Math. Z. 212, 313–336 (1993)

Mathematische Zeitschrift

An analogue of the Gauss summation formula for hypergeometric functions related to root systems

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Received September 19, 1991; in final form March 16, 1992

1 Introduction

A remarkable feature of the theory of the ordinary hypergeometric differential equation is that one can explicitly determine the analytic continuation of a basis of local solutions at a given singular point towards another singular point. For example, one has the following Gauss-Kummer formula:

$$(1.1)\qquad F(a,b;c;z)=\frac{\Gamma(c)\,\Gamma(b-a)}{\Gamma(b)\,\Gamma(c-a)}(1-z)^{-a}F\left(a,c-b;a-b+1;\frac{1}{1-z}\right)+\frac{\Gamma(c)\,\Gamma(a-b)}{\Gamma(a)\,\Gamma(c-b)}(1-z)^{-b}F\left(b,c-a;b-a+1;\frac{1}{1-z}\right).$$

(Note that this formula in particular expresses that the exponents at ∞ are a and b). If one uses Kummer's list of 24 solutions of the hypergeometric equation, formula (1.1) is a direct consequence of the following special value of F (the Gauss summation formula):

$$(1.2)\qquad F(a,b;c;1)=\frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}\quad \text{if } c\notin \mathbf{Z}_-,\qquad \operatorname{Re}(c-a-b)>0.$$

For instance when $\operatorname{Re}(a-b)>0$ we multiply both sides of (1.1) by $(1-z)^a$ and use the linear transformation $F(a,b;c;z)=(1-z)^{-a}F\left(a,c-b;c;\frac{z}{1-z}\right)$ on the left hand side. Now take the limit $z\to-\infty$ and use (1.2) in order to verify the first coefficient on the right hand side of (1.1). A similar argument applies to the second coefficient on the right hand side of (1.1). Clearly, formula (1.1) determines the monodromy representation of the multivalued function $F(a,b;c;.)$, but it contains more information than that. Formula (1.2) goes back to Gauss [7], who proved this using a certain contiguacy relation for F.

In the theory of hypergeometric functions associated with a root system R we study a system of differential equations on the complex torus H

$=Q^\vee \otimes_{\mathbf{Z}} \mathbf{C}^\times$ (where $\mathbf{Q}^\vee$ denotes the coroot lattice (see Sect. 2)). Instead of the parameters $(a, b; c)$ the equations involve two sets of parameters of a different nature: there is the spectral parameter $\lambda \in \mathbf{C}R$ and the multiplicity parameter k which is nothing but a Weyl group invariant function on R (see Sect. 2 for details and references). This setup is of course inspired by the theory of spherical functions on a noncompact Riemannian symmetric space and their differential equations. In the case where $2k$ is the multiplicity function for a certain Riemannian symmetric space X with restricted root system R we recover the restrictions to a Cartan subspace of spherical functions on X. The exponents at ∞ of our equations are the elements $w\lambda + \rho(k)$, with $w \in W$ (the Weyl group) and $\rho(k) = \frac{1}{2} \sum_{\alpha \in R_+} k_\alpha \alpha (R_+ \subset R$ a fixed choice of positive roots) (see (2.7)). Corresponding to these exponents one has the following basis of solutions "at infinity": $\{\phi(w\lambda + \rho(k), k; h)\}_{w \in W}$, with $\phi(w\lambda + \rho(k), k;\ h)$ the solution which is analytic on $A_- \subset H$ (the negative Weyl chamber) and has a series expansion of the form (see (2.9))

$$\phi(\lambda + \rho(k), k; h) = h^{\lambda + \rho(k)} \sum_{\kappa \in Q_+} \Delta_\kappa(\lambda, k)\, h^\kappa.$$

On the other hand there is the situation at the most singular point of the hypergeometric equations, namely the identity element e of H. The singularities of the equations have nonnormal crossings at this point. One can show however that generically there exists a one dimensional space of solutions which extend holomorphically in a neighbourhood of e [13, Sect. 6]. Hence we may determine a solution $\hat{F}(\lambda, k)$ in a neighbourhood of e by the requirement that $\hat{F}$ is holomorphic at e and that $\hat{F}(\lambda, k; e) = 1$ (in the case of a symmetric space this would be a spherical function). The analog of (1.1) would be a formula of the form:

$$\hat{F}(\lambda, k; h) = \sum_{w \in W} \hat{c}(w\lambda, k)\, \phi(w\lambda + \rho(k), k; h) \tag{1.3}$$

and again it suffices to calculate the limit ($\mathrm{Re}(\lambda, \alpha) < 0\, \forall \alpha \in R_+$):

$$\lim_{A_- \ni h \to \infty} h^{-\lambda - \rho(k)}\, \hat{F}(\lambda, k; h) = \hat{c}(\lambda, k). \tag{1.4}$$

In the theory of spherical functions formula (1.3) is known as Harish-Chandra's asymptotic expansion for the spherical function [9] and in that situation it is possible to obtain an explicit expression in terms of Γ-functions for the limit in (1.4). This is the well known product formula of Gindekin and Karpelevič for Harish-Chandra's c-function [8]. In this paper we will extend this formula to the general case by showing that $\hat{c} = c$, which is given by an explicit expression (see (3.2)). (This is the content of Theorem 6.1). It should be remarked that it is relatively easy to determine the ratios of the coefficients $\hat{c}$ in (1.3) by means of rank reduction, which was already done in [13, Theorem 6.7]. Also, Theorem 6.1 was conjectured in [13, Conjecture 6.11]. We will use Theorem 6.1 in order to obtain several other results on the values of solutions of the hypergeometric equations at special values of the argument. Especially pleasing is the result Theorem 6.3 which gives (under certain restrictions on the parameters of course) an elegant explicit expression for the value of the series $\phi(\lambda + \rho(k), k; h)$

at e, which is at the boundary of the polydisk of convergence. This is related to a well known combinatorial result for root systems due to Macdonald (see formula (4.8)). For its proof we make use of a pairing between the solutions of the hypergeometric equations with so called dual parameters. The matrix of this pairing with respect to the bases of series solutions at infinity can be obtained explicitly (see Theorem 4.9). On the other hand it is possible to interpret the evaluation of solutions of the hypergeometric equations in terms of this pairing. Corollary 6.10 gives a generalization of the Gauss summation formula for hypergeometric functions of matrix argument due to Macdonald (see Remark 6.11).

The paper is organized as follows. In the Sects. 2 and 3 we mainly review some facts of the theory of hypergeometric functions that will be used in the subsequent sections. In Sect. 4 we study in detail the pairing mentioned above and we derive some results that we will apply in Sect. 6. Section 5 is devoted to the proof of a technical result on the growth rate of $F(\lambda, k; e)$ as an entire function of the variable k. This result plays a decisive role in the proof of the main result Theorem 6.1. Finally in Sect. 6 we put everything together in order to obtain our evaluation formulas.

2 Preliminaries

In this section we will review the basic setup of the theory of hypergeometric differential equations associated with a root system as developed in the papers [13, 10, 11, 12, 19, 20, 21]. Hopefully this will help the reader to find his way through these papers, and to understand those properties of these equations that are relevant in this paper.

Let $\mathfrak{a}$ be a Euclidean space of dimension n and $R \subset \mathfrak{a}^*$ (the dual of $\mathfrak{a}$) be an integral root system. We do not assume that R is reduced, and we will write R^0 for the inmultiplicable roots in R and R_0 for the indivisible roots in R. Denote by W the associated Weyl group. If $\alpha \in R$ then we use the notation $\alpha^\vee \in \mathfrak{a}$ for the element in $\mathfrak{a}$ that satisfies $\lambda(\alpha^\vee) = \frac{2(\alpha, \lambda)}{(\alpha, \alpha)}$. The set $R^\vee = \{\alpha^\vee\} \subset \mathfrak{a}$ is called the coroot system (and its elements are called coroots). We define $Q = \mathbf{Z}.R$, the root lattice of R, and $Q^\vee = Q(R^\vee)$. We will also need the so-called weight lattice $P = P(R) = \mathrm{Hom}_{\mathbf{Z}}(Q^\vee, \mathbf{Z}) \subset \mathfrak{a}^*$. Let us denote by $\mathfrak{h}$ the complexification $\mathbf{C} \otimes_{\mathbf{R}} \mathfrak{a}$ of $\mathfrak{a}$. The complex torus H is given by $H = Q^\vee \otimes_{\mathbf{Z}} \mathbf{C}^\times$. We write A for the real split part of H, and T for the compact part of H, so that we have the decomposition $H = AT$. The Weyl group acts on H in a natural way (via the W-action on $Q^\vee$). If we put

$$(2.1) \qquad \begin{aligned} h^\lambda \colon & H \to \mathbf{C}^\times \\ h = \kappa \otimes z & \to z^{\lambda(\kappa)} \end{aligned}$$

(where $\lambda \in \mathfrak{h}^*$, the dual of $\mathfrak{h}$), then this defines a single valued function if and only if $\lambda \in P(R^\vee)$. The set $\{h^\lambda\}_{\lambda \in P}$ exhausts the algebraic characters of H, and the $\mathbf{C}$-linear span of these characters is the ring of regular functions $\mathbf{C}[H]$ of H. The regular points of H for the action of W are $H^{\mathrm{reg}} = \{h \in H \mid \Delta(h)$

$=\prod_{\alpha\in R_+^0}(h^{-\frac{\alpha}{2}}-h^{\frac{\alpha}{2}})\neq 0\}$, where R_+ is a choice of positive roots (the function Δ is called the Weyl denominator). We also choose a basis $\{\alpha_1, \ldots, \alpha_n\}$ for R_+, and let $\{\lambda_1, \ldots, \lambda_n\}\subset P$ be the corresponding basis of dual weights. The subset $Q_+\subset Q(P_+\subset P)$ is by definition the $\mathbf{Z}_+$-span of $\{\alpha_i\}_{i=1}^n(\{\lambda_i\}_{i=1}^n)$ (where $\mathbf{Z}_+ = 0, 1, 2, \ldots$) and is referred to as "the positive roots (weights)". Corresponding to these notions of positivity we also have $\mathfrak{a}_+(=\mathbf{R}_+\cdot P(R^\vee)_+)$, $\mathfrak{a}_+^*(=\mathbf{R}_+\cdot P_+)$, A_+ etcetera. It is well known that

(2.2) $$\mathbf{C}[H]^W=\mathbf{C}[z_1, \ldots, z_n]$$

where

(2.3) $$z_i=\sum_{w\in W/W^{\lambda_i}} h^{-w\lambda_i}.$$

The map

(2.4) $$\begin{aligned}\pi\colon\ & H\to\mathbf{C}^n\\ & h\to(z_1(h), \ldots, z_n(h))\end{aligned}$$

parametrizes the W-orbits in H, and is ramified along the discriminant of R: $\{z\in\mathbf{C}^n \mid d(z)=\Delta^2(h)=0\}$. Let $\mathbf{A}$ be the so-called Weyl algebra of polynomial differential operators on $W\backslash H\simeq\mathbf{C}^n$. Via the finite map π (see (1.4)) we will often regard elements of $\mathbf{A}$ as W-invariant differential operators on H^{reg} and vice versa, an algebraic W-invariant differential operator on H^{reg} can be regarded as element of $d^{-1}\mathbf{A}$.

We are now in the position to introduce the foremost important differential operator in this theory. Let $\mathscr{K}$ denote the vector space of W-invariant complex functions on R, and let $k\in\mathscr{K}$. Then $L(k)\in\mathbf{A}$ is given by (as operator on H^{reg}):

(2.5) $$L(k)=\sum_{i=1}^n\partial(X_i)^2-\sum_{\alpha\in R_+}k_\alpha(1+h^\alpha)(1-h^\alpha)^{-1}\partial(X_\alpha).$$

Here we used the convention to write $\partial(p)(p\in\mathbf{C}[\mathfrak{h}^*])$ to indicate the constant coefficient differential operator on H which comes from p by considering p as an element of the universal enveloping algebra $\mathrm{Sym}(\mathfrak{h})\simeq\mathbf{C}[\mathfrak{h}^*]$ of $\mathfrak{h}=\mathrm{Lie}(H)$. Moreover, we have chosen an orthonormal basis $\{X_i\}$ in $\mathfrak{a}$, and if $\lambda\in\mathfrak{h}^*$ then X_λ denotes the element of $\mathfrak{h}$ that corresponds to λ when we identify $\mathfrak{h}$ and $\mathfrak{h}^*$ via the bilinear form that comes from the inner product on $\mathfrak{a}$. This differential operator L has the following series expansion, converging on A_-T:

(2.6) $$L(k)=\sum_{i=1}^n\partial(X_i)^2-2\partial(X_{\rho(k)})-2\sum_{\alpha\in R_+}k_\alpha\sum_{j=1}^\infty h^{j\alpha}\partial(X_\alpha),$$

where

(2.7) $$\rho(k)=\frac{1}{2}\sum_{\alpha\in R_+}k_\alpha\,\alpha.$$

Formula (2.6) gives rise to the following important observation: we can find many eigenfunctions for the operator L in the space of formal exponential series on A_-T. More precisely, if we consider the eigenfunction equation (see [13, formula (3.12)])

$$L(k)\,\phi=(\lambda+\rho(k),\lambda-\rho(k))\,\phi \qquad (\lambda\in\mathfrak{h}^* \text{ and } k\in\mathscr{K} \text{ fixed}) \tag{2.8}$$

then there exists a unique solution $\phi=\phi(\lambda+\rho(k),k)$ of the form

$$\phi(\lambda+\rho(k),k,h)=h^{\lambda+\rho(k)}\sum_{\kappa\in Q_+}\Delta_\kappa(\lambda,k)\,h^\kappa \tag{2.9}$$

provided that

$$2(\lambda,\kappa)+(\kappa,\kappa)\neq 0 \qquad \forall_\kappa\in Q_+\backslash\{0\}. \tag{2.10}$$

The verification of these statements is not hard using formula (2.6), since one obtains the following recurrence relation for the coefficients Δ_κ of $\phi(\lambda+\rho(k),k)$ (see [13], and [20, formula (2.1)]).

$$(2\lambda+\kappa,\kappa)\,\Delta_\kappa(\lambda,k)=2\sum_{\alpha\in R_+}k_\alpha\sum_{j=1}^{\infty}(\lambda+\rho(k)+\kappa-j\alpha,\alpha)\,\Delta_{\kappa-j\alpha}(\lambda,k). \tag{2.11}$$

In fact one can show that these formal series converge locally uniformly in (λ,k) on the domain A_-U, where $U\subset T$ denotes a connected, simply connected open subset of T and this yields the following lemma (see [20, Corollary 2.3]).

Lemma 2.1 *The function*

$$\begin{aligned}\phi:\ &\mathfrak{h}^*\times\mathscr{K}\times A_-U\to\mathbf{C}\\ &(\lambda,k,h)\to\phi(\lambda+\rho(k),k;h)\end{aligned} \tag{2.12}$$

is a meromorphic function with poles only along the hyperplanes of the form $H_\kappa\times\mathscr{K}\times A_-U$, $\kappa\in Q_+\backslash\{0\}$, *where*

$$H_\kappa=\{\lambda\in\mathfrak{h}^*\,|\,2(\lambda,\kappa)+(\kappa,\kappa)=0\}. \tag{2.13}$$

Another interesting property of these series $\phi(\lambda+\rho(k),k)$ is the following fact. If one takes $\mu=\lambda+\rho(k)\in P_-$ then the series terminates, and results in a W-invariant Fourier polynomial on H, which we denote by $P(\mu,k)$ (see [13, Lemma 3.12]). By formula (2.2) these so-called Jacobi polynomials on H can also be considered as elements of $\mathbf{C}[z_1,\dots,z_n]$. Actually, the set $\{P(\mu,k)\}_{\mu\in P_-}$ is a $\mathbf{C}$-basis for $\mathbf{C}[z_1,\dots,z_n]$.

One of the miracles in the theory of the differential operator (2.5) is the fact that its commutant within the algebra $\mathbf{A}$ is very large. This commutant has the structure of a polynomial algebra in n generators (so in particular it is a commutative subalgebra of $\mathbf{A}$). For certain values of $l\in\mathscr{K}$ one can even prove the existence of a slightly more general type of operators, the so-called shift operators with shift l, which, by definition, are elements $S\in\mathbf{C}[\mathscr{K}]\otimes\mathbf{A}$ such that:

$$S(k)\cdot(L(k)+(\rho(k),\rho(k)))=(L(k+l)+(\rho(k+l),\rho(k+l)))\cdot S(k) \tag{2.14}$$

(and thus the elements in the commutant of L are shift operators with shift 0). Let us define $\mathbf{S}(l)$ as the $\mathbf{C}[\mathcal{K}]$-module of all such shift operators with shift l. It is not difficult to see that (see [19, Proposition 2.5, and formula (2.4)]) there exists a $\mathbf{C}[\mathcal{K}]$ linear map (the generalized Harish-Chandra map):

$$\eta = \eta(l)\colon\ \mathbf{S}(l) \to \mathbf{C}[\mathfrak{h}^*] \otimes \mathbf{C}[\mathcal{K}] \tag{2.15}$$

such that

$$S(k)\,\phi(\lambda+\rho(k), k) = \eta(S)(\lambda, k)\,\phi(\lambda+\rho(l+k), k+l), \qquad \forall S \in \mathbf{S}(l). \tag{2.16}$$

Note that the map $\eta(\lambda)$ is injective since (2.16) determines in particular the image under $S(k)$ of the Jacobi polynomials and the Jacobi polynomials form a basis for the space $\mathbf{C}[z_1, \ldots, z_n]$. In order to describe the basic existence theorem for shift operators we introduce the following lattice inside $\mathcal{K}$. Let $R = \coprod_{i=1}^{n} C_i$ be the decomposition of R into minimal orbits for the action of W, and denote by $\chi_i \in \mathcal{K}$ the characteristic function of C_i. Denote by $b_i \in \mathcal{K}$ the following vector: $b_i = \chi_i$ if $2C_i \not\subset R$ and $b_i = (2\chi_i - \chi_j)$ if $2C_i = C_j$. Then $\mathcal{B} \subset \mathcal{K}$ is the $\mathbf{Z}$-lattice generated by the $\{b_i\}$. The following theorem was proven in [20, Theorem 3.6] and later by completely elementary means in [12].

Theorem 2.2 (1) *The map $\eta(0)$ is an isomorphism from $\mathbf{S}(0)$ onto $\mathbf{C}[\mathfrak{h}^*]^W \otimes \mathbf{C}[\mathcal{K}]$.* (2) *The space $\mathbf{S}(l)$ is nonzero if and only if $l \in \mathcal{B}$. In this case $\mathbf{S}(l)$ is a free rank one $\mathbf{S}(0)$-module via multiplication on the right hand side.*

The above theorem has a lot of consequences for the theory of the eigen function equation (2.8) and its solutions. For example, we have (see [13, Corollary 3.9])

Theorem 2.3 *Let $\lambda \in \mathfrak{h}^*$ and $k \in \mathcal{K}$ be arbitrary. The system of equations*

$$D(k)\,\phi = \eta(0)(D)(\lambda, k)\,\phi \qquad \forall D \in \mathbf{S}(0) \tag{2.17}$$

is holonomic of rank $|W|$ on the space $W \backslash H^{\mathrm{reg}}$.

Definition 2.4 For $(\lambda, k) \in \mathfrak{h}^* \times \mathcal{K}$ we denote by $\mathcal{L}(\lambda, k)$ the rank $|W|$ local system of local holomorphic solutions to the Eq. (2.17) on the space $W \backslash H^{\mathrm{reg}}$.

Corollary 2.5 *If $\lambda \notin H_\kappa$ $\forall \kappa \in Q \backslash \{0\}$ and λ is regular then $\{\phi(w\lambda + \rho(k), k)\}_{w \in W}$ is a basis of $\mathcal{L}(\lambda, k)(A_- U)$.*

Proof. If λ satisfies the above conditions then the series $\phi(w\lambda + \rho(k), k)$ is well defined and convergent on $A_- U$ (see Lemma 2.1). These series are solutions of (2.17) as a result of (2.16) and the fact that $\eta(0)(\lambda)$ is W invariant in λ (Theorem 2.2). By looking at the leading exponents of the series we see that $\{\phi(w\lambda + \rho(k), k)\}_{w \in W}$ is a linearly independent set if λ is regular. But the local solution space has dimension $|W|$ by Theorem 2.3, so we are done. □

3 The monodromy representation and regular singularities

In this section we briefly review some basic facts of the monodromy representation of the local system $\mathscr{L}(\lambda, k)$ which are used later on in this paper. Let us first of all give a description of the fundamental group of the space $W\backslash H^{\mathrm{reg}}$, due to van der Lek [14, p. 69]. (In fact we will content ourselves with a simplified version here). Choose a base point $X_0 \in \mathfrak{a}_-$ and let $h_0 \in A_-$ and $z_0 \in W\backslash H^{\mathrm{reg}}$ be its images. Clearly the fundamental group of H with respect to the base point z_0 is isomorphic to $Q^\vee$ if we map $Z \in Q^\vee$ to the loop

$$T_Z(t) = (Z \otimes e^{2\pi\sqrt{-1}t}) \cdot h_0, \qquad t \in [0, 1].$$

This group is mapped injectively into $\Pi_1(W\backslash H^{\mathrm{reg}}, z_0)$ and the image is denoted by $\Pi_1(\infty)$. The image of T_Z is denoted by t_Z. Furthermore, let s_i be the image in $\Pi_1(W\backslash H^{\mathrm{reg}}, z_0)$ of the following curve S_i in $\mathfrak{h}$ (r_i denotes the simple reflection in the simple root α_i):

$$S_i(t) = (1-t)\, X_0 + t\, r_i\, X_0 - \left(\frac{\pi}{4}\sqrt{-1}\, \sin \pi t\right) \alpha_i^\vee, \qquad t \in [0, 1].$$

(This loop s_i is the inverse of the one used in [13]. This is the cause of some minor differences in the presentation of results on the monodromy representation between [13] and this paper). We will use the notation $\Pi_1(e)$ for the subgroup of $\Pi_1(W\backslash H^{\mathrm{reg}}, z_0)$ generated by the loops s_i, $i \in \{1, \ldots, n\}$.

Theorem 3.1 (1) *The fundamental group $\Pi_1(W\backslash H^{\mathrm{reg}}, z_0)$ is generated by the subgroups $\Pi_1(\infty)$ and $\Pi_1(e)$.*
(2) *The subgroup $\Pi_1(e)$ is the local fundamental group of $W\backslash H^{\mathrm{reg}}$ at the identity element e of H.*
(3) *A complete set of relations for $\Pi_1(e)$ is given by: $s_i s_j s_i \ldots = s_j s_i s_j \ldots$ (m_{ij} factors on both sides, where m_{ij} is the order of $r_i r_j \in W$). Hence $\Pi_1(e)$ is isomorphic to the braid group related to W.*

Proof. See [14, p. 69] for an even more detailed description. It should be noted that (3) goes back to Brieskorn (see [5]). □

The following elementary property of the monodromy of $\mathscr{L}(\lambda, k)$ is a direct result of the fact that the defining differential equations can be rewritten as a system of first order equations depending polynomially on $(\lambda, k) \in \mathfrak{h}^* \times \mathscr{K}$. Its precise proof can be found in [22, Corollary 3.8].

Lemma 3.2 *Let us denote by $\mu(\lambda, k)$ the monodromy representation of the local system $\mathscr{L}(\lambda, k)$. With respect to a suitable basis for $\mathscr{L}$, the matrices $\mu(s)$ ($\forall s \in \Pi_1(W\backslash H^{\mathrm{reg}}, z_0)$) are $|W| \times |W|$-matrices with coefficients in the ring $\mathscr{O}(\mathfrak{h}^* \times \mathscr{K})$* of entire functions on $\mathfrak{h}^* \times \mathscr{K}$.

The monodromy representation of $\mathscr{L}(\lambda, k)$ has the property that it behaves nicely with respect to reduction to lower rank subsystems, and this elementary fact was exploited in [13] to derive the following result (loc. cit. Corollary 6.8).

Lemma 3.3 *The images $\mu(\lambda, k)(s_i)$ satisfy the following quadratic relations:*

$$(\mu(\lambda, k)(s_i) - 1)(\mu(\lambda, k)(s_i) - q_i) = 0.$$

Here $q_i = q_{\alpha_i} = \exp 2\pi\sqrt{-1}\,(\tfrac{1}{2} - k_{\alpha_i} - k_{\frac{1}{2}\alpha_i})$.

Therefore one is led to consider the Hecke algebra $H_W(q)$ (where $q=q(k)$ is the element of $\mathscr{K}$ as described in Lemma 3.3), which is by definition the complex algebra with generators δ_i, $i\in\{1, \ldots,\}$ and relations:

(1) $\delta_i \delta_j \delta_i \ldots = \delta_j \delta_i \delta_j \ldots$, m_{ij} factors on both sides.
(2) $(\delta_i - 1)(\delta_i - q_i) = 0$.

By Lemma 3.3 the monodromy representation gives rise to a representation $v(\lambda, k)$ of $H_W(q)$. The next result is an easy corollary of the description of the monodromy $\mu(\lambda, k)$ for generic parameters as given by Heckman (see [11]) or could be proved similar to [22, Corollary 5.5] using the Lemmas 3.2 and 3.3.

Lemma 3.4 *For generic parameters (λ, k) the representation $v(\lambda, k)$ is equivalent to the regular representation.*

In view of Corollary 2.5 it is interesting to consider the monodromy with respect to the basis $\{\phi(w\lambda+\rho(k), k)\}_{w\in W}$ of $\mathscr{L}(\lambda, k)$. In order to do so we introduce the Harish-Chandra c-function: Let $\forall \alpha \in R_+$,

$$\tilde{c}_\alpha(\lambda, k) = \frac{\Gamma(-(\lambda, \alpha^\vee) + \frac{1}{2} k_{\frac{\alpha}{2}})}{\Gamma(-(\lambda, \alpha^\vee) + \frac{1}{2} k_{\frac{\alpha}{2}} + k_\alpha)},$$

and define the following meromorphic function on $\mathfrak{h}^* \times \mathscr{K}$:

$$\tilde{c} = \prod_{\alpha \in R_+} \tilde{c}_\alpha. \tag{3.1}$$

Finally Harish-Chandra's c-function is given by:

$$c(\lambda, k) = \frac{\tilde{c}(\lambda, k)}{\tilde{c}(-\rho(k), k)}. \tag{3.2}$$

The next result is also proven by reduction to rank one subsystems (see [13, Theorem 6.7]).

Proposition 3.5 *Suppose that the conditions of Corollary 2.5 are fulfilled. Then:*
(1) *Clearly the series $\phi(w\lambda+\rho(k), k)$ are eigenvectors for $\mu(\lambda, k)(t_Z)$ with eigenvalue* $\exp(2\pi\sqrt{-1}(w\lambda+\rho(k), Z))$.
(2) *The functions*

$$\tilde{c}(\lambda, k)\,\phi(w\lambda+\rho(k), k) + \tilde{c}(r_i\lambda, k)\,\phi(r_i w\lambda + \rho(k), k)$$

($r_i = r_{\alpha_i}$ is the reflection in the simple root α_i) are invariant for the action of $\mu(\lambda, k)(s_i)$.

For the goals we have in mind, it will be quite important to have some insight in the behaviour of the sections of our local system $\mathscr{L}(\lambda, k)$ in the vicinity of the singular locus. The following lemma is of fundamental importance. Its proof is analogous to that of [22, Corollary 4.2] as far as the points of the discriminant are concerned, and is standard for the points at infinity (see [13, Sect. 3]).

Lemma 3.6 *The system of hypergeometric differential equations* (2.17) *has regular singularities on* $W\backslash H^{\text{reg}}$.

Combining this and Lemma 3.5 gives the following result (also see [13, Theorem 6.9] and [10, Theorem 7.5]).

Corollary 3.7 *If* (λ, k) *are generic then the function*

$$F(\lambda, k)=\sum_{w\in W} c(w\lambda, k)\,\phi(w\lambda+\rho(k), k)$$

extends to a W-invariant holomorphic function on $A\cdot U$.

Let us recall some useful properties of F.

Proposition 3.8 (1) *The function* $\tilde{F}(\lambda, k; h)=\tilde{c}(-\rho(k), k)\, F(\lambda, k; h)$ *is a holomorphic function on* $\mathfrak{h}^*\times\mathscr{K}\times A\cdot U$.
(2) *The function* $F(\lambda, k; e)$ *is an entire function on* $\mathfrak{h}^*\times\mathscr{K}$, *and is periodic with period lattice* $\mathscr{B}$ *with respect to the variable* $k\in\mathscr{K}$.

Proof. For (1) we refer the reader to [20, Theorem 2.8]. (Note that the remaining poles along S_2 (notation as in [20]) are cancelled by the multiplication with $\tilde{c}(-\rho(k), k)$). In order to prove the assertion on periodicity in (2) one uses the existence of shift operators for shifts in the lattice $\mathscr{B}$, see Theorem 2.2. We refer the reader to [21, Theorem 5.1] for a similar argument. Once the periodicity is established it is easy to see that $F(\lambda, k; e)$ is entire using (1). □

We want to close this section with a review of the notion of exponents of the differential equations (2.17) at the identity element $e\in H$. This subject was studied in [22, Sect. 7] for a somewhat simpler system of differential equations. The above lemmas make sure that we may conduct exactly the same analysis for the Eq. (2.17). Let $\xi=(\xi_2, \ldots, \xi_n)$ be coordinates in some ball B that is contained in an affine hyperplane in $\mathfrak{h}$ that does not contain the origin, and let $\mathbf{D}^\times=\{t\in\mathbf{C}\,|\,0<|t|<1\}$. Let C be the open complex "cone" $C=\{tb\,|\,t\in\mathbf{D}^\times, b\in B\}$. Then we consider $(t, \xi)=(t, \xi_2, \ldots, \xi_n)$ as a set of coordinates on C via the map $\mathbf{D}^\times\times B\to\mathfrak{h}$ defined by $(t, b)\to tb$. Assume that 0 is the only singular point of (2.17) inside the closure of C. We will refer to such a system of coordinates as a set of conical coordinates near the origin, and we will often think of these as coordinates on H (in the natural way).

Definition 3.9 A complex number ε is called an *exponent at* $e\in H$ *for the system* (2.17) if there exists a set of conical coordinates (t, ξ) and a (multivalued) solution ϕ of (2.17) on the corresponding cone such that ϕ has the following form (for some n):

$$\phi(t, \xi)=t^\varepsilon((\log t)^n\phi_n(t, \xi)+\ldots+\phi_0(t, \xi))$$

where $\phi_i(t, \xi)$ is holomorphic at $t=0$ and $(\phi_n(0, \xi), \ldots, \phi_0(0, \xi))\not\equiv(0, \ldots, 0)$.

If p denotes a point close to $e\in H$ then we denote by $\mathscr{L}_p^\varepsilon(\lambda, k)$ the closure in $\mathscr{L}_p(\lambda, k)$ of the set of solutions of (2.17) at p that have the exponent ε at the identity. Clearly this is a linear subspace of $\mathscr{L}_p(\lambda, k)$ and $\mathscr{L}_p^\varepsilon(\lambda, k)\subset\mathscr{L}_p^{\varepsilon'}(\lambda, k)$ if $\varepsilon, \varepsilon'$ are exponents such that $\varepsilon-\varepsilon'\in\mathbf{N}$. Define the multiplicity of an exponent

ε by $\mu(\varepsilon)=\dim(\mathscr{L}_p^\varepsilon(\lambda,k))-\dim(\bigcup_{i\in\mathbf{N}}\cdot\mathscr{L}_p^{\varepsilon+i}(\lambda,k))$. The following result was proven in [22, Corollary 7.12 and Remark 7.17].

Proposition 3.10 *Let $\mathscr{H}=\{q\in\mathbf{C}[\mathfrak{h}]\,|\,\partial(p)\,q=0\ \forall p\in\mathbf{C}[\mathfrak{h}^*]^W$ such that $p(0)=0\}$ be the $|W|$ dimensional space of harmonic polynomials. The exponents of (2.17) at the identity are equal to the eigenvalues of the linear operator (r_α denotes the reflection in α)*

$$\sum_{i=1}^{n} X_i^*\,\partial(X_i)-\sum_{\alpha\in R_+} k_\alpha(1-r_\alpha)$$

acting on $\mathscr{H}$ (counted with multiplicity).

Note that this operator is equivariant for the action of W on $\mathscr{H}$, so that the eigenspaces are themselves W-invariant. Therefore there is precisely one exponent that does not depend on k, namely the one that corresponds to the trivial representation contained in $\mathscr{H}$. Hence (since the trivial representation occurs only in degree 0 in $\mathscr{H}$) this exponent is equal to 0 and has multiplicity 1. We refer to this as the trivial exponent.

Definition 3.11 We write $\mathscr{K}_\pm=\{k\in\mathscr{K}\,|\,\mp\mathrm{Re}(\varepsilon(k))>0\ \forall\varepsilon$ a nontrivial exponent$\}$.

Remark 3.12 *Note that* $\mathscr{K}_-\supset\{k\in\mathscr{K}\,|\,\mathrm{Re}(k_\alpha)<0\ \forall\alpha\in R^0\cap R_0$, *and* $\mathrm{Re}(k_\alpha+k_{\frac{\alpha}{2}}<0\ \forall\alpha\in R^0\backslash R_0\}$. It is also not hard to see that $\mathscr{K}_+=\chi_{R^0}-\mathscr{K}_-$. Less obvious is the following fact (that can be concluded from the results of [22, Sect. 9]: if $k=a\cdot\chi_{R^0}$ then the condition for k to be in $\mathscr{K}_-$ is that $\mathrm{Re}(a)<\frac{1}{h}$, where h is the Coxeter number of W.

From the above we see that if $k\in\mathscr{K}_-$ then any solution of (2.17) has a finite limit towards the identity element e when it is continued analytically along any curve in H^{reg} with end point at e. From [22, Proposition 8.2] we see that this limit is independent of the choice of the curve locally in a neighbourhood of e. Hence this limit towards e defines a local section $E_e(\lambda,k)\in\mathscr{L}^*(\lambda,k)(V^{\mathrm{reg}})$ of the dual local system $\mathscr{L}^*(\lambda,k)$ where V is some open neighbourhood of e. Finally note that the system (2.17), when considered on H, is invariant for translations along the lattice $2\pi\sqrt{-1}P^\vee$ where $P^\vee=P(R^\vee)$ is the coweight lattice. Hence we may define local sections $E_p(\lambda,k)$ of $\mathscr{L}^*(\lambda,k)$ if $p\in 2\pi\sqrt{-1}P^\vee$ in a similar way, provided of course that $k\in\mathscr{K}_-$.

Definition 3.13 The local section $E_p(\lambda,k)$ of $\mathscr{L}^*(\lambda,k)$ is called *evaluation at* p ($\forall p\in 2\pi\sqrt{-1}P^\vee$).

Remark 3.14 It seems likely that *evaluation at p* defines a local section of $\mathscr{L}^*(\lambda,k)$ in a neighbourhood of p for any $p\in W\backslash H$ when the parameter k is in some suitable region (depending on p) of $\mathscr{K}$. For instance, when p is regular then this is obviously the case for any k.

4 Evaluation and duality

In the last section we have seen that evaluation of local sections of the local system $\mathscr{L}(\lambda,k)$ at special singular points p of (2.17) defines a local section in the dual $\mathscr{L}^*(\lambda, k)$ in a neighbourhood of p. It turns out to come in handy later on to study this in more detail.

Definition 4.1 *If* $k \in \mathscr{K}$ *then we call* $k' = \chi_{R^0} - k$ *the dual parameter.* The dual of a pair $(\lambda, k) \in \mathfrak{h}^* \times \mathscr{K}$ is given by $(-\lambda, k')$.

Let us introduce some notations. We put

$$\delta(k; h) = \prod_{\alpha \in R_+} (h^{-\frac{\alpha}{2}} - h^{\frac{\alpha}{2}})^{2k_\alpha}. \tag{4.1}$$

Let, if D is a differential operator on H^{reg}, ${}^tD$ denote the formal transpose of D, (i.e. the map $D \to {}^tD$ is the unique anti automorphism generated by ${}^t(f) = f$ for functions f on H^{reg}, and ${}^t(\partial(X_i)) = -\partial(X_i)$). Let θ be the anti automorphism of the algebra of differential operators on H^{reg} defined by $\theta(D) = \delta(-\frac{1}{2}\chi_{R^0}) \circ {}^tD \circ \delta(\frac{1}{2}\chi_{R^0})$ (θ is the transpose on $W \backslash H$).

Proposition 4.2 *With the above notations we have:*

(1) $D(k') = \delta\left(-\frac{(2k)'}{2}\right) \circ D(k) \circ \delta\left(\frac{(2k)'}{2}\right) \ \forall D \in \mathbf{S}(0)$.

(2) $\forall D \in \mathbf{S}(0)$: $\theta(D(k)) \in \mathbf{S}(0, k')$ *and has the property* $\eta(0)(\theta(D))(-\lambda, k') = \eta(0)(D)(\lambda, k)$.

Proof. Using the Harish-Chandra isomorphism $\eta(0)$ (see Theorem 2.2(1)) it is clear that it suffices to prove the asserted transformation properties in both (1) and (2) in the special case where $D = L$. These properties for the operator $L(k)$ are consequences of the following formula, which was proven in [13, Proposition 2.2]:

$$\begin{aligned} &\delta\left(\frac{k}{2}\right) \circ (L(k) + (\rho(k), \rho(k))) \circ \delta\left(-\frac{k}{2}\right) \\ &= \sum_{i=1}^{n} \partial^2(X_i) + \frac{1}{4} \sum_{\alpha \in R_+} k_\alpha(1 - k_\alpha - 2k_{2\alpha})(\alpha, \alpha) \sinh^{-2}\left(\frac{\alpha}{2}\right). \end{aligned}$$

Now use that the operator on the right hand side is symmetric on $\mathfrak{h}$ and invariant for the substitution $(k \to k')$. □

As an immediate consequence of (1) we have:

Corollary 4.3 *The series solutions* (2.9) *of* (2.17) *satisfy*

$$\delta\left(-\frac{(2k)'}{2}\right) \phi(\lambda + \rho(k'), k') = \phi(\lambda + \rho(k), k).$$

Another application of Proposition 4.2 is the existence of a pairing of the solution spaces of (2.17) associated with dual parameters. This can be proved using some elementary homological algebra in combination with (2) of Proposition 4.2. The proofs of the similar statements [22, Corollary 3.11 and Corollary 3.12] can be applied here almost without change. Let us introduce one more notation. Denote by $\mathscr{H}^*$ the linear subspace of $\mathbf{C}[\mathfrak{h}]$ that corresponds to $\mathscr{H}$ when we identify $\mathbf{C}[\mathfrak{h}^*]$ and $\mathbf{C}[\mathfrak{h}]$ using the bilinear form on $\mathfrak{h}$.

Corollary 4.4 *Choose a basis $\{q_i\}$ of $\mathscr{H}^*$ consisting of homogeneous elements. There exist elements $\alpha_{ij} \in \mathbf{C}[H^{\mathrm{reg}} \times \mathfrak{h}^* \times \mathscr{K}]$ such that for any pair of local sections ϕ_1 of $\mathscr{L}(\lambda, k)$, ϕ_2 of $\mathscr{L}(-\lambda, k')$ the expression*

$$\{\phi_1, \phi_2\} = \sum_{i,j} a_{ij}\, \partial(q_i)(\phi_1)\, \partial(q_j)(\phi_2) \tag{4.2}$$

is constant on H^{reg}, and such that $\{\cdot,\cdot\}$ gives rise to a nondegenerate pairing of $\pi^{-1}\mathscr{L}(\lambda, k)$ with $\pi^{-1}\mathscr{L}(-\lambda, k')$ $\forall(\lambda, k) \in \mathfrak{h}^* \times \mathscr{K}$. Moreover, the coefficients α_{ij} are uniquely determined up to a common complex multiplicative constant by these conditions.

Proof. The proof of the existence of $\{\cdot,\cdot)$ is similar to [22, Corollary 3.11 and Corollary 3.12], but with the minor difference that we work on H^{reg} rather than on $W \backslash H^{\mathrm{reg}}$ this time (in order to avoid some technical difficulties). As a consequence we have to use θ instead of the ordinary transpose when we are converting left and right modules for the algebra of differential operators on H^{reg}. The uniqueness follows from the fact that generically $\pi^{-1}\mathscr{L}(\lambda, k)$ is an irreducible local system, as one easily concludes from Proposition 3.5 combined with Corollary 4.3. □

Using this we see that there should exist a local section of $\pi^{-1}\mathscr{L}(-\lambda, k')$ that corresponds to $E_e(\lambda, k)$ via $\{\cdot,\cdot\}$. Of course this local section should be, up to a multiplicative constant, the hypergeometric function $F(\lambda, k)$. With a little bit of extra effort we obtain the following result (see [22, Theorem 8.3]).

Corollary 4.5 *There exists a rational function $a \in \mathbf{C}(\mathscr{K})$, having neither poles nor zeros inside the set $\mathscr{K}_-$, such that if $k \in \mathscr{K}_-$ then*

$$\{\phi, F(-\lambda, k')\} = a(k)\, F(-\lambda, k'; e)\, E_e(\phi)$$

for any local section ϕ of $\mathscr{L}(\lambda, k)$ defined near e.

Corollary 4.6 *The pairing $\{\cdot,\cdot\}$ is in fact a pairing of $\mathscr{L}(\lambda, k)$ with $\mathscr{L}(-\lambda, k')$.*

Proof. In view of Corollary 4.4 we only need to show that W acts trivially on $\{\cdot,\cdot\}$ in the following sense: let $\phi \in \mathscr{L}(\lambda, k)(U)$ and $\psi \in \mathscr{L}(-\lambda, k')(U)$ for some open set $U \in H^{\mathrm{reg}}$, and let $w \in W$. Then $\{\phi, \psi\} = \{\phi \circ w, \psi \circ w\}$. By the uniqueness assertion of Corollary 4.4 this equality certainly holds up to some constant depending on w only. But by Corollary 4.5 this constant has to be 1 since both $F(\lambda, k)$ and $E_e(\lambda, k)$ are invariant for this action of W. □

Example 4.7 (The case of the ordinary hypergeometric function) The hypergeometric function associated with the rank one root system BC_1 is, up to some notational differences, equal to the ordinary hypergeometric function. The differential equation (2.8) is classically written as follows (the ordinary hypergeometric equation for the parameters (a, b, c)):

$$z(z-1)\,\partial_z^2\phi + (c-(1+a+b)z)\,\partial_z\phi - ab\phi = 0. \tag{4.3}$$

We refer the reader to [13, Sect. 4] for all the details. The relation between the parameters (a, b, c) and the parameters (λ, k) is given by:

$$a=(\lambda+\rho(k), \alpha^\vee)$$
$$b=(-\lambda+\rho(k), \alpha^\vee)$$
$$c=\tfrac{1}{2}+k_{\frac{1}{2}\alpha}+k_\alpha.$$

The notation of dual parameters in the classical notation is given by the following: $(a, b, c)'=(1-a, 1-b, 2-c)$. Now let ϕ be a (local) solution of (4.3) and ψ for the same equation but with (a, b, c) replaced by its dual. Then it is not hard to verify that one may define the pairing in this situation by the expression:

$$\{\phi, \psi\}=((1-c)+(a+b-1)\,z)\,\phi(z)\,\psi(z) \\ +z(z-1)(\phi'(z)\,\psi(z)-\phi(z)\,\psi'(z)).$$

Theorem 4.8 *It is possible to normalize the bracket* $\{\cdot,\cdot\}$ *in such a way that:*
(1) $\{\phi(w\lambda+\rho(k), k), \phi(-w'\lambda+\rho(k'), k')\}=\delta_{w,w'}\prod_{\alpha\in R_+^0}(\lambda, \alpha^\vee)$.
(2) *The rational function* $a\in\mathbf{C}(\mathscr{K})$ *as described in Corollary* 4.5 *is given by the formula*

$$a(k)=\frac{\tilde{c}(\rho(k), -k)}{|W|\,\tilde{c}(-\rho(k'), k')}.$$

This is actually a polynomial.

Proof. Using Proposition 3.5(1) it is obvious that $\{\phi(w\lambda+\rho(k), k), \phi(-w'\lambda+\rho(k'), k')\}=\delta_{w,w'}f(w\lambda, k)$ for some function f. Now take the limit for $h\to\infty$ with $h\in A_-$ and use the formulas (2.9) and (2.11) together with expression (4.2) in order to obtain that f is rational in λ and polynomial in k. We claim that f is in fact W-anti invariant. Namely if we use the action of $\mu(\lambda, k)(s_i)$ and $\mu(-\lambda, k')(s_i)$ and Proposition 3.5 and Corollary 4.3 we see that:

$$\{\tilde{c}(w\lambda, k)\,\phi(w\lambda+\rho(k), k)+\tilde{c}(r_i w\lambda, k)\,\phi(r_i w\lambda+\rho(k), k), \\ \tilde{c}(-w\lambda, k)\,\phi(-w\lambda+\rho(k'), k')+\tilde{c}(-r_i w\lambda, k)\,\phi(-r_i w\lambda+\rho(k'), k')\}=0.$$

This yields (since $\tilde{c}(w\lambda, k)\,\tilde{c}(-w\lambda, k)$ is W-invariant):

$$\tilde{c}(w\lambda, k)\,\tilde{c}(-w\lambda, k)\,(f(w\lambda, k)+f(r_i w\lambda, k))=0$$

proving our claim. But then it is also clear that f is in fact independent of k since the nondegeneracy of the pairing now implies that $f(\lambda, k)$ cannot have zeros as a function of k when λ is fixed, $\lambda\notin H_\kappa\ \forall\kappa\in Q\setminus\{0\}$. In view of Corollary 3.7 this result leads to the following formula:

$$\{F(\lambda, k), F(-\lambda, k')\}=f(\lambda)\sum_{w\in W}(-1)^{l(w)}c(w\lambda, k)\,c(-w\lambda, k').$$

On the other hand one has, as a result of Corollary 4.5:

$$\{F(\lambda, k), F(-\lambda, k')\}=a(k)\,F(\lambda, k; e)\,F(-\lambda, k'; e).$$

Hence:

$$\sum_{w\in W}(-1)^{l(w)}c(w\lambda,k)\,c(-w\lambda,k')=\frac{a(k)}{f(\lambda)}F(\lambda,k;e)\,F(-\lambda,k';e). \tag{4.4}$$

This formula can be simplified as follows. A straightforward computation (using the definition of the c-function and the reflection formula for the Γ-function) yields:

$$\tilde{c}(\lambda,k)\,\tilde{c}(-\lambda,k')=\prod_{\alpha\in R^0_-}(\lambda,\alpha^\vee)^{-1}\prod_{\alpha\in R_+}\frac{\sin\pi((-\lambda,\alpha^\vee)+\frac{1}{2}k_{\frac{\alpha}{2}}+k_\alpha)}{\sin\pi((-\lambda,\alpha^\vee)+\frac{1}{2}k_{\frac{\alpha}{2}}}. \tag{4.5}$$

In a similar but slightly different way we rewrite

$$\tilde{c}(-\rho(k),k)\,\tilde{c}(-\rho(k'),k')=A(k)^{-1}\,\tilde{c}(-\rho(k),k)\,\tilde{c}(\rho(k),-k) \tag{4.6}$$

where $A(k)=\dfrac{\tilde{c}(\rho(k),-k)}{\tilde{c}(-\rho(k'),k')}$. This is a polynomial, as one may see for example from the fact that this expression is equal to the constant term of the shift operator $G(-\chi_{R^0},k')$ (see [21, Proposition 3.2, formula (3.5) and Corollary 3.5]). From (4.6) we now get, again using the reflection formula for the Γ-function:

$$\tilde{c}(-\rho(k),k)\,\tilde{c}(-\rho(k'),k)=|W|\,A(k)^{-1}\prod_{\alpha\in R_+}\frac{\sin\pi((\rho(k),\alpha^\vee)+\frac{1}{2}k_{\frac{\alpha}{2}}+k_\alpha)}{\sin\pi((\rho(k),\alpha^\vee)+\frac{1}{2}k_{\frac{\alpha}{2}})}. \tag{4.7}$$

(here we also used the formula (see [21, formula (4.6)]):

$$\prod_{\alpha\in R_+}\frac{((\rho(k),\alpha^\vee)+\frac{1}{2}k_{\frac{\alpha}{2}}+k_\alpha)}{((\rho(k),\alpha^\vee)+\frac{1}{2}k_{\frac{\alpha}{2}})}=|W|).$$

Putting (4.5) and (4.7) together we obtain the following expression for the left hand side of (4.4):

$$\left\{\frac{A(k)}{|W|\prod_{\alpha\in R^0_+}(\lambda,\alpha^\vee)}\right\}\cdot\left\{\sum_{w\in W}\prod_{\alpha\in R_+}\frac{\sin\pi((-w\lambda,\alpha^\vee)+\frac{1}{2}k_{\frac{\alpha}{2}}+k_\alpha)}{\sin\pi((-w\lambda,\alpha^\vee)+\frac{1}{2}k_{\frac{\alpha}{2}})}\right\}\cdot\left\{\prod_{\alpha\in R_+}\frac{\sin\pi((\rho(k),\alpha^\vee)+\frac{1}{2}k_{\frac{\alpha}{2}}+k_\alpha)}{\sin\pi((\rho k),\alpha^\vee)+\frac{1}{2}k_{\frac{\alpha}{2}})}\right\}^{-1}. \tag{4.8}$$

Now invoke the following formula due to Macdonald [17, formula (2.4 nr) and (2.8 nr)]): $(u\in\mathcal{K})$

$$\sum_{w\in W}\prod_{\alpha\in R_+}\frac{1-u_{\frac{\alpha}{2}}^{\frac{1}{2}}u_\alpha\exp(-w\alpha^\vee)}{1-u_{\frac{\alpha}{2}}^{\frac{1}{2}}\exp(-w\alpha^\vee)}=\prod_{\alpha\in R_+}\frac{1-u_{\frac{\alpha}{2}}^{\frac{1}{2}}u_\alpha u^{\mathrm{ht}(\alpha^\vee)}}{1-u_{\frac{\alpha}{2}}^{\frac{1}{2}}u^{\mathrm{ht}(\alpha^\vee)}}. \tag{4.9}$$

(In Beerends' thesis [1, Theorem 1 of Chap. 2]) it was observed that this formula is in fact a disguised version of the denominator formula of Weyl. This gives a very short proof). Formula (4.9) reduces the left hand side of (4.4) (see 4.8)) to the expression $|W|^{-1} A(k) \prod_{\alpha\in R^0_1} (\lambda, \alpha^\vee)^{-1}$. Therefore (4.4) becomes:

$$\frac{A(k)}{|W| \prod_{\alpha\in R^0_1} (\lambda, \alpha^\vee)} = \frac{a(k)}{f(\lambda)} F(\lambda, k; e)\, F(-\lambda, k'; e).$$

But in view of Proposition 3.5(2) this implies that $F(\lambda, k; e)\, F(-\lambda, k'; e)$ is independent of k, and thus (when we take $k=0$) we get: $F(\lambda, k; e)\, F(-\lambda, k'; e)=1$. Now it is clear that we may take $f(\lambda)=\prod_{\alpha\in R^0_1} (\lambda, \alpha^\vee)$ and $a(k)=|W|^{-1} A(k)$, proving both (1) and (2). □

Remark 4.9 The polynomial $a(k)$ has a very simple explicit form when $k=\xi\chi_{R^0}$, namely (see the remark after (4.6), and [21, formula (7.2)]):

$$a(\xi\chi_{R^0}) = \prod_{i=1}^{n} \prod_{j=1}^{d_i-1} (j-d_i\,\xi). \tag{4.10}$$

(Where the integers d_i denote the primitive degrees, as usual).

Corollary 4.10 *The entire function $F(\lambda, k; e)$ satisfies the functional equation*

$$F(\lambda, k; e)\, F(-\lambda, k'; e)=1.$$

In particular, this function is nonvanishing.

Remark 4.11 In Sect. 6 we will prove that in fact $F(\lambda, k; e)=1$. Unfortunately we will need more serious analysis than the above formal manipulations in this proof. The nonvanishing of $F(\lambda, k; e)$ (Corollary 4.10) will be one of the ingredients. This could have been proven without the use of the pairing $\{\cdot,\cdot\}$ in Sect. 3 but since this would not have simplified the proof of Theorem 4.8 we have omitted this in Sect. 3.

Remark 4.12 Suppose that the conditions of Corollary 2.5 are fulfilled. Then one may define a nondegenerate pairing of $\mathcal{L}(\lambda, k)$ and $\mathcal{L}(-\lambda, k')$ by Theorem 4.8(1). Using Proposition 3.5 it is then completely obvious that this pairing is invariant for the monodromy action. It is quite surprising however that this pairing extends to give a nondegenerate pairing for all choices of (λ, k).

5 Order of growth

In this section we will be dealing with a preparatory result. Namely we will give an upper bound for the order of growth of the entire function $F(\lambda, k; e)$ (see Proposition 3.8) as a function of the variable $k\in\mathcal{K}$. This result will be used in the next section in order to prove that (see [13, Conjecture 6.11]) $F(\lambda, k; e)=1$. We will proceed as follows. First, we will obtain an upper bound for the growth order of the series $\phi(\lambda+\rho(k), k; h)$ as a function of k and with $h\in A_- U$. Then we will use a result from the theory of analytic functions of several complex variables to obtain the required upper bound for the growth order of $F(\lambda, k; e)$.

First let us recall the basic definitions and facts of the theory of growth scales for entire functions. An excellent reference for this material is the book of Lelong and Gruman [15].

Definition 5.1 Let $\Omega \subset \mathbf{C}^n$ be a domain. A function $\phi: \Omega \to \mathbf{R} \cup \{-\infty\}$ is called *plurisubharmonic* (PSH) on Ω if:

61) ϕ is upper semi-continuous, and $\phi \not\equiv -\infty$.
(2) For any complex line $L \subset \mathbf{C}^n$, $\phi|_L$ is subharmonic on every component of $L \cap \Omega$.

Note that $\mathrm{PSH}(\Omega) \subset \mathrm{S}(\Omega)$, the subharmonic functions on Ω. The most important example of a subharmonic function on a domain Ω is the function $z \to \log|f(z)|$, where $f \in \mathcal{O}(\Omega)$, the holomorphic functions on Ω. Based on this example one has the following well known lemma.

Lemma 5.2 *Let $\Omega \subset \mathbf{C}^n$ be a domain, and let $f \in \mathcal{O}(\Omega \times \mathbf{C}^m)$, $f \not\equiv 0$. For $z \in \Omega$, $r \in \mathbf{R}_+$ we define $M_f(z, r) = \sup_{\|z'\| \leqq r} \log|f(z, z')|$. Then the function*

$$\Omega \times \mathbf{C} \to \mathbf{R} \cup \{-\infty\}$$
$$(z, s) \to M_f(z, |s|)$$

is in $PSH(\Omega \times \mathbf{C})$.

Proof. The upper semi-continuity is clear, so we only have to look at (2) of Definition 5.1. Observe that

$$M_f(z, s) = \sup_{\|\alpha\| \leqq 1} \log|f(z, s\alpha)|$$

and that for all $\alpha \in \mathbf{C}^m$, the function $\log|f(z, s\alpha)|$ is PSH in $(z, s) \in \Omega \times \mathbf{C}$. Hence $M_f(z, s)$ satisfies the required mean value inequality on complex lines $L \subset \Omega \times \mathbf{C}$ because it is the supremum of a locally bounded family of functions with this property. □

Definition 5.3 Let $\Omega \subset \mathbf{C}^n$ be a domain. A subset $E \subset \Omega$ is called *Ω-pluripolar* if $\exists \phi \in \mathrm{PSH}(\Omega)$ such that $E \subset \phi^{-1}(-\infty)$.

Corollary 5.4 *Pluripolar sets have Lebesque measure* 0.

Proof. It is well known that PSH-functions are locally Lebesque integrable. □

We now come to the definition of the scale of growth orders.

Definition 5.5 The *order* ρ of a positive real valued function a on $\mathbf{C}^n$ is given by

$$\rho = \limsup_{r \to \infty} \frac{\log M_a(r)}{\log r}$$

where $M_a(r) = \sup_{\|z\| \leqq r} |a(z)|$.

By abuse of language we will say that an entire function $f \in \mathcal{O}(\mathbf{C}^n)$ has order of growth $\rho \in [0, \infty]$ if the PSH-function $\log^+|f| = \sup(0, \log|f|)$ is of the order

ρ. We can now state the theorem on exceptional sets for growth conditions (see [15, Theorem 1.41]).

Theorem 5.6 *Let $\Omega \subset \mathbf{C}^n$ be a domain and let $f \in \mathrm{PSH}(\Omega \times \mathbf{C})$. For a fixed $z \in \Omega$ we denote $\rho(z) =$ order of $(s \to f(z, s))$. If for some $\Omega' \subset \Omega$ such that $\overline{\Omega'} = \Omega$ there exists a Ω'-pluripolar subset $M \subset \Omega'$ such that $\rho(z) \leqq \infty$, $\forall z \in M$, then ρ is locally bounded on Ω and, in addition, the regularization $\rho^*(z) := \limsup_{z' \to z} \rho(z')$ is PSH on Ω.*

We will use this theorem in the following form:

Corollary 5.7 *Let Ω be a domain, and $F \in \mathcal{O}(\Omega \times \mathbf{C}^m)$. Denote the growth order of $(z' \to f(z, z')) \in \mathcal{O}(\mathbf{C}^m)$ by $\rho(z)$, $z \in \Omega$. Suppose that $\rho(z) \leqq R \in \mathbf{R}_{\geqq 0}$ outside an analytic subset of Ω. Then $\rho(z) \leqq R$ $\forall z \in \Omega$.*

Proof. Apply Theorem 5.6 to the function $((z, s) \to M_f(z, |s|)) \in \mathrm{PSH}(\Omega \times \mathbf{C})$ (see Lemma 5.2). We get that $\rho^* \in \mathrm{PSH}(\Omega) \subset \mathrm{S}(\Omega)$ and of course we still have the inequality $\rho^* \leqq R$ in the complement of an analytic subset of Ω. But then, as a consequence of the mean value inequality, we have $\rho^* \leqq R$ everywhere in Ω. But $\rho \leqq \rho^*$ so we are done. □

We not return to the theory of the hypergeometric function. The next lemma is of crucial importance in this section.

Lemma 5.8 *Let $\lambda \notin H_\kappa$, $\forall_\kappa \in Q_+ \setminus \{0\}$ and let $h \in A_-U$ (see Sect. 2). The function $(k \to \phi(\lambda + \rho(k), k; h))$ is entire (see Lemma 2.1) with growth order at most 1.*

Proof. In Lemma 2.1 we already saw that this function is entire. Recall that (see (2.9))

$$\phi(\lambda + \rho(k), k, h) = h^{\lambda + \rho(k)} \sum_{\kappa \in Q_+} \Delta_\kappa(\lambda, k)\, h^\kappa$$

where the coefficients $\Delta_\kappa(\lambda, k)$ are determined by (see (2.11))

$$(5.1) \qquad (2\lambda + \kappa, \kappa)\, \Delta_\kappa(\lambda, k) = 2 \sum_{\alpha \in R_+} k_\alpha \sum_{j=1}^{\infty} (\lambda + \rho(k) + \kappa - j\alpha, \alpha)\, \Delta_{\kappa - j\alpha}(\lambda, k).$$

In particular, if λ satisfies the conditions stated in the lemma, then the coefficients Δ_κ are polynomials in k. Now let $a \in A_-$ be fixed. We claim that there exists a constant $D \in \mathbf{R}_+$, depending on the choices of λ and a only, such that (where $\|\cdot\|$ denotes some norm on $\mathscr{K}$):

$$(5.2) \qquad |\Delta_\kappa(\lambda, k)| \leqq (D(1 + \|k\|)^2)^{2D\|k\|} \cdot a^{-\kappa} \qquad (\forall \kappa \in Q_+).$$

It is clear that this proves our lemma. In order to prove (5.2), note that we can choose positive constants C_1 and C_2, depending on λ only, such that

$$(5.3) \qquad |(\lambda + \rho(k) + \tau, \alpha)| \leqq C_1(1 + \|k\| + \mathrm{ht}(\tau)) \qquad \forall \alpha \in \mathbf{R}_+, \tau \in Q_+, k \in \mathscr{K}$$

and

$$(5.4) \qquad |(2\lambda + \kappa, \kappa)| \geqq C_2(\mathrm{ht}(\kappa))^2 \qquad \forall \kappa \in Q_+ \setminus \{0\}.$$

Here, as usual, $\mathrm{ht}\left(\sum_{i=1}^{n} n_i \alpha_i\right) := \sum_{i=1}^{n} n_i$ (ht is the height function on Q_+). Hence, if we put $C := \frac{2C_1}{C_2}$ we get from (5.1) (for $\kappa \neq 0$):

$$(5.5) \qquad |\Delta_\kappa(\lambda, k)| \leqq C\,\|k\|\,(\|k\|\,\mathrm{ht}(\kappa)^{-2} + \mathrm{ht}(\kappa)^{-1}) \sum_{\alpha \in R_+,\, j \geqq 1} |\Delta_{\kappa - j\alpha}(\lambda, k)|.$$

The constant $D \in \mathbf{R}_+$ mentioned in the formulation of our claim (5.2) is defined by

$$(5.6) \qquad D = \max\left(C \sum_{\alpha \in R_+,\, j \geqq 1} a^{j\alpha}, 1\right).$$

As an intermediate step towards proving (5.2), we show by induction on $\kappa \in Q_+$ (with respect to the partial ordering $\leqq$)

$$(5.7) \qquad |\Delta_\kappa(\lambda, k)| \leqq (D(1 + \|k\|)^2)^{\mathrm{ht}(\kappa)} \cdot a^{-\kappa} \qquad (\forall \kappa \in Q_+).$$

Indeed, (5.7) is true for $\kappa = 0$, and the necessary induction step can be made using (5.5):

$$\begin{aligned} |\Delta_\kappa(\lambda, k)| &\leqq C\,\|k\|\,(\|k\|\,\mathrm{ht}(\kappa)^{-2} + \mathrm{ht}(\kappa)^{-1}) \sum_{\alpha \in R_+,\, j \geqq 1} |\Delta_{\kappa - j\alpha}(\lambda, k)| \\ &\leqq C(1 + \|k\|)^2 \sum_{\alpha \in R_+,\, j \geqq 1} (D(1 + \|k\|)^2)^{\mathrm{ht}(\kappa) - j} a^{j\alpha - \kappa} \\ &\leqq \frac{C}{D} (D(1 + \|k\|)^2)^{\mathrm{ht}(\kappa)} \Big(\sum_{\alpha \in R_+,\, j \geqq 1} \alpha^{j\alpha} \Big) a^{-\kappa} \\ &\leqq (D(1 + \|k\|)^2)^{\mathrm{ht}(\kappa)} a^{-\kappa} \end{aligned}$$

as desired. On the other hand, from (5.5) it is also clear that

$$(5.8) \qquad |\Delta_\kappa(\lambda, k)| \leqq \frac{C}{D} \sum_{\alpha \in R_+,\, j \geqq 1} |\Delta_{\kappa - j\alpha}(\lambda, k)| \qquad \text{if } \kappa \text{ satisfies } \mathrm{ht}(\kappa) \geqq 2D\,\|k\|.$$

Now we prove (5.2) by induction on $\mathrm{ht}(\kappa)$. Namely, if $\mathrm{ht}(\kappa) \leqq 2D\,\|k\|$ then we see that (5.2) is true by using formula (5.7) so that we may assume that κ is such that $\mathrm{ht}(\kappa) > 2D\,\|k\|$ in the induction step. But then we can use formula (5.8) in combination with the induction hypothesis:

$$\begin{aligned} \Delta_\kappa(\lambda, k)| &\leqq \frac{C}{D} \sum_{\alpha \in R_+,\, j \geqq 1} (D(1 + \|k\|)^2)^{2D\|k\|} a^{j\alpha - \kappa} \\ &\leqq (D(1 + \|k\|)^2)^{2D\|k\|} a^{-\kappa}. \quad \square \end{aligned}$$

Corollary 5.9 *For all $\lambda \in \mathfrak{h}^*$, $h \in AU$ fixed, the entire function $(k \to \tilde{F}(\lambda, k; h))$ has growth order at most* 1.

Proof. From the explicit formula (3.1) for the $\tilde{c}$-function it is clear that for λ outside some locally finite set of hyperplanes this is an entire function of k of growth order 1. Using Corollary 3.7, Proposition 3.8(1) and the previous lemma we therefore obtain that $(k \to \tilde{F}(\lambda, k; h))$ has growth order 1 as a function of k at least when $(\lambda; h)$ is outside a certain analytic subset of $\mathfrak{h}^* \times AU$. Now apply Corollary 5.7. □

Corollary 5.10 *For any complex line* $L \subset \mathcal{K}$, *the growth order of* $(k \to F(\lambda, k; e))|_L \in \mathcal{O}(L)$ *does not exceed* 1.

Proof. From the definition of order it is clear that the order of the product of two entire functions can not exceed the larger of the orders of the factors. Also true, but (far) less obvious, is the same statement for an entire function of one variable that can be written as the quotient of two entire functions (see [16, corollary of Theorem 12]). We obtain the desired result if we apply this fact to the identity $F(\lambda, k; e) = \tilde{c}(-\rho(k), k)^{-1} \tilde{F}(\lambda, k; e)$, restricted to L, and use the explicit definition of $\tilde{c}$ in combination with Corollary 5.9. □

6 Summation formulas

In this final section we will combine the results of the previous sections in order to obtain explicit formulas for the values of certain solutions of the hypergeometric differential equations at certain singular points. The main formula is (also see [13], Conjecture 6.11):

Theorem 6.1 $F(\lambda, k; e) = 1$.

Proof. Let $L \subset \mathcal{K}$ be a complex line which is rational with respect to the lattice of (real) periods $\mathcal{B}$, and let z denote a coordinate on L. For the sake of simplicity we write, by abuse of language, $F(z) := F(\lambda, z; e)$ (where $\lambda \in \mathfrak{a}^*$ is arbitrary but fixed). Let us list the properties of $F(z)$ that we have derived so far:
(1) F is periodic (see Proposition 3.8). Let us choose z in such a way that the period is real.
(2) $F(z) \neq 0 \ \forall z \in \mathbf{C}$ (see Corollary 4.11).
(3) $F(z) \in \mathbf{R} \ \forall z \in \mathbf{R}$.
(4) The growth order of F is at most 1 (see Corollary 5.10).
(Note that (3) is simply due to the fact that for real values of λ and k, $F(\lambda, k)$ is a real function on A). If we now use Hadamard's factorization theorem for entire functions of finite growth order (see for instance [6, Chap. XI]) together with (2) and (4) we obtain that we can write $F = \exp(f)$ where f is a polynomial in z of degree at most 1. But then the combination of (1) and (3) implies that F is constant. □

Hence we obtain from Corollary 4.6 and Theorem 4.9:

Corollary 6.2 *If* $k \in \mathcal{K}_-$ *then*

$$\{\phi, F(-\lambda, k')\} = \frac{\tilde{c}(\rho(k), -k)}{|W| \, \tilde{c}(-\rho(k'), k')} E_e(\phi)$$

for any local section ϕ *of* $\mathcal{L}(\lambda, k)$ *defined near* e.

In particular we have:

Theorem 6.3 (The generalized Gauss summation formula) *Let $k \in \mathcal{K}_-$. Then*

$$E_e(\phi(\lambda+\rho(k),k)) = \lim_{h\to e} \phi(\lambda+\rho(k),k;h) = |W|\,(\prod_{\alpha\in R_+^0} (\lambda,\alpha^\vee))\,\frac{\tilde{c}(-\lambda,k')}{\tilde{c}(\rho(k),-k)}.$$

Proof. Combine the previous corollary with Theorem 4.9(1). □

Example 6.4 (The case of the ordinary hypergeometric function) In this case one can express the series solution (2.9) as an ordinary Gaussian hypergeometric function by using one of the well known linear transformation formulas. Explicitly (see [13, Sect. 4):

$$\phi(\lambda+\rho(k),k;h) = 2^{-2a}(-z)^{-a} F\left(a, 1+a-c; 1+a-b; \frac{1}{z}\right)$$
$$= 2^{-2a}(1-z)^{-a} F\left(a, c-b; 1+a-b; \frac{1}{1-z}\right).$$

(The relation between $(a, b; c)$ and (λ, k) as in Example 4.8, and $z = \frac{1}{2} - \frac{1}{4}(h+h^{-1})$). By using the ordinary Gauss summation formula (1.2) we obtain (if $\mathrm{Re}(1-c) = \mathrm{Re}(\frac{1}{2} - k_\alpha - k_{\frac{\alpha}{2}}) > 0$:

$$\lim_{h\to 1} \phi(\lambda+\rho(k),k;h) = 2^{-2a} F(a, c-b; 1+a-b; 1) \tag{6.1}$$
$$= 2^{-2a} \frac{\Gamma(1+a-b)\,\Gamma(1-c)}{\Gamma(1+a-c)\,\Gamma(1-b)}.$$

On the other hand, by using the relation between (λ, k) and $(a, b; c)$ and the duplication formula for the Γ-function we obtain that:

$$\tilde{c}(-\lambda,k') = \frac{2^{2(c-a)}\sqrt{\pi}\,\Gamma(a-b)}{\Gamma(1+a-c)\,\Gamma(1-b)}$$

and

$$\tilde{c}(\rho(k),-k) = \frac{2^{2c}\sqrt{\pi}}{\Gamma(1-c)}.$$

When we substitute these formulas in the expression given in Theorem 6.3 and use that $|W|(\lambda,\alpha^\vee) = a-b$ we indeed recover formula (6.1).

We may use Theorem 6.3 in order to evaluate the hypergeometric function itself at certain special points. Let $\forall p \in \mathfrak{h}$ the point on H that corresponds to p be denoted by $\exp p$.

Corollary 6.5 *Let $p \in 2\pi\sqrt{-1}P^\vee$. If we continue $F(\lambda,k)$ analytically along a path inside $\mathfrak{a}_- + \sqrt{-1}\,\mathfrak{a}$ joining the origin in $\mathfrak{h}$ with p then:*

$$F(\lambda,k;\exp p) = \frac{\displaystyle\sum_{w\in W} e^{(w\lambda,p)} \prod_{\alpha\in R_+} \frac{\sin\pi((-w\lambda,\alpha^\vee)+\frac{1}{2}k_{\frac{\alpha}{2}}+k_\alpha)}{\sin\pi((-w\lambda,\alpha^\vee)+\frac{1}{2}k_{\frac{\alpha}{2}})}}{\displaystyle e^{-(\rho(k),p)} \prod_{\alpha\in R_+} \frac{\sin\pi((\rho(k),\alpha^\vee)+\frac{1}{2}k_{\frac{\alpha}{2}}+k_\alpha)}{\sin\pi((\rho(k),\alpha^\vee)+\frac{1}{2}k_{\frac{\alpha}{2}})}} \tag{6.2}$$

Proof. In order to prove this result one has to use the following obvious formula:

$$\phi(\lambda+\rho(k), k; h \exp p) = e^{(\lambda+\rho(k), p)}\phi(\lambda+\rho(k), k; h).$$

Hence by using Corollary 3.7 and Theorem 6.3 we get:

$$F(\lambda, k; \exp p) = |W| \prod_{\alpha\in R_+^0} (\lambda, \alpha^\vee) \sum_{w\in W} (-1)^{l(w)} e^{(w\lambda+\rho(k), p)} \cdot \frac{\tilde{c}(w\lambda, k)\,\tilde{c}(-w\lambda, k')}{\tilde{c}(-\rho(k), k)\,\tilde{c}(\rho(k), -k)}. \tag{6.3}$$

The desired result now follows by using the formulas (4.5) and (4.6), (4.7). □

Remark 6.6 As a function of λ the apparent poles in formula (6.2) all cancel. Indeed, when we multiply the numerator of (6.2) by the Weyl denominator $\prod_{\alpha\in R_+^0}\left(e^{\frac{\alpha^\vee}{2}} - e^{-\frac{\alpha^\vee}{2}}\right)$ then the result is a W-anti invariant polynomial in the elements e^q, $q\in 2\pi\sqrt{-1}P^\vee$. Hence this can be divided by the Weyl denominator again, and the resulting quotient (the original numerator) is a W-invariant polynomial in the functions e^q, $q\in 2\pi\sqrt{-1}P^\vee$. Moreover it is easy to see that the lowest term occurring in this numerator is $e^{w_0 p}$ (with w_0 such that $w_0 p\in\sqrt{-1}\,\mathfrak{a}_-$) and the coefficient of this term is $\exp(\sqrt{-1}\pi \sum_{\alpha\in R_+} k_\alpha)$. The denominator is obtained from the numerator by substituting $\lambda = -\rho(k)$.

When $p\in 2\pi\sqrt{-1}P^\vee$ is a point of the fundamental alcove for the action of the affine Weyl group $\tilde{W}$ on $\sqrt{-1}\,\mathfrak{a}$ then formula (6.2) simplifies considerably. These special $p\in 2\pi\sqrt{-1}P_+^\vee$ are "minuscule" (see [4, chap. VI, par. 2, exercise 5]). They correspond bijectively to the elements of the so-called central subgroup of H (see [13, formula 3.8]).

Corollary 6.7 *Let $k\in K_-$ and let $p\in 2\pi\sqrt{-1}P^\vee$ be minuscule. If we continue $F(\lambda, k)$ analytically via a path inside the regular part of the fundamental alcove from the origin to p then*:

$$F(\lambda, k; \exp p) = \frac{\sum_{w\in W} e^{(w\lambda, p)}}{\sum_{w\in W} e^{(-w\rho(k), p)}}. \tag{6.4}$$

Proof. This easily follows from the arguments used in Remark 6.6 since a saturated set of coweights having a minuscule coweight p as maximal element consists only of the elements Wp.

Remark 6.8 Note that (6.4) applies to all the vertices of the fundamental alcove in the case of the root system A_n.

Finally we consider the somewhat peculiar case of the root system BC_n. Although the central subgroup is trivial in this case it turns out that the series solutions $\phi(\lambda+\rho(k), k)$ have a nice transformation property with respect to the action of the central subgroup of the reduced subsystem $C_n^\vee \subset BC_n^\vee$ (recall that $(BC_n)^0 = C_n$ and $(BC_n)_0 = B_n$).

Lemma 6.9 *Let R be of type BC_n and choose an orthonormal basis $(e_1, \ldots, e_n)$ in $\mathfrak{a}$ such that the roots are given by $\pm x_i$, $\pm 2x_i$, $\pm x_i \pm x_j$ (where $(x_1, \ldots, x_n)$ are the coordinates associated with the basis $(e_1, \ldots, e_n)$). Let $p_0 = \sqrt{-1}\pi(e_1 + \ldots + e_n)$ be the minuscule element in $2\pi\sqrt{-1}P^\vee(C_n) \supset 2\pi\sqrt{-1}P^\vee(BC_n)$. One now has:*

$$\phi(\lambda + \rho(k), k; h \exp p_0) = e^{(\lambda + \rho(k), p_0)} \phi(\lambda + \rho(k), \tilde{k}; h)$$

where $(k_1, k_2, k_3)^\sim = (k_1, k_2 + k_3, -k_3)$ if $k_1 = k_{x_i - x_j}$, $k_2 = k_{2x_i}$, and $k_3 = k_{x_i}$.

Proof. It suffices to prove the appropriate transformation property for the second order operator $L(k)$ (see (2.11)). This is a straightforward calculation. □

Corollary 6.10 *Use the same notations as in Lemma 6.9. If $k \in \mathscr{K}_-$ then:*

$$F(\lambda, k; \exp p_0) = \prod_{i=1}^{n} \frac{\Gamma(\rho_i(k) + \frac{1}{2}k_3 + \frac{1}{2})\, \Gamma(-\rho_i(k) + \frac{1}{2}k_3 + \frac{1}{2})}{\Gamma(\lambda_i + \frac{1}{2}k_3 + \frac{1}{2})\, \Gamma(-\lambda_i + \frac{1}{2}k_3 + \frac{1}{2})}.$$

Proof. By the previous lemma it is clear that (similar to formula (6.3)) (note that $(\tilde{k})' = (k')^\sim$):

$$\text{(6.5)} \quad F(\lambda, k; \exp p_0) = |W| \prod_{\alpha \in C_{n,+}} (\lambda, \alpha^\vee) \cdot \sum_{w \in W} (-1)^{l(w)} e^{(w\lambda + \rho(k), p_0)} \frac{\tilde{c}(w\lambda, k)\, \tilde{c}(-w\lambda, \tilde{k}')}{\tilde{c}(-\rho(k), k)\, \tilde{c}(\rho(k), -\tilde{k})}.$$

We are going to relate this expression to formula (6.3) for the root system $R = C_n = (BC_n)^0$, endowed with the multiplicity function $\bar{k}$ defined by $\bar{k}_1 = k_1$ and $\bar{k}_2 = k_2 + \frac{1}{2}k_3$. Let us denote by $\tilde{c}^0$ the $\tilde{c}$-function of the root system C_n. A straightforward calculation shows that

$$\text{(6.6)} \quad \tilde{c}(w\lambda, k)\, \tilde{c}(-w\lambda, \tilde{k}') = \prod_{i=1}^{n} \frac{\pi 2^{-2k_3}}{\cos \pi\lambda_i\, \Gamma(\lambda_i + \frac{1}{2}k_3 + \frac{1}{2})\, \Gamma(-\lambda_i + \frac{1}{2}k_3 + \frac{1}{2})} \cdot \tilde{c}^0(w\lambda, \bar{k})\, \tilde{c}^0(-w\lambda, \bar{k}').$$

Using (6.6) we also get:

$$\begin{aligned}
\tilde{c}(-\rho(k), k)\, \tilde{c}(\rho(k), -\tilde{k}) &= \tilde{c}(-\rho(k), k)\, \tilde{c}(\rho(k), \tilde{k}') \left(\frac{\tilde{c}(\rho(k), -\tilde{k})}{\tilde{c}(\rho(k), \tilde{k}')} \right) \\
&= \tilde{c}(-\rho(k), k)\, \tilde{c}(\rho(k), \tilde{k}') \left(\frac{\tilde{c}^0(\rho(\bar{k}), -\bar{k})}{\tilde{c}^0(\rho(\bar{k}), \bar{k}')} \right) \\
&= \prod_{i=1}^{n} \frac{\pi 2^{-2k_3}}{\cos \pi\rho_i(k)\, \Gamma(\rho_i(k) + \frac{1}{2}k_3 + \frac{1}{2})\, \Gamma(-\rho_i(k) + \frac{1}{2}k_3 + \frac{1}{2})} \\
&\quad \cdot \tilde{c}^0(-\rho(\bar{k}), \bar{k})\, \tilde{c}^0(\rho(\bar{k}), -\bar{k}).
\end{aligned}$$

Substituting this and (6.6) in (6.5) gives:

$$F(\lambda, k; \exp p_0) = |W| \prod_{i=1}^{n} \frac{\cos \pi \rho_i(k)\, \Gamma(\rho_i(k)+\frac{1}{2}k_3+\frac{1}{2})\, \Gamma(-\rho_i(k)+\frac{1}{2}k_3+\frac{1}{2})}{\cos \pi \lambda_i\, \Gamma(\lambda_i+\frac{1}{2}k_3+\frac{1}{2})\, \Gamma(-\lambda_i+\frac{1}{2}k_3+\frac{1}{2})}$$
$$\cdot \prod_{\alpha \in C_{n,+}} (\lambda, \alpha^\vee) \sum_{w \in W} (-1)^{l(w)} e^{(w\lambda+\rho(k), p_0)}$$
$$\cdot \frac{\tilde{c}^0(w\lambda, \bar{k})\, \tilde{c}^0(-w\lambda, \bar{k}')}{\tilde{c}^0(-\rho(k), \bar{k})\, \tilde{c}^0(\rho(k), -\bar{k})}.$$

Now apply formula (6.3) and (6.4) to finish the proof:

$$F(\lambda, k; \exp p_0) = F_{C_n}(\lambda, \bar{k}; \exp p_0) \prod_{i=1}^{n} \frac{\cos \pi \rho_i(k)}{\cos \pi \lambda_i}$$
$$\cdot \frac{\Gamma(\rho_i(k)+\frac{1}{2}k_3+\frac{1}{2})\, \Gamma(-\rho_i(k)+\frac{1}{2}k_3+\frac{1}{2})}{\Gamma(\lambda_i+\frac{1}{2}k_3+\frac{1}{2})\, \Gamma(-\lambda_i+\frac{1}{2}k_3+\frac{1}{2})}$$
$$= \frac{\sum_{w \in W} e^{(w\lambda, p_0)}}{\sum_{w \in W} e^{(-w\rho(k), p_0)}} \prod_{i=1}^{n} \frac{\cos \pi \rho_i(k)}{\cos \pi \lambda_i}$$
$$\cdot \frac{\Gamma(\rho_i(k)+\frac{1}{2}k_3+\frac{1}{2})\, \Gamma(-\rho_i(k)+\frac{1}{2}k_3+\frac{1}{2})}{\Gamma(\lambda_i+\frac{1}{2}k_3+\frac{1}{2})\, \Gamma(-\lambda_i+\frac{1}{2}k_3+\frac{1}{2})}$$
$$= \prod_{i=1}^{n} \frac{\Gamma(\rho_i(k)+\frac{1}{2}k_3+\frac{1}{2})\, \Gamma(-\rho_i(k)+\frac{1}{2}k_3+\frac{1}{2})}{\Gamma(\lambda_i+\frac{1}{2}k_3+\frac{1}{2})\, \Gamma(-\lambda_i+\frac{1}{2}k_3+\frac{1}{2})}. \qquad \square$$

Remark 6.11 The result Corollary 6.10 was conjectured by Beerends [2]. It generalizes the Gauss summation formula for so-called hypergeometric functions of matrix argument due to Macdonald [18]. (Namely by Theorem 4.2 of [3] this summation formula of Macdonald corresponds to the case where $\lambda \in \mathbf{C}(x_1 + \ldots + x_n)$.)

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