Quantum double symmetries of the even dihedral groups and their breaking

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Abstract

In the physical setting of 2+1D gauge theories broken down to a finite group, the quantum double is the natural mathematical construct to describe the physical excitations, which may carry both topological and regular charges.

Furthermore, when taking this quantum double symmetry as a starting point, one is led to symmetry breaking in such theories, providing a description of condensates of electrical, magnetic or dyonic particles.

We have applied this formalism to almost all possible condensates of quantum doubles of the even dihedral groups. Most of these condensates follow a general scheme in their discription of the residual symmetry and confinement, but we also present new results for some special cases, for which the standard analysis does not apply. Although we are able to find particular solutions in these cases, it has not yet led to a general extension of the existing framework.

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Introduction

Symmetry and symmetry breaking The concept of symmetry is widely regarded as a very fundamental principle in many areas of physics. A system is symmetric if its physical properties are unchanged after a certain symmetry transformation. All transformations that leave the system invariant together form a group, the symmetry group.

There are many kinds of symmetry. We may distinguish between external symmetry, in which the system is left invariant by transformations in spacetime, or equivalently by coordinate transformations, and internal symmetry, when a system carries a so-called internal space which is acted on by a symmetry group; this internal symmetry is not directly measurable, but allows for example for the classification of particles in multiplets, leading to physical understanding of particles found in nature and created in particle accelerators.

Another important notion is that of global and local symmetry. For global symmetry, the transformation is the same in all of spacetime. A local symmetry transformation may differ from point to point. In (quantum) field theory, local symmetry gives rise to gauge fields, which are generally interpreted as carriers of the fundamental forces.

Now symmetry may be explicitly broken by an external potential, by which we mean that the system (to be more precise the action or Lagrangian) is no longer invariant under the full symmetry group, but rather under one of its subgroups. Another phenomenon is *spontaneous symmetry breaking*, where the action is left invariant under the full group, but the groundstate has become degenerate, and one of many groundstates will be 'spontaneously chosen'. This is sometimes referred to as the 'vacuum acquiring an expectation value'. The symmetry transformations that still leave this groundstate invariant form a subgroup, called the residual symmetry group.

What we will do in this thesis is to extend the concept of a symmetry group to a Hopf algebra, which can be viewed as a generalization of a group, through the use of the so-called quantum double construction. This yields a natural realization of braid statistics and allows for prescriptions for multiple-particle states and anti-particles, a formalism for symmetry breaking, which will also incorporate other features of the theories described below.

Topological interactions and 2+1-dimensional physics A topological defect can be viewed as an excitation of the system whose quantum number does not fit in the symmetry describing that system. It arises as a collective behaviour of the particles. To be more precise, the defects are discontinuities of the *order parameter*, which is some vector-valued function defined on every point in space. They can be zero-, one- or higher-dimensional. Some examples are the eye of a whirlwind, flux tubes in type-II superconductors and domain walls in ferromagnets. In theoretical high energy physics, there have been many proposals for topological defects such as skyrmions and magnetic monopoles.

In a system with topological defects, there can be so-called topological interactions between the defects, or between a defect and a fundamental (gauge) charge. These topological interactions do not manifest themselves as interaction terms in the Hamiltonian, but as consequences of the topological properties of the configuration space of the system.

The typical example is the Aharonov–Bohm effect: the vector potential induced by an infinitely long solenoid can influence the interference pattern of two electrically charged particles, even if these particles travel only through regions where the magnetic field is zero. The solenoid in this case constitutes a 'hole' in the plane on which the particles move, which makes the configuration space topological non-trivial.

Defects are classified using homotopy theory. Basically, this describes in what way loops of surfaces in configuration space are non-contractible. A loop can determine defects of dimension two smaller than the dimension of the space (e.g. so point defects in two-dimensional and line defects in three-dimensional space). We turn to discrete gauge theories in 2+1 dimensions, because in that case the inequivalent point defects will be labelled by elements of the residual symmetry group.

Using this classification, we can think of defects as particles carrying topological charges. The topological charge may affect other defects or particles with fundamental charge through topological interactions. Because of the structure imposed by these interactions, they can be described by representations, just as we do for gauge transformations on fundamental charges.

If the group labelling the defects is non-Abelian, topological interactions may cause a defect that circumvents another to change its topological charge: it gets conjugated by the group value of the circumvented defect. This is called *flux metamorphosis* [4]. The result is that topological charges should be organized in conjugacy classes, rather than group elements. We can say that the particle has an internal space of dimension equal to the number of elements in the conjugacy class.

Braid statistics Because of topological interactions, the interchange of two particles need not just be given by a factor of +1 (Bose–Einstein statistics) or -1 (Fermi–Dirac statistics, which are the values obtained by the two one-dimensional representations of the permutation group. The particles (and defects) obey braid statistics, that is, interchanges are governed by the action of the braid group (of which the permutation group is a subgroup).

If the gauge group is Abelian, we get one-dimensional representations of the braid group, which can just give any phase factor under interchange, hence the name *anyons*. For non-Abelian gauge groups, the braiding is more complex.

The details of discrete gauge theories, with topological interactions and braid statistics are summarized in chapter 1.

The quantum double construction treats topological and fundamental charges on the same level (they label the representations of the symmetry algebra) and also tells us how to perform braiding of the different particles. It is then also straightforward to describe particles carrying both topological and fundamental charge, which are called dyons. In analogy with quantum electrodynamics, we will say that particles with fundamental charge form the electric sector, and topological charges form the magnetic sector. This is complemented by the dyonic sector.

Braid statistics seems to capture the essence of physics in 2+1 dimensions. Since the first proposal by Wilczek [50] of Abelian anyons, the concept has been studied thoroughly up to this day. Fractional quantum Hall systems are currently regarded as the most promising to show (quasi-)particles obeying fractional an non-Abelian statistics. Some references to recent work are given in the concluding remarks (chap. 7).

Hopf symmetry breaking To summarize, in discrete gauge theory in 2+1 dimensions, we find particles with fundamental and/or topological charge, which have topological interactions, leading to certain braid statistics. The insight reviewed in [52] was that the mathematical construction of the quantum double $D(H)$ of the unbroken gauge group H, which is a special form of a Hopf algebra, contains all information to describe such theories: its irreducible representations, which label the spectrum of particles, depend on both the fundamental and the topological charge. Furthermore, it provides a description for braiding, and for multi-particle states and fusion of particles. This is explained in chapter 2.

The next step, taken in [6, 7], was to consider the breaking of these Hopf symmetries and the formation of the equivalent of Bose–Einstein condensates. When a group symmetry is broken, the residual symmetry group is simply the subgroup of transformations that leave the spontaneously chosen groundstate, which can be regarded as a condensate of particles in that state, invariant.

We want to have an analogous description of the breaking of a Hopf symmetry algebra $\mathscr{A}$. However, for an algebra the demand that a symmetry transformation leaves the groundstate invariant breaks down, because two such transformations may add up to zero. An adapted demand takes this into account, making use of the additional structure of a Hopf algebra. The result is a sub-Hopf algebra, the residual symmetry algebra $\mathscr{T}$.

The irreducible representations of the residual symmetry algebra form the particle spectrum in the condensate state. Some of these particles will be confined, namely when they do not braid trivially with the condensate particles, as in that case 'moving around in the condensate' will cost energy proportional to the distance travelled due to topological interactions. The particles that are not confined are representations of another algebra, the unconfined algebra $\mathscr{U}$.

Many properties of the original algebra $\mathscr{A}$ will be carried over to $\mathscr{T}$ and $\mathscr{U}$. The formalism of Hopf symmetry breaking and subsequent confinement will be the subject of chapter 3.

Quantum doubles of even dihedral groups The first discrete groups to consider are the finite subgroups of the gauge group of proper rotations $SO(3)$. Apart from three special cases, these are just the Abelian cyclic groups $\mathbb{Z}_n$ (rotations over $\frac{2\pi}{n}$) and the dihedral groups D_n (rotations over $\frac{2\pi}{n}$ and an equal number of reflections). The cyclic groups and the dihedral groups for n odd were treated in [7], and in chapter 4 we look at the cases with n even.

We first consider condensates in the electric and magnetic sectors, and then in the dyonic sector. As the magnetic (topological) charges are categorized into conjugacy classes, we face several options for defining a magnetic condensate, depending on the state vector. There is one gauge-invariant option, the class sum. When considering other states, there is a condition, which states that the condensate particles braid trivially with themselves, that eliminates many linear combinations. However, several others remain, of which we treat one case, a pure, single flux, in detail, and give an example of another.

For pure electric or pure magnetic particles, we find a general form to describe the symmetry breaking.

In the dyonic sector, there are condensates that deviate from that description, and in chapter 5 we work out such a case in a $D(D_4)$-theory, in which the residual symmetry algebra turns out to be twice as large as one would naively expect. Furthermore, as we perform those calculations, it will become clear that there is a special basis of $D(D_4)$ that has the structure of a group. This is explored in chapter 6.

Notation

- We often use the notation $a \rightharpoonup \mathbf{v}$ for the action of a on a vector $\mathbf{v}$, which respects all the structure of the object to which a belongs. It is identical to the representation notation $\pi(a)\mathbf{v}$, but the representation in question is left implicit in the form of the representation space of which $\mathbf{v}$ is an element.
- The acronym $\gcd(m,n)$ is used to denote the greatest common divisor of the integers m and n. E.g. $\gcd(12,8) = 4$.
- Cyclic groups of order n are denoted by $\mathbb{Z}_n$, dihedral groups of order $2n$ by D_n.
- A group generated by an element g will be denoted by $\langle g \rangle$; for example if r generates $\mathbb{Z}_n$ with n even, then $\langle r^2 \rangle$ is a subgroup isomorphic to $\mathbb{Z}_{n/2}$.
- The imaginary element is denoted by $\imath$, to avoid confusion with i as an index label.
- Group and algebra representations will always be taken over the field of complex numbers.

Chapter 1

2+1D discrete gauge theories

In this chapter we recall some features of a gauge theories in 2+1 dimensions, broken down to a discrete subgroup. When topological interactions are present in such a system, the topological charges, which are labelled by the elements of that subgroup, can be transformed by interaction with each other and by gauge transformations. Accordingly, these charges are organized into conjugacy classes, and carry a higher-dimensional internal state space.

Topological interactions lead to the description of *braid statistics*: the interchange of two particles may introduce factors beyond just 1 (Bose–Einstein statistics) or -1 (Fermi–Dirac statistics).

The representations, denoting particles carrying both topological and fundamental (gauge) charge, and also the braiding of these particles are described by the quantum double construction of chapter 2.

We will briefly mention the properties of these discrete gauge theories, which were developed in [52] and extended in [34, 6, 7].

1.1 Yang–Mills–Higgs gauge theory

Gauge symmetry The concept of *gauge freedom* arose in classical electrodynamics, where the scalar and vector potentials have a degree of freedom: these potentials can undergo *gauge transformations* which alter the potential but do not affect the physically measurable electric and magnetic fields. Such gauge transformations always have the mathematical structure of a group, because two consecutive transformations are equivalent to another single transformation.

A system that possesses gauge freedom is said to be *gauge symmetric* under a certain *gauge group*, borrowing the terminology from other symmetry phenomena such as Lorentz group spacetime symmetry or point group symmtery in crystals.

A symmetry group G acts on a state $|\phi\rangle$ of the system via one of its representations. The state is a vector in the base space of this representation. In other words, we classify particles according to their transformation properties.

In most cases the representation will be *completely reducible*, so that it may be written as a direct sum of irreducible representations of that group. The inequivalent particles in that theory will be labelled by these irreducible representations, and we can restrict our classification to finding out what the irreducible representations are.

The last important concept is the commutativity of the gauge group. The aforementioned gauge group of electrodynamics $U(1)$ is Abelian, which means that the order of two consecutive transformations has no influence on the resulting transformation. As a consequence all complex† irreducible representations are one-dimensional, and the result of a gauge transformation will always be just a phase factor.

In contrast, a non-Abelian gauge group also has higher-dimensional irreducible representations. A particle that is labelled by such a representation has a higher-dimensional internal state space, so that gauge transformations may alter the state of that particle. We usually choose a convenient basis in the state space according to respective inequivalent states. If the symmetry were an external symmetry, this basis would reflect the eigenvectors of some measurement operation. But as gauge transformations (almost by definition) do not affect the system, such considerations do not apply.

Theories with non-Abelian gauge bosons coupled to massive particle fields are generally called Yang–Mills theories.

Spontaneous symmetry breaking A theory is said to be spontaneously broken when the Langrangian (or the action, or the Hamiltonian) of the theory is invariant under all gauge transformations, but the groundstate of the system is not. In that case the groundstate becomes degenerate, and nature has to 'spontaneously choose' in which groundstate the system will manifest itself. Mostly, one encounters systems which are already broken, and we say that these systems possess a hidden symmetry.

The gauge group is usually a Lie group, which is generated by a finite number of generators. Goldstone's theorem states that for each broken symmetry generator, there will appear a massless particle, now called Goldstone bosons (see e.g. [19]). These Goldstone bosons appear when a *global* symmetry is spontaneoulsy broken. Many light particles, such as pions, can be interpreted as Goldstone bosons.

Next, look at local gauge symmetry. Now we have gauge fields, (vector) functions on a spacetime manifold. These fields, such as the familiar electromagnetic A-field, are in general massless. When the local symmetry is spontaneously broken, the Goldstone bosons 'conspire' with the gauge fields, which results in the gauge bosons acquiring mass. This phenomenon, which is sometimes referred to as the gauge fields 'eating' the Goldstone boson, is called the Higgs mechanism. This will be important in our theories, because now the interactions mediated by the massive vector bosons are short-ranged.

We refer to standard text books ([38, §20.1] or [12, §5.2]) for further details.

It is also possible to have a Higgs mechanism in theories with no apparent local gauge symmetry. For instance, in smectic and hexatic liquid crystals, a gauge field may be 'dynamically generated' and can cause some hydrodynamic modes, which are the equivalents of Goldstone bosons, to become suppressed (massive) [31, §§3.2.2,3.3.1].

A common example is of a broken local gauge symmetry is superconductivity: the electromagnetic gauge group $U(1)$ is broken down to $\mathbb{Z}_2$ as two electrons are bound together in momentum space to form a Cooper pair. The photon gauge field acquires a mass, which is inversely proportional to its penetration depth. We observe this phenomenon as magnetic fields are expelled from the superconductor, which is called the Meissner effect.

† As mentioned in the introduction, we only concern ourselves with complex representations.

1.2 Discrete gauge theories

We will consider systems that are governed by a Yang–Mills–Higgs gauge theory with matter fields that are covariantly coupled to the gauge fields.

The assumption is that the original continuous gauge group G will be spontaneously broken down to a finite subgroup H by the Higgs field. The system assumes one of the, now degenerate, groundstates, and the gauge fields become massive through the Higgs mechanism.

Now, we are interested in the long-distance physics, or equivalently the low energy regime, of the system. In this regime, the Higgs field cannot be excited, so it is condensed in one of its groundstates. We can speak of a Higgs medium, filled with consensed Higgs particles.

The massive gauge bosons have a mean free path inversely proportional to their mass, in other words: the field strength decays exponentially with the mass, so that the interaction between two widely separated particles by the exchange of gauge bosons is (exponentially) suppressed. It is explained in [52, §1.3.1] that for a electromagnetic gauge field, this may be regarded as the Coulomb screening of the gauge field by the Higgs medium.

We will see in the next section that when topological interactions are also present in the theory, they are not screened, and lead to the special characteristics, in particular braid statistics, of these discrete gauge theories.

