Each block is by construction equivalent to the category of modules over a two sided ideal $C_c^{\infty}(G)_{\mathfrak{s}}$ of the convolution algebra $C_c^{\infty}(G)$ of compactly supported, locally constant complex valued functions on G (for the sake of this exposition I will refrain from calling $C_c^{\infty}(G)$ the Hecke algebra of G, although it is customary to so).

It is a major question how to describe the blocks $\mathfrak{R}_{\mathfrak{s}}$. Bushnell and Kutzko [16], [17] introduced the notion of $\mathfrak{s}$-types. A pair (K, σ) is called an $\mathfrak{s}$-type if the block $\mathfrak{R}_{\mathfrak{s}}$ consists precisely of those smooth representations of G which are generated by their (K, σ) isotypic component. In that case the functor $V \rightarrow \mathrm{Hom}_K(V_\sigma, V|_K)$ is an equivalence from $\mathfrak{R}_{\mathfrak{s}}$ to the category of $\mathcal{H}_{\mathbb{C}}(G, K, \sigma)$-modules. This notion of types originates from the work of Borel [14], who showed that an irreducible smooth G-module V is a subquotient of the unramified principal series iff V contains fixed vectors with respect to an Iwahori subgroup $B \subset G$ (the corresponding component of Σ is called the "Borel component"). Through the work of Bushnell and Kutzko (loc. cit.), Morris [51], [52], and Moy and Prasad [53] $\mathfrak{s}$-types are known to exist in many cases. For instance $\mathfrak{s}$-types always exist for "level 0" components $\mathfrak{s}$.

The algebra $H_{\mathbb{C}}(G, K, \sigma)$ comes equipped with a trace $\mathrm{tr}\colon f \rightarrow \mathrm{trace}(f(e))$ and a $*$ structure $f^*(g) = f(g^{-1})^*$. This defines a canonical C^*-algebra closure $C_r^*(H, K, \sigma)$ of $H_{\mathbb{C}}(G, K, \sigma)$ (the reduced C^*-algebra) which is of type I and comes with the distinguished faithful trace tr. The equivalence between a block $\mathfrak{R}_{\mathfrak{s}}$ and the module category of the Hecke algebra $H_{\mathbb{C}}(G, K, \sigma)$ of an $\mathfrak{s}$ respects this structure, and the Plancherel measure of G restricted to the irreducible representations in the block $\mathfrak{R}_{\mathfrak{s}}$ coincides with the spectral measure of the trace tr. We normalize tr to obtain a tracial state τ on $C_r^*(H, K, \sigma)$, and we refer to its spectral measure as the "Plancherel measure of $H_{\mathbb{C}}(G, K, \sigma)$". The theory of types seeks to decompose the harmonic analysis on G essentially in two separate parts: (1) knowledge of the supercuspidal representations of all Levi subgroups $L \subset G$ and (2) knowledge of the Plancherel measure of the Hecke algebras $H_{\mathbb{C}}(G, K, \sigma)$ [15].

As a complement to these results, the structure of the Hecke algebra $H_{\mathbb{C}}(G, K, \sigma)$ of an $\mathfrak{s}$-type can be described fairly explicitly in various cases. In the case of the Borel component this algebra is the *Iwahori–Hecke algebra* $\mathcal{H}(W, q)$ [32], where W is an extended affine Weyl group which can be attached to G by Bruhat–Tits theory, and where q is a label function on W which is defined in terms of the structure of a (generalized) affine BN-pair for G and the cardinality $\boldsymbol{q}$ of the residue field of k. By results of Morris (loc. cit.) and Lusztig [41] the Hecke algebra $H_{\mathbb{C}}(G, K, \sigma)$ of a type (K, σ) of level 0 is always a twisted crossed product of an Iwahori–Hecke algebra of the form $\mathcal{H}(W', q')$ (for a certain affine Weyl group W' and label function q') and a group $C(K, \sigma)$ (where the 2-cocycle lives on $C(K, \sigma)$). These results draw heavily on the work of Howlett and Lehrer [31] who successfully followed a similar approach for the representation theory of finite groups of Lie type.

The above exposition makes a quite compelling case for the study of an Iwahori–Hecke algebra $\mathcal{H}(W, q)$ as an object of (harmonic) analysis and the spectral problem described above. Indeed, this point of view did not go unnoticed and in some sense was already promoted by Matsumoto in [47]. But it turns out that it is quite difficult to carry it out. The description of the support of the Plancherel measure amounts to the description of the tempered dual of $\mathcal{H}(W, q)$. Using geometric methods of a completely different nature this problem was solved explicitly by Kazhdan and Lusztig in their profound paper [34], in the special case of the "Borel component" when in addition G is split semisimple and of adjoint type. Lusztig [41], [42] has in principle solved such classification problems in greater generality, when G splits over an unramified extension of k, and σ is a cuspidal unipotent representation. These methods do not give information on the Plancherel measures.

Iwahori–Hecke algebras also play a fundamental role in a wide range of other areas for some of which the aforementioned spectral problems are of immediate interest, such as integrable models in mathematical physics (the Calogero–Moser systems [29], [54], and also the generalized quantum Bose gas with delta function potential and the nonlinear Schrödinger equation [27], [24]), and the theory of multivariable orthogonal polynomials and special functions [46], [19]. These applications have led to interesting new directions in the theory of Hecke algebras (most notably Ivan Cherednik's double affine Hecke algebra [18], [19]) and this in fact raises challenging new questions in harmonic analysis. We also mention the role of the Hecke algebra for unitarizability of Iwahori-spherical representations [5], [6].

Therefore it is a problem of considerable interest to describe the Plancherel measure of the Iwahori–Hecke algebras $\mathcal{H}(W, q)$ of affine type (simply called "affine Hecke algebras" in the sequel) explicitly, and it is this problem that we will address in this paper. The paper has three parts. In the sections 2–4 we review results of [56] on the L^2-completion of the affine Hecke algebra. The main results are: (1) An algebraic characterization of the central support of the tempered spectrum. (2) The Plancherel density depends up to constants independent of $\boldsymbol{q}$ only on the central character. (3) An explicit product formula for the formal dimensions of the discrete series, up to constants independent of $\boldsymbol{q}$. The sections 5–6 give an overview of the joint work of Patrick Delorme and myself [22], [23] on the Schwartz algebra completion of the affine Hecke algebra. Here we discuss the geometric structure of the tempered dual by means of the analogue of results of Harish-Chandra [25], [26] and Knapp–Stein [36] on analytic R-groups. Finally in Sections 7–8 we discuss various natural conjectures on the K-theory of the Schwartz algebra. In the three parameter example of type C_n^{aff} we indicate how these conjectures lead in fact to complete description of the tempered dual. In striking contrast to the geometric methods mentioned above, the affine Hecke algebras with generic unequal parameters should be considered as the most basic cases from this point of view. Using the conjectures, all non-generic cases are understood by deformation to the generic case.

2. Affine Hecke algebras

The structure of an affine Hecke algebra $\mathcal{H} = \mathcal{H}(\mathcal{R}, q)$ is determined by an affine root datum (with basis) $\mathcal{R}$ together with a label function q defined on the extended affine Weyl group W associated to $\mathcal{R}$. We refer the reader to [39], [56], [22] for the details of the definition of the algebra $\mathcal{H}(\mathcal{R}, q)$, which we will only briefly review here.

Let $\mathcal{R} = (X, R_0, Y, R_0^\vee, F_0)$ be a root datum (with basis $F_0 \subset X$ of simple roots of $R_0 \subset X$). This means that R_0 is a (reduced, integral) root system with basis of simple roots F_0, that $R_0^\vee$ is the coroot system of R_0, and that X, Y are lattices in duality such that $R_0 \subset X$ and $R_0^\vee \subset Y$. For example take $X = P(R_0)$, the weight lattice of R_0, and $Y = Q(R_0^\vee)$, the root lattice of $R_0^\vee$. If $\mathfrak{a} := \mathbb{R} \otimes_{\mathbb{Z}} Y$ is spanned by the coroots we call $\mathcal{R}$ semisimple.

Let $W_0 = W(R_0)$ denote the Weyl group of the reduced integral root system R_0. The extended affine Weyl group W associated with $\mathcal{R}$ is by definition $W = W_0 \ltimes X$. The affine root system R is equal to $R := R_0^\vee \times \mathbb{Z} \subset Y \times \mathbb{Z}$. We view elements of $Y \times \mathbb{Z}$ as affine linear functions on X with values in $\mathbb{Z}$. Observe that R is closed for the natural action of W on the set of integral affine linear functions $Y \times \mathbb{Z}$ on X. Furthermore R is the disjoint union of the sets of positive and negative affine roots $R = R_+ \cup R_-$ as usual, and we define the length function l on W by

$$l(w) := |R_+ \cap w^{-1} R_-|. \tag{2.1}$$

A label function $q\colon W \to \mathbb{C}^\times$ is a function which is length multiplicative (i.e. $q(uv) = q(u)q(v)$ if $l(uv) = l(u) + l(v)$) and which in addition satisfies $q(\omega) = 1$ if $l(\omega) = 0$. Thus a label function is completely determined by its values on the set S^{aff} of affine simple reflections in W. It follows easily that its restriction to S^{aff} is constant on W-conjugacy classes of simple reflections. Conversely, any $\mathbb{C}^\times$-valued function on S^{aff} with this property extends uniquely to a label function.

For the purpose of this analytic approach to affine Hecke algebras we will work with positive real label functions only.

Definition 2.1. We denote the set of all positive real label functions for $\mathcal{R}$ by $\mathcal{Q} = \mathcal{Q}_{\mathcal{R}}$. For later reference, we choose a base $\boldsymbol{q} > 1$ and define $f_s \in \mathbb{R}$ such that $q(s) = \boldsymbol{q}^{f_s}$ for all $s \in S^{\mathrm{aff}}$.

Definition 2.2. Given a root datum $\mathcal{R}$ and a positive real label function $q \in \mathcal{Q}$ there exists a unique complex associative unital algebra $\mathcal{H}$ with $\mathbb{C}$-basis N_w $(w \in W)$ subject to the following relations (here $q(s)^{1/2}$ denotes the positive square root of $q(s)$):

(a) $N_{uv} = N_u N_v$ for all $u, v \in W$ such that $l(uv) = l(u) + l(v)$.

(b) $(N_s + q(s)^{-1/2})(N_s - q(s)^{1/2}) = 0$ for all $s \in S^{\mathrm{aff}}$.

We call $\mathcal{H} = \mathcal{H}(\mathcal{R}, q)$ the affine Hecke algebra associated with the pair $(\mathcal{R}, q)$.

Remark 2.3. We equip $\mathcal{Q}$ in the obvious way with the structure of the vector group $\mathbb{R}_+^N$ where N denotes the number of W-conjugacy classes in S^{aff}. Given the base $\boldsymbol{q} > 1$ we identify $\mathcal{Q}$ with the finite dimensional real vector space of real functions $s \to f_s$ on S^{aff} which are constant on W-conjugacy classes (see Definition 2.1). In this sense we speak of (linear) *hyperplanes* in $\mathcal{Q}$ (this notion is independent of $\boldsymbol{q}$). By a *half line* in $\mathcal{Q}$ we mean a family of label functions $q \in \mathcal{Q}$ in which the $f_s \in \mathbb{R}$ are kept fixed and are not all equal to 0 and $\boldsymbol{q}$ is varying in $\mathbb{R}_{>1}$. As we will see later, for many problems it is interesting to consider the family of Hecke algebras when q varies in a half line in $\mathcal{Q}$ ("changing the base").

2.1. Root labels for the non-reduced root system. The label function q on W can also be defined in terms of root labels for a certain possibly non-reduced root system which is associated with $\mathcal{R}$. We define R_{nr} associated with $\mathcal{R}$ by

$$R_{\mathrm{nr}} := R_0 \cup \{2\alpha \mid \alpha^\vee \in R_0^\vee \cap 2Y\}. \tag{2.2}$$

Observe that $a + 2 \in Wa$ for all $a \in R$, but that $a + 1 \in Wa$ iff $a = \alpha^\vee + n$ with $2\alpha \notin R_{\mathrm{nr}}$. For affine simple roots $a \in F^{\mathrm{aff}}$ (and thus in particular for $a \in F_0^\vee$) we define

$$q_{a+1} := q(s_a), \tag{2.3}$$

and we extend this to a W-invariant function $a \to q_a$ on the affine root system R (this is possible in a unique fashion). Now for $\alpha = 2\beta \in R_{\mathrm{nr}} \backslash R_0$ we define

$$q_{\alpha^\vee} := \frac{q_{\beta^\vee+1}}{q_{\beta^\vee}}. \tag{2.4}$$

In this way the set of label functions q on W corresponds bijectively to the set of positive W_0-invariant functions $R_{\mathrm{nr}} \ni \alpha \to q_{\alpha^\vee}$.

2.2. Bernstein presentation. There is another, extremely important presentation of the algebra $\mathcal{H}$, due to J. Bernstein (unpublished) and Lusztig [39]). Since the length function is additive on the dominant cone X^+, the map $X^+ \ni x \to N_x$ is a homomorphism of the commutative monoid X^+ with values in $\mathcal{H}^\times$, the group of invertible elements of $\mathcal{H}$. Thus there exists a unique extension to a homomorphism $X \ni x \to \theta_x \in \mathcal{H}^\times$ of the lattice X with values in $\mathcal{H}^\times$.

The abelian subalgebra of $\mathcal{H}$ generated by θ_x, $x \in X$, is denoted by $\mathcal{A}$. Let $\mathcal{H}_0 = \mathcal{H}(W_0, q_0)$ be the finite type Hecke algebra associated with W_0 and the restriction q_0 of q to W_0. Then the Bernstein presentation asserts that both the collections $\theta_x N_w$ and $N_w \theta_x$ ($w \in W_0$, $x \in X$) are bases of $\mathcal{H}$ over $\mathbb{C}$, subject only to the cross relation (for all $x \in X$ and $s = s_\alpha$ with $\alpha \in F_0$):

$$\theta_x N_s - N_s\theta_{s(x)} =$$
$$\begin{cases} (q_{\alpha^\vee}^{1/2} - q_{\alpha^\vee}^{-1/2})\frac{\theta_x-\theta_{s(x)}}{1-\theta_{-\alpha}} & \text{if } 2\alpha \notin R_{\mathrm{nr}}, \\ ((q_{\alpha^\vee/2}^{1/2}q_{\alpha^\vee}^{1/2} - q_{\alpha^\vee/2}^{-1/2}q_{\alpha^\vee}^{-1/2}) + (q_{\alpha^\vee}^{1/2} - q_{\alpha^\vee}^{-1/2})\theta_{-\alpha})\frac{\theta_x-\theta_{s(x)}}{1-\theta_{-2\alpha}} & \text{if } 2\alpha \in R_{\mathrm{nr}}. \end{cases} \tag{2.5}$$

2.3. The center $\mathcal{Z}$ of $\mathcal{H}$. From the Bernstein presentation of $\mathcal{H}$ one easily derives the following fundamental result, the description of the center of $\mathcal{H}$.

Theorem 2.4 (Bernstein). *The center $\mathcal{Z}$ of $\mathcal{H}$ is equal to $\mathcal{A}^{W_0}$. In particular, $\mathcal{H}$ is finitely generated over its center.*

As an immediate consequence we see that irreducible representations of $\mathcal{H}$ are finite dimensional by application of (Dixmier's version of) Schur's lemma.

We denote by T the complex algebraic torus $T = \mathrm{Hom}(X, \mathbb{C}^\times)$ of complex characters of the lattice X. The space $\mathrm{Spec}(\mathcal{Z})$ of complex homomorphisms of $\mathcal{Z}$ is thus canonically isomorphic to the (categorical) quotient $W_0\backslash T$. By Bernstein's theorem and Schur's lemma we obtain a continuous, finite, surjective map

$$z\colon \mathrm{Irr}(\mathcal{H}) \to \mathrm{MaxSpec}(\mathcal{Z}) = W_0\backslash T, \quad [\pi] \mapsto z(\pi), \tag{2.6}$$

where $\mathrm{Irr}(\mathcal{H})$, the set of equivalence classes of irreducible representations of $\mathcal{H}$, is given the usual Jacobson topology via its identification with the primitive ideal spectrum of $\mathcal{H}$. We call this map the (algebraic) *central character*.

3. L^2-theory and abstract Plancherel theorem

We will study $\mathcal{H}$ via certain topological completions of $\mathcal{H}$. In this section we will study the L^2-completion of $\mathcal{H}$ and the associated reduced C^*-algebra of $\mathcal{H}$.

3.1. $\mathcal{H}$ as a Hilbert algebra. It is a basic fact that the anti-linear map $*$ on $\mathcal{H}$ defined by

$$\Big(\sum_{w\in W} c_w N_w\Big)^* = \sum_{w\in W} \overline{c_w} N_{w^{-1}} \tag{3.1}$$

is an anti-involution of $\mathcal{H}$, making $(\mathcal{H}, *)$ into an involutive algebra. In addition, the linear functional τ defined by

$$\tau\Big(\sum_{w\in W} c_w N_w\Big) = c_e \tag{3.2}$$

is a positive trace on $(\mathcal{H}, *)$. In particular, the sesquilinear pairing $(x, y) := \tau(x^* y)$ defines a pre-Hilbert structure on $\mathcal{H}$.

Definition 3.1. We call $L^2(\mathcal{H})$ the Hilbert space completion of $\mathcal{H}$. Observe that the elements N_w $(w \in W)$ form a Hilbert basis for $L^2(\mathcal{H})$.

It is easy to see that the regular representation of $\mathcal{H}$ extends to a representation of $\mathcal{H}$ in $B(L^2(\mathcal{H}))$, the algebra of bounded linear operators on $L^2(\mathcal{H})$. This gives $\mathcal{H}$ the structure of a unital Hilbert algebra, with its Hermitian form defined by the finite positive trace τ.

3.2. The reduced C^*-algebra $\mathfrak{C}$ of $\mathcal{H}$. The following results on the reduced C^*-algebra of $\mathcal{H}$ go back to [47].

Definition 3.2. We define the reduced C^*-algebra $\mathfrak{C}$ of $\mathcal{H}$ as the norm closure of $\lambda(\mathcal{H}) \subset B(L^2(\mathcal{H}))$, where λ denotes the left regular representation of $\mathcal{H}$. We identify $\mathfrak{C}$ with a dense subspace of $L^2(\mathcal{H})$ via the continuous injection $\mathfrak{C} \ni x \to x(1) \in L^2(\mathcal{H})$.

Let λ (resp. ρ) denote the left (resp. right) regular representation of $\mathfrak{C}$ on $L^2(\mathcal{H})$. One has the following basic statements:

Corollary 3.3. *The C^*-algebra completion $\mathfrak{C}$ of $\mathcal{H}$ has type I, and τ extends to a finite tracial state of $\mathfrak{C}$ such that $\lambda = \lambda_\tau$ (resp. $\rho = \rho_\tau$), where λ_τ (resp. ρ_τ) denotes the left (resp. right) GNS-representation of $\mathfrak{C}$ associated with τ.*

Standard results in the spectral theory of C^*-algebras of type I yield the following:

Corollary 3.4. *There exists a unique positive Borel measure μ_{Pl} on $\hat{\mathfrak{C}}$, the Plancherel measure of $\mathcal{H}$, such that we have the following decomposition of τ in irreducible characters of $\mathfrak{C}$:*

$$\tau = \int_{\pi \in \hat{\mathfrak{C}}} \chi_\pi \, d\mu_{\mathrm{Pl}}(\pi). \tag{3.3}$$

4. The Plancherel measure

We will now address the problem to describe the spectrum $\hat{\mathfrak{C}}$ of $\mathfrak{C}$ and the Plancherel measure μ_{Pl}. The spectrum of $\mathfrak{C}$ is a rather complicated topological space. But it turns out that μ_{Pl}-almost everywhere it can be described by a simpler structure, namely a compact orbifold. This orbifold is represented by (in the sense of [50]) a groupoid of unitary standard induction data $\mathcal{W}_{\Xi_u}$ which is canonically associated with the affine Hecke algebra $\mathcal{H}$ (see [56], [22]). Its space of objects Ξ_u consists of induction data of $\mathcal{H}$ and the arrows $\mathcal{W}_{\Xi_u}$ are twisting isomorphisms between induction data. We will exhibit an explicit (up to some positive real multiplicative constants which are independent of the base $\boldsymbol{q}$)) positive measure μ on the compact orbifold $|\Xi_u| := \mathcal{W} \backslash \Xi_u$ such that a suitable open dense subset of $(|\Xi_u|, \mu)$ with a complement of measure 0 describes $(\hat{\mathfrak{C}}, \mu_{\mathrm{Pl}})$ almost everywhere.

The method in [56] to find this almost explicit Plancherel formula is a calculation of residues, starting from a basic complex analytic representation of τ as an integral over a certain rational n-form with values in the linear dual of $\mathcal{H}$, over a coset $pT_u \subset T$ of the compact real form T_u with $p \in T_{\mathrm{rs}} := \mathrm{Hom}(X, \mathbb{R}_+)$ far in the negative chamber [55]. Although such residue computations are certainly not new (see e.g. [1], [2], [38], [49]), the treatment of the uniqueness of residue data is new and is based on a simple geometric lemma in distribution theory which goes back to joint work with Gert Heckman [27]. This improved treatment of the residues is surprisingly powerful. It is sufficient to compute the Plancherel measure of the center $\mathcal{Z}$ of $\mathcal{H}$ explicitly, and in particular the central projection of the support of the Plancherel measure follows exactly [56] (see also [30]). In combination with Lusztig's results on the structure of completions of affine Hecke algebras at central characters (see [39]) we reorganize in [56] the residues according to parabolic induction and we *derive* the Maass–Selberg relations, the unitarity of the normalized intertwining operators, and finally the explicit (up to positive real factors) product formula for the Plancherel density.

4.1. The discrete series representations. Let us first recall the definition of the discrete series and of tempered representations:

Definition 4.1. An irreducible representation (V, π) of $\mathcal{H}$ is called a *discrete series representations* if it is equivalent to a subrepresentation of $(L^2(\mathcal{H}), \lambda)$. Equivalently, (V, π) is a discrete series representation if its character χ_π extends continuously to $L^2(\mathcal{H})$.

Remark 4.2. As an immediate consequence of this definition, a discrete series representation (V, π) can be equipped with an Hermitian inner product $\langle \cdot, \cdot \rangle$ with respect to which $\pi(h^*) = \pi(h)^*$ for all $h \in \mathcal{H}$. Such a Hilbert space representation of $\mathcal{H}$ is called unitary.

We will describe an *algebraic criterion* for a central character $W_0 t \in W_0 \backslash T$ to be the central character of a discrete series representation. For this we need to introduce the Macdonald c-function (see [45], [55]). This c-function is introduced as an element of the field of fractions of $\mathcal{A}$. The ring $\mathcal{A}$ can be interpreted as the ring of regular functions on T via $\theta_x \to x$, and thus the c-function can be interpreted as a rational function on T. Explicitly, we put

$$c := \prod_{\alpha \in R_{1,+}} c_\alpha, \tag{4.1}$$

where c_α is defined for $\alpha \in R_1$ by

$$c_\alpha := \frac{(1 + q_{\alpha^\vee}^{-1/2} \theta_{-\alpha/2})(1 - q_{\alpha^\vee}^{-1/2} q_{2\alpha^\vee}^{-1} \theta_{-\alpha/2})}{1 - \theta_{-\alpha}}. \tag{4.2}$$

The square roots here are positive square roots; observe that this formula makes sense: if $\alpha/2 \notin X$ then we have $q_{2\alpha^\vee} = 1$ (since $2\alpha^\vee \notin R_{\mathrm{nr}}$) and thus the numerator reduces to $(1 - q_{\alpha^\vee} \theta_{-\alpha})$.

We remark that there is no problem in defining the pole order of the rational function

$$\nu(t) := (c(t)c(t^{-1}))^{-1} \tag{4.3}$$

at a point $t_0 \in T$, since $\nu(t)$ is equal to a product of the rational functions of the form $(c_\alpha(t)c_\alpha(t^{-1}))^{-1}$ (with $\alpha \in R_1$). This function is the pull back via $\alpha/2$ (or α) of a rational function on $\mathbb{C}^\times$ and so it has a well defined pole order at t_0. The pole order of $\nu(t)$ at t_0 is defined as the sum of these pole orders.

The following theorem is of crucial importance.

Theorem 4.3 ([56, Corollary A.12]). *For any point $t_0 \in T$, the pole order of $\nu(t)$ at t_0 is at most equal to the rank* $\mathrm{rk}(R_0)$ *of R_0.*

Definition 4.4. We call $t_0 \in T$ a *residual point* if the pole order of $\nu(t)$ at t_0 is equal to the rank $\mathrm{rk}(X)$ of X.

Theorem 4.3 was proved in [56] by reducing it to a case by case inspection using the classification of residual points for graded Hecke algebras in [27], Section 4 [1]. The following result follows easily from Theorem 4.3.

Corollary 4.5. *For any root datum $\mathcal{R}$ and positive real label function q the set of residual points in T is a finite union of W_0-orbits. This set is nonempty only if* $\mathrm{rk}(R_0) = \mathrm{rk}(X)$.

Example 4.6 (The split adjoint case). *By the split adjoint case we mean that $f_s = 1$ for all $s \in S^{\mathrm{aff}}$ and $X = P$. The work of Kazhdan and Lusztig* [34] *implies* (*see* [56], *Appendix B for the translation*) *that the residual points are the points of T of the following form. Let G be the Langlands dual group, i.e. the complex semisimple group with root datum $\mathcal{R}$* (*G is simply connected*). *Then T is a maximal torus for G. Let $s \in T_u$ be such that $G_s \subset G$ is semisimple. Let $\mathcal{O}$ be a distinguished unipotent orbit of G_s and choose a homomorphism $\phi\colon \mathrm{SL}_2(\mathbb{C}) \to G_s$ with the property that*

$$\phi\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathcal{O} \quad \text{and} \quad c := \phi\begin{pmatrix} \boldsymbol{q}^{1/2} & 0 \\ 0 & \boldsymbol{q}^{-1/2} \end{pmatrix} \in T.$$

Then $r = sc \in T$ is a residual point, and all residual points in this case are of this form.

In general there exists an effective algorithm to classify the set of residual points for any root datum $\mathcal{R}$ with indeterminate positive real label function q (see [56, Theorem A.7]). The residual points come in finitely many generic families of residual points:

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[1] The notion of a residual point in [27] seems more restrictive at first sight since it involves the existence of a full flag of intermediate “residual subspaces”. In [56], Lemma A.11 we show however that this technical condition is always fulfilled. The existence of such full flags is the main tool for the classification of residual points in [27].

Proposition 4.7. *Let $\mathcal{R}$ be a semisimple root datum. There exists a nonempty finite W_0-invariant set* Res *of generic residual points $r\colon \mathcal{Q} \to T$ of $\mathcal{R}$. If $r \in$* Res *then $r(q) = s.c(q)$ where $s \in T_u$ (T_u denoting the compact real form of T) is independent of q, and such that* $\operatorname{rank}(R_{0,s}) = \operatorname{rank}(R_0)$, *and where $c\colon \mathcal{Q} \to T_{\mathrm{rs}}$ (T_{rs} being the connected component of the split real form of T) is a group homomorphism such that for all $\alpha \in R_0$, $\alpha(c)^2$ is a monomial in the root labels $q_{\beta^\vee}$ ($\beta \in R_{\mathrm{nr}}$). For each generic residual point $r \in$* Res *there exists an open set $\mathcal{Q}_r \subset \mathcal{Q}$ (depending only on the orbit $W_0 r$) which is the complement of finitely many (rational) hyperplanes in $\mathcal{Q}$, such that $r(q_0)$ is residual iff $q_0 \in \mathcal{Q}_r$ (see* [56, *Theorem* A.14]).

Remark 4.8. For each $\mathcal{R}$ one can explicitly determine the generic residual points r and the sets $\mathcal{Q}_r$. From the classification one can check that all residual points $r \in T$ of $\mathcal{R}$ have the important property that $\overline{r^{-1}} \in W_0 r$.

The following theorem expresses the central support of the discrete series representations of $\mathcal{H}$ in terms of residual points of T.

Theorem 4.9 ([56, Theorem 3.29]). *An orbit $W_0 r \in W_0\backslash T$ is the algebraic central character of a discrete series representation of $\mathcal{H}$ if and only if $r \in T$ is a residual point. In particular, the set $\Delta_{\mathcal{R}}$ of equivalence classes of discrete series representations of $\mathcal{H}$ is finite, and is nonempty only if* $\operatorname{rk}(R_0) = \operatorname{rk}(X)$.

As a consequence of the residue calculus one obtains an almost explicit *product formula* for the formal dimension (the Plancherel mass) of the discrete series as a function of the base $\boldsymbol{q}$.

Theorem 4.10 ([56, Corollary 3.32, Theorem 5.6]). *In this theorem we fix $f_s \in \mathbb{R}$ and we denote the corresponding half line in $\mathcal{Q}$ by $\mathcal{L} \subset \mathcal{Q}$ (see Remark* 2.3).

Notice that for each $r \in$ Res *we have either $\mathcal{L} \subset \mathcal{Q}_r$ or $\mathcal{L} \cap \mathcal{Q}_r = \emptyset$. Let $r \in$* Res *be such that $\mathcal{L} \subset \mathcal{Q}_r$. Via scaling isomorphisms* [56, *Theorem* 5.6] *there exists a finite (trivial) fibration $\Delta_{W_0 r} \to \mathcal{L}$ whose fiber at $q \in \mathcal{L}$ is $\Delta_{W_0 r(q)}$, the finite set of discrete series representations of $\mathcal{H}(\mathcal{R}, q)$ with algebraic central character $W_0 r(q)$.*

Recall that we view $\boldsymbol{q} > 1$ as coordinate on $\mathcal{L}$. The expressions $\alpha(r) = \alpha(r(q))$ and $q_{\alpha^\vee}$ ($\alpha \in R_{\mathrm{nr}}$) with $q \in \mathcal{L}$ are thus viewed as functions of $\boldsymbol{q} > 1$. For each connected component $\delta \subset \Delta_{W_0 r}$ there exists a nonzero real constant $d_\delta \in \mathbb{R}^\times$ independent of $\boldsymbol{q}$ *such that for all $q \in \mathcal{L}$,*

$$\mu_{\mathrm{Pl}}(\{\delta(q)\}) = d_\delta \frac{q(w_0) \prod'_{\alpha \in R_1} (\alpha(r) - 1)}{\prod'_{\alpha \in R_1} (q_{\alpha^\vee}^{1/2} \alpha(r)^{1/2} + 1) \prod'_{\alpha \in R_1} (q_{\alpha^\vee}^{1/2} q_{2\alpha^\vee} \alpha(r)^{1/2} - 1)} \tag{4.4}$$

where $\prod'$ denotes the product in which we omit the factors which are equal to 0.

Remark 4.11. (a) Observe that only the constant d_δ depends on δ; the other factors only depend on the central character $W_0 r$ of δ.

(b) It was shown by Mark Reeder [57] that this leads to an effective way to determine L-packets of square integrable unipotent representations for exceptional

p-adic Chevalley groups (also see [28]) by considering the almost explicit product formulae for the Plancherel densities of the unipotent representations as a function of $\boldsymbol{q}$, the cardinality of the residue field [2]. Lusztig [41], [42] has given a more general parameterization of the unipotent L packets from a different point of view. In the cases considered the partitions of the set of square integrable unipotent representations coincide [57].

(c) This formula is explicit up to the nonzero real constant d_δ. We remark that if the restrictions $\chi_\delta|_{\mathcal{A}}$ of the characters χ_δ with $\delta \in \Delta_{W_0r}$ are linearly independent then $d_\delta \in \mathbb{Q}$ for all $\delta \in \Delta_{W_0r}$ ([56, Remark 3.35]). However we do not know this linear independence, and in fact for the more general class of tempered representations of $\mathcal{H}(\mathcal{R}, q)$ it is easy to find counterexamples for this. In any case, we conjecture that $d_\delta \in \mathbb{Q}$ (see [56], Conjecture 2.27).

(d) The constants d_δ were computed for the exceptional root systems (in the case $q(s) = \boldsymbol{q}$ for all $s \in S^{\mathrm{aff}}$, and $X = P$) in terms of the Kazhdan–Lusztig parameters of δ by Mark Reeder [57].

(e) An irreducible representation of $\mathcal{H}$ is a discrete series if and only if its matrix coefficients have exponential decay with respect to the norm function $\mathcal{N}$ on W (see Definition 5.1); this follows from the Casselman conditions (see [56, Lemma 2.22, Theorem 6.1(ii)]). In particular these matrix coefficients belong to the Schwartz algebra $\mathcal{S} \subset \mathfrak{C}$ (see Definition 5.1), and this shows that the discrete series are isolated points in $\hat{\mathfrak{C}}$.

4.2. Standard parabolic structures. We will now concentrate on the higher dimensional spectral series. Let $\mathcal{P}$ denote the power set of F_0. Let $R_P \subset R_0$ be the root subsystem $\mathbb{R}P \cap R_0$. Notice that P is a basis of simple roots for R_P. With $P \in \mathcal{P}$ we associate the sub root datum $\mathcal{R}^P := (X, R_P, Y, R_P^\vee, P)$. The associated nonreduced root system $R_{\mathrm{nr}}(\mathcal{R}^P)$ is equal to $R_{\mathrm{nr}}(\mathcal{R}^P) = \mathbb{R}P \cap R_{\mathrm{nr}}$ hence we can restrict the root labels on R_{nr} to $R_{P,\mathrm{nr}}$. This defines a label function q^P for the affine Weyl group of the root datum $\mathcal{R}^P$. We now define the subalgebra

$$\mathcal{H}^P := \mathcal{H}(\mathcal{R}^P, q^P) \hookrightarrow \mathcal{H}. \tag{4.5}$$

This affine Hecke algebra will typically not be semisimple. Its semisimple quotient is called $\mathcal{H}_P$, the quotient of $\mathcal{H}^P$ by the two sided ideal generated by $\theta_x - 1$ where $x \in Z^P := \{x \in X \mid x(\alpha^\vee) = 0 \forall \alpha \in P\}$. Notice that the elements θ_x with $x \in Z^P$ are central in $\mathcal{H}^P$. Let us denote by X_P the quotient $X_P = X/Z^P$, and by $Y_P = Y \cap \mathbb{R}P^\vee \subset Y$ its dual lattice. This gives us a semisimple root datum $\mathcal{R}_P := (X_P, R_P, Y_P, R_P^\vee, P)$. Again we see that $R_{\mathrm{nr}}(\mathcal{R}_P) = \mathbb{R}P \cap R_{\mathrm{nr}}$, so from the restriction of the root labels on R_{nr} to $R_{\mathrm{nr}}(\mathcal{R}^P)$ we can define a label function q_P on

[2] At the time when [57] was written, (4.4) was only conjectural for general discrete series representations (see [28]). Instead Reeder used a general Euler–Poincaré type formula for the formal dimension of a discrete series representation of a semisimple p-adic group (see [59]), which requires case-by-case considerations and presents serious computational difficulties. With (4.4) at hand some of these aspects of [57] can be simplified.

the affine Weyl group of $\mathcal{R}_P$. It is now easy to check that the semisimple quotient $\mathcal{H}_P$ of $\mathcal{H}^P$ is equal to

$$\mathcal{H}^P \twoheadrightarrow \mathcal{H}_P = \mathcal{H}(\mathcal{R}_P, q_P). \tag{4.6}$$

4.3. Twisting. Let $T_P = \mathrm{Hom}(X_P, \mathbb{C}^\times) \subset T$ be the character torus of X_P. Let T^P be the connected component of e of the subgroup $\{t \in T \mid \alpha(t) = 1\} \subset T$. We see that $T = T_P T^P$, that $K_P := T_P \cap T^P$ is a finite abelian group, and that T^P is pointwise fixed by the action of W_P on T.

Using the cross relations (2.5) we check that $t \in T^P$ gives rise to an automorphism

$$\phi_t \colon \mathcal{H}^P \to \mathcal{H}^P, \quad \phi_t(\theta_x N_w) = t(x)\theta_x N_w \tag{4.7}$$

of $\mathcal{H}^P$ (where $w \in W_P$).

Definition 4.12. Let Δ_P denote the set of equivalence classes of discrete series representations of $\mathcal{H}_P$. For $\delta \in \Delta_P$ and $t \in T^P$ we write $\delta_t = \tilde{\delta} \circ \phi_t$ for the twist by ϕ_t of the lift $\tilde{\delta}$ of δ to $\mathcal{H}^P$. Observe that for $k \in K_P$, ϕ_k descends to an automorphism ψ_k of $\mathcal{H}_P$.

Let $\mathfrak{W}_{P,P'} = \{w \in W_0 \mid w(P) = P'\}$. If $w \in \mathfrak{W}_{P,P'}$ then w induces an isomorphism of root data $w \colon \mathcal{R}^P \to \mathcal{R}^{P'}$ and $w \colon \mathcal{R}_P \to \mathcal{R}_{P'}$ which is compatible with the label functions. This defines corresponding isomorphisms

$$\phi_w \colon \mathcal{H}^P \to \mathcal{H}^{P'}; \quad \psi_w \colon \mathcal{H}_P \to \mathcal{H}_{P'} \tag{4.8}$$

of affine Hecke algebras.

Definition 4.13. Let $\delta \in \Delta_P$. We denote by $\delta^w = \delta \circ \psi_w^{-1} \in \Delta_{P'}$ the twist of δ by the isomorphism ψ_w. We define $(\delta_t)^w$ (with $t \in T^P, w \in \mathfrak{W}_{P,P'}$) similarly. Observe that $(\widetilde{\delta^w})_{wt} = (\delta_t)^w$. We denote by $\delta^k = \delta \circ \psi_k^{-1} \in \Delta_P$ the twist of δ by the automorphism ψ_k of $\mathcal{H}_P$. Observe that $(\widetilde{\delta^k})_{kt} = \delta_t$ if $k \in K_P$ and $t \in T^P$.

4.4. The groupoid of standard induction data. First of all, we denote by $\mathfrak{W}$ the (standard) Weyl groupoid, the groupoid whose set of objects is $\mathcal{P}$ and whose space of arrows from P to P' is equal to $\mathfrak{W}_{P,P'}$. This groupoid acts in an obvious way on the groupoid $\mathcal{K}$ whose set of objects is also $\mathcal{P}$ and whose space of arrows is described by $\mathcal{K}_{P,P'} = \emptyset$ if $P \neq P'$, and $\mathcal{K}_{P,P} = K_P$. We denote by $\mathcal{W}$ the semidirect product $\mathcal{W} = \mathcal{K} \rtimes \mathfrak{W}$, i.e. $\mathcal{W}$ is the finite groupoid whose set of objects is $\mathcal{P}$ and whose set of arrows from P to P' equals $\mathcal{W}_{P,P'} = K_{P'} \times \mathfrak{W}_{P,P'}$. The composition of arrows is given by $(k \times u)(l \times v) = ku(l) \times uv$.

The above twisting action on induction data coming from $\mathcal{H}_P$ naturally gives rise to an action of the groupoid $\mathcal{W}$ on the set Ξ of induction data of the various subquotient algebras $\mathcal{H}_P$ with $P \in \mathcal{P}$. Let Ξ be the set of triples (P, δ, t) with $P \in \mathcal{P}$, $\delta \in \Delta_P$ and $t \in T^P$. We see that Ξ is a finite union of complex algebraic tori of the form

$\Xi_{(P,\delta)} = \{(P, \delta, t) \mid t \in T^P\}$. In particular Ξ is fibered over $\mathcal{P}$ in a way compatible with the twisting action of $\mathcal{W}$.

The groupoid of standard induction data $\mathcal{W}_\Xi$ is the translation groupoid arising from the action of $\mathcal{W}$ on Ξ. Explicitly, if $\xi = (P, \delta, t)$, $\xi' = (P', \delta', t') \in \Xi_u$ then the set of arrows $\mathcal{W}_{\xi,\xi'}$ in $\mathcal{W}_\Xi$ between these induction data consists of the set of $g = k \times w \in K_{P'} \times \mathfrak{W}_{P,P'}$ such that $P' = w(P)$, $\delta' = \delta^g$, and $t' = gt$. One easily verifies that this forms an orbifold groupoid in the sense of [50]. We remark that the action of $\mathcal{K}$ on Ξ is free. Thus the quotient $\mathcal{W}_\Xi \to \mathcal{K}\backslash\mathcal{W}_\Xi = \mathfrak{W}_{\mathcal{K}\backslash\Xi}$ is a Morita equivalence and defines the same orbifold structure on $|\Xi| = \mathcal{W}\backslash\Xi$.

We denote by $\mathcal{W}_{\Xi_u}$ the full subgroupoid whose set of objects Ξ_u consists of the unitary standard induction data, i.e. the induction data of the form (P, δ, t) with $t \in T_u^P$, the compact real form of T^P. Hence Ξ_u is a finite disjoint union of compact tori, and $\mathcal{W}_{\Xi_u}$ is a compact orbifold groupoid.

4.5. The induction-intertwining functor. *We choose explicit representatives* (V_δ, δ) *for the equivalence classes* $\delta \in \Delta = \coprod_{p\in\mathcal{P}} \Delta_P$. Given $\xi = (P, \delta, t) \in \Xi$ we denote by $\pi(\xi)$ the induced representation $\mathrm{Ind}_{\mathcal{H}^P}^{\mathcal{H}}(\delta_t)$, realized on the finite dimensional vector space

$$i(V_\delta) = \mathcal{H} \otimes_{\mathcal{H}^P} V_\delta = \bigoplus_{w\in W^P} N_w \otimes V_\delta \tag{4.9}$$

where W^P denotes the set of shortest length representatives in W_0 for the left cosets of $W_P = W(R_P)$. It is not very difficult to show (see [6] or [56]) that for $\xi \in \Xi_u$, the induced representation $\pi(\xi)$ is unitary with respect to the Hermitian inner product on $i(V_\delta)$ defined by (with $x, y \in W^P$ and $u, v \in V_\delta$)

$$\langle N_x \otimes u, N_y \otimes v\rangle = \delta_{x,y}\langle u, v\rangle. \tag{4.10}$$

Theorem 4.14 ([56, Theorem 4.38]). *The assignment* $\Xi_u \ni \xi \to \pi(\xi)$ *extends to a functor* π *(the "induction intertwining" functor) from* $\mathcal{W}_{\Xi_u}$ *to* $\mathbb{P}\mathrm{Rep}(\mathcal{H})_{\mathrm{unit}}$, *the category of unitary modules of* $\mathcal{H}$ *in which the morphisms are unitary* $\mathcal{H}$*-intertwiners modulo scalars. This functor assigns a projectively unitary intertwining isomorphism* $\pi(g, \xi)\colon \pi(\xi) \to \pi(\xi')$ *to each* $g \in \mathcal{W}_{\xi,\xi'}$. *For all* $h \in \mathcal{H}$, *the map* $\xi \to \pi(\xi)(h)$ *extends to a regular function on* Ξ, *and for all* $g \in \mathcal{W}_{\xi,\xi'}$ *the map* $\xi \to \pi(g, \xi)$ *is rational but regular at* Ξ_u. *The functor* π *is independent of the choices of the realizations* (V_δ, δ) *of the* $\delta \in \Delta$ *up to natural isomorphisms.*

Remark 4.15. In fact the representations $\pi(\xi)$ with $\xi \in \Xi_u$ are known to be tempered, see below (see [56, Proposition 4.20]).

The projective representation π of $\mathcal{W}_{\Xi_u}$ canonically determines a 2-cohomology class $[\eta] \in H^2(\mathcal{W}_{\Xi_u}, S^1)$ of the groupoid $\mathcal{W}_{\Xi_u}$, namely the pull back via π of the 2-cohomology class of the standard central extension of the category of finite dimensional projective Hilbert spaces by S^1. Notice that $[\eta]$ is obviously a torsion class since the dimensions of the representations $\pi(\xi)$ are bounded by $|W_0|$.

In fact we can be a bit more precise here. Let $\mathcal{W}_\Delta$ be the finite groupoid which is defined like $\mathcal{W}_\Xi$ but with the finite set of objects Δ instead of Ξ, and its morphisms given by twisting. Then the assignment $(P,\delta) \to V_\delta$ can be upgraded to a projective representation of $\mathcal{W}_\Delta$ by *choosing*, for each arrow $g \in \mathcal{W}_\Delta^1$ with source δ, unitary intertwining isomorphisms $\delta_g^i \colon V_\delta \to V_{\delta^g}$ such that for all $h \in \mathcal{H}_{P'}$:

$$\delta_g^i \circ \delta(\psi_g^{-1} h) = \delta^g(h) \circ \delta_g^i. \tag{4.11}$$

One easily checks that such a choice defines a 2-cocycle η_Δ with values in S^1 by

$$\delta_g^i \circ (\delta')_{g'}^i = \eta_\Delta(g,g')(\delta')_{gg'}^i \tag{4.12}$$

where g, g' are composable arrows of $\mathcal{W}_\Delta$. Its class $[\eta_\Delta] \in H^2(\mathcal{W}_\Delta, S^1)$ is independent of the chosen representatives and intertwining morphisms. Then the details of the construction of the projective representation π actually show that $[\eta] \in H^2(\mathcal{W}_{\Xi_u}, S^1)$ is the pull back of $[\eta_\Delta] \in H^2(\mathcal{W}_\Delta, S^1)$ via the natural homomorphism of groupoids

$$\mathcal{W}_{\Xi_u} \to \mathcal{W}_\Delta, \quad (P,\delta,t) \mapsto (P,\delta). \tag{4.13}$$

Let D denote the order of $[\eta_\Delta]$. Then we can choose the 2-cocycle η_Δ so that it has its values in μ_D, the group of complex D-th roots of unity. Then the above amounts to

Proposition 4.16. *Let $\tilde{\mathcal{W}}_{\Xi_u}$ denote the central extension of $\mathcal{W}_{\Xi_u}$ by μ_D determined by $[\eta]$. The lifting $\tilde{\pi}$ to $\tilde{\mathcal{W}}_{\Xi_u}$ of the projective representation π of $\mathcal{W}_{\Xi_u}$ splits.*

4.6. The Plancherel decomposition of τ. We finally have everything in place to formulate the Plancherel theorem for the spectral decomposition of $L^2(\mathcal{H})$ as a (type I) representation of $\mathfrak{C}$. In order to describe the Plancherel density, we introduce relative c-functions for the spectral series of the form $\pi(\xi)$ with $\xi \in \Xi_{(P,\delta),u} := \{(P,\delta,t) \mid \delta \in \Delta_P, t \in T_u^P\}$.

Definition 4.17. We adopt the notation (P,α) to denote the restriction of $\alpha \in R_0 \backslash R_P$ to $T^P \subset T$. Let X^P denote the character lattice of T^P. We write $R^P \subset X^P \backslash \{0\}$ for the set of restrictions (P,α) of roots $\alpha \in R_0 \backslash R_P$ which are in addition primitive in the sense that if $\beta \in R_0 \backslash R_P$ and $(P,\alpha) \in R^P$ such that (P,α) and (P,β) are proportional, then $(P,\beta) = c(P,\alpha)$ with $c \in \mathbb{Z}$. We write R_+^P for the primitive restrictions corresponding to the positive roots $\alpha \in R_{0,+} \backslash R_{P,+}$. An element (P,α) is called *simple* if (P,α) is indecomposable in $\mathbb{Z}_+ R_+^P$. This is equivalent to saying that (P,α) is the restriction of an element of $F_0 \backslash P$. To each $(P,\alpha) \in R^P$ we denote by $(P,H_\alpha) \subset T^P$ the connected component of e of $\mathrm{Ker}(P,\alpha) \subset T^P$. It is a codimension 1 subtorus of T^P. The real hyperplanes $(P,H_\alpha) \cap T_{\mathrm{rs}}^P$ are called the *walls* in T_{rs}^P. The positive chamber $T_{\mathrm{rs}}^{P,+}$ is the connected component of the complement of the walls on which $(P,\alpha) > 1$ for all $(P,\alpha) \in R_+^P$.

Let $(P,\alpha) \in R^P$. Then the rational function on T defined by

$$c_{(P,\alpha)}(t) := \prod_{\beta \in R_{0,+} \setminus R_P : (P,\beta) \in \mathbb{Z}_+(P,\alpha)} c_\beta(t) \tag{4.14}$$

is clearly $W_P = W(R_P)$-invariant. Hence it can be viewed as a rational function on $W_P \backslash T$. We compose this rational function with the algebraic central character map $z_P \colon \operatorname{Irr}(\mathcal{H}^P) \to W_P \backslash T$ and write $c_{(P,\alpha)}(\sigma)$ for any $\sigma \in \operatorname{Irr}(\mathcal{H}^P)$. We define a rational function $c_{(P,\alpha)}$ on $\Xi_{(P,\delta)}$ by putting $c_{(P,\alpha)}(\xi) := c_{(P,\alpha)}(\delta_t)$. Observe that its poles and zeroes are orbits of (P, H_α) acting on $\Xi_{(P,\delta)}$.

Definition 4.18. (i) We define the c-function on $\xi \in \Xi_{(P,\delta)}$ by means of the formula $c(\xi) = \prod_{(P,\alpha)\in R_+^P} c_{(P,\alpha)}(\xi)$.

(ii) We put $\mu_{(P,\alpha)}(\xi) = (c_{(P,\alpha)}(\xi) c_{(P,-\alpha)}(\xi))^{-1} = |c_{(P,\alpha)}(\xi)|^{-2}$ (see Remark 4.8).

(iii) If $\xi \in \Xi_{(P,\delta)}$ we put $\mu(\xi) = \prod_{(P,\alpha)\in R_+^P} \mu_{(P,\alpha)}(\xi) = (c(\xi)c(w^P\xi))^{-1}$ (where w^P denotes the longest element in W^P).

The following main theorem of [56] makes the abstract Plancherel formula (3.3) almost (up to the constants $d_\delta \in \mathbb{R}_+$ for $\delta \in \Delta$) explicit.

Theorem 4.19 ([56, Theorem 4.39, Proposition 4.42, Theorem 4.43]). (i) *For all $(P,\alpha) \in R_+^P$ the functions $\mu_{(P,\alpha)}$ are smooth and nonnegative on $\Xi_{(P,\delta),u}$. Moreover μ is $\mathcal{W}$-invariant on Ξ.*

(ii) *The function $\Delta \ni (P,\delta) \to \mu_{(\mathcal{R}_P,q_P),\mathrm{Pl}}(\{\delta\})$ (given by the product formula* (4.4) *applied to $\mathcal{H}(\mathcal{R}_P, q_P)$) is $\mathcal{W}$-invariant on Δ.*

(iii) *Outside a $\mathcal{W}$-invariant subset Ξ_u^{reg} whose complement in Ξ_u consists of finitely many components of codimension at least* 1*, the representations $\pi(\xi)$ are irreducible. This defines a homeomorphism $[\xi] \to [\pi(\xi)]$ from $|\Xi_u^{\mathrm{reg}}| := \mathcal{W} \backslash \Xi_u^{\mathrm{reg}}$ onto a subset of $\hat{\mathfrak{C}}$ whose complement has measure zero.*

(iv) *For ξ in the compact torus $\Xi_{(P,\delta),u}$, let $d\xi$ denote the normalized Haar measure on $\Xi_{(P,\delta)}$. The Plancherel measure μ_{Pl} is the push forward to the quotient $|\Xi_u| = \mathcal{W} \backslash \Xi_u$ of the absolutely continuous, $\mathcal{W}$-invariant measure on Ξ_u given by* (*with $\xi = (P,\delta,t) \in \Xi_{(P,\delta),u}$*)

$$d\mu_{\mathrm{Pl}}(\pi(\xi)) = q(w^P)^{-1} |\mathfrak{W}_P|^{-1} \mu_{(\mathcal{R}_P,q_P),\mathrm{Pl}}(\{\delta\}) \mu(\xi) d\xi \tag{4.15}$$

where $\mathfrak{W}_P$ denotes the set $\{w \in W_0 \mid w(P) \subset F_0\}$.

5. The structure of the Schwartz algebra $\mathcal{S}$

In joint work with Patrick Delorme [22], [23] we studied the refinement of the Fourier isomorphism where the L^2-functions are replaced by smooth functions. Following Harish-Chandra and Langlands this step miraculously leads to deep insights in the structure of $\hat{\mathfrak{C}}$ and of $\operatorname{Irr}(\mathcal{H})$. It also brings into play methods from noncommutative geometry.

5.1. The Schwartz algebra $\mathcal{S}$. We define a sub-multiplicative function $\mathcal{N}: W \to \mathbb{R}_+$ as follows. Let $Z = X^+ \cap X^- \subset X$ be the lattice of central translations in W. Given $v \in \mathfrak{a}^*$ we denote by $\overline{v} \in \mathbb{R} \otimes Z$ the projection of v onto $\mathbb{R} \otimes Z$ along $\mathbb{R} \otimes Q(R_0)$. We choose a Euclidean norm $\|\cdot\|$ on $\mathbb{R} \otimes Z$ and put for any $w \in W$:

$$\mathcal{N}(w) := l(w) + \|\overline{w(0)}\|. \tag{5.1}$$

Now we come to the definition of the Schwartz space completion $\mathcal{S}$ of $\mathcal{H}$.

Definition 5.1. The Schwartz space completion $\mathcal{S} \subset L^2(\mathcal{H})$ of $\mathcal{H}$ is the nuclear Fréchet space consisting of the elements $\sum_{w \in W} c_w N_w \in L^2(\mathcal{H})$ such that $w \to |c_w|$ is of rapid decay with respect to the function $\mathcal{N}$.

As an application of the explicit knowledge on $\hat{\mathfrak{C}}$ we obtained in the previous section one can actually prove that

Theorem 5.2 ([56, Theorem 6.5]). *The multiplication of $\mathcal{H}$ extends continuously to $\mathcal{S}$, giving $\mathcal{S}$ the structure of a nuclear Fréchet algebra. Moreover, matrix coefficients of discrete series representations are elements of $\mathcal{S}$.*

5.2. Tempered representations

Definition 5.3. A finite dimensional representation (V, π) of $\mathcal{H}$ is called tempered if it extends continuously to $\mathcal{S}$. Equivalently, (V, π) is tempered if χ_π extends continuously to $\mathcal{S}$.

The next theorem explains the relation of tempered representations with L^2-theory:

Theorem 5.4 ([22, Theorem 3.19]). *If $\xi \in \Xi_u$ then $\pi(\xi)$ is unitary and tempered. Conversely, let (V, π) be an irreducible tempered $\mathcal{H}$ representation. Then there exists a $\xi \in \Xi_u$ such that (V, π) is a direct summand of $\pi(\xi)$.*

Corollary 5.5. *Irreducible tempered representations of $\mathcal{H}$ are unitarizable. The set $\hat{\mathcal{S}}$ of irreducible tempered representations is equal to the support $\hat{\mathfrak{C}}$ of the Plancherel measure.*

5.3. The Fourier isomorphism. Let us denote by $\mathcal{V}_{\Xi_u}$ the trivial fibre bundle over Ξ_u whose fibre at $\xi = (P, \delta, t) \in \Xi_u$ is $V_\xi := i(V_\delta)$ (the representation space of $\pi(\xi)$), thus

$$\mathcal{V}_{\Xi_u} := \coprod_{(P,\delta)} \Xi_{(P,\delta),u} \times i(V_\delta). \tag{5.2}$$

We denote by $\mathrm{End}(\mathcal{V}_{\Xi_u})$ the corresponding bundle of endomorphism algebras. Let $\mathrm{Pol}(\Xi_u, \mathrm{End}(\mathcal{V}_{\Xi_u}))$ denote the algebra of polynomial sections in $\mathrm{End}(\mathcal{V}_{\Xi_u})$, i.e. sections such that the matrix coefficients are Laurent polynomials on each component $\Xi_{(P,\delta),u}$ of Ξ_u.

There is a natural action of $\mathcal{W}$ on $\mathrm{End}(\mathcal{V}_{\Xi_u})$ by algebra homomorphisms as follows. Suppose that $\xi \in \Xi_u$ and that $A \in \mathrm{End}(V_\xi)$. Given $g \in \mathcal{W}_{\xi,\xi'}$ we define $g(A) := \pi(g,\xi)\circ A\circ\pi(g,\xi)^{-1} \in \mathrm{End}(V_{\xi'})$. A section of f of $\mathrm{End}(\mathcal{V}_\Xi)$ is called $\mathcal{W}$-equivariant if we have $f(g(\xi)) = g(f(\xi))$ for all $\xi \in \Xi_u$ and all $g \in \mathcal{W}_{\Xi_u}$ with source ξ.

Observe that for all $h \in \mathcal{H}$, the polynomial section $\xi \to \pi(\xi)(h)$ on Ξ_u is $\mathcal{W}$-equivariant. Let us denote the algebra of $\mathcal{W}$-equivariant polynomial sections by $\mathrm{Pol}(\Xi_u, \mathrm{End}(\mathcal{V}_\Xi))^{\mathcal{W}}$. The Fourier transform on $\mathcal{H}$ is the canonical algebra homomorphism

$$\begin{aligned} \mathcal{F}_{\mathcal{H}}\colon \mathcal{H} &\to \mathrm{Pol}(\Xi_u, \mathrm{End}(\mathcal{V}_\Xi))^{\mathcal{W}}, \\ h &\mapsto \{\xi \to \pi(\xi)(h)\}. \end{aligned} \tag{5.3}$$

The image of $\mathcal{F}_{\mathcal{H}}$ is difficult to describe. But it turns out that the situation becomes very intelligible upon extension to $L^2(\mathcal{H})$ or to $\mathcal{S}$.

Let $L^2(\Xi_u, \mathrm{End}(\mathcal{V}_{\Xi_u}), d\mu_{\mathrm{Pl}})$ denote the Hilbert space of L^2-sections of $\mathrm{End}(\mathcal{V}_{\Xi_u})$ with respect to the inner product

$$\langle \sigma, \tau\rangle = \int_{\Xi_u} \mathrm{trace}(\sigma(\xi)^*\tau(\xi))d\mu_{\mathrm{Pl}}(\pi(\xi)). \tag{5.4}$$

Let the Fréchet algebra of smooth sections of $\mathrm{End}(\mathcal{V}_{\Xi_u})$ on Ξ_u be denoted by $C^\infty(\Xi_u, \mathrm{End}(\mathcal{V}_{\Xi_u}))$.

Theorem 5.6 ([56, Theorem 4.43], [22, Theorem 5.3]). (i) *The Fourier transform $\mathcal{F}_{\mathcal{H}}$ extends to an isometric isomorphism*

$$\mathcal{F}\colon L^2(\mathcal{H}) \to L^2(\Xi_u, \mathrm{End}(\mathcal{V}_{\Xi_u}), d\mu_{\mathrm{Pl}})^{\mathcal{W}}. \tag{5.5}$$

(ii) *The Fourier transform $\mathcal{F}$ restricts to an isomorphism $\mathcal{F}_{\mathcal{S}}$ of Fréchet algebras*

$$\mathcal{F}_{\mathcal{S}}\colon \mathcal{S} \to C^\infty(\Xi_u, \mathrm{End}(\mathcal{V}_{\Xi_u}))^{\mathcal{W}}. \tag{5.6}$$

5.4. The center of $\mathcal{Z}_{\mathcal{S}}$ of $\mathcal{S}$. As a consequence of Theorem 5.6 we see:

Corollary 5.7. *The center $\mathcal{Z}_{\mathcal{S}}$ of $\mathcal{S}$ is, via the Fourier Transform $\mathcal{F}_{\mathcal{S}}$, isomorphic to the algebra $C^\infty(\Xi_u)^{\mathcal{W}}$ of $\mathcal{W}$-invariant C^∞-functions on Ξ_u. In particular the algebra $\mathcal{S}$ is a $\mathcal{Z}_{\mathcal{S}}$-algebra of finite type.*

This gives, for irreducible tempered representations, a finer notion of central character which we call the *tempered central character*.

Definition 5.8. We denote by $\hat{\mathcal{S}}$ the set of irreducible tempered representations, equipped with its Jacobson topology.

It is easy to see that in our situation $\hat{\mathcal{S}}$ is naturally in bijection with the set $\mathrm{Prim}(\mathcal{S})$ if primitive ideals of $\mathcal{S}$. In view of Corollary 5.5 we have a bijection $\hat{\mathcal{S}} \to \hat{\mathfrak{C}}$, and it is not difficult to show that this bijection is in fact a homeomorphism.

Proposition 5.9. *We have a surjective, finite, continuous map*

$$z_{\mathcal{S}}\colon \hat{\mathcal{S}} \to |\Xi_u| = \mathcal{W}\backslash\Xi_u. \tag{5.7}$$

We call this map the tempered central character. *The irreducible summands of* $\pi(\xi)$ $(\xi \in \Xi_u)$ *all have central character* $[\xi] = \mathcal{W}_\xi \xi$.

5.5. Analytic *R*-groups. The structure of the fibres of the tempered central character map $z_{\mathcal{S}}$ is completely determined by the geometry of the orbifold $\mathcal{W}_{\Xi_u}$ in combination with the behaviour of the Plancherel density μ and the cohomology class η. This is revealed by studying the explicit inversion of the Fourier isomorphism $\mathcal{F}_{\mathcal{S}}$ by means of the wave packet operator $\mathcal{J}_{\mathcal{S}}$. Let $\mathcal{J}$ be the adjoint of $\mathcal{F}$ defined on the space $L^2(\Xi_u, \mathrm{End}(\mathcal{V}_{\Xi_u}), d\mu_{\mathrm{Pl}})$ of non-invariant L^2-sections. Then $p_{\mathcal{W}} := \mathcal{F} \circ \mathcal{J}$ is equal to taking the $\mathcal{W}$-average, and [22, Theorem 5.3] shows that $p_{\mathcal{W}}$ maps the subspace of $L^2(\Xi_u, \mathrm{End}(\mathcal{V}_{\Xi_u}), d\mu_{\mathrm{Pl}})$ consisting of sections of the form $c\sigma$ (with c the c-function and $\sigma \in C^\infty(\Xi_u, \mathrm{End}(\mathcal{V}_{\Xi_u})))$ to the space $C^\infty(\Xi_u, \mathrm{End}(\mathcal{V}_{\Xi_u}))^{\mathcal{W}}$ of *smooth* invariant sections.

Definition 5.10. We call the connected components of the poles of $c_{(P,\alpha)}$ on $\Xi_{(P,\delta),u}$ the set of (P,α)-mirrors (since a root (P,α) is real, this set is empty or of codimension one in $\Xi_{(P,\delta),u}$). We define the set of mirrors on Ξ_u to be the union of all (P,α)-mirrors (with $P \in \mathcal{P}$ and $(P,\alpha) \in R^P$). This set is $\mathcal{W}$-invariant.

Using the above properties of $p_{\mathcal{W}}$ one shows that:

Proposition 5.11. (i) *For any* (P,α)*-mirror* $M \subset \Xi_{(P,\delta)}$ *there exists a mirror reflection* $\mathfrak{s}_M \in \mathcal{W}_{(P,\delta),(P,\delta)}$*, i.e. an involution in* $\mathcal{W}_{P,P}$ *fixing* δ *such that* $\mathfrak{s}_M$ *fixes* M *pointwise.*

(ii) *Let* $\xi \in \Xi_u$*, say* $\xi = (P,\delta,t)$*. The set* R_ξ *of roots* $(P,\alpha) \in R^P$ *such that* ξ *lies in a* (P,α)*-mirror is a reduced integral root system. We denote by* $R_{\xi,+}$ *the set of roots in* R_ξ *which are element of* $(P,\alpha) \in R_+^P$.

(iii) *For each mirror* $M \in \Xi_u$ *and for all* $\xi \in M$*, the intertwining operator* $\pi(\mathfrak{s}_M, \xi)$ *is scalar.*

Definition 5.12. Denote by $\mathcal{W}_\xi^m = W(R_\xi) \subset \mathcal{W}_{\xi,\xi}$ the normal reflection subgroup generated by the reflections in the mirrors through ξ. The analytic R-group $\mathfrak{R}_\xi$ at ξ is the group of $\mathfrak{r} \in \mathcal{W}_{\xi,\xi}$ such that $\mathfrak{r}(R_{\xi,+}) = R_{\xi,+}$. Then $\mathcal{W}_{\xi,\xi} = \mathcal{W}_\xi^m \rtimes \mathfrak{R}_\xi$.

The above implies that the restriction η_ξ of the 2-cocycle η to $\mathcal{W}_{\xi,\xi}$ is cohomologous to the pull-back of a 2-cocycle (also denoted η_ξ) of $\mathfrak{R}_\xi$. We have the following analog of results of Harish-Chandra ([25], [26]), Knapp–Stein ([36]) and Silberger [61] on analytic R-groups.

Theorem 5.13 ([22], [23]). *The restriction of the induction-intertwining functor* π *to* $\mathfrak{R}_\xi$ *gives rise to an isomorphism* $\mathrm{End}_{\mathcal{H}} V_\xi \simeq \mathbb{C}[\mathfrak{R}_\xi, \eta]$*, the* η_ξ*-twisted group ring of* $\mathfrak{R}_\xi$*. In particular, the functor* E_ξ *defined by* $\psi \to \mathrm{Hom}_{\mathbb{C}[\mathfrak{R}_\xi,\eta]}(\psi, V_\xi)$ *defines* *an equivalence between the category of* $\mathbb{C}[\mathfrak{R}_\xi, \eta]$*-modules and the category of finite dimensional unitary tempered* $\mathcal{H}$*-modules with tempered central character* $\mathcal{W}\xi$*.*

Corollary 5.14. *Let* $w \in \mathcal{W}_\xi$*. There is a unique decomposition* $w = xm$ *with* $m \in \mathcal{W}_\xi^m$ *and* $x(R_{\xi,+}) = R_{w\xi,+}$*. Hence* $x\mathfrak{R}_\xi x^{-1} = \mathfrak{R}_{w\xi}$*. Consider the isomorphism of algebras* $c_w \colon \mathbb{C}[\mathfrak{R}_\xi, \eta] \to \mathbb{C}[\mathfrak{R}_{w\xi}, \eta]$ *defined by* η*-twisted conjugation with* x*, i.e.* $c_w(r) = \eta(x, r)\eta(xr, x^{-1})\eta(x, x^{-1})^{-1}xrx^{-1}$*. For any* $\mathbb{C}[\mathfrak{R}_\xi, \eta]$*-module* ρ *we have* $E_\xi(\rho) = E_{w\xi}(\rho^w)$ *where* $\rho^w = \rho \circ c_w^{-1}$*.*

Recall that the action of $\mathcal{K}$ on Ξ_u is free. We identify the R-groups $\mathfrak{R}_{k\xi}$ $(k \in \mathcal{K})$ via conjugation by k, and write $\mathfrak{R}_{\mathcal{K}\xi}$. We identify via the twisted conjugations c_k $(k \in \mathcal{K})$ the rings $\mathbb{C}[\mathfrak{R}_{k\xi}, \eta]$ $(k \in \mathcal{K})$ and write $\mathbb{C}[\mathfrak{R}_{\mathcal{K}\xi}, \eta_{\mathcal{K}}]$. We consider the sheaf $\mathrm{G}(\mathfrak{R}, \eta)$ on Ξ_u whose fiber at ξ is the complex representation space $\mathrm{G}(\mathfrak{R}_\xi, \eta)$ of $\mathbb{C}[\mathfrak{R}_\xi, \eta_\xi]$ (the argument that this defines a sheaf is the same as for the usual representation ring sheaf of the orbifold $\mathcal{W}_{\Xi_u}$, see [7]). By the above there exists a $\mathcal{W}$-sheaf $\underline{\mathrm{G}}(\mathfrak{R}, \eta)$ of complex vector spaces on Ξ_u (we identify $\mathfrak{R}_\xi$ with the quotient $\mathcal{W}_{\xi,\xi}/\mathcal{W}_\xi^m$, so that the action on characters by conjugation with $w \in \mathcal{W}_\xi$ is equal to c_w as in Proposition 5.14):

Definition 5.15. The $\mathcal{W}$-sheaf $\mathrm{G}(\mathfrak{R}, \eta)$ on Ξ_u descends to a sheaf of complex vector spaces on $|\Xi_u|$ denoted by $\underline{\mathrm{G}}(\mathfrak{R}, \eta)$, and to a sheaf $\mathrm{G}(\mathfrak{R}_{\mathcal{K}}, \eta_{\mathcal{K}})$ on $\mathcal{K}\backslash\Xi_u$.

6. Smooth families of tempered representations

We investigate the geometry of the orbifold structure of $|\Xi_u|$ in relation to the structure of the Grothendieck group of (finite dimensional) tempered representations of $\mathcal{H}$ in this subsection. The approach is comparable with [12] (but working with $\mathcal{S}$ instead of $\mathcal{H}$). Much of the material in this section and the next is joint work in progress with Maarten Solleveld.

As was explained by Arthur [3], it is a basic fact that the functor E_ξ is compatible with the geometric structure of $\mathcal{W}_{\Xi_u}$ in the following sense. The same arguments apply in the present situation. Let $\xi = (P, \delta, t)$ and let $P \subset Q$. Let $\mathfrak{R}_\xi^Q$ be the subgroup of elements of $\mathfrak{R}_\xi$ which pointwise fix the T_u^Q-orbit through ξ. Let π^Q denote the induction-intertwining functor of the Hecke algebra $\mathcal{H}^Q = \mathcal{H}(\mathcal{R}^Q, q^Q) \subset \mathcal{H}$, and let E_ξ^Q be the corresponding equivalence. Let Ξ_u^Q denote the space of unitary standard induction data for $\mathcal{H}^Q$. Then $\mathfrak{R}_\xi^Q$ is the R-group at $\xi \in \Xi_u^Q \subset \Xi_u$ for induction to $\mathcal{H}^Q$.

Proposition 6.1 ([3], [23]). *We have the following equality of functors:*

$$\mathrm{Ind}_{\mathcal{H}^Q}^{\mathcal{H}} \circ E_\xi^Q = E_\xi \circ \mathrm{Ind}_{\mathbb{C}[\mathfrak{R}_\xi^Q, \eta]}^{\mathbb{C}[\mathfrak{R}_\xi, \eta]}. \tag{6.1}$$

For any $\mathfrak{r} \in \mathfrak{R}_\xi$ the set of fixed points is of the form $T_u^Q\xi$ for some parabolic subsystem R_Q where Q is not necessarily standard, but *compatible* in the sense that there exists a $w \in \mathcal{W}_\xi$ such that $w(Q)$ is standard and $w(R_{\xi,+}) = R_{w\xi,+}$ (compare with [3, Section 2]). In combination with basic aspects of the structure theory of the (analytic or formal) completion of $\mathcal{H}$ at an algebraic central character (see [39] or [56]) we can draw several conclusions. Let $\mathrm{G}(\mathcal{S}_{\mathcal{W}\xi})$ denote the Grothendieck group of $\mathcal{S}$-modules with tempered central character $\mathcal{W}\xi$ where $\xi = (P, \delta, t)$, and let $\mathrm{G}^{\mathbb{Q}}(\mathcal{S}_{\mathcal{W}\xi}) = \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{G}(\mathcal{S}_{\mathcal{W}\xi})$ (viewed as virtual tempered characters of $\mathcal{H}$).

Corollary 6.2. *Let $\chi \in \mathrm{G}^{\mathbb{Q}}(\mathcal{S}_{\mathcal{W}\xi})$ where $\xi = (P, \delta, t) \in \Xi_{(P,\delta),u}$, and let $Q \in \mathcal{P}$ be such that $P \subset Q$. The following are equivalent:*

(i) $\chi = E_\xi(\rho)$ *with ρ induced from* $\mathbb{C}[\mathfrak{R}_\xi^Q, \eta]$.

(ii) $\chi = E_\xi(\rho)$ *with ρ supported on the $\mathfrak{R}_\xi$-conjugacy classes meeting* $\mathfrak{R}_\xi^Q$.

(iii) χ *is induced from a virtual tempered character of* $\mathcal{H}^Q$.

(iv) $\chi = (\chi_t)|_{t=1}$ *for a* (*weakly*) *smooth family* $T_u^Q \ni t \to \chi_t \in \mathrm{G}^{\mathbb{Q}}(\mathcal{S}_{\mathcal{W}(t\xi)})$.

Proof. It is elementary (using the formula for induced characters in twisted group rings) that (i) $\Leftrightarrow$ (ii) and (i) $\Leftrightarrow$ (iii) follows from (6.1). It is trivial that (iii) $\Rightarrow$ (iv). The step (iv) $\Rightarrow$ (iii) needs explanation. By the (weak) smoothness of the family it enough to prove that χ_t is induced from $\mathcal{H}^Q$ for generic $t \in T_u^Q$. For generic $t \in T_u^Q$ the algebraic central character $z(\chi_t)$ is "R_Q-generic" in the sense of [56, Definition 4.12]. By [56, Corollary 4.15] all characters with an R^Q-generic central character are induced from $\mathcal{H}^Q$. □

Definition 6.3. We say that $\chi \in \mathrm{G}^{\mathbb{Q}}(\mathcal{S}_{\mathcal{W}\xi})$ is *d-smooth and pure* if there exists an open neighborhood $U \subset \Xi_u$ of ξ, a d-dimensional smooth submanifold $S \subset U$ with $\xi \in S$, and a weakly smooth family $S \ni s \to \chi_s \in \mathrm{G}^{\mathbb{Q}}(\mathcal{S}_{\mathcal{W}s})$ (weakly smooth means that $S \ni s \to \chi_s(h)$ is smooth for all $h \in \mathcal{H}$) such that the tempered central character of χ_s equals $\mathcal{W}s$ and $\chi = \chi_\xi$. We say that χ is *d-smooth* if χ is a $\mathbb{Q}$-linear combination of pure d-smooth characters.

We obtain a descending filtration $F^i_{\mathcal{W}\xi} = \mathbb{C} \otimes_{\mathbb{Q}} \{\chi \in \mathrm{G}^{\mathbb{Q}}(\mathcal{S}_{\mathcal{W}\xi}) \mid \chi \text{ is } i\text{-smooth}\}$ of $\mathrm{G}^{\mathbb{C}}(\mathcal{S}_{\mathcal{W}\xi})$. Observe that the filtration is stationary at least until the rank i_Z of the central lattice $Z \subset X$ of $\mathcal{R}$.

Definition 6.4. The vector space $\mathrm{Ell}^{\mathrm{temp}}_{\mathcal{W}\xi}$ of *elliptic tempered characters* of $\mathcal{S} = \mathcal{S}(\mathcal{R}, q)$ with tempered central character $\mathcal{W}\xi$ (with $\xi \in \Xi_u$) is defined as $F^0_{\mathcal{W}\xi}/F^1_{\mathcal{W}\xi}$. We call an irreducible tempered representation (with tempered central character $\mathcal{W}\xi$) elliptic if its character has a nonzero image in $\mathrm{Ell}^{\mathrm{temp}}_{\mathcal{W}\xi}$.

Proposition 6.5. *The complex vector space* $\mathrm{Ell}^{\mathrm{temp}}_{\mathcal{W}\xi}$ *is zero for all but finitely many orbits* $\mathcal{W}\xi \in |\Xi_u|$*. There exist orbits* $\mathcal{W}\xi \in |\Xi_u|$ *such that* $\mathrm{Ell}^{\mathrm{temp}}_{\mathcal{W}\xi} \neq 0$ *only if* $\mathcal{R}$ *is semisimple.*

One can show the following result:

Proposition 6.6. *Let* $F^d(\mathfrak{R}^Q_\xi, \eta)$ *denote the complex vector space of virtual* η*-twisted characters of* $\mathfrak{R}^Q_\xi$ *whose support consists of elements* $\mathfrak{r} \in \mathfrak{R}^Q_\xi$ *whose fixed point set in* $T_{Q,u}$ *has dimension at least* d*. In the case* $Q = F_0$ *this defines a filtration of* $\underline{\mathrm{G}}^{\mathbb{C}}_{\mathcal{W}\xi}(\mathfrak{R}, \eta)$ *(the complex span of irreducible* η*-twisted characters of* $\mathfrak{R}_\xi$*). Put* $I^Q_\xi := \mathrm{Ind}^{\mathbb{C}[\mathfrak{R}_\xi,\eta]}_{\mathbb{C}[\mathfrak{R}^Q_\xi,\eta]}$ *and define* $\mathrm{Ell}(\mathfrak{R}^Q_\xi, \eta) = F^0(\mathfrak{R}^Q_\xi, \eta)/F^1(\mathfrak{R}^Q_\xi, \eta)$*. We have an isomorphism of graded vector spaces*

$$\bigoplus_Q I^Q_\xi : \bigoplus_Q \mathrm{Ell}(\mathfrak{R}^Q_\xi, \eta) \xrightarrow{\sim} \mathrm{gr}(\underline{\mathrm{G}}^{\mathbb{C}}_{\mathcal{W}\xi}(\mathfrak{R}, \eta)) \tag{6.2}$$

where Q *runs over a complete set of representatives of the* $\mathcal{W}$*-association classes of compatible parabolic subgroups, and the degree of* $\mathrm{Ell}(\mathfrak{R}^Q_\xi, \eta)$ *is defined as the depth of* Q*. The functor* E_ξ *induces an isomorphism of graded vector spaces* $\mathrm{gr}(\underline{\mathrm{G}}^{\mathbb{C}}_{\mathcal{W}\xi}(\mathfrak{R}, \eta)) \to \mathrm{gr}(\underline{\mathrm{G}}^{\mathbb{C}}(\mathcal{S}_{\mathcal{W}\xi}))$*. If* $w \in \mathcal{W}_{\xi,\xi}$ *satisfies* $w(Q) = Q$ *then twisting by* w *acts trivially on* $\mathrm{Ell}(\mathfrak{R}^Q_\xi, \eta)$*.*

7. K-theory of the Schwartz algebra $\mathcal{S}$

In the last two sections we will discuss results and conjectures for the K-theory and noncommutative geometry of affine Hecke algebras and we indicate some of the evidence in support of these conjectures. These conjectures are not new, certainly not their analogues in the context of reductive algebraic groups. The point is however that these conjectures give a detailed guideline for representation theory of affine Hecke algebras with (unequal) continuous positive real parameters. In this sense the discussion below is a natural extension of the harmonic analysis questions that have been studied in this paper.

First of all recall the following result.

Proposition 7.1 (Corollary 5.9, [22]). *The dense* $*$*-subalgebra* $\mathcal{S} \subset \mathfrak{C}$ *is closed for holomorphic functional calculus. Hence the inclusion* $i\colon \mathcal{S} \to \mathfrak{C}$ *induces an isomorphism* $\mathrm{K}_*(\mathcal{S}) \xrightarrow{\sim} \mathrm{K}_*(\mathfrak{C})$*.*

7.1. The Chern character for $\mathcal{S}$. The result in this subsection is due to Maarten Solleveld. Proposition 7.1 shows that we can study the K-theory of the reduced C^*-algebra $\mathfrak{C}$ of $\mathcal{H}$ in terms of $\mathcal{S}$. By Theorem 5.6(ii) it follows easily that $\mathcal{S}$ is a Fréchet m-algebra, and for this type of topological algebras Cuntz [20] has shown the existence of a unique functorial Chern character map $\mathrm{ch}\colon \mathrm{K}_* \to \mathrm{HP}_*$ with values in Connes' periodic cyclic homology HP_* and which is subject to certain natural compatibility properties. In [64] it is shown for a quite general class of Fréchet m-algebras that the Chern character becomes a natural isomorphism after tensoring with $\mathbb{C}$. Theorem 5.6(ii) shows that the Schwartz algebra $\mathcal{S}$ of $\mathcal{H}$ always falls in the class of algebras to which Solleveld's Theorem applies. Thus we have

Theorem 7.2 (Corollary 9, [64]). *The abelian group* $\mathrm{K}_*(\mathcal{S})$ *is finitely generated, and upon tensoring by* $\mathbb{C}$ *the Chern character* $\mathrm{Id} \otimes \mathrm{ch}\colon \mathrm{K}_*^{\mathbb{C}}(\mathcal{S}) \to \mathrm{HP}_*(\mathcal{S})$ *becomes an isomorphism, where we have used the notation* $\mathrm{K}_*^{\mathbb{C}}(\mathcal{S}) := \mathbb{C} \otimes_{\mathbb{Z}} \mathrm{K}_*(\mathcal{S})$.

7.2. Comparison between $\mathcal{S}$ and $\mathcal{H}$. The material this subsection is joint work in progress with Maarten Solleveld.

Conjecture 1 (Conjecture 8.9, [9]). The inclusion homomorphism $i\colon \mathcal{H} \to \mathcal{S}$ induces an isomorphism $\mathrm{HP}_*(\mathcal{H}) \to \mathrm{HP}_*(\mathcal{S})$.

There is quite solid evidence in support of this conjecture. Under certain additional assumptions we in fact have a proof of the statement (in fact, Solleveld has recently announced a general proof (private communication)). This proof is based on the philosophy explained in the important paper [35] and the Langlands classification for general affine Hecke algebras [23].

7.3. Independence of the parameters. Let us introduce the notation $\mathcal{S}(\boldsymbol{q}) := \mathcal{S}(\mathcal{R}, q)$ to stress the dependence on the base $\boldsymbol{q} > 1$ of the function q defined by $q(s) = \boldsymbol{q}^{f_s}$ while keeping the $f_s \in \mathbb{R}$ fixed. (In the terminology of Remark 2.3, we vary q in a half-line.) Let $\mathcal{S}_W = \mathcal{S}(1)$ denote the limiting case, the Schwartz algebra of functions $f\colon W \to \mathbb{C}$ of rapid decay. Observe that the Fréchet space $\mathcal{S}_W$ is isomorphic to $\mathcal{S}(\boldsymbol{q})$ as a Fréchet space via the naive linear isomorphism $f \to \sum_{w\in W} f(w)N_w$.

Solleveld has shown that there exists a family of isomorphisms (unique up to homotopy) $\psi_\varepsilon\colon \mathcal{S}(\boldsymbol{q}) \to \mathcal{S}(\boldsymbol{q}^\varepsilon)$ of pre C^*-algebras depending continuously on $\varepsilon \in (0, 1]$ such that $\psi_1 = \mathrm{id}_{\mathcal{S}}$. Now consider the family of linear isomorphisms $\phi_\varepsilon\colon \mathcal{S}_W \to \mathcal{S}(\boldsymbol{q})$ (with $\varepsilon \in (0, 1]$) consisting of the composition of the naive linear isomorphism $\mathcal{S}_W \to \mathcal{S}(\boldsymbol{q}^\varepsilon)$ with the inverse of ψ_ε. Solleveld has shown that the family ϕ_ε behaves as an asymptotic morphism $\mathcal{S}_W \to \mathcal{S}(\boldsymbol{q})$ as $\varepsilon \downarrow 0$ and defines a homomorphism $\mathrm{K}_*(\phi)\colon \mathrm{K}_*(\mathcal{S}_W) \to \mathrm{K}_*(\mathcal{S}(\boldsymbol{q}))$.

Conjecture 2. The map $\mathrm{K}_*^{\mathbb{Q}}(\phi)$ is an isomorphism.

This conjecture is the natural extension of [8, Conjecture 6.21] in the present context (after throwing away torsion), and in this sense it is entirely in the spirit of the Baum–Connes–Kasparov conjecture for p-adic reductive groups. In our situation (working with general continuous, unequal Hecke algebra parameters) it would be a very powerful principle for understanding the geometry of the irreducible spectrum and the tempered irreducible spectrum of affine Hecke algebras, especially in combination with certain geometric refinements to be described below (see Conjectures 2b and 2c).

Let us first consider a weaker statement and the evidence in support of it:

Conjecture 2a (weaker version). $\dim(\mathrm{K}_*^{\mathbb{Q}}(\mathcal{S}(\boldsymbol{q}))) = \dim(\mathrm{K}_*^{\mathbb{Q}}(\mathcal{S}_W))$.

By Solleveld's Theorem 7.2 this statement is equivalent to $\dim(\mathrm{HP}_*(\mathcal{S}(\boldsymbol{q}))) = \dim(\mathrm{HP}_*(\mathcal{S}_W))$, and thus in cases where Conjecture 1 is known (certainly including the split case $f_s = 1, \forall s \in S^{\mathrm{aff}}$) it is equivalent to $\dim(\mathrm{HP}_*(\mathcal{H}(\boldsymbol{q}))) = \dim(\mathrm{HP}_*(\mathbb{C}[W]))$. In the split case this is a theorem of Baum and Nistor [10]. The proof in [10] is very interesting and is based on Lusztig's asymptotic morphism in combination with techniques from [35]. For unequal label Hecke algebras one may connect this with Lusztig's conjectures from [43], see [4].

7.4. Equivariant K-theory. The conjecture 2 can be expressed more geometrically in terms of equivariant K-theory using well known results (see [7], [10], [67]). First of all $\mathcal{S}_W = C^\infty(T_u) \rtimes W_0$, and this gives the identification $\mathrm{K}_*(\mathcal{S}_W) = \mathrm{K}^*_{W_0}(T_u)$. Recall that for any finite group G acting on a compact topological space X the equivariant Chern character defines an isomorphism $\mathrm{Id} \otimes \mathrm{ch_G} : \mathbb{C} \otimes_{\mathbb{Z}} \mathrm{K}^*_G(X) \xrightarrow{\sim} \mathrm{H}^{[*]}(G\backslash\hat{X}, \mathbb{C})$ where $\hat{X} = \cup_{g\in G}(g, X^g)$ is the disjoint union of the fixed point spaces X^g of the elements of G, and where $\mathrm{H}^{[*]}$ denotes the $\mathbb{Z}/2\mathbb{Z}$-periodic Čech cohomology groups (see [7]). The topological space $G\backslash\hat{X}$ is called the extended quotient of X. It is the orbit space of the inertia orbifold $\Lambda(G_X) := G_{\hat{X}}$ of the translation orbifold $G_X = X \rtimes G$. In this geometric form the conjecture gives rise to a natural refinement of the conjecture for $\mathrm{K}_0(\mathcal{S})$ which is the main reason for this reformulation.

Definition 7.3. Given $\xi \in \mathcal{W}\backslash\Xi_u$ we have a quotient homomorphism $\pi_\xi : \mathcal{S} \to \mathcal{S}/m_\xi\mathcal{S}$ where m_ξ denotes the maximal ideal of $\mathcal{Z}_{\mathcal{S}}$ at ξ (the ring $\mathcal{S}/m_\xi\mathcal{S}$ is finite dimensional and semisimple by Theorem 5.6). For $\alpha \in \mathrm{K}_0(\mathcal{S})$ we define $\mathrm{Supp}(\alpha) = \{\mathcal{W}\xi \in \mathcal{W}\backslash\Xi_u \mid \mathrm{K}_0(\pi_\xi)(\alpha) \neq 0\}$ (a closed set in $\mathcal{W}\backslash\Xi_u$, as one checks easily).

Conjecture 2b (geometric refinement). There exists a natural isomorphism (take $\mathrm{K}_*^{\mathbb{C}}(\phi)$ composed with the inverse of the equivariant Chern character for the action of W_0 on T_u):

$$\kappa : \mathrm{H}^{[*]}(W_0\backslash\widehat{T_u}, \mathbb{C}) \xrightarrow{\sim} \mathrm{K}_*^{\mathbb{C}}(\mathcal{S}(\boldsymbol{q})). \tag{7.1}$$

The ascending filtration $F_i(\mathrm{K}_0^{\mathbb{C}}(\mathcal{S}(\boldsymbol{q}))) = \{\alpha \mid \dim(\mathrm{Supp}(\alpha)) \leq i\}$ of $\mathrm{K}_0^{\mathbb{C}}(\mathcal{S}(\boldsymbol{q}))$ coincides via this isomorphism with the filtration of the left hand side whose i-th filtered piece consist of the sum of the even cohomologies of the components of the extended quotient $W_0\backslash\widehat{T_u}$ of dimension at most i.

This form of the conjecture guides us to a further reduction to the discrete part $F_0(\mathrm{K}_0^{\mathbb{C}}(\mathcal{S}(\boldsymbol{q})))$ of the vector space $\mathrm{K}_0^{\mathbb{C}}(\mathcal{S}(\boldsymbol{q}))$. This is best understood in the context of a conjecture concerning index theory.

8. Index functions

The material in this section was shaped in its present form in the course of various conversations with Mark Reeder and Joseph Bernstein. I am much indebted for their insightful comments. The ideas in this section have their origin in the theory of the Selberg trace formula. We want to construct "index functions" using the Euler–Poincaré principle in K-theory (see e.g. [37], [59], [21]).

We assume throughout in this section that the root datum $\mathcal{R}$ is semisimple unless stated otherwise. Every finite dimensional tempered module π gives rise to a $\mathbb{Z}$-valued "local index" function Ind_π on $\mathrm{K}_0(\mathcal{S})$. Namely $\pi\colon \mathcal{S} \to \mathrm{End}_{\mathbb{C}}(V_\pi)$ is a continuous algebra homomorphism, and given $\alpha = [p] \in \mathrm{K}_0(\mathcal{S})$ (with $p \in M_N(\mathcal{S})$ an idempotent) we define $\mathrm{Ind}_\pi(\alpha) = \mathrm{rank}(\pi(p)) \in \mathbb{Z}$. This naturally descends to bilinear pairing $[\,\cdot\,,\cdot]\colon\ \mathrm{K}_0(\mathcal{S})\times\mathrm{G}(\mathcal{S}) \to \mathbb{Z}$ given by $[\alpha, [\pi]] := \mathrm{Ind}_{[\pi]}(\alpha)$. Moreover, by the structure theory of $\mathcal{S}$ (Theorem 5.6), it is clear that if the local index function $[\pi] \to \mathrm{Ind}_{[\pi]}(\alpha)$ vanishes identically on $\mathrm{G}(\mathcal{S})$ then $\alpha = 0$.

Now suppose that $\alpha \in F_0(\mathrm{K}_0^{\mathbb{C}}(\mathcal{S}))$. By definition of the support of α and of our notion of smooth families this means that $\mathrm{Ind}_{[\pi]}(\alpha) = 0$ for all π which are 1-smooth (it must be constant along the family on the one hand, but on the other hand its support must be finite). Therefore $\pi \to \mathrm{Ind}_{[\pi]}(\alpha)$ factors through a function on $\mathrm{Ell}^{\mathrm{temp}}$, and $[\,\cdot\,,\cdot]$ factors through a bilinear pairing $[\,\cdot\,,\cdot]$ on $F_0(\mathrm{K}_0^{\mathbb{C}}(\mathcal{S})) \times \mathrm{Ell}^{\mathrm{temp}}$ which is non-degenerate on the left.

Next we consider the change of base ring homomorphism $\beta\colon\ \mathrm{K}_0(\mathcal{H}) \to \mathrm{K}_0(\mathcal{S})$ defined by $[P] \to [S \otimes_{\mathcal{H}} P]$ if P denotes a finitely generated projective $\mathcal{H}$-module. One would like to complement this base change homomorphism β with a base change homomorphism $\beta\colon\ \mathrm{K}(\mathrm{Mod}_{fg}(\mathcal{H})) \to \mathrm{K}(\mathrm{Mod}(\mathcal{S}))$ but there seems no obvious way to do this. First of all $\mathcal{S}$ is *not* flat over $\mathcal{H}$ (this problem already occurs in rank 1) and it is also not quite clear which category of $\mathcal{S}$-modules one should consider. The conjecture we are about to make precisely states that this can anyway be done if one restricts in some sense to the tempered modules of finite length. To this end we will first show that the projective dimension of $\mathcal{H}$-modules of finite length is bounded by the rank [3] (which is a joint result with Mark Reeder). Let (π, V) be an $\mathcal{H}$-module of finite length. Let $C \subset \mathfrak{a}^* = \mathbb{R} \otimes_{\mathbb{Z}} X$ be the fundamental alcove for the action of $W^a = W_0 \ltimes Q \lhd W$ (the normal subgroup generated by reflections in W). Let $\Omega \subset W$ be the finite abelian subgroup of elements of length 0, then $W = W^a \rtimes \Omega$. We denote by $\mathcal{H}^a \subset \mathcal{H}$ the unital subalgebra of $\mathcal{H}$ spanned by the elements $\{N_w\}_{w\in W^a}$, then $\mathcal{H} = \mathcal{H}^a \rtimes \Omega$ (a crossed product).

Given a nonempty facet $\emptyset \neq f \subset C$ of C we have the corresponding subset $S_f \subset S^{\mathrm{aff}}$ of affine simple reflections fixing f. Let $\mathcal{H}_f \subset \mathcal{H}^a$ be the finite type Hecke subalgebra $\mathcal{H}_f = \mathcal{H}(S_f, q_{S_f})$. For any subset $I \subset S^{\mathrm{aff}}$ we denote by $\mathbb{C}^I$ the

[3] One can deduce from this the fact that the category of $\mathcal{H}$-modules is of finite cohomological dimension, as was explained to me by Joseph Bernstein. One should compare this to Bernstein's result that the category of smooth representations of a reductive p-adic group has finite cohomological dimension, see [65, Prop. 37].

complex vector space which has as a basis the set I. Put

$$\tilde{C}_i(V) = \bigoplus_{f:\dim(f)=i} \mathcal{H} \otimes_{\mathcal{H}_f} (V|_{\mathcal{H}_f}) \otimes_{\mathbb{C}} \bigwedge^{n-i} \mathbb{C}^{S_f}. \tag{8.1}$$

Since $\mathcal{H}_f \subset \mathcal{H}$ is a finite dimensional semisimple subalgebra this is clearly a projective, finitely generated $\mathcal{H}$-module. We define $\mathcal{H}$-linear maps $\tilde{d}_i : \tilde{C}_i \to \tilde{C}_{i-1}$ by

$$\tilde{d}_i(h \otimes_{\mathcal{H}_f} v \otimes \lambda) := \bigoplus_{\substack{f' \subset f \\ \dim(f')=i-1}} h \otimes_{\mathcal{H}_{f'}} v \otimes (\lambda \wedge s_{f,f'}) \tag{8.2}$$

where $s_{f,f'} \in S^{\mathrm{aff}}$ is defined by $S_{f'} = S_f \cup \{s_{f,f'}\}$. Observe that there is a natural left Ω-action on $\tilde{C}_i(V)$ by means of $\mathcal{H}$-intertwining operators via

$$j_\omega(h \otimes_{\mathcal{H}_f} v \otimes \lambda) = h\omega^{-1} \otimes_{\mathcal{H}_{\omega(f)}} \pi(\omega)v \otimes \omega(\lambda). \tag{8.3}$$

This action commutes with the action of the operators $\tilde{d}_i$. Finally, we define $C_i(V) = (\tilde{C}_i(V))^{j(\Omega)}$ and we denote by d_i the restriction of $\tilde{d}_i$ to $C_i(V)$.

Proposition 8.1 (E. Opdam and M. Reeder, unpublished). *Let (V, π) be an $\mathcal{H}$-module of finite length. The graded $\mathcal{H}$-linear operator $d = \{d_i\}$ is a differential on $C_*(V)$, making $C_*(V)$ into a bounded complex of finitely generated projective $\mathcal{H}$-modules. The $\mathcal{H}$-linear map $d_0 : C_0(V) \to C_{-1}(V) \simeq V$ extends this to a finite projective resolution $0 \leftarrow V \xleftarrow{d_0} C_*(V)$ of V.*

The proof of this proposition reduces simply to the case $\mathcal{H} = \mathcal{H}^a$, and there the proof is a variation of Kato's proof [33] of a statement about a similar "restriction-induction" complex for finite type Hecke algebras.

Definition 8.2. By the previous result all $\mathcal{H}$-modules of finite length have finite projective dimension. Hence there is a well defined Euler–Poincaré homomorphism $\varepsilon\colon \mathrm{G}(\mathcal{H}) \to \mathrm{K}_0(\mathcal{H})$. It has an explicit realization $\varepsilon([V]) := \sum_i (-1)^i [C_i(V)]$.

Let $\rho\colon \mathrm{G}(\mathcal{S}) \to \mathrm{G}(\mathcal{H})$ be the homomorphism which corresponds to the forgetful functor (forgetting temperedness). We have now altogether constructed a homomorphism $\gamma = \beta \circ \varepsilon \circ \rho\colon \mathrm{G}(\mathcal{S}) \to \mathrm{K}_0(\mathcal{S})$. We extend this to an *anti-linear* map $\gamma\colon \mathrm{G}^{\mathbb{C}}(\mathcal{S}) \to \mathrm{K}_0^{\mathbb{C}}(\mathcal{S})$. It is not difficult to show that this map vanishes on modules induced from a proper parabolic subalgebra (e.g. by using the Koszul resolution for a regular sequence of parameters for the smooth family of induced representations). Thus we obtain an anti-linear map

$$\gamma\colon \mathrm{Ell}^{\mathrm{temp}} \to F_0(\mathrm{K}_0^{\mathbb{C}}(\mathcal{S})). \tag{8.4}$$

Using γ the previously defined bilinear pairing $[\cdot,\cdot]$ on $F_0(\mathrm{K}_0^{\mathbb{C}}(\mathcal{S})) \times \mathrm{Ell}^{\mathrm{temp}}$ gives rise to a sesquilinear form $\langle U, V\rangle_{\mathrm{ell}} := [\gamma(U), V]$ on $\mathrm{Ell}^{\mathrm{temp}}$. This is the precise analog of the elliptic pairing of tempered characters as defined by Schneider and Stuhler [59]: suppose that U and V are finite dimensional tempered modules of $\mathcal{H}$, then

$$\langle U, V\rangle_{\mathrm{ell}} = [\gamma(U), V] = \sum_{i\geq 0}(-1)^i \dim \mathrm{Ext}^i_{\mathcal{H}}(U, V). \tag{8.5}$$

In this context we remark that the natural map $\mathrm{Ell}^{\mathrm{temp}} \to \mathrm{Ell}^{\mathrm{alg}}$ (the elliptic virtual representations of $\mathcal{H}$) is a linear isomorphism [23], by the Langlands parametrization [23] for affine Hecke algebras.

By the Euler–Poincaré principle applied to our standard resolution we can also express this explicitly (following [59], [58]) in terms of an "index function" for U. Let $\Omega_f \subset \Omega$ be the stabilizer of f in Ω, and let ε_f be the character of Ω_f on $\mathbb{C}^{S_f}$. We define the index function $f_U \in \mathcal{H}$ by

$$f_U = \sum_f (-1)^{\dim(f)} \sum_{\sigma\in\mathrm{Irr}(\mathcal{H}_f\rtimes\Omega_f)} \dim(\sigma)^{-1}[U|_{(\mathcal{H}_f\rtimes\Omega_f)} \otimes \varepsilon_f : \sigma]e_\sigma \in \mathcal{H} \tag{8.6}$$

where f runs over a complete set of representatives of the Ω-orbits of faces of C, and where $e_\sigma \in \mathcal{H}_f \rtimes \Omega_f$ denotes the central idempotent corresponding to σ in the finite dimensional complex semisimple algebra $\mathcal{H}_f \rtimes \Omega_f$. Then

$$\langle U, V\rangle_{\mathrm{ell}} = \chi_V(f_U) \tag{8.7}$$

By (8.5) it is clear that this pairing is Hermitian, and that (virtual) tempered representations U and V with distinct tempered central characters are orthogonal. Moreover, the pairing is integral with respect to the lattice generated by the elliptic (true) characters.

Conjecture 3. The pairing $\langle U, V\rangle_{\mathrm{ell}}$ on $\mathrm{Ell}^{\mathrm{temp}}_{W\xi}$ corresponds, via the functor E_ξ of Theorem 5.13, with the elliptic paring on $\mathrm{Ell}(\mathfrak{R}_\xi, \eta)$ given by

$$\langle \phi, \chi\rangle_{\mathrm{ell}} = |\mathfrak{R}_\xi|^{-1} \sum_{\mathfrak{r}\in\mathfrak{R}_\xi} |\det(1-\mathfrak{r})|\overline{\phi(\mathfrak{r})}\chi(\mathfrak{r}) \tag{8.8}$$

(see [3] and [58]). In particular, this pairing is positive definite (since the support of the function $\det(1-\mathfrak{r})$ is exactly equal to the set of elliptic conjugacy classes of $\mathfrak{R}_\xi$).

This conjecture is the natural analog of results of Arthur in the theory of the local trace formula [3]. In the split case with $X = P$ it was shown by Mark Reeder [58] using the Kazhdan–Lusztig parameterization and a comparison between geometric and analytic R-groups. The formula (8.6) for the index functions f_U is due to Mark Reeder [57], based on work of Schneider and Stuhler [59]. Recently a related and very general result was obtained by R. Meyer [48] for Schwartz algebra's of reductive p-adic groups. In order to explain this, first observe that Theorem 5.6 implies that a discrete series representation (U, δ) of $\mathcal{S}$ is a projective $\mathcal{S}$-module. Therefore it defines a class $[U]_{\mathcal{S}} \in K_0(\mathcal{S})$. On the other hand we have defined the class $\gamma([U]) \in K_0(\mathcal{S})$ above. In our context Meyer's result would mean that $[U]_{\mathcal{S}} = \gamma([U])$, and in particular that all higher extensions $\mathrm{Ext}^i_{\mathcal{H}}(U, V)$ $(i > 0)$ vanish if V is tempered (this is remarkable since $\mathcal{S}$ is not flat over $\mathcal{H}$). Meyer's Theorem actually implies the validity of this statement for all affine Hecke algebras which arise in connection with a reductive p-adic group G via the theory of types of equivalence classes of G-inertial cuspidal data (of course, we conjecture that it holds for general $\mathcal{H}$). This proves that in those cases Conjecture 3 holds for the discrete series representations of $\mathcal{H}$, thus providing strong evidence in support of this conjecture.

Corollary 8.3 (L^2-index for discrete series representations of $\mathcal{H}$). *We have the following explicit Euler–Poincaré formula for the formal dimension of a discrete series representation (U, δ) of $\mathcal{H}$, expressed in terms of its "K-types":*

$$\begin{aligned}\mu_{\mathrm{Pl}}(\{\delta(q)\}) &= \tau(f_U(q)) \\ &= \sum_f (-1)^{\dim(f)} \sum_{\sigma \in \mathrm{Irr}(\mathcal{H}_f \rtimes \Omega_f)} [U|_{(\mathcal{H}_f \rtimes \Omega_f)} \otimes \varepsilon_f : \sigma] d_\sigma(q) \end{aligned} \tag{8.9}$$

where f runs over a complete set of representatives of the Ω-orbits of faces of C, and where $d_\sigma(q)$ denotes the formal dimension of σ in the finite dimensional Hilbert algebra $\mathcal{H}_f \rtimes \Omega_f$ whose trace is the restriction of the trace τ of $\mathcal{H}$ (these are rational functions in the parameters $q_s^{1/2}$, $q_s^{-1/2}$ with rational coefficients).

Please compare this statement with the product formula (4.4) for the formal dimensions. Observe that the rationality of the constant d_δ in (4.4) is an immediate consequence. The Euler–Poincaré formula for f_U (as obtained by Schneider and Stuhler [59]) was used by Reeder [57] for the computation of all formal dimensions of the square integrable unipotent representations of Chevalley groups of exceptional type. The main computational work in [57] consists in the reduction of the Euler–Poincaré alternating sum to the product formula.

A second consequence of Conjecture 3 is:

Corollary 8.4. *The map $\gamma\colon \mathrm{Ell}^{\mathrm{temp}} \to F_0(\mathrm{K}_0^{\mathbb{C}}(\mathcal{S}))$ is an anti-linear isomorphism.*

Let us write $\mathrm{Ell}^{\mathrm{temp}}(q)$ in order to stress the dependence on q. Observe that $\mathrm{Ell}^{\mathrm{temp}}(q)$ is a semisimple $\mathcal{Z}$-module via the algebraic central character z map. The combination of Corollary 8.4 with Conjecture 2b implies that the dimension of the finite dimensional space (see Proposition 6.5) $\mathrm{Ell}^{\mathrm{temp}}(q)$ is independent of $q \in \mathcal{Q}$. We conjecture a stronger statement:

Conjecture 2c. The $\mathcal{Q}$-family of finite dimensional semisimple $\mathcal{Z}$-modules $q \to \mathrm{Ell}^{\mathrm{temp}}(q)$ is continuous (i.e. isomorphic to a direct sum of one-dimensional $\mathcal{Z}$-modules, each depending continuously on q).

Let us denote by $j\colon |\Xi_u| \to W_0\backslash T_u$ the map that sends $\mathcal{W}\xi$ (with $\xi = (P, \delta, t)$) to $W_0(|r|^{-1}rt)$ where $z_P(\delta) = W_P r$ is the central character of δ. Let $q = 1$ and let

$t \in T_u = \Xi_u(1)$. Then $\mathfrak{R}_t$ is the isotropy group $W_{0,t}$ of t in W_0. Using Proposition 6.6, Conjecture 2c implies an isomorphism of sheaves

$$j_*(\underline{\mathrm{G}}(\mathfrak{R}, \eta)) \simeq \underline{\mathrm{G}}(W_0) \tag{8.10}$$

on $W_0 \backslash T_u$, where the sheaf on the right hand side is the usual complex representation ring sheaf for the action of W_0 on T_u.

8.1. Discussion and examples. We discuss the implications of Conjecture 2c for the problem of understanding the tempered spectrum of non-simply laced affine Hecke algebras. The results in this section are joint with Maarten Solleveld. In these cases, Proposition 4.7, Theorem 4.9 and Conjecture 2c suggest to use deformations to generic points $q \in \mathcal{Q}$ in order to approach this problem.

Let us consider the three parameter case $\mathcal{R} = (\mathbb{Z}^n, B_n, \mathbb{Z}^n, C_n, F_0)$ with $F_0 = \{e_1 - e_2, \ldots, e_{n-1} - e_n, e_n\}$ (thus $Q(B_n) = \mathbb{Z}^n$; we refer to this case as "type C_n^{aff}") with parameters $q_0 = q(s_0)$, $q_1 = q(s_{e_i})$, $q_2 = q(s_{e_i - e_{i+1}})$. The case $n = 1$ will be considered as a special case (but with two parameters q_0, q_1); the discussion below applies to this degenerate case without modifications). The set of distinct W_0-orbits of generic residual points is easily seen to be parametrized (using [56, Appendix A]) by ordered pairs (μ, ν) of partitions of total weight n. The orbit of the unitary part of (μ, ν) only depends on $i := |\mu| \in \{0, 1, \ldots, n\}$. Let $W_0 s_i$ denote this orbit. The orbits $W_0 s_i \in W_0 \backslash T_u$ are mutually distinct, and the stabilizer group of s_i is isomorphic to the Weyl group of type $B_i \times B_{n-i}$.

We see that for each i the cardinality of the set of orbits of generic residual points whose corresponding orbit of unitary parts equals $W_0 s_i$ is precisely equal to the number of elliptic conjugacy classes of the stabilizer group $W(B_i) \times W(B_{n-i})$ of s_i, which is the R-group $\mathfrak{R}_{s_i}$ for $\mathcal{S}_W = \mathcal{S}(1)$. By Theorem 4.9 each orbit $W_0 r$ of residual points carries *at least one* discrete series representation of $\mathcal{H}$. We call $q \in \mathcal{Q}$ *generic* if $q \in \cap \mathcal{Q}_r$ (intersection over all $r \in \mathrm{Res}$) and if for all $r, r' \in \mathrm{Res} : W_0 r(q) = W_0 r'(q) \Rightarrow W_0 r = W_0 r'$. By Conjecture 2c and the above description of the set of orbits of generic residual points we conclude that for a generic parameter q each residual orbit $W_0 r(q)$ must carry *precisely one* discrete series representation. So for generic $q \in \mathcal{Q}$ the discrete series characters are separated by their algebraic central character. Moreover, the complex linear span of these discrete series characters is isomorphic to the space $\mathrm{Ell}^{\mathrm{temp}}(q)$.

By the continuity aspect of Conjecture 2c we are now in principle able to understand the spaces $\mathrm{Ell}^{\mathrm{temp}}(q_0)$ for an *arbitrary* parameter $q_0 \in \mathcal{Q}$ by deformation to the generic case. The central support of the discrete series representations of $\mathcal{H}(q_0)$ is given by Theorem 4.9. For a given orbit of residual points $W_0 r$ for $\mathcal{H}(q_0)$ the set of discrete series representations which it carries is, in view of the above, parametrized by the set of orbits of generic residual points $\mathcal{W}_0 r(q)$ such that $W_0 r(q_0) = W_0 r$. This determines the set $\Delta_{F_0}(q_0)$, with complete information about the action of $\mathcal{Z}$. We can repeat this for any standard parabolic subset P, since the associated affine Hecke algebra $\mathcal{H}_P$ is a tensor product of at most one factor of type C_m^{aff} (with $m \leq n$) (which we can handle as above) and factors of type A_{λ_j-1} (with lattice $X_j = P(A_{\lambda_j-1})$, the corresponding weight lattice) such that $\sum \lambda_j = n - m$. The group K_P is a product of cyclic groups C_{λ_j} of order λ_j. We also get complete information on the action of $\mathcal{W}$ on $\Delta : \mathcal{W}$ is generated by permutations of the type A-factors, by twisting with $-w$ if w is the longest element in the Weyl group of one of the type A-factors, and by twisting of one of the type A factors A_{k-1} by the corresponding cyclic group C_k. Observe that the action of $\mathcal{W}$ only affects the type A-factors of $\mathcal{H}_P$.

From this information we can reconstruct $\mathcal{W}_{\Xi_u}$, and we can verify directly that the cocycle η_Δ (and thus η itself) *is in fact always trivial.* Using Theorem 4.19 we can in principle compute the R-groups (this was actually carried out by Klaas Slooten [63] in the case of real central characters for all $q_0 \in \mathcal{Q}$). By Theorem 5.13 this gives essentially complete information about the structure of the tempered dual $\hat{\mathcal{S}}$.

If $q \in \cap \mathcal{Q}_r$ (intersection over all $r \in \mathrm{Res}$) it is not hard to show that the R-groups are all trivial. This implies that $\mathcal{S}(q)$, in view of Theorem 5.13 and Theorem 5.11, is Morita equivalent to the algebra of $\mathcal{W}$-invariant C^∞-functions on $\Xi_u(q)$. We see that the components of $|\Xi_u(q)|$ are parametrized by the set of ordered triples of partitions (λ, μ, ν) of total weight n. If $\lambda = (1^{m_1}, 2^{m_2}, \dots, k^{m_k})$ then the component corresponding to (λ, μ, ν) is the product of the quotients $W_0(B_{m_i})\backslash(S^1)^{m_i}$ $(i = 1, \dots, k)$, and thus homeomorphic to $[0, 1]^{|\lambda|}$. Hence $\mathrm{K}_1(\mathcal{S}(q)) = 0$, and $\mathrm{K}_0(\mathcal{S}(q))$ is the free abelian group generated by the above set of components.

Let us compare this result with the other extreme case $q = 1$. Now $\mathcal{S}(1) = \mathcal{S}_W \simeq C^\infty(T_u) \rtimes W_0$ and thus $\mathrm{K}_*(\mathcal{S}(1)) \simeq \mathrm{K}^*_{W_0}(T_u)$. Application of the equivariant Chern character [7] to this last group yields an isomorphism (after killing torsion) with the periodized cohomology of the extended quotient $W_0\backslash\widehat{T_u}$ of T_u with respect to the action of $W_0 = W(B_n)$. The extended quotient $W_0\backslash\widehat{T_u}$ can be computed directly in this case. It turns out that this space is actually homeomorphic to the orbit space $|\Xi_u(q)|$ which we have just computed in the case where $q \in \cap\mathcal{Q}_r$. This is in complete accordance with Conjecture 2b (but of course, we already used the "discrete part" of this conjecture in order to conclude that each generic residual orbit carries precisely one discrete series representation).

Conjecture 2c gives in this example complete information on the classification problem of the irreducible tempered representations, but not on the internal structure of these representations. In the case of type C_n^{aff} Klaas Slooten [62] defined (for real central character) a "generalized Springer correspondence" in terms of certain symbols (in the sense of Malle [45]) and conjectured that the restriction of these tempered representations to $\mathcal{H}(W_0)$ is precisely given by the generalized Green functions [40], [60] attached to these symbols. These conjectures were verified for $n = 3, 4$.

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