Particles in the broken theory In gauge theories, the particle spectrum is determined by the properties of particles under gauge transformations. The gauge group acts on a Hilbert space via one of its representations. As mentioned earlier, the description reduces to classifying the inequivalent particles by the irreducible representations of the group. We now want to know what happens to such particles when the symmetry is spontaneously broken.

One can readily see that a representation of a group is also a representation of each of its subgroups, because a subgroup is closed under multiplication and a group representation respects this multiplication. However, an irreducible representation of the full group need no longer be irreducible under the action of the subgroup.

Because the subgroup is finite, all its representations are completely reducible. An irreducible representation of the full group, which may be reducible under the action of the subgroup, can therefore always be written as a direct sum of irreducible representations of the subgroup. The decompositions of the restriction to H of irreducible G-representations Π_α into irreducible H-representations Ω_β are called *branching rules*. These are similar to the familiar Clebsch–Gordan decomposition, and are determined by coefficients N_α^β:

$$\Pi_\alpha|_H = \bigoplus_\beta N_\alpha^\beta \Omega_\beta. \tag{1.1}$$

Naturally, the dimensions of the representations in the decomposition have to add up to the dimension of Π_α.

1.3 Topological interactions

Apart from the fundamental, gauge charges, there may be other possible excitations in the system which are of a completely different nature. These excitations, called topological defects, can interact with each other or with fundamental charges in a way that is not described by explicit interaction terms in the action (Hamiltonian, Lagrangian) governing the system. Rather, such interactions arise as a consequence of the geometry (the multiply-connectedness to be specific) of the configuration space. For these reasons, they are called *topological interactions*.

Many examples of topological defects in different physical theories can be found in the introduction of [52], and references therein. Most of these have been observed in experiments as well.

Terminology In analogy with electromagnetism, we will often refer to fundamental, gauge charges as "matter particles" carrying "electric charge", and to topological excitations as "vortices" carrying "magnetic flux" (topological charge). One can also envisage particles carrying both fundamental and topological charge, and these will be called dyons. An electric particle may then be regarded as a dyon carrying trivial magnetix flux, and a magnetic particle as a dyon carrying trivial electric charge.

1.3.1 The Aharonov–Bohm effect

The most well-known example of topological interaction is the Aharonov–Bohm effect. Originally devised as a thought-experiment [1], it has now been verified by many different experiments, see [37] for a detailed account, and [47] for the 1998 verification of the so-called electric Aharonov–Bohm effect.

The Aharonov–Bohm effect, in its typical formulation, shows that a topological defect, in this case a solenoid generating a magnetic vector potential, can influence a fundamental charge, in this case an electron, *even when the electron travels only through regions where the electric and magnetic fields vanish*. That is, the vector potential, long thought to be just a convenient auxiliary quantity, has physical meaning even when its curl, the magnetic field, is zero in the regions through which the electron travels (when it is *locally pure gauge*).

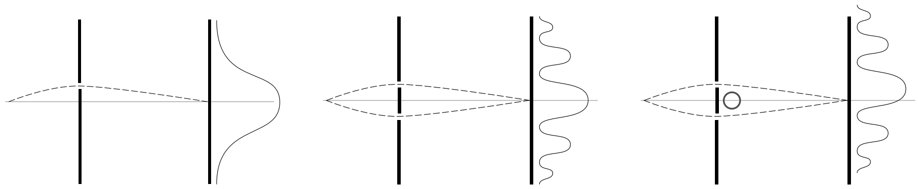
This effect is purely quantum-mechanical, having no classical analogue. One important property of this interaction is that it can only be detected when the particle interacting with the defect completely circumvents the defect. We will now discuss how this works.

The magnetic Aharonov–Bohm effect Consider the double-slit experiment: electrons are 'fired' one-at-a-time at a screen, but they first pass through an obstruction with two parallel slits in it. When one of the slits is closed off, we will see a Gaussian distribution of impacts on the screen, in the direction perpendicular to the slit on the screen (figure 1.1(a)). When both slits are open, we will see an interference pattern, confirming the wave-like nature of electrons (figure 1.1(b)).

Now we place a solenoid between the two slits, in such a way that the electron beams only travel trough regions of space where the magnetic field generated by the solenoid is zero, which can be accomplished by making the solenoid very long. What we will observe is that the interference pattern is shifted relative to the situation without a solenoid (figure 1.1(c)).

An explanation is that the wave functions of the two electron beams each undergo a phase shift in such a way that the total phase shift is non-zero. This phase shift is due to the non-zero vector potential. To be more precise, write the total wave function as a superposition of the two beams:

$$\psi_{\text{tot}}(x) = \psi_{\text{upper}}(x) + \psi_{\text{lower}}(x). \tag{1.2}$$



(a) Electrons scatter when sent through a small slit

(b) Single electrons show interference

(c) The interference pattern is shifted when a solenoid generates a vector potential

Figure 1.1: The magnetic Aharonov–Bohm effect

Each beam undergoes a phase shift $e^{\imath\phi(x)}$ due to the vector potential $\mathbf{A}(x)$:

$$\phi(x) = \frac{e}{\hbar c}\int_{C=0}^{x} \mathbf{A}(x')dx', \tag{1.3}$$

where C denotes the path around the solenoid, ending in point x. When the beams recombine (interfere) there is a difference in phase shifts, which corresponds to a phase shift of the total wave function obtained by traversing a closed loop around the solenoid. But we know that $\oint \mathbf{A}dx = \Phi$, with Φ the magnetic flux through the solenoid. The total phase shift between the two beams is then given by

$$\phi_{\mathrm{tot}} = \frac{e}{\hbar c}\Phi. \tag{1.4}$$

This phase shift is visible as the interference pattern on the screen is also shifted, unless the phase shift is a multiple of $2\pi\frac{\hbar c}{e}$, in which case we cannot discern the shift.

The important thing we learn from this is that the actual path the electron beams take is of no importance: the only measurable quantity, the total phase shift, is solely dependent on circumventing the solenoid—if we would be able to encircle the solenoid twice, the phase shift would be twice as large. In other words, the solenoid constitutes a defect, and the *winding number* is a discrete number describing the topological interaction.

1.3.2 Topological interactions in discrete gauge theories

It is known that certain theories with massive gauge fields allow magnetic vortices carrying flux (see e.g. [52, §1.3.2]). We from this point on assume that such vortices can arise in each discrete gauge theory we consider.

Classification of stable defects When vortices are present in discrete gauge theories, they lead to interactions similar to the Aharonov–Bohm effect: because the gauge fields are all massive, they are pure gauge at large distances from the defect. It can be argued that these topological interactions are not suppressed by Coulomb screening due to the condensed Higgs particles [52, §1.3.2]. We now show that the vortices are labelled by elements of the residual symmetry group H.

We have seen in §1.3.1 that the quantity of importance is the exponential of the contour integral of the gauge field. Now, assuming our (broken) gauge group G to be a Lie group, the gauge field A can be decomposed into generators T_a of the Lie algebra corresponding to the gauge group. Because this group can be non-Abelian, we must now take the path-ordered integral, denoted by the operator P. Now a particle traversing a loop undergoes a *holonomy*:

$$w(C) = P\exp(\oint_C A) = P\exp(\oint_C dx^i A_i^a T_a) = h \in H, \tag{1.5}$$

where the identification with a group element is made using the Lie algebra–Lie group correspondence, and the fact that it must take values in the unbroken group H, because the holonomy has to leave the groundstate invariant.

Order parameter Let's discuss this a little more explicitly. The most convenient way to describe defects is by introducing an *order parameter*, some vector valued function on every point in space. This order parameter usually varies continuously through the medium, but discontinuities can occur at some points, lines or surfaces, and these discontinuities constitute the defects.

As an example, think of spins, dipoles, in a three-dimensional medium. The order parameter can be represented by evenly-sized arrows pointing in a certain direction. In an unordered system, the arrows point in random directions. In a uniform medium, all spins are aligned, and the order parameter is constant. One can think of other cases, for example a point around which all arrows point outward, as if it were a source. This is an example of a *point defect*.

It can be shown (e.g. [52, §1.4.1]) that for discrete gauge theories, the order parameter space of a continuous symmetry G broken down to a finite subgroup H is isomorphic to the coset space G/H.

What we need is a way to detect discontinuities in the order parameter, and a way to describe whether two defects are equivalent. This is done using homotopy theory.

The fundamental group classifies defects Because the order parameter is a continuous function everywhere in space except on points where there is a defect, a loop in space around a defect corresponds to a continuous path in order parameter space. A continuous deformation of the loop in real space will correspond to an continuous deformation of the path in order parameter space. Recall that we are still working in two-dimensional space, so that a loop around a defect cannot be continuously contracted to a point.

We say that two loops that can be continuously deformed into each other are homotopically equivalent. Because the order parameter is discontinuous at points where there is a defect, a loop that circumvents a defect can never be deformed into one that does not. This is the essence of the defect classification.

So the problem of classifying different vortices in a discrete gauge theory reduces to the question: when can two loops in G/H be continuously deformed into each other? It can be shown that G/H is broken up into disconnected components, which each contain precisely one element of H. Inequivalent loops correspond to paths that begin on different disconnected components (see e.g. [33, §V]). Therefore the defects are classified by elements of H.

The mathematical formulation is that the fundamental group π_1, or first homotopy group, of the order parameter space G/H is isomorphic to H. The fundamental group classifies all non-equivalent loops in a space and can be shown to indeed possess a group structure by composition of loops. We will use the fundamental group again in §1.4.

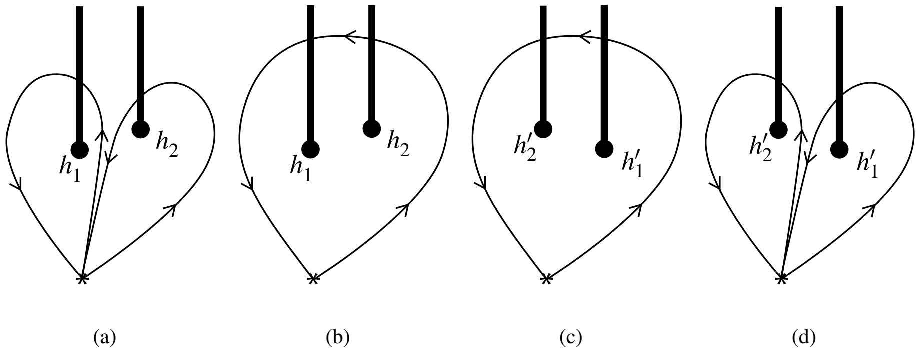


Figure 1.2: Flux metamorphosis Two fluxes labelled by h_1 and h_2 (a) fuse into a flux h_1h_2 (b). This fusion product, which corresponds to the long-range properties of the two-particle state, should not change under a local interchange (c). When we choose the flux h'_1 to still have h_1 for its flux value after the interchange, the other flux must then be $h'_2 = h_1h_2h_1^{-1}$ (d), so that the fusion product remains h_1h_2. The Dirac strings are denoted by black vertical lines; we can say that h_2 undergoes flux metamorphosis when passing through the Dirac string of h_1.

1.3.3 Flux metamorphosis

Consider a global symmetry transformation $g \in H$ on a defect. This transformation leaves the groundstate invariant, and is therefore a true symmetry of the system. The gauge field transforms as

$$A(x) \mapsto gA(x)g^{-1}, \tag{1.6}$$

and we see immediately that (1.5) changes accordingly:

$$w(C) \mapsto gw(C)g^{-1}. \tag{1.7}$$

So we now see that a global symmetry transformation, which may have no effect on any measurable properties, will send a defect to an element in its conjugacy class. This leads us to the notion that topological charge must in fact be labelled by conjugacy classes, not elements, of a group.

It was shown in [4] that another phenomenon, dubbed *flux metamorphosis*, also causes the defects to be organized into conjugacy classes of H instead of single elements: take two vortices with fluxes h_1 (the left vortex) and h_2 (the right vortex). The long-range interactions of this two-particle system are dependent on the holonomy circumventing both fluxes, which is equal to the group product h_1h_2 as shown in figure 1.2(a)-(b). Now a local interchange of the fluxes can never affect the long-distance properties of the flux-couple. We can choose the flux of the vortex h_1 on the left to remain fixed, so that the other vortex must now have flux $h_1h_2h_1^{-1}$ in order to leave the product h_1h_2 unchanged. This is also denoted pictorially in figure 1.2(c)-(d).

Dirac string So the interchange of two fluxes leads to transformation of one of them, which we can also call a topological interaction. It is convenient to make the following choice: we have seen that a particle or flux circumventing a non-trival vortex undergoes some transformation. We can perform local symmetry transformations so that all of this non-trivial action is located in a narrow wedge, which is called the Dirac string. This enables us to say that a particle or flux is transformed when it passes through the Dirac string of the vortex. This choice makes it easier to consider many-particle systems, when the vortices are described using their Dirac strings.

The Dirac string is pictured as a black line in figure 1.2.

1.3.4 Quantization

So far, we have talked about charges and vortices as classical particles, and both global symmetry transformations and local interchanges of fluxes should leave the total flux of the configuration invariant. We now want to turn this into a quantized description, by representing the fluxes and charges as vectors in a Hilbert space, on which we can perform symmetry transformations and other operations.

A fundamental particle is a representation Γ of the gauge group, which for discrete gauge theories is the residual symmetry group H. The particle is in a state $|v\rangle$ in the representation space V_Γ, and symmetry transformations act through the representation:

$$g \rightharpoonup |v\rangle = \Gamma(g)|v\rangle. \tag{1.8}$$

As we have seen in the previous section, a vortex should be labelled by the conjugacy class A of the gauge group H, and it has an internal structure labelled by the elements of A. This internal structure is of physical importance. For instance, a flux h_1 may annihilate its anti-flux h_1^{-1}. However, when we first take h_1 around another flux h_2, it takes the value $h_2 h_1 h_2^{-1}$, which only equals h_1 when h_1 and h_2 commute.

One can also envision a double slit interference experiment, where a vortex with flux h_2 is placed behind and between the slits. We send fluxes h_1 through the slits. The vortex between the slits is transformed into $h_1 h_2 h_1^{-1}$, and the projectile particle hits the screen carrying flux $h_1 h_2 h_1 h_2^{-1} h_1^{-1}$. Therefore we do not observe interference unless h_1 and h_2 commute. This is discussed in more detail in [27, §II-IV].

So a vortex is in a state in the representation space V_A, the basis of which is labelled by the elements of A, and on which there is an inner product

$$\langle h|h'\rangle = \delta_{h,h'}. \tag{1.9}$$

This basis and inner product have the same structure are the canonical basis and multiplication of the *algebra* of functions from A to $\mathbb{C}$. We will return to this in the next chapter. For now, it will suffice to say that a global symmetry transformation $g \in H$ acts on a basis vector $|h\rangle$ by conjugation:

$$g \rightharpoonup |h\rangle = |ghg^{-1}\rangle. \tag{1.10}$$

Topological interactions The next thing we want to describe are the topological interactions, in terms of operations on states $|h\rangle$ and $|v\rangle$. What we will use is the operator $\mathscr{R}$, called the *braid operator*, which interchanges two particles counter-clockwise. The action of taking the particle on the right around the particle on the left counter-clockwise is then denoted by $\mathscr{R}^2$, and this is called the *monodromy* operator.

Let's first look at taking a charge around a flux. When the charge circumvents the flux, it passes through its Dirac string once, which leads to a transformation relative to the flux:

$$\mathscr{R}^2|h\rangle|v\rangle = |h\rangle\big(\Gamma(h)|v\rangle\big). \tag{1.11}$$

Now, take a look at two fluxes. By interchanging once, the flux on the right gets conjugated by the flux on the left. By interchanging once again, the fluxes return to their original position, where the conjugated flux now acts on the flux that moves to the left:

$$\mathscr{R}|h_1\rangle|h_2\rangle = |h_1h_2h_1^{-1}\rangle|h_1\rangle, \tag{1.12}$$

$$\mathscr{R}^2|h_1\rangle|h_2\rangle = |h_1h_2h_1h_2^{-1}h_1^{-1}\rangle|h_1h_2h_1^{-1}\rangle. \tag{1.13}$$

This reproduces what we found above.

Centralizer representation Interference experiments can be used to determine flux $|h\rangle$ of a particle, as an incoming charge in state $|v\rangle$ of the representation Γ gets transformed by the representation value of the flux through (1.11) and we get interference amplitudes

$$\langle v|\langle h|\mathscr{R}^2|h\rangle|v\rangle = \langle h|h\rangle\langle v|\Gamma(h)v\rangle = \langle v|\Gamma(h)v\rangle. \tag{1.14}$$

By repeating this experiment for a complete set of states $|v_i\rangle$, we find all matrix elements of $\Gamma(h)$; if Γ is a faithful[†] irreducible representation of the residual symmetry group H, we can determine the element h itself.

We may also place an unknown charge Γ between the slits, and perform the experiments with all fluxes $h \in H$, by which we can determine the irreducible representation Γ.

However, if the unknown charged particle also carries flux, because of flux metamorphosis we will only be able to determine the charge by incoming fluxes that commute with the flux of the charged particle; otherwise no interference is observed. The charge will no longer be a representation of the full residual symmetry group, but of the *centralizer* subgroup N_h, consisting of all elements of H that commute with the flux h of the charged particle.

This can already be seen without even referring to any experiment, but by trying to perform a global symmetry transformation (1.7). The transformations that commute with the flux of the particle can still act on the charge part in the tensor product representing the state of the particle. This still forms an internal degree of freedom: the representation space is that of a representation of the centralizer N_h of the flux.

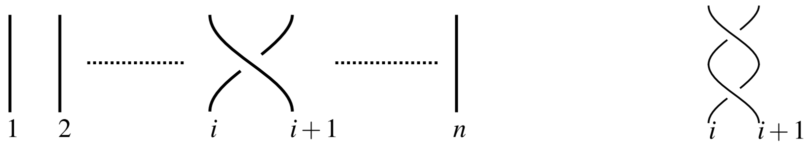
All of this will become much clearer by applying the mathematical framework to be developed in chapter 2, but do remember that there is a physical reasoning behind it. For an exploration of subtleties arising in measurement experiments of these discrete gauge theories, see [35].

1.4 Braid statistics

We can extend the description of interchanging particles to a system with any number n of particles. Because of topological interactions, we may, apart from ordinary (Bose–Einstein or Fermi–Dirac) statistics corresponding to the (ordinary) spin of the particles, have to deal with *braid statistics*. In particular, an n-particle configuration no longer gives rise to a representation of the permutation group S_n, but rather of the braid group B_n. We will now discuss this.

Quantization and configuration space For a system of n indistinguishable (identical) particles moving on a manifold M, the classical configuration space $\mathscr{C}_n$ is given

[†]A representation ρ is faithful if $\rho(h_1) = \rho(h_2) \Rightarrow h_1 = h_2 \quad \forall h_1, h_2 \in H$.



(a) The generator τ_i

(b) τ_i^2 corresponds to taking one particle around another

Figure 1.3: Braid group generators

by

$$\mathscr{C}_n(M) = (M^n - D)/S_n. \tag{1.15}$$

Here $M^n - D$ represents all possible particle positions minus all configurations where two or more particles coincide. Furthermore all permutations of particles are divided out because they are indistinguishable. This configuration space is in general multiply-connected.

If we now quantize the system, all configurations become states in the representation space of an irreducible representation of the fundamental group of the configuration space $\pi_1(\mathscr{C}_n(M))$. It can be shown that for manifolds of dimension larger than two

$$\pi_1(\mathscr{C}_n(M)) = \pi_1((M^n - D)/S_n) \simeq S_n \quad \dim M > 2. \tag{1.16}$$

There are two one-dimensional representations of S_n: the trivial, completely symmetric representation, correpsonding to Bose–Einstein statistics, and the completely anti-symmetric representation, corresponding to Fermi–Dirac statistics.

Braid groups Now we turn to 2+1 dimensions. We consider our particles to move in a flat plane $\mathbb{R}^2$. It may be calculated that

$$\pi_1(\mathscr{C}_n(\mathbb{R}^2)) = \pi_1((\mathbb{R}^{2n} - D)/S_n) \simeq B_n. \tag{1.17}$$

Here B_n is the braid group defined by $n-1$ generators τ_i and the relations (see figure 1.3(a))

$$\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1} \tag{1.18}$$

$$\tau_i \tau_j = \tau_j \tau_i \qquad |i-j| \geq 2. \tag{1.19}$$

When we label the n particles with index (i), the action of generator τ_i on the n-particle state is that of locally interchanging particles (i) and $(i+1)$ counter-clockwise, so corresponding to the operation $\mathscr{R}$ on those two particles, while leaving the others unchanged.

The nth permutation group S_n is actually defined by the same generators and relations plus an additional relation $\tau_i^2 = 1$. This relation assures that interchanging two particles twice will bring back the original configuration. For the braid group this is not the case: the particles 'wind around' one another, leading to a different configuration; if the system contains topological interactions as in §1.3.4, this winding around is not trivial (figure 1.4), and is therefore not equivalent to the identity. In principle we could go on and on applying τ_i, each time obtaining inequivalent configurations. For this reason, the braid group is infinite.

So an n-particle configuration in a discrete gauge theory with residual symmetry group H will be a state in the representation space of some irreducible representation of the direct product of H and the braid group B_n. The abstract generators τ_i of B_n are represented by the braid operators $\mathscr{R}_i$, which physically interchange particles (i) and $(i+1)$.

Yang–Baxter equation Let $\mathscr{R}$ as before denote the process of braiding two adjacent particles, so

$$\mathscr{R}_i = \underbrace{1 \otimes \cdots \otimes 1}_{i-1 \text{ times}} \otimes \mathscr{R} \otimes \underbrace{1 \otimes \cdots \otimes 1}_{n-i-1 \text{ times}}. \tag{1.20}$$

Because the $\mathscr{R}_i$ represent the τ_i they obey the relation (1.18), which leads to

$$(1 \otimes \mathscr{R})(\mathscr{R} \otimes 1)(1 \otimes \mathscr{R}) = (\mathscr{R} \otimes 1)(1 \otimes \mathscr{R})(\mathscr{R} \otimes 1). \tag{1.21}$$

This equation identifies operations on three-particle states, and is known as the Yang–Baxter equation (A.15). The braid operator $\mathscr{R}$ is in this context called an R-matrix. Figure 1.4 represents the Yang–Baxter equation on the group generator level.

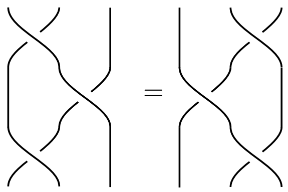


Figure 1.4: The Yang–Baxter equation (1.18)

It is precisely this relation that causes many special properties of 2+1-dimensional physics. It is also a main reason to turn to the quantum double construction of chapter 2, as this structure automatically provides R-matrices for every representation space of its representations.

Truncated and coloured braid groups Recall that the effect of a monodromy in a system is determined by the kind of topological interactions it features. In the systems discussed in §1.3, these interactions were dependent on the residual symmetry group H. Because this group is finite, the repeated action of interchanging two particles in a certain state will eventually produce the same state[†]. In other words, for any two particles there exists some integer m for which $\mathscr{R}^m = 1$.

When a configuration of n indistinguishable particles is in a state of a representation of B_n that obeys this equation, it corresponds effectively to introducing an extra relation to the braid group:

$$\tau_i^m = 1 \quad \forall i. \tag{1.22}$$

The group thus obtained is called the *truncated braid group* $B(n,m)$.

[†]The argument is as follows: by winding the particles around each other, their fluxes may change through conjugation, and their charges are transformed by the representation value of fluxes. The values obtained through the conjugation and representation are dependent on the group action. Because the group is finite, we will inevitably come upon a state identical to a previous one after a finite number of monodromies. This can then be generalized to any state. The number of monodromies required will depend on the particular representations of the particles.

Next, we can consider distinguishable particles, or particles "carrying a different colour". This has nothing to do with colour charge of quantum chromodynamics, but just means that we can distinguish particles because of some property (mass, charge etc.). In this case, the interchange of two particles of different colour does not lead to an identical configuration, and we can only consider those operations where the particles are carried back to their original position. In this case we can no longer speak of a braid operation $\mathscr{R}$, but monodromies $\mathscr{R}^2$ do still exist. The group describing these operations is called the coloured braid group $P(n)$, or $P(n,m)$ if it is also truncated.

When a system contains multiple particles of each colour, we are left with the subgroup of the truncated braid group consisting of all operations for which the final position of a particle is the initial position of a particle of the same colour.

Braid statistics We have seen that statistics of a discrete gauge theory with topological interactions depend upon the particular form of the braid matrix $\mathscr{R}$, which in turn depends upon the nature of the topological interactions.

If the topological interactions are Abelian, the action of $\mathscr{R}$ will also be Abelian, being a one-dimensional representation of the braid group; interchanging two identical particles will give additional phase factors. This is called *Abelian braid statistics*, and the particles transforming in this way have been named *anyons* by Wilczek because their interchange can give *any* phase [50], not just 0 or π like bosons and fermions. This also leads to a generalized notion of spin, discussed further in §2.5.

Claims have been made very recently that anyons have been measured directly in a fractional quantum Hall system [10].

The other interesting case arises when the topological interactions are non-Abelian, leading to higher-dimensional irreducible representations of the braid group. Then the interchange of two particles will give a matrix-valued phase factor, and two interchanges may not commute. We call this *non-Abelian braid statistics*, and it opens up possibilities for many interesting physical models.

In the next chapter, we will argue that the best way to treat non-Abelian statistics is the ribbon Hopf algebra formalism, the structure of which comprises all physical properties of these systems.

Chapter 2

Quantum double symmetry

The conventional description of symmetry in physics makes extensive use of the mathematical structure of groups. Symmetry transformations are given by group elements acting on physical states denoted as vectors in a Hilbert space via the group representations. Particles are labelled by these representations, as they contain the characteristics under transformations.

As we will show, the proper way to describe the symmetry in a theory featuring interactions of both fundamental charges and topological defects is to extend the residual gauge group to its so-called quantum double. The quantum double is an example of a mathematical construction called *ribbon algebra*, a particular form of a *Hopf algebra*, which has a lot of structure, mathematically speaking. The beauty is now, that every single piece of this structure corresponds directly to some physical property in the theory we discussed in chapter 1.

Because of this correspondence, we will build up both the mathematical structure and the physical concepts side-by-side, each time introducing a physical property and its structure in mathematics. Another motivation for this approach is that in [52, 34] the theory is developed from a physical point of view, whereas in [7] the mathematics were presented first, followed by the relation to the physics. We therefore employ a third method to present the formalism.

For several mathematical constructions, references are given to the definitions in appendix A.1.

2.1 The structure of flux–charge composites

In the previous chapter we have seen that in discrete gauge theories in 2+1 dimensions, the excitations (particles) can carry two types of charge: fundamental and topological. As always, we want to describe these excitations by their properties under symmetry transformations, which act on the excitations through representations. In this section we will discuss how we can implement these ideas for particles carrying both flux and charge, following the argument in [52, §2.1].

2.1.1 Identifying the state of a flux–charge composite

Given a particle with unknown flux and charge, we can perform global symmetry transformations (1.8) and (1.10). These transformations are elements of the group H.

Independently we can also perform interference experiments using charges, which result in monodromies of the form (1.11) and (1.13). Through a series of interference experiments, we may establish the flux of the unidentified particle, so these combined operations can be described as projecting out a certain flux h, which we denote by the projector P_h. As mentioned in (1.9), flux projections are orthogonal, so for two consecutive projections P_h and $P_{h'}$ we have:

$$P_h \circ P_{h'} = \delta_{h,h'} P_h. \tag{2.1}$$

This relation is not compatible with a group structure—in particular the zero element cannot exist in any multiplicative group—so we cannot speak of a symmetry group any longer. But we can describe the particles in this system as representations of a structure called an associative algebra.

Algebras An assoctiative algebra (see p.102) is a vector space with an associative multiplication. When there is a unit element for this multiplication, which we will always assume, we speak of a unital algebra. The multiplication and the unit of an algebra A can also be defined in terms of maps $\mu : A \otimes A \to A, \quad a \otimes b \mapsto ab$ and $\eta : \mathbb{C} \to A, \quad \lambda \mapsto \lambda 1$ (see p.102). We will use these to clarify the structure of Hopf algebras in §2.2.4.

The algebra of flux projections and symmetry transformations A flux projection P_h can be seen as a function $P_h : H \to \mathbb{C}$, defined by $P_h(h') = \delta_{h,h'}$. In fact the set of all flux projections form the canonical basis of the vector space of functions on a group $F(H)$. When we define multiplication of elements of this space by composition (2.1), we obtain an algebra with unit $\sum_{h \in H} P_h$. The multiplication for general functions can also be written $(f \cdot f')(h) = f(h) f'(h)$, and is called pointwise multiplication.

Note that the group multiplication structure in H does not play any role here; we could have taken any set of $|H|$ elements just as well. It will become important later on, as we will see in §2.2.1.

The group of symmetry transformations H can be naturally extended to an algebra called the group algebra $\mathbb{C}H$, by taking each group element as a basis vector, and defining the multiplication of these basis vectors by the group multiplication. This algebra is also $|H|$-dimensional.

We will now classify the particles in our theory by their properties under a global symmetry transformation $g \in \mathbb{C}H$, followed by a flux projection $P_h \in F(H)$. Because these operations are independent, any combination of g and P_h is possible, and this is denoted by the tensor product $(P_h \otimes g) \in F(H) \otimes \mathbb{C}H$. From now on we will write such an element as (P_h, g).

Twisted multiplication Now the tensor product is a way to create a new vector space out of two others. This can be carried over to algebras by defining the multiplication of the new algebra as

$$(a \otimes b) \cdot (a' \otimes b') = (aa' \otimes bb'). \tag{2.2}$$

However, from (1.10) we know that a global symmetry transformation does affect a flux. In other words, the outcome of a flux projection P_h will be different whether performed before or after the symmetry transformation. In fact, we demand

$$g \circ P_h = P_{ghg^{-1}} \circ g. \tag{2.3}$$

So we define our algebra, denoted by $F(H)\tilde{\otimes}\mathbb{C}H$ as in [7], by the basis vectors $(P_h, g)_{h,g\in H}$ and the 'twisted' multiplication

$$(P_h, g)\cdot(P_{h'}, g') = \delta_{h,ghg^{-1}}(P_h, gg'). \tag{2.4}$$

From now on we will often suppress the $\cdot$ notation for multiplication.

It is precisely this twisting that accounts for the peculiar properties of the quantum double construction we are now building up. The other (Hopf algebra) structure is actually the same as it would be for a regular tensor product. In anticipation of this, from now on we will denote this algebra by $D(H)$, the quantum double of H (p.105).

2.1.2 Representations

Now we wish to talk about particles as states in a Hilbert space of a representation of this algebra $D(H)$, similar to what is done for ordinary symmetry groups. Mathematically, a representation is a family of linear maps that can be associated (homomorphically, see p.102) to every element of a certain structure, which has to respect all rules for combining elements. Mathematicians prefer to use to word *module* (p.103).

Algebra representations For groups, the only rule that we have to respect is multiplication: then the inverse and unit will be carried over as well. For algebras we demand that the multiplication is conserved, and that the representation is linear, which is a different way of saying that addition and scalar multiplication must also be conserved. Denoting a representation of an algebra A by Π, we demand that, $\forall\, a,b\in A$,

$$\Pi(ab) = \Pi(a)\Pi(b), \tag{2.5}$$

$$\Pi(\lambda_a a + \lambda_b b) = \lambda_a\Pi(a) + \lambda_b\Pi(b). \tag{2.6}$$

Just as for a group, an algebra can have irreducible representations, which are called *simple modules*. An algebra is called semisimple if every module can be written as a direct sum of simple modules. This is similar to the statement that every representation of a group is completely reducible, which holds for instance for all finite groups. In particular, if an algebra is semisimple, we need only concern ourselves with the irreducible representations, as these will generate all representations.

Representations of $\mathbb{C}H$ and $F(H)$ First, we state that the algebras $\mathbb{C}H$ and $F(H)$ are semisimple†. Therefore, we wish to find all their irreducible representations.

The irreducible representations of the group algebra $\mathbb{C}H$ are given by the linear extension of the irreducible representations of the group H itself, see e.g. [43, §6.1]. We stress that these can be higher-dimensional.

The irreducible representations of the function algebra $F(H)$ are all one-dimensional, because it is an Abelian algebra. They are given by $E_g \quad g\in H$, with action $E_g(P_h) = P_h(g) = \delta_{h,g}$.

†The semisimplicity of the group algebra follows from the correspondence from its irreducible representions with those of the group H. The function algebra is also semisimple, because it is the dual of the group algebra, which is cosemisimple.

2.1.3 Representations of $D(H)$

Because of the twisted multiplication we saw in the previous section, the representations of $D(H)$ are not just given by the tensor product of the irreducible representations of $F(H)$ and $\mathbb{C}H$.

Instead of diving into mathematical technicalities, let us turn to our physical models to see if we can predict the form of the irreducible representations, which, as we know, represent the inequivalent particles which can be found in such systems.

Indications from physics Firstly, we know from §1.3.3 that the fluxes are organized in conjugacy classes. That is, globally we cannot distinguish between fluxes h and $ghg^{-1} \quad \forall\, g \in H$. Global symmetry transformations $(1,g)$ should act on the internal space by conjugating the flux. This internal space is therefore spanned by the elements in the conjugation class of the flux h of the particle.

Next, we know that the symmetry transformations commuting with the flux h, the centralizer N_h, can still act on the 'charge part' of the particle. More precisely, we can categorize our group elements in *cosets* H/N_h labelled by $[k]$. In other words we can separate each group element in a representative of the coset and an element of the centralizer:

$$g \in H = k_g n_g \quad \text{for some } n_g \in N_h \text{ and } k_g \text{ the coset representative of } [g]. \tag{2.7}$$

Now k_g will conjugate the flux, and n_g will transform the charge-state, according to a representation of N_h.

The flux measurements P_h project out the flux h, and do not perform any action on a charge. It will only signal whether it faces the specific flux h. If we are only interested in the 'global description' of the flux, we should measure with $\sum_{h' \in A_h} P_{h'}$, where the sum runs over the conjugacy class of h; we can then be sure to measure this flux, regardless of what internal state it will be in.

Irreducible $D(H)$-representations The irreducible representations of $D(H)$ were described in [15]. We state the results.

The irreducible representations of the quantum double $D(H)$ of a finite group H are labelled by a conjugacy class $A = \{a_1, \ldots, a_m\}$ and an irreducible representation[†] α of the centralizer N_a of a distinguished element a of this class. We denote this representation by Π^A_α. The representation space V^A_α is the tensor product of the space spanned by elements of the conjugacy class and the representation space of α:

$$V^A_\alpha = \{|a_i, v_j\rangle\} \quad i = 1, \ldots, |A|;\ j = 1, \ldots, \dim\alpha. \tag{2.8}$$

Recall that the N_a-coset representatives k_i correspond to the elements of the conjugacy class $[a]$ by

$$a_i = k_i a k_i^{-1}. \tag{2.9}$$

Defining $a_x = g a_i g^{-1} \Rightarrow k_x a k_x^{-1} = g k_i a k_i^{-1} g^{-1}$, the action of an element (P_h, g) on the state $\{|a_i, v_j\rangle\}$ is now given by

$$\begin{aligned} \Pi^A_\alpha(P_h, g)\{|a_i, v_j\rangle\} &= \Pi^A_\alpha(P_h, e)\{|g a_i g^{-1}, \alpha(k_x^{-1} g k_i) v_j\rangle\} \\ &= \delta_{h, g a_i g^{-1}} \{|g a_i g^{-1}, \alpha(k_x^{-1} g k_i) v_j\rangle\}. \end{aligned} \tag{2.10}$$

[†]For finite groups, every representation is equivalent to a unitary representation, and we shall always choose them to bee so. See e.g. [28, §III.1].

We see that indeed the flux a_i gets conjugated by the global symmetry transformation g and subsequently projected out by P_h. Some freedom to act on the charge-part may be left, although this must be calculated as in the formula given, because the part of g that commutes with a_i will be different from the part that commutes with a: the centralizer N_a of a need not be identical to the centralizer of a_i, although they are isomorphic (lemma A.7).

These representations can be shown to be orthogonal in the sense that their characters are orthogonal with respect to the canonical inner product. Using this property and the fact that $D(H)$ is semisimple†, we have found all irreducible representations, and the following holds:

$$\sum_{A,\alpha} (\dim \Pi^A_\alpha)^2 = \sum_{A,\alpha} (|A| \dim \alpha)^2 = \dim D(H). \tag{2.11}$$

Particle sectors Through the representations (2.10) we have classified all possible particles that may exist in our system. We can identify four sectors: the vacuum, charges, fluxes and dyons, as mentioned in §1.3.

The *vacuum* is represented by the representation Π^e_1, corresponding to the trivial conjugacy class, carrying the trivial flux, and the trivial H-representation. This representation is one-dimensional and sends every transformation by basis vectors (P_h, g) to 1.

The *charges* are the representations Π^e_α, corresponding to the trivial conjugacy class, carrying the trivial flux. All symmetry transformations commute with this flux, and the representations α are irreducible representations of the full group H. These particles behave as regular particles transforming under H.

The *fluxes* are the representations Π^A_1. They carry flux, but have no further internal state that may transform under residual symmetry transformation commuting with the flux. These particles behave as regular fluxes and show flux metamorphosis.

The most interesting particles are the *dyons*, made up by all other representations Π^A_α $A \neq [e]$; $\alpha \neq 1$. They transform states according to (2.10). If the group H is Abelian, all conjugacy classes consist of one element, and all irreducible representations are one-dimensional. Then the Π^A_α are also one-dimensional and represent anyons. If H is non-Abelian the representations may be higher-dimensional.

Electric–magnetic duality As the charges are related to the symmetry transformations of $\mathbb{C}H$, we call the group algebra the 'electric part' of $D(H)$. The fluxes are similarly related to the function algebra $F(H)$, which is therefore called the 'magnetic part' of $D(H)$. In fact, the magnetic part is dual to the electric part, as the space $F(H)$ is isomorphic to the dual vector space $\mathbb{C}H^*$ of functions on $\mathbb{C}H$.

With the quantum double construction, we are able to treat the electric and magnetic parts on equal footing. The fundamental particles and the topological defects are both represented by irreducible representations of $D(H)$. Moreover, we are provided with a description of all possible dyons using the same formalism.

†In [15], $D(H)$ was shown to be semisimple because it is "a based ring in the sense of Lusztig". Although valid, this seems as bringing in some unwanted external theory. Alternatively, in [40, prop.7] it was shown that $D(H)$ is semisimple if $\mathbb{C}H$ and $\mathbb{C}H^* \simeq F(H)$ are semisimple. Another approach is calculating that the sum of the squares of the dimensions of the irreducible representation equals the dimension of the algebra, which implies semisimplicity. I thank Vincent van der Noort for pointing this out; his proof can be found in [46, cor.1.44].

2.2 Multiparticle states – Coalgebra

Now that we know what particles may exist in our systems, we want to explore multi-particle configurations. We may ask ourselves is: "what happens when we perform a global symmetry transformation followed by a projection onto a flux, so (P_h, g), on a two-particle state?" But first, we have to consider what this means.

What is fusion? For flux metamorphosis (§1.3.3), we have regarded interactions between two fluxes in a sense as local processes, which have no effect on the global properties of the two-particle system. We can generalize this to any two-particle state: by the *fusion* of two particles Π^A_α and Π^B_β, we mean the process of measuring the long-range properties of the two-particles near each other, which we henceforth consider as a single localized particle-like object.

Mathematically, we denote this by the tensor product $\Pi^A_\alpha \otimes \Pi^B_\beta$. The coproduct, developed in this section, will allow us to split the multi-particle Hilbert space in this way into factors corresponding to irreducible reprsentations of the quantum double. In general, $\Pi^A_\alpha \otimes \Pi^B_\beta$ will be different from $\Pi^B_\beta \otimes \Pi^A_\alpha$, due to flux metamorphosis. We'll elaborate on this later on.

We can now also regard dyons as the fusion product of a flux and a charge, but this will make no difference in our formalism.

2.2.1 Transformation on composites – Comultiplication

Symmetry transformations of composites Now that we have defined our composite by $\Pi^A_\alpha \otimes \Pi^B_\beta$, we want to perform a global symmetry transformation followed by a flux projection on it by (P_h, g). The global symmetry transformations form a group H, and the natural tensor product for group algebras is $g \otimes g$. We see that this correctly corresponds to the physical situation as such a transformations works on both particles independently.

Flux measurements on composites If we try to measure the flux of the composite state by interference experiments, we will measure the total flux, given by the product of the fluxes of the constituents. In other words, the process of traversing a loop around the two fluxes counter-clockwise is equivalent to traversing first a loop around the particle on the right followed by traversing a loop around the particle on the left, so $h = h'h''$ (see figure 1.2 on page 7).

The process of projecting out the flux h is in this setting equivalent to projecting out all combinations of h' and h'' of which the group product gives h. That is

$$P_h \simeq \sum_{\substack{h',h'' \\ h'h''=h}} P_{h'} \otimes P_{h''}. \tag{2.12}$$

At this point, the group multiplication of H does play a role in the structure of $F(H)$ (cf. §2.1.1). It follows that the process of performing a global symmetry transformation g and then projecting out a flux P_h on a two-particle system is given by

$$\sum_{\substack{h',h'' \\ h'h''=h}} (P_{h'}, g) \otimes (P_{h''}, g). \tag{2.13}$$

Comultiplication The mathematical language to describe these processes is to say that our space $D(H)$ is equipped with a *comultiplication* or *coproduct* $\Delta : D(H) \to D(H) \otimes D(H)$. This comultiplication generates the correct tensor product expressions for probing composite states, by $(\Pi^A_\alpha \otimes \Pi^B_\beta) \circ \Delta$:

$$(\Pi^A_\alpha \otimes \Pi^B_\beta)(P_h, g) := (\Pi^A_\alpha \otimes \Pi^B_\beta)\Delta(P_h, g) = \sum_{\substack{h',h'' \\ h'h''=h}} \Pi^A_\alpha(P_{h'}, g) \otimes \Pi^B_\beta(P_{h''}, g). \quad (2.14)$$

Please note that this is the 'regular' comultiplication of the tensor product $F(H) \otimes \mathbb{C}H$; there is no twisting of tensorands as there is in the product (2.4).

Coassociativity and cocommutativity A nice feature of the comultiplication is that it is coassociative

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta, \quad (2.15)$$

which assures that composites of more than two particles are uniquely defined. That is, the representation spaces $(V^A_\alpha \otimes V^B_\beta) \otimes V^C_\gamma$ and $V^A_\alpha \otimes (V^B_\beta \otimes V^C_\gamma)$ can be identified.

The (mathematical) operation of switching the two tensorands in a tensor product expression is called the *flip*: $\tau : a \otimes b \mapsto b \otimes a$. The comultiplication is called *cocommutative* (p. 104) if it is identical to the operation of the comultiplication followed by the flip, that is $\Delta = (\tau \circ \Delta) \equiv \Delta^{\mathrm{op}}$. The comultiplication in $D(H)$ is not cocommutative if H is not Abelian.

2.2.2 Vacuum – Counit

The comultiplication provides us with a formalism to describe fusion. In §2.1.3, we mentioned that there is a vacuum sector, consisting of particles Π^e_1 which cannot actually be measured. In particular, we want non-trivial excitations, particles, to behave in the same way before and after fusion with a vacuum particle. That is, we demand

$$\Pi^e_1 \otimes \Pi^A_\alpha \simeq \Pi^A_\alpha \simeq \Pi^A_\alpha \otimes \Pi^e_1. \quad (2.16)$$

A function $\varepsilon : D(H) \to \mathbb{C}$ with the property $(\varepsilon \otimes \mathrm{id})\Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon)\Delta$ is called the *counit* for the comultiplication Δ. We see that the counit is precisely the vacuum representation. Together, the comultiplication and the counit equip a vector space with a *coalgebra* structure (p.103).

2.2.3 Fusion rules – Tensor product decomposition

We have seen how the properties of a two-particle composite state can be described using the coproduct on the tensor product of states. In fact, when probing only the long-range properties, we cannot distinguish between a composite state and a point particle. We may regard the two particles to have 'fused' into another single particle-like object.

Starting out with the particles Π^A_α and Π^B_β, we are actually able to calculate into which particles a composite of these two may fuse. The possible outcomes are called the *fusion rules*, given by the *tensor product decomposition*, which is reminiscent of the Clebsch–Gordan decomposition for group representations:

$$\Pi^A_\alpha \otimes \Pi^B_\beta = \bigoplus N^{AB\gamma}_{\alpha\beta C} \Pi^C_\gamma, \quad (2.17)$$

(co)multiplication	$\mu : D(H) \otimes D(H) \to D(H)$	$\Delta : D(H) \to D(H) \otimes D(H)$
(co)unit	$\eta : \mathbb{C} \to D(H)$	$\varepsilon : D(H) \to \mathbb{C}$
(co)unitality	$\mu(\eta \otimes \mathrm{id}) = \mathrm{id} = \mu(\mathrm{id} \otimes \eta)$	$(\varepsilon \otimes \mathrm{id})\Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon)\Delta$
(co)associativity	$\mu(\mu \otimes \mathrm{id}) = \mu(\mathrm{id} \otimes \mu)$	$(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$
(co)commutativity	$\mu = \mu \circ \tau$	$\Delta = \tau \circ \Delta$

Table 2.1: Duality of algebra and coalgebra structures

where $N^{AB\gamma}_{\alpha\beta C}$ gives the mulitplicity† of the irreducible representation Π^C_γ found in this decomposition.

Using this formula, we can determine what the so-called *fusion channels* of bringing two particles together may be. Reversely, this gives the possible *decay channels* of a particle which we may consider as a composite. In the end, they are determined by $(\Pi^A_\alpha \otimes \Pi^B_\beta) \circ \Delta$.

Please keep in mind that we do not give a treatment of the dynamics of any system at all. When considering a particular theory these must come from other considerations, for instance the action of the system in question. Nevertheless, we are already able to determine the fusion and decay channels from the representation theory.

2.2.4 Algebra–coalgebra duality

We should inform the reader here that each concept of the coalgebra is dual to a concept of an algebra. This is listed in table 2.1.

One can check that the algebra and coalgebra structure of $D(H)$ are compatible in the sense that

$$\Delta\big((P_h, g)(P_{h'}, g')\big) = \Delta(P_h, g)\Delta(P_{h'}, g'), \tag{2.18}$$

$$\varepsilon\big((P_h, g)(P_{h'}, g')\big) = \varepsilon(P_h, g)\varepsilon(P_{h'}, g'). \tag{2.19}$$

Because of this compatibility, such a construction is called a *bialgebra* (p.104).

Dual bialgebras One particularly appealing property of (finite-dimensional) bialgebras is that it allows one to put a corresponding bialgebra structure on its dual vector space. Here, the dual multiplication is dependent on the comultiplication and vice versa, in the following manner ($f, f' \in D(H)^*$):

$$\mu^*(f, f') = (f \otimes f') \circ \Delta, \tag{2.20}$$

$$\Delta^*(f) = f \circ \mu. \tag{2.21}$$

The quantum double and its dual seem to have more connections, and we will use some of these in chapter 3.

2.3 Braiding – Universal *R*-matrix

We know from §§1.3-1.4 that our particles do not interchange commutatively. That is, the interchange of a state $\Pi^A_\alpha \otimes \Pi^B_\beta$ is not given by just $\tau \circ (\Pi^A_\alpha \otimes \Pi^B_\beta)$. Instead, our particles obey braid statistics due to topological interactions.

†These multiplicities can be expressed in terms of so-called modular *S*-matrices. This was developed by Verlinde [48], see also [15] and [52, §2.3].

The braid operator From what we have discussed before, we know how an interchange will affect the particles in question: one particle will pass through the Dirac string of the other, and undergoes a symmetry transformation dependent on the particular value of the flux carried by that particle.

So our braid operator $\mathscr{R}$ must signal the flux of the first particle, perform a transformation according to that flux on the second particle, and then interchange the two particles. Signalling the flux is done by the projection P_h, acting on the particle state via its representation. When a particular flux h has been identified, it must then act on the second particle. This gives us

$$\sum_{h\in H} \Pi^A_\alpha(P_h,e)\otimes\Pi^B_\beta(1,h). \tag{2.22}$$

This process must then be followed by interchanging the particle representations by τ, which leads us to the braid operator $\mathscr{R}^{AB}_{\alpha\beta}$ for particles Π^A_α and Π^B_β

$$\mathscr{R}^{AB}_{\alpha\beta} = \tau\circ(\Pi^A_\alpha\otimes\Pi^B_\beta)(R), \tag{2.23}$$

where $R = \sum_{h\in H}(P_h,e)\otimes(1,h)\in D(H)\otimes D(H)$ is called the *universal R-matrix* of $D(H)$ (p.104). It is the particular combination of transformations of a two-particle state that implements the topological interaction of one particle on another.

Quasi-cocommutativity The universal R-matrix has some desirable properties. First of all we have

$$\Delta^{\mathrm{op}}(P_h,g)R = R\Delta(P_h,g), \tag{2.24}$$

ensuring that braiding and the action of the quantum double on two-particle states commute:

$$\mathscr{R}\Delta(P_h,g) = \Delta(P_h,g)\mathscr{R}, \tag{2.25}$$

where the element provided by the comultiplication acts on a state via the appropriate representation. This can be verified by calculating the action on a general two-particle state (cf. [52, §2.1]). In other words: the local interchange of two particles does not affect the long-range properties of the two-particle state.

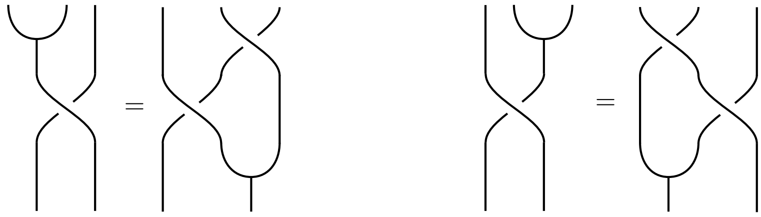
This property is called *quasi-cocommutativity* (p.104) because R determines the manner in which the coproduct is unequal to its opposite (cf. §2.2.1). For a cocommutative bialgebra $R = 1\otimes 1$.

Quasi-triangularity Furthermore, the operations of braiding two particles and then letting one decay is equivalent to letting the particle decay and then braiding the other particle with both decay products. If we write $R = \sum_k R^k_l\otimes R^k_r$ and then denote by R_{ij} the triple tensor product with R^k_l as the ith tensorand, R^k_r as the jth tensorand and 1 as the other tensorand (so $R_{32} = \sum_k 1\otimes R^k_r\otimes R^k_l$), this condition is expressed by

$$(\Delta\otimes\mathrm{id})(R) = R_{13}R_{23} \tag{2.26}$$

$$(\mathrm{id}\otimes\Delta)(R) = R_{13}R_{12}. \tag{2.27}$$

This can be verified by direct calculation. If we define the action of R on elements of $D(H)\otimes D(H)$ by left multiplication, the conditions (2.26) and (2.27) give us maps from $D(H)\otimes D(H)\to D(H)\otimes D(H)\otimes D(H)$, which ensure the claim that braiding and decay (or fusion) commute. Drinfeld has named this *quasi-triangularity* (see p.105 and figure 2.1) [17].



(a) Relation (2.26) (b) Relation (2.27)

Figure 2.1: Quasi-triangularity

Yang–Baxter equation The condition (2.24) together with either (2.26) or (2.27) give the equality

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}, \tag{2.28}$$

which leads to the *Yang–Baxter equation* (recall figure 1.4 on p.11) for three-particle states (1.21), assuring that braiding of three particles is uniquely defined.

2.4 Anti-particles – Antipode

From high-energy physics we know that for every particle in a theory there exists an anti-particle carrying the opposite quantum numbers. In particular, a particle and its anti-particle are able to fuse into the vacuum sector.

Recall that for a representation π of a group G, the anti-particle is given by the dual representation $\overline{\pi}$, defined by

$$\overline{\pi}(g) = \pi^{\mathrm{t}}(g^{-1}) \quad \forall\, g \in G \quad \Rightarrow \quad \overline{\pi} = \pi^{\mathrm{t}} \circ s, \tag{2.29}$$

where $s : G \to G$ is the group operation of taking the inverse of an element, and the superscript t denotes matrix transposition.

For our bialgebra $D(H)$ we are now looking for an operation S which will be the analogue of taking a group inverse. This operation is called the *antipode* (p. 104) and it is given by the linear map $S : D(H) \to D(H)$ satisfying

$$\mu \circ (S \otimes \mathrm{id}) \circ \Delta = 1\varepsilon = \mu \circ (\mathrm{id} \otimes S) \circ \Delta. \tag{2.30}$$

From the requirement (2.30), one can deduce that the antipode is an anti-algebra morphism, i.e. $S\big((P_h, g)(P_{h'}, g')\big) = S(P_{h'}, g')S(P_h, g)$, and an anti-coalgebra morphism, i.e. $(S \otimes S) \circ \Delta = \Delta^{\mathrm{op}} \circ S$.

The antipode for $D(H)$ is defined by

$$S(P_h, g) = (P_{g^{-1}h^{-1}g}, g^{-1}). \tag{2.31}$$

Note that the antipode does not have to be invertible, but it always is for semisimple Hopf algebras (lemma A.3). Furthermore, although its corresponds to the inverse of a group in the above mentioned fashion, there are some important differences: for example $S\big(\lambda (P_h, g)\big) = \lambda S(P_h, g)\ \forall\, \lambda \in \mathbb{C}$ by linearity, where we might have expected $\lambda^{-1}S(a)$.

Anti-particles The anti-particle $\overline{\Pi}$ of Π is now defined by $\overline{\Pi} = \Pi^{\mathrm{t}} \circ S$. If we decompose the two-particle tensor product $(\overline{\Pi} \otimes \Pi)$ according to the fusion rules (2.17), we are certain† to get at least one copy of the trivial (vacuum) representation ε.

Hopf algebras A bialgebra with an antipode is called a *Hopf algebra* (p.104), and it is the structure which is the closest analogue of a group in the setting of algebras. We have seen that we need the coproduct to properly define tensor products, and the antipode to resemble the group inverse.

Some Hopf algebras can be extended to braided Hopf algebras. In fact, the quantum double construction turns *any* Hopf algebra into another, non-trivially braided Hopf algebra. This was exactly the reason why much attention was given to these non-commutative, non-cocommutative Hopf algebras, which were and are very interesting to both mathematicians and physicists.

Hopf algebras have been used in a wide range of areas in physics. Mainly the coproduct seems to be the natural way to define many-particle states, or more generally the combination of possible outcomes of a certain process. See for instance [49] for applications in particle physics, [9] for an approach to quantization, and [30, 11] for uses in combinatorics.

2.5 Spin – ribbon element

We have one additional piece of data left in our quantum doubles. It is the generalized spin, a scalar number identifying the behaviour under a 2π clockwise rotation of a particle. We envisage this operation by considering a particle as a flux–charge composite, where the charge-part travels through the Dirac string of the flux. It thereby undergoes a symmetry transformation dependent upon the particular value of the flux. That is, we act on the internal particle state $|a_i, v_j\rangle$ of Π^A_α with the element

$$c = \sum_{h \in H} (P_{h^{-1}}, h).^{\ddagger} \tag{2.32}$$

One can calculate that

$$\Pi^A_\alpha(c)|a_i, v_j\rangle = \alpha(a)|a_i, v_j\rangle \quad \forall i, \tag{2.33}$$

where a is the distinguished element of A. The element c is called the *ribbon element* and it is central in (commutes with every element of) the Hopf algebra $D(H)$: because a commutes by definition with every element of its centralizer, its representation value is proportional to the unit matrix by Schur's lemma:

$$\Pi^A_\alpha(c) \sim \alpha(a) = \exp(2\pi\imath s^A_\alpha)\mathbf{1}. \tag{2.34}$$

The exponential will always be a root of unity, because the representation is unitary, and it can be identified as the spin factor of the particle Π^A_α.

Please note that in this definition the exchange of two particles is not incorporated. This can be seen by noting that for electric particles, carrying trivial flux, the exponential in (2.34) will always be unity, although we know such particles can be fermions. So

†For a proof by splitting a general vector of the tensor product representation space in a traceless part and the trace, see [31, §B.1.3].

‡Note that in [52] the ribbon element was defined as $\sum_h (P_h, h)$, which leads to the complex conjugate value of the spin, corresponding to *counter-clockwise* rotations.

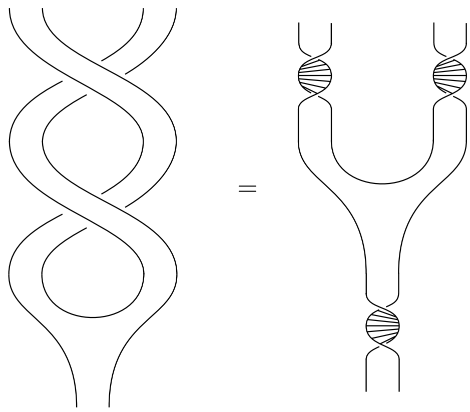


Figure 2.2: Generalized spin–statistics connection (2.35). Particle trajectories are pictured as ribbons, and the spin factor given by the action of the ribbon element c is shown as a double twist in a ribbon. On the left-hand side, we see the process of a particle decaying and then braiding these particles twice. This picture is continuously deformable into the picture on the right-hand side, which denotes rotating the particle clockwise, letting it decay and rotating the decay products counter-clockwise

we are only describing bosons, and the spin we are talking about is due to topological interactions. If we want to include fermions, we have to put in an additional factor of $\exp(\imath\pi) = -1$.

2.5.1 Generalized spin–statistics connection

In ordinary quantum mechanics, we are faced with particles of two kinds: bosons and fermions. It can be calculated using quantum field theory that particles that interchange without a sign change, the bosons, always have an intergral value of spin. On the other hand, the fermions, which pick up a sign unter interchange of two of them, always carry half-integral spin. This is called the *spin–statistics connection*, a theorem introduced by Pauli [36].

We can also put up a 'canonical' spin–statistics theorem for 2+1 dimensions by saying that the interchange two particles in identical states, carrying the same quantum numbers for both flux and charge, will give a factor $\exp(\imath 2\pi s)$, with s the spin of the particle, which may take any non-zero integer value, according to the system at hand. By canonical we mean that this is only valid for particles in exactly the same state.

For different particles, or identical particles in different states, this no longer holds, but there is still a relation between spin and braiding, which we call the *generalized spin–statistics connection*. It is due to the following property of the ribbon element (A.17):

$$\tau(R)R = (c \otimes c)\Delta(c)^{-1}. \tag{2.35}$$

This can be interpreted as follows (figure 2.2): on the right-hand side of the equation, we perform a clockwise rotation on a two-particle composite, which then decays, and we rotate the resulting particles counter-clockwise. On the left-hand side we can see the process of braiding the two particles twice, which we considered as having decayed from a single particle.

2.6 Summary of the quantum double

In this chapter we have constructed the quantum double of our residual symmetry group H, the representations of which correctly describe the spectrum of particles, accounting for both global symmetry transformations and topological interactions. It is a $|H|^2$-dimensional vector space equipped with compatible algebra and coalgebra structures, so we can compose elements and take tensor products of them. The respective axioms ascertain that the behaviour of representations is well-defined.

This bialgebra is provided with a non-trivial braiding, which yields solutions to the Yang–Baxter equation for all of its representations. Also, the actions of braiding and fusion commute. Furthermore, we are given tools to define anti-particles and spin by the antipode and the ribbon element.

We can employ the ribbon diagrams to visualize some of the processes the particles may undertake. Remarkably, such diagrams are used in the mathematics of the highly abstract *tensor categories* to aid in constructing certain proofs (see e.g. [20, ch.14]).

2.6.1 Connection between fusion and braiding

At first, one may think that fusion and braiding are two unrelated processes, although they feature in the same theory. Indeed, how should probing the quantum numbers of a two-particle composite be connected to its braid properties?

However, they do seem to be related, by the Verlinde formula mentioned on p.20. The multiplicities in (2.17) of the decomposition of a tensor product of two particles, defined by the comultiplication, are dependent as a kind of relative probability of decaying into a particular particle representation:

$$N^{AB\gamma}_{\alpha\beta C} = \sum_{D,\delta} \frac{S^{AD}_{\alpha\delta} S^{BD}_{\beta\delta} (S^{CD}_{\gamma\delta})^*}{S^{eD}_{1\delta}}. \tag{2.36}$$

These probabilities depend on the modular S-matrix given by

$$S^{AB}_{\alpha\beta} = \frac{1}{|H|} \mathrm{tr}\, \mathscr{R}^{-2}{}^{AB}_{\alpha\beta}. \tag{2.37}$$

This shows that there is a (deep) connection between fusion and braiding. For the quantum double, this relation can be proven quite straightforwardly using a duality property of the algebra [23].

In this light it might be clarifying to note that in the language of modular transformations [16, 15], the operator T is related to the spin of a particle, and the charge conjugation operator $\mathscr{C}$ transforms a particle into its anti-particle. Some properties of the particles may be derived from this, for instance, the spin of a particle and its anti-particle are the same, because the operators T and $\mathscr{C}$ commute.

Chapter 3

Condensates and confinement

In the previous chapter we classified which inequivalent particles may arise in a theory featuring topological interactions, and built up the machinery to describe their properties under braiding, 2π-rotations and also the fusion rules. The particles form representations of the quantum double of the residual symmetry group, and this space may be viewed as all combinations of global symmetry transformations followed by projecting out a certain flux.

The next step in exploring these models is to look at the formation of a condensate in some state. That is, we assume that the groundstate, instead of being the vacuum, is now formed by a background of particles in a certain state $|\phi\rangle$. Because of the non-trivial braiding, not all particles in the original theory will be able to travel freely through this condensate, and such particles will be confined.

In this chapter we give the proper definitions and some more algebraic tools to deal with these phenomena, summarizing the results of [7, ch.6-8],[6]. Some proofs will be given when they are clarifying, others are omitted and can be found in the cited literature.

3.1 Spontaneous breaking of Hopf symmetry

3.1.1 What are condensates?

In general, the term *condensate* is used when a macroscopic number of a certain excitation, i.e. particle, assume the same state, thus forming a new groundstate of the system. This may come about either through spontaneous symmetry breaking or by an external force (§1.1).

The term *condensation* stems from the phenomenon of *Bose–Einstein condensation*, which can take place in dense quantum gases or liquids and in dilute gases, the particles of which all fall back into the groundstate below a critical temperature. Other phenomena such as superconductivity and superfluidity are also forms of Bose–Einstein condensation: the groundstate is a background of particles. In other words, instead of a system where the groundstate is the vacuum, and particles enter this system as excitations of this vacuum, the groundstate is now an indefinite number of particles in a certain state. Other particles may exist in this new system, but they can have interactions with the background particles. Therefore, they manifest themselves differently, and may for example acquire a mass or be confined.

3.1.2 Setting in Hopf-symmetric theories

Symmetry breaking (1) Recall from §§1.1-1.3 that our theory arose when a continuous gauge group G was broken down to a discrete subgroup H, so that the inequivalent topological defects are labelled by elements of H. We stated that this spontaneous symmetry breaking was due to a potential leading to a vacuum expectation value, so that the groundstate turned degenerate, and one particular state would be picked 'at random'.

Symmetry breaking (2) In the present case, we consider the residual gauge group H, elevated to the quantum double symmetry $D(H)$ with its particle spectrum Π^A_α, as the unbroken theory. We contemplate a situation in which a vector $|\phi\rangle$ in the representation space of a certain irreducible representation Π^A_α is condensed, so there is a background of particles in the state $|\phi\rangle \in V^A_\alpha$. Again, we do not specify why this would happen, but assume that this is achieved by varying certain parameters in an effective potential; or by varying external parameters, such as the temperature.

Residual symmetry Firstly, we may ask ourselves what the elements (P_h, g) of our $D(H)$-theory mean in the presence of a condensate. We expect that some symmetry transformations will leave the condensate invariant, while others will change the condensate vector. It is natural to define a residual symmetry algebra, and label the possible excitations in the condensate by its irreducible representations. In this view, we are repeating the process of the initial symmetry breaking from G to H.

Confinement Now, we know that particles in these theories exhibit topological interactions leading to non-trivial braiding. One can already imagine that particles that do not braid trivially with the background particles may not travel as freely through the condensate as particles that do. When this happens, such a particle causes a half line discontinuity in the order parameter of the condensate, and its energy increases linearly with the system size. Therefore such particles are *confined*; we will treat confinement in §3.3.

General Hopf symmetry In [7], instead of looking only at $D(H)$-models, it was argued that the residual symmetry could also be a general Hopf algebra. In fact, there is a natural way to define the residual symmetry algebra using only the bialgebra structure, there is no need to restrict ourselves to the quantum double. Because of this, we will speak about a ribbon Hopf algebra $\mathscr{A}$, and only specialize to $D(H)$ when this is necessary or as an illustration.

3.2 The Hopf-symmetry breaking formalism

3.2.1 Definition of the residual symmetry algebra

When a group symmetry is broken down, the residual symmetry is defined as the subgroup that leaves the favoured groundstate invariant. For groups, this all works out nicely, as two symmetry transformations that leave the groundstate invariant will always multiply to a single transformation that does so as well.

But now we are dealing with algebras, which also possess an addition. Then, two elements whose action leave the condensate vector invariant may add up to the zero element, which obviously does not. We therefore need another definition of invariant action.

We can see what this definition should be by reverting to a different point of view: in our condensate of particles in the state $|\phi\rangle$, we can still consider interactions with the original vacuum representation ε (which is isomorphic to Π_1^e for $D(H)$-models). The vacuum can be thought of as a particle which can fuse with other particles at will, the fusion product being identical to the particles fusing with the vacuum as in (2.16).

Now we define an element $a \in \mathscr{A}$ to leave the condensate $|\phi\rangle$ invariant, if it acts on $|\phi\rangle$ in the same way as it would on the vacuum:

$$a \rightharpoonup |\phi\rangle = \varepsilon(a)|\phi\rangle. \tag{3.1}$$

This specializes to groups correctly, as the counit then reverts to the trivial representation. We see that this definition is well-defined with respect to vector space addition, as the counit is a linear map.

Clearly, the set of all elements leaving the condensate invariant is a subalgebra of $\mathscr{A}$, because $a \rightharpoonup (a' \rightharpoonup |\phi\rangle) = (aa') \rightharpoonup |\phi\rangle$. However, this set is in general not a sub-Hopf algebra, moreover, it may even not be possible to define a coproduct. For this reason, the residual symmetry algebra leaving the condensate invariant was defined in [7] to be the maximal sub-Hopf algebra satisfying condition (3.1)†.

We denote this Hopf algebra by $\mathscr{T}$.

We would like the residual symmetry algebra to be a Hopf algebra, as we are as always interested in many-particle states, and therefore need a tensor product definition. That a such a composite state will be left invariant is guaranteed by

$$(\varepsilon \otimes \varepsilon) \circ \Delta = \varepsilon \circ (\mathrm{id} \otimes \varepsilon) \circ \Delta = \varepsilon \circ \mathrm{id} = \varepsilon, \tag{3.2}$$

using counitality (A.3).

Remark In [31, ch.5] it is shown that for certain physical models, this definition is too strict. The subtlety arises due to the fact that there may be confined degrees of freedom. Then the demand that the residual symmetry algebra be a Hopf algebra must be abandoned, but there is a way to define tensor products of representations, which is, after all, what we are physically interested in. We will stick to the formalism of [7].

3.2.2 Residual symmetry algebra structure

Now it is time to carefully consider which properties described in chapter 2 the residual symmetry algebra $\mathscr{T}$ possesses.

Particle classification The residual symmetry algebra $\mathscr{T}$ is defined as those symmetry operations (including certain linear combinations of flux projections) that leave the condensate invariant, and that form a Hopf algebra. The particles that can exist in this background of condensate particles should be labelled by algebra representations of $\mathscr{T}$: they can only be distinguished by their properties under the subset of symmetry operations in $\mathscr{T}$.

Just as for groups (see §1.2), the representations of the unbroken theory $\mathscr{A}$ branch into irreducible representations of the residual symmetry algebra $\mathscr{T}$. This is worked out in the next section.

†By maximal we mean the unique sub-Hopf algebra that is not a proper sub-Hopf algebra of another proper sub-Hopf algebra of $\mathscr{A}$ satisfying the same condition. It is a subset of the set of all elements satisfying (3.1), in general a smaller one.

Many-particle states We already mentioned in §3.2.1 that we need a way to define many-particle states. This is performed by the coproduct, accompanied by the counit, which denotes the vacuum sector. These maps are now defined only on $\mathscr{T}$, and the demand that it be a *sub*-Hopf algebra states that it is closed under these maps. For example $\Delta : \mathscr{T} \to \mathscr{T} \otimes \mathscr{T}$. However, consider the remark at the end of the previous section.

Anti-particles It is natural to expect that if a particle is present in the residual system, its anti-particle will as well. Fortunately, when $\mathscr{T}$ is a finite-dimensional sub-bialgebra of $\mathscr{A}$, so when it is closed under both multiplication and comultiplication, it is closed under the antipode as well by lemma A.4. Therefore, we can rightfully speak of a sub-Hopf algebra.

Braiding It would be nice if there were a description of braiding two particles in a condensate, preferably one that would derive from the corresponding braiding of some particles originating from $\mathscr{A}$. However, we can already see that we run into problems with this statement. An irreducible representation Π of $\mathscr{A}$ has a unique decomposition into irreducible representations Ω of $\mathscr{T}$. But for a single representation Ω of $\mathscr{T}$, there can be different representations Π which branch to Ω. Which ones are we to choose if we want to braid them?

But there is also a rigorous proof by Radford [40], which states that the quantum double of a finite-dimensional Hopf algebra does not have any proper sub-Hopf algebras which are braided by the restriction of the original R-matrix. This of course does not exclude that there may be another element that provides a quasi-triangular structure for $\mathscr{T}$, but even if we are able to find it, we have to ask ourselves what it means in relation to $\mathscr{A}$.

For example, many of the condensates coming from $\mathscr{A} = D(H) \simeq F(H)\,\tilde{\otimes}\,\mathbb{C}H$, which we are discussing in later chapters, will have the form $\mathscr{T} \simeq F(H_1)\,\tilde{\otimes}\,\mathbb{C}H_2$, where H_1, H_2 are certain subgroups of H. In [31, §B.2] it was shown that an R-matrix can be defined for such algebras. However, the relation with $R_{D(H)}$ is obscure and we therefore do not know whether this concept describes the physics of the broken phase correctly.

The question arises whether we really need a formalism for braiding at the condensate level. As mentioned earlier, we will argue that some of the possible particle excitations will be confined, that is, they cannot exist independently in the system. In that case we have no need of a description of braiding of all of these particles, just the ones which are unconfined. In most cases, the unconfined algebra in an $D(H)$-condensate will have the form of a quantum double of a subgroup of H. In that case, the braiding is well-defined, and relates to that of $D(H)$ in a natural way. But we are running ahead of things now, and will return to this issue later on.

3.2.3 Determination of the residual symmetry algebra

We have defined that $\mathscr{T}$ be the largest (maximal) sub-Hopf algebra of $\mathscr{A}$ that leaves $|\phi\rangle$ invariant. For semisimple Hopf algebras, there is a method to classify all its sub-Hopf algebras, which we will now describe.

Classification of sub-Hopf algebras Recall from §2.4 that the conjugate $\overline{\pi}$ of a representation π is given by $\overline{\pi} := \pi^{\mathrm{t}} \circ S$, and from §2.2.1 that the tensor product of two representations is defined by $\pi_a \otimes \pi_b = (\pi_a \otimes \pi_b) \circ \Delta$.

We will call a set X of representations of a Hopf algebra *closed under tensor products* if all decomposition factors in $\pi_a \otimes \pi_b$ are contained in $X \quad \forall\, \pi_a, \pi_b \in X$.

Proposition 3.1. *Let $\mathscr{A}$ be a semisimple Hopf algebra. Each sub-Hopf algebra of $\mathscr{A}$ is spanned by the matrix elements of a certain set of irreducible representations of the dual Hopf algebra $\mathscr{A}^*$* (see §2.2.4) *that closes under tensor products. And each such a set constitutes a sub-Hopf algebra of $\mathscr{A}$.*

The proof is given in [7, §6.2]. However, there the stronger condition that the set be closed under tensor products *and* conjugation was imposed. Because of the, earlier mentioned, result of lemma A.4, every sub-bialgebra of a Hopf algebra is a sub-Hopf algebra, and we do not have to additionally demand this.

To see that these elements actually form an algebra, observe the following:

Firstly, note that $a \in \mathscr{A}^* : \mathscr{A} \to \mathbb{C}$; then a matrix element of a representation of $\mathscr{A}^*$ is a function which assigns a number in $\mathbb{C}$ to each element in $\mathscr{A}^*$, i.e. it is an element of[†] $\mathscr{A}^{**} \simeq \mathscr{A}$. Next, the multiplication of such elements is defined by the comultiplication of $\mathscr{A}^*$, which in turn depends on the multiplication of $\mathscr{A}$. To be precise:

$$\mu^{**}\big((\pi_a)_{ij}, (\pi_b)_{kl}\big) = (\pi_a \otimes \pi_b \circ \Delta^*)_{(ij),(kl)}, \tag{3.3}$$

and we know that $\Delta^*(a) = a \circ \mu$. Similarly the comultiplication and antipode are also obtained by these duality considerations.

The dual quantum double We now turn to the specific case of the quantum double $D(H)$. Its dual $D(H)^*$ is also a Hopf algebra, and its structure, determined by $D(H)$, can be found on p.106. In particular, as a vector space $D(H)^* \simeq \mathbb{C}H \otimes F(H)$, its multiplication is the regular multiplication for this tensor product, and its comultiplication is given by

$$\Delta^*(g, P_h) = \sum_{h' \in H} (g, P_{h'}) \otimes (h'^{-1} g h', P_{h'^{-1}h}). \tag{3.4}$$

Because the multiplication is not twisted in the sense of (2.4), the irreducible representations are just the tensor product of the irreducible representations of $\mathbb{C}H$ and $F(H)$ (which are given in §2.1.2):

$$(\rho_a, E_x) : \mathbb{C}H \otimes F(H) \to GL(V_{\rho_a} \otimes v), \tag{3.5}$$

$$(g, P_h) \mapsto \rho_a(g) P_h(x), \tag{3.6}$$

where ρ_a is an irreducible representation of H with representation space V_{ρ_a}, E_x is labelled by elements $x \in H$ and v is a one-dimensional representation space.

So now the sub-Hopf algebras of $D(H)$ are given by the matrix elements of a set of these representations that is closed under tensor products. The multiplication of matrix elements is determined by (3.3):

$$\mu\big((\rho_a, E_x)_{ij}, (\rho_b, E_y)_{kl}\big) = \big((\rho_a \otimes \rho_b^x)_{(ij)(kl)}, E_{xy}\big), \tag{3.7}$$

where ρ_b^x is the H-representation ρ_b with the argument conjugated by x, and the tensor product on the right hand side is the regular tensor product for group representations.

[†] In general, $\mathscr{A}^{**}$ is not isomorphic to $\mathscr{A}$. In particular, the dual vector space of an algebra has a coalgebra structure only when the algebra is finite-dimensional. When the algebra is infinite-dimensional, we can define a *restricted dual*, which allows for a coalgebra structure (see [20, §III.9]). We are considering only finite-dimensional Hopf algebras. In fact, semisimplicity of a Hopf algebra implies finite-dimensionality by lemma A.1.

Implementation of the invariance condition Using the condition (3.1), we can now determine the residual symmetry algebra:

Proposition 3.2. *The residual symmetry algebra $\mathscr{T}$ of a vector $|\phi\rangle \in V_\alpha^A$ is spanned by the matrix elements of those representations (ρ, E_g) of $D(H)^*$ for which*

$$\Pi_\alpha^A(1, E_{g^{-1}})|\phi\rangle = \frac{\chi_\rho(a)}{d_\rho}|\phi\rangle, \tag{3.8}$$

where χ_ρ is the character and d_ρ the dimension of ρ, and a denotes the distinguished element of A.

Proof. The condition that a matrix element $(\rho, E_g)_{ij}$ leave $|\phi\rangle$ invariant is given by

$$\Pi_\alpha^A(\rho_{ij}, E_g)|\phi\rangle = \rho(kak^{-1})_{ij}\Pi_\alpha^A(1,g)|\phi\rangle = \varepsilon(\rho_{ij}, E_g)|\phi\rangle = \rho(e)_{ij}|\phi\rangle = \delta_{ij}|\phi\rangle, \tag{3.9}$$

where the last equality holds because ρ is a group representation; k is some A-coset representative determined by g and $|\phi\rangle$.

Now take the trace, which is invariant under cyclic permutation of its argument factors, on both sides and we are left with

$$\chi_\rho(a)\Pi_\alpha^A(1,g)|\phi\rangle = d_\rho|\phi\rangle, \tag{3.10}$$

which is equivalent to (3.8). So all elements leaving $|\phi\rangle$ invariant satisfy this equation.

On the other hand, all elements satisfying (3.8) leave $|\phi\rangle$ invariant. The argument is as follows:

The equation $g^{-1} \rightharpoonup |\phi\rangle = \frac{\chi_\rho(a)}{d_\rho}|\phi\rangle$ shows that $|\phi\rangle$ is an eigenvector of the action of g^{-1} with eigenvalue $\frac{\chi_\rho(a)}{d_\rho}$. Because H is a finite group, there is an $m \in \mathbb{N}$ so that $g^m = e$, so this eigenvalue has to be a root of unity. Independently, ρ is a unitary irreducible representation of H, so *its* eigenvalues are non-zero and roots of unity. The trace of $\rho(a)$ is then the sum of these eigenvalues. So we compare

$$qd_\rho = \sum_{i=1}^{d_\rho} \lambda_i, \tag{3.11}$$

where q denotes the particular eigenvalue of the action of g, and the λ_i are the eigenvalues of $\rho(a)$. This equation can only be satisfied if all the λ_i are equal to q, which one can see by considering vector addition in the complex plane.

Therefore ρ must be equal to q times the unit matrix, and (3.9) is satisfied: these (ρ, g) leave the condensate invariant.

We now show that these elements close under tensor products, i.e. that all products also leave the condensate invariant: we have, for all pairs $(\rho_i, E_g), (\rho_j, E_{g'})$ satisfying (3.8),

$$\begin{aligned}
\mathrm{tr}\Big(\big((\rho_i,E_g)\otimes(\rho_j,E_{g'})\big)\rightharpoonup|\phi\rangle\Big) &= \mathrm{tr}\big((\rho_i\otimes\rho_j^g)(kak^{-1})(gg')\rightharpoonup|\phi\rangle\big) && (3.12)\\
&= \mathrm{tr}\big(\rho_i(kak^{-1})\big)\mathrm{tr}\big(\rho_j(gkak^{-1}g^{-1})\big)g\rightharpoonup(g'\rightharpoonup|\phi\rangle) && (3.13)\\
&= \chi_{\rho_i}(a)\chi_{\rho_j}(a)\frac{d_{\rho_j}}{\chi_{\rho_j}(a)}(g\rightharpoonup|\phi\rangle) && (3.14)\\
&= \chi_{\rho_i}(a)\chi_{\rho_j}(a)\frac{d_{\rho_i}}{\chi_{\rho_i}(a)}\frac{d_{\rho_j}}{\chi_{\rho_j}(a)}|\phi\rangle && (3.15)\\
&= d_{\rho_i}d_{\rho_j}|\phi\rangle = \mathrm{tr}\Big(\varepsilon\big((\rho_i,g)(\rho_j,g')\big)\Big)|\phi\rangle, && (3.16)
\end{aligned}$$

where we have used (3.7) in the first equality, and in the second the fact that the character of the tensor product of two representation is the product of the two characters. □

Form of the residual symmetry algebra Now, the set defined by (3.8) can yield any sub-Hopf algebra, but in a lot of cases it will have an appealing form. In fact, many sub-Hopf algebras of $D(H)$ are of the form $F(H/K)\tilde{\otimes}\mathbb{C}N$, where K is a normal subgroup of H, and N is a subgroup of H. This is specified by the following corollary to proposition 3.2, the proof of which can be found in [7, §6.3]:

Corollary 3.3. *The equation* (3.8) *can only be satisfied if* $\frac{\chi_\rho}{d_\rho}$ *is a root of unity. The residual symmetry algebra* $\mathscr{T}$ *is of the form* $F(H/K)\tilde{\otimes}\mathbb{C}N$ *if and only if this root of unity equals* 1 *for all* (ρ,g) *leaving the condensate invariant.*

The normal subgroup K divided out is usually related to the magnetic part of the condensate vector $|\phi\rangle$. Similarly, the subgroup N relates to the unbroken electric symmetry, usually depending on which elements of H have trivial action under α. So, in many cases, the residual symmetry algebra will be quite easy to find.

However, sometimes, and in particular when $\mathscr{T}$ is not of the form $F(H/K)\tilde{\otimes}\mathbb{C}N$ due to the corollary, we will have to manually apply (3.8) to all representations of $D(H)^*$ in order to find $\mathscr{T}$. Indeed, such situations do arise and an example will be worked out in detail for a dyonic condensate of $D(D_4)$ (chap. 5).

3.2.4 Particles in the condensate

We have already mentioned that particles in the broken phase should be labelled by irreducible representations of $\mathscr{T}$. Now we ask ourselves how these relate to particles in the original $\mathscr{A}$-theory. After all, we can envisage an $\mathscr{A}$-particle 'entering' the condensate from an unbroken vacuum region. However, an element of $\mathscr{A}$ outside $\mathscr{T}$ no longer constitutes a valid symmetry transformation, because it changes the condensate vector.

Note again that $\mathscr{T}$ is a subalgebra, so the multiplication of elements is inherited from $\mathscr{A}$, and it contains the the unit element 1. But it is in general a different algebra, having different irreducible representations, which we label by Ω.

Representation decomposition Now, every irreducible representation Π of $\mathscr{A}$ will also be a representation of the subalgebra $\mathscr{T}$ by restriction, because $\mathscr{T}$ is closed under multiplication. However, Π need no longer be irreducible. So in general, Π decomposes into irreducible representations of $\mathscr{T}$:

$$\Pi|_{\mathscr{T}} = \oplus_i N_i \Omega_i, \tag{3.17}$$

where the direct sum runs over all irreducible representations of $\mathscr{T}$ and N_i gives the multiplicity (as components in the direct sum) of Ω_i in this decomposition.

So we see that we have *branching* from $\mathscr{A}$-particles to $\mathscr{T}$-particles. If we were talking about finite groups and subgroups, we would determine this branching by using characters of group representations, which are orthogonal to one another. For Hopf algebras, a similar notion exists, see e.g. [53].

Decomposition for the quantum double For the quantum double $D(H)$, the characters χ_α^A of the representations Π_α^A are (just) given by the matrix trace of those representations. This results in

$$\chi_\alpha^A(P_h, g) = 1_A(h) 1_{N_h}(g) \chi_\alpha(k_h^{-1} g k_h), \tag{3.18}$$

where $1_A(h) = 1$ if h is an element of the conjugacy class A and 0 otherwise; $1_{N_h}(g) = 1$ if g commutes with h and 0 otherwise; k_h is a N_a-coset representative, such that $h = k_h a k_h^{-1}$, where a is the distinguished element of A.

For these characters, we have the orthogonality relation [15]:

$$\sum_{h,g \in H} \chi_\alpha^A(P_h, g) \chi_\beta^B(P_h, g)^* = |H| \delta_{AB} \delta_{\alpha\beta}. \tag{3.19}$$

For sub-Hopf algebras of the form $\mathscr{T} \simeq F(H/K) \tilde{\otimes} \mathbb{C}N$ the character is defined analogously: elements of $\mathbb{C}N$ can act on elements of $F(H/K)$ by conjugation of the argument. Picking a distinguished element o in H/K, we obtain a subset $\mathscr{O}$ of H/K, called the orbit, consisting of all elements which can be reached by conjugating o by an element of N. The irreducible representations are then labelled by the orbits $\mathscr{O}$, and irreducible representations α of the stabilizer N_o of o in N, similar to what is discussed in §2.1.3†.

To determine the decomposition components of a representation Π_α^A of $D(H)$, we should apply the orthogonality relation (3.19) to χ_α^A and the character of each irreducible representation of $\mathscr{T}$, but with the sum restricted to (basis) elements of $\mathscr{T}$ only.

In the next chapter, we will see some explicit examples of what is discussed here, which may help to gain insight in the particular structure one encounters in these kinds of models.

3.2.5 Additional requirements on condensates

In what is discussed in this chapter, the formalism allows for condensates of any state vector in the representation space of any particle. However, we must impose the condition that the condensate state have *trivial self-braiding*:

$$\mathscr{R}(|\phi\rangle \otimes |\phi\rangle) = \tau \circ \left((\Pi_\alpha^A \otimes \Pi_\alpha^A)(R) \right) (|\phi\rangle \otimes |\phi\rangle) = |\phi\rangle \otimes |\phi\rangle. \tag{3.20}$$

†This construction can be generalized to any action of N on any set X. This is then called a transformation group algebra, and the representation structure was worked out in [22, 21]

This demand states that there should be trivial topological interactions between particles in the condensate state. Otherwise, the particles introduce discontinuities in the order parameter everywhere, and we cannot speak of a true condensate.

Furthermore, this condition is required in order to properly define a Hopf algebra structure on the *unconfined algebra*, treated in the next section (cf. [31, §5.2.4]).

Similar is the notion of trivial spin, that is $\Pi^A_\alpha(c) = 1$, which is equivalent to $\alpha(a) = 1$. This is related to the the requirement of having Bose-condensates; Fermi-condensates are now studied both theoretically and experimentally, and we could introduce an extra minus sign on interchanging to particles, as mentioned at the end of §2.5. But as we do not know what to make of an 'anyon-condensate', we for the moment discard states with non-trivial spin as possible condensate candidates.

In many cases, these requirements can be calculated at once for a whole class of representations, for instance, a vector in the representation space of a representation Π^e_α in the electric sector (see p.17). Such a representation carries trivial flux and it can be easily seen that the conditions of both trivial self-braiding and trivial spin are always satisfied. We will perform more of these calculations in chapter 4.

In other cases, we may have to manually verify these conditions.

3.3 Confinement

3.3.1 Braiding with the condensate

In the previous section, we have treated the irreducible representations of the residual symmetry algebra $\mathscr{T}$ after symmetry breaking by the formation of a condensate. We then stated that these representations are the possible particles that could exist in the condensate.

But this is not the whole story: although each such particle transforms correctly under the residual symmetry transformations, corresponding to external probing by global symmetry transformations and topological interactions, the particles may also undergo topological interactions with the condensate particles.

If these topological interactions are non-trivial, a domain wall will be attached to the particle, because the vacuum state will be different on each side of the wall: a vacuum state will be transformed due to the topological interactions when crossing this wall.

Because of these considerations, such particles cannot exist freely in the condensate and are *confined*. However, it is possible that the string drawn by one particle ends on another particle, like the quark–anti-quark meson-composites. Continuing this analogy, one may also imagine many-particle (baryonic) composites, which when bound together braid trivially with the background, and can therefore exist as effectively 'single' particles.

3.3.2 Properties of non-confined particles

Again, before we mathetically derive the formalism describing the non-confined particles, we predict its structure by looking at the properties we want it to possess from physical considerations.

Hopf algebra structure Firstly, we wish the particles to be representations of some algebra $\mathscr{U}$ of symmetry transformations which transform a state vector braiding trivially with the condensate particle, into another that does as well. Clearly, such symmetry transformations should also leave the condensate invariant, so $\mathscr{U}$ should be a subspace of $\mathscr{T}$.

Furthermore, we want unconfined particles to be able to fuse into another unconfined particle, so $\mathscr{U}$ should be equipped with a comultiplication which closes on this subspace. As mentioned before, two confined particles may form a composite which is unconfined, but for now, this is not of our concern. The vacuum representation given by the counit ε should be included in $\mathscr{U}$ as well, since this 'particle' is of course always unconfined.

It also seems natural to demand that when a particle is unconfined, its anti-particle be unconfined as well. If this were not the case, vacuum fluctuations could cause the creation of an unconfined and a confined particle. So $\mathscr{U}$ needs an antipode map, under which it should close.

All together this gives us the structure of a Hopf algebra.

Braiding Unlike $\mathscr{T}$ (cf.§3.2.2), we do want $\mathscr{U}$ to be equipped with a braiding. The unconfined particles should be able to move freely in the condensate, by which we mean that they still could have topological interactions with each other as described in §1.4.

It also makes sense that this braiding should correspond with braiding in the original symmetry algebra $\mathscr{A}$. For example, recall that the braid matrix R of the $D(H)$-models lets the charge of one particle be conjugated by the flux of the other; the unconfined particles also have 'flux' and 'charge', which should obey the same braiding rules.

Because $\mathscr{T}$ is a subalgebra of $\mathscr{A}$, the natural definition of the restriction of the R-matrix to $\mathscr{T}$ is by orthogonal projection denoted by P (see also the next section). Although $\mathscr{U}$ can be thought of as a subset of $\mathscr{T}$ (by choosing equivalence classes according to the map Γ described in the next few pages), the orthogonal projection onto $\mathscr{U}$ is not the correct way to define a universal R-matrix. Rather, we expect that it be of the following form:

Conjecture 3.4. *The universal R-matrix of the unconfined algebra $\mathscr{U}$ is given by*

$$(\Gamma \otimes \Gamma) \circ (P \otimes P)(R_{\mathscr{A}}). \tag{3.21}$$

It has not been proven yet that this constitutes a valid universal R-matrix in all cases, but we will see in §5.2.2 that for a certain dyonic condensate in a $D(D_4)$-theory this leads to the correct solution, whereas the orthogonal projection on $\mathscr{U}$ does not. This is in contrast to the remark at the end of §7.1 in [7].

Hopf quotient One might expect that $\mathscr{U}$ should be a sub-Hopf algebra of $\mathscr{T}$, just as $\mathscr{T}$ is of $\mathscr{A}$. But this is not the case; we will present some evidence now, and will give a rigorous mathematical treatment later.

First of all, when taking a subalgebra, one 'throws away' all elements not contained in the subalgebra. Consequently, the representations of the original algebra branch into representations of the subalgebra. But in the case of $\mathscr{T}$, we already know how the particles behave, only now some of them are confined while others are not. That is, we are reclassifying the representations of $\mathscr{T}$. Therefore, the unconfined particles, presented as irreducible representations of $\mathscr{U}$, should be in one-to-one correspondence with those irreducible representations of $\mathscr{T}$ which are unconfined.

Next, we have seen that a sub-Hopf algebra of a braided Hopf algebra need not have a braiding. Moreover, a quantum double possesses no sub-Hopf algebras which have a braiding derived from the original braiding (see §3.2.2).

From these considerations, we suspect that $\mathscr{U}$ is not a sub-Hopf algebra of $\mathscr{T}$, but of another form, in particular one which carries the unconfined representations from $\mathscr{T}$ over to $\mathscr{U}$. The structure we need turns out to be a *Hopf quotient*[†].

3.3.3 Algebraic structure of the unconfined algebra

We will now derive the structure of $\mathscr{U}$, following the argument in [7, §7.1].

Hopf quotients A Hopf quotient of a Hopf algebra $\mathscr{A}$ is a Hopf algebra $\mathscr{B}$, such that there exists a surjective Hopf morphism $\Gamma : \mathscr{A} \to \mathscr{B}$. This is of course analogous to the definition of a quotient space Y of a space X, where each element of x is sent to an element of y, and elements of X satisfying some equivalence relation are sent to the same element in Y.

One property of a Hopf quotient $\mathscr{B}$ of $\mathscr{A}$ is that its irreducible representations ρ are in one-to-one correspondence with the irreducible representations τ of $\mathscr{A}$ which *factor over* Γ, that is, $\tau = \rho \circ \Gamma$ for some τ [7, prop.2 in §6.2]. This is exactly what we would like them to be, as mentioned in the previous section.

Furthermore, if $\mathscr{A}$ is braided with R-matrix $R_{\mathscr{A}}$, then $\mathscr{B}$ is braided by $(\Gamma \otimes \Gamma)(R_{\mathscr{A}})$ (lemma A.6). However, this is of no direct use to us, as we have seen that $\mathscr{T}$ is not necessarily braided.

It turns out that if $\mathscr{B}^*$ is a sub-Hopf algebra of $\mathscr{A}^*$, the dual of $\mathscr{A}$, then $\mathscr{B}^{**} \simeq \mathscr{B}$ is a Hopf quotient of $\mathscr{A}$ [7, §6.2]. Because the irreducible representations of $\mathscr{T}$ which are unconfined are closed under tensor products, their matrix elements span a sub-Hopf algebra $\mathscr{U}^*$ of $\mathscr{T}^*$ by proposition 3.1. This sub-Hopf algebra is then the dual of the Hopf quotient $\mathscr{U}$, the symmetry algebra of unconfined particles.

Implementation of the braid condition Now we want to determine which representations of $\mathscr{T}$ braid trivially with the condensate vector. In fact we have to define what we mean by this braiding: the condensate vector is an element of the representation space of a representation Π of $\mathscr{A}$, whereas the particle is a representation Ω of the residual symmetry algebra $\mathscr{T}$, *for which we have not defined any braiding*.

The most natural choice is to braid these particles by making use of the braid description in $\mathscr{A}$, so by using $R = R_{\mathscr{A}}$. Naively, we would write $\mathscr{R}|\omega\rangle \otimes |\phi\rangle = \tau\big((\Omega \otimes \Pi)(R)|\omega\rangle \otimes |\phi\rangle\big)$. However, R is an element of $\mathscr{A} \otimes \mathscr{A}$ and the action of Ω is only defined for elements of $\mathscr{T}$. To solve this, we project this part of the braid matrix orthogonally on $\mathscr{T}$, denoted by the projection operator P.

Now we state that a particle braids trivially with the condensate if it does so in the same way as the vacuum representation ε. The conditions on Ω then have the form

$$\begin{aligned}\big((\Omega \circ P) \otimes \Pi\big)(R)(|\omega\rangle \otimes |\phi\rangle) &= \big(\Omega(1)(\varepsilon \circ P) \otimes \Pi\big)(R)(|\omega\rangle \otimes |\phi\rangle) \\ \big(\Pi \otimes (\Omega \circ P)\big)(R)(|\phi\rangle \otimes |\omega\rangle) &= \big(\Pi \otimes \Omega(1)(\varepsilon \circ P)\big)(R)(|\phi\rangle \otimes |\omega\rangle).\end{aligned} \tag{3.22}$$

[†] At the end of §3.2.1 it was remarked that extensions to the theory show that in some physical situations, $\mathscr{T}$ is not a Hopf algebra, because there is no well-defined comultiplication, which is due exactly to the presence of confined particle representations. In that case, the considerations given here still apply, and the unconfined algebra should have a braided Hopf algebra structure.

Next we translate these conditions on irreducible representations of $\mathscr{T}$ into conditions on their matrix elements. We therefore define left and right actions of elements of $\mathscr{T}^*$ on the representation space V_Π of Π, turning it into a left and right $\mathscr{T}^*$-module (see p.103), by

$$f \rightharpoonup |v\rangle = \sum_k (f \circ P)(R_l^k) \otimes \left(\Pi(R_r^k)|v\rangle\right) \tag{3.23}$$

$$|v\rangle \leftharpoonup f = \sum_k \left(\Pi(R_l^k)|v\rangle\right) \otimes (f \circ P)(R_r^k), \tag{3.24}$$

where $f \in \mathscr{T}^*$, $|v\rangle \in V_\Pi$ and $R = \sum_k R_l^k \otimes R_r^k$. It is shown that these actions respect multiplication in $\mathscr{T}^*$ in [7, §7.1], making use of the fact that the comultiplication in $\mathscr{A}$ commutes with the projection onto $\mathscr{T}$. The braiding is now implicitly defined in these actions.

The conditions (3.22) now become

$$\Omega_{ij} \rightharpoonup |\phi\rangle = \Omega_{ij}(1)\varepsilon \rightharpoonup |\phi\rangle \tag{3.25}$$

$$|\phi\rangle \leftharpoonup \Omega_{ij} = \Omega_{ij}(1)|\phi\rangle \leftharpoonup \varepsilon, \tag{3.26}$$

on matrix elements Ω_{ij} of irreducible representations of $\mathscr{T}$. Because the subalgebras of $\mathscr{T}^*$ are spanned by matrix elements of a set of irreducible representations that closes under tensor products, when one matrix element of a certain representation satisfies (3.25) and (3.26), the others do as well.

Conditions for $F(H/K)\tilde{\otimes}\mathbb{C}N$ We mentioned that many residual symmetry algebras $\mathscr{T}$ of some quantum double $D(H)$ have the form $F(H/K)\tilde{\otimes}\mathbb{C}N$, where K is a normal subgroup, and N a subgroup of H. For these cases, the conditions (3.25) and (3.26) reduce to more explicit formulae.

The irreducible representations for such algebras are labelled by N-orbits $B \subset H/K$, and irreducible representations β of the centralizer N_b of a distinguished element $b \in B$, as mentioned in §3.2.4. The action is defined by (2.10). We can expand $|\phi\rangle = \sum_{p,q} \omega_{pq}|a_p v_q\rangle$; by the supp($|\phi\rangle$) (the support of $|\phi\rangle$) we mean those $|a_p\rangle$ for which $\exists q : \omega_{pq} \neq 0$.

Proposition 3.5. *Let $\mathscr{T}$ be of the form $F(H/K)\tilde{\otimes}\mathbb{C}N$, with irreducible representations Ω^B_β. Then the condition* (3.25) *reduces to*

$$\frac{1}{|K|}\sum_{k\in K} \Pi^A_\alpha(1,\eta k)|\phi\rangle = \frac{1}{|K|}\sum_{k\in K} \Pi^A_\alpha(1,k)|\phi\rangle \quad \forall \eta \in B. \tag{3.27}$$

The condition (3.26) *reduces to*

$$a_p \notin N \vee \left(\beta(x_\eta^{-1} a_p x_\eta) = 1 \quad \forall \eta \in B\right) \quad \forall a_p \in \mathit{supp}(|\phi\rangle), \tag{3.28}$$

where $x_\eta \in N$ is defined by $x_\eta b x_\eta^{-1} = \eta$, so that the set $\{x_\eta b x_\eta^{-1}\} = B$.

We state the proof, translating the one given in [7, §7.3] to our notation.

Proof. Firstly, note that the orthogonal projection $P_{\mathscr{T}}$ on $\mathscr{T}$ is given by

$$P_{\mathscr{T}}(P_h, g) = \frac{1}{|K|}\sum_{k\in K}(P_{hk}, g)1_N(g). \tag{3.29}$$

The representation space of Ω^B_β is spanned by $|\eta w_i\rangle$, with $\eta \in B$ and $|w_i\rangle$ basis vectors of β. We therefore label its matrix elements by $(\Omega^B_\beta)^{\eta,\zeta}_{i,j}$. We can then write (cf. (2.10))

$$(\Omega^B_\beta)^{\eta,\zeta}_{i,j}(f,g) = f(x_\eta b x_\eta^{-1}) 1_{N_b}(x_\eta^{-1} g x_\zeta)\beta_{ij}(x_\eta^{-1} g x_\zeta) \quad \forall (f,g) \in \mathscr{T} \simeq F(H/K)\tilde{\otimes}\mathbb{C}N. \tag{3.30}$$

The left action of such a matrix element on the condensate vector is given by (3.23):

$$\begin{aligned}
(\Omega^B_\beta)^{\eta,\zeta}_{i,j} \rightharpoonup |\phi\rangle &= \sum_h (\Omega^B_\beta)^{\eta,\zeta}_{i,j}\big(P_{\mathscr{T}}(P_h,e)\big) \otimes \Pi^A_\alpha(1,h)|\phi\rangle \\
&= \frac{1}{|K|}\sum_h \sum_{k\in K} (\Omega^B_\beta)^{\eta,\zeta}_{i,j}(P_{hk},e) \otimes \Pi^A_\alpha(1,h)|\phi\rangle \\
&= \sum_h \sum_k 1_{N_b}(x_\eta^{-1}x_\zeta)\beta_{i,j}(x_\eta^{-1}x_\zeta)\delta_{hk,\eta} \otimes \Pi^A_\alpha(1,h)|\phi\rangle \\
&= 1_{N_b}(x_\eta^{-1}x_\zeta)\beta_{i,j}(x_\eta^{-1}x_\zeta)\sum_k \Pi^A_\alpha(1,\eta k)|\phi\rangle.
\end{aligned}$$

We in (3.25) require that this be equal to

$$\begin{aligned}
(\Omega^B_\beta)^{\eta,\zeta}_{i,j}(1,e)\varepsilon \rightharpoonup |\phi\rangle &= 1_{N_b}(x_\eta^{-1}x_\zeta)\beta_{i,j}(x_\eta^{-1}x_\zeta)\sum_h\sum_k \delta_{e,hk} \otimes \Pi^A_\alpha(1,h)|\phi\rangle \\
&= 1_{N_b}(x_\eta^{-1}x_\zeta)\beta_{i,j}(x_\eta^{-1}x_\zeta)\sum_k \Pi^A_\alpha(1,k)|\phi\rangle,
\end{aligned}$$

which leads to (3.27).

For the second part, we expand $|\phi\rangle = \sum_{p,q}\omega_{pq}|a_p v_q\rangle$. The right action of a matrix element is given by (3.23):

$$\begin{aligned}
|\phi\rangle \leftharpoonup (\Omega^B_\beta)^{\eta,\zeta}_{i,j} &= \sum_h \sum_{p,q} \omega_{pq}\Pi^A_\alpha(P_h,e)|a_p,v_q\rangle \otimes (\Omega^B_\beta)^{\eta,\zeta}_{i,j}\big(P_{\mathscr{T}}(1,h)\big) \\
&= \sum_h \sum_{p,q} \omega_{pq}P_h(a_p)|a_p,v_q\rangle 1_{N_b}(x_\eta^{-1}hx_\zeta)\beta_{i,j}(x_\eta^{-1}hx_\zeta)1_N(h) \\
&= \sum_{p,q} 1_N(a_p)\omega_{pq}|a_p,v_q\rangle 1_{N_b}(x_\eta^{-1}a_p x_\zeta)\beta_{i,j}(x_\eta^{-1}a_p x_\zeta).
\end{aligned}$$

We demand in (3.26) that $\forall \eta,\zeta \in B$ this be equal to the expression obtained by acting on the right by the vacuum representation:

$$\begin{aligned}
(\Omega^B_\beta)^{\eta,\zeta}_{i,j}(1,e)|\phi\rangle \leftharpoonup \varepsilon &= 1_{N_b}(x_\eta^{-1}x_\zeta)\beta_{i,j}(x_\eta^{-1}x_\zeta)\sum_h \Pi^A_\alpha(P_h,e)|\phi\rangle\varepsilon(1,h)1_N(h) \\
&= 1_{N_b}(x_\eta^{-1}x_\zeta)\beta_{i,j}(x_\eta^{-1}x_\zeta)\sum_{p,q} 1_N(a_p)\omega_{pq}|a_p,v_q\rangle.
\end{aligned}$$

When $a_p \notin \operatorname{supp}(|\phi\rangle)$ or $a_p \notin N$ this relation is trivially satisfied. The demand then reduces to

$$a_p \notin N \vee \big(1_{N_b}(x_\eta^{-1}a_p x_\zeta)\beta_{i,j}(x_\eta^{-1}a_p x_\zeta) = 1_{N_b}(x_\eta^{-1}x_\zeta)\beta_{i,j}(x_\eta^{-1}x_\zeta)\forall \eta,\zeta\big) \qquad \forall a_p \in \operatorname{supp}(|\phi\rangle). \tag{3.31}$$

When $\eta = \zeta$ this further reduces to

$$a_p \notin N \vee 1_{N_b}(x_\eta^{-1}a_p x_\eta)\beta_{i,j}(x_\eta^{-1}a_p x_\eta) = \delta_{i,j} \quad \forall a_p \in \operatorname{supp}(|\phi\rangle). \tag{3.32}$$

If this holds, then (3.31) will hold for all η,ζ because then

$$\begin{aligned}\delta_{i,j} &= 1_{N_b}(x_\eta^{-1}a_px_\eta)\beta_{i,j}(x_\eta^{-1}a_px_\eta)\\ &= 1_{N_b}(x_\eta^{-1}a_px_\eta)1_{N_b}(x_\eta^{-1}x_\zeta)\beta_{i,j}(x_\eta^{-1}a_px_\eta)\beta_{i,j}(x_\eta^{-1}x_\zeta)\beta_{i,j}(x_\zeta^{-1}x_\eta)\\ &= 1_{N_b}(x_\eta^{-1}a_px_\eta)\beta_{i,j}(x_\eta^{-1}a_px_\zeta)\,\cdot\,1_{N_b}(x_\eta^{-1}x_\zeta)\beta_{i,j}(x_\zeta^{-1}x_\eta).\end{aligned}$$

If $a_p \in N$ then $x_\eta^{-1}a_px_\eta \in N$, because by definition $x_\eta \in N$. Now we will show that $x_\eta^{-1}a_px_\eta \in K$, and that the action of $N\cap K$ on H/K is trivial so that $x_\eta^{-1}a_px_\eta \in N_b$, in which case (3.32) reduces to $a_p \notin N \vee \beta(x_\eta a_p x_\eta^{-1}) = 1 \quad \forall a_p \in \text{ supp}(|\phi\rangle)$.

The orthogonal projection (3.29) of (P_e,e) is proportional to $\sum_{k\in K}(P_k,e)$. This is an element of $\mathscr{T}$, so we must have

$$\sum_k (P_k,e) \rightharpoonup |\phi\rangle = \varepsilon\big(\sum_k (P_k,e)\big)|\phi\rangle = \sum_k \delta_{e,k}|\phi\rangle = |\phi\rangle.$$

When we expand $|\phi\rangle$, this gives the relation

$$\sum_k\sum_{p,q}\omega_{pq}P_k(a_p)|a_pv_q\rangle = \sum_{p,q}\omega_{pq}|a_pv_q\rangle,$$

which is only true if all $a_p \in \text{ supp}(|\phi\rangle)$ are elements of K.

We have seen that $x_\eta^{-1}a_px_\eta \in N\cap K \quad \forall a_p \in \text{ supp}(|\phi\rangle)$. We will now show that $N\cap K \subset N_b \subset N$. Take an element $n \in N\cap K$. Its action on $hK \in H/K$ is given by conjugation: $nhKn^{-1}$. But since $n\in K$ we have $Kn^{-1} = K$. Furthermore $nh = hh^{-1}nh$, and $h^{-1}nh \in K$, because K is a normal subgroup of H. All this gives $nhKn^{-1} = hK$. The action of $N\cap K$ on H/K is trivial, b is an element of H/K, so $N\cap K\subset N_b$.

We find that (3.32) reduces to (3.28), which concludes the proof. □

It can be shown that the set $D\big(N_{|\phi\rangle}/(K\cap N_{|\phi\rangle})\big)$ always satisfies the relations of the proposition above [7, prop. 9]. However, we will find examples of condensates in which there are more solutions, even when the residual algebra $\mathscr{T}$ is of the form $F(H/K)\,\tilde{\otimes}\,\mathbb{C}N_{|\phi\rangle}$. But when we demand that the unconfined algebra $\mathscr{U}$ be braided, this quantum double does give the correct result.

3.3.4 Domain walls

We now return to the issue of the confined particles. When such a particle travels through the condensate, due to topological interactions it will transform condensate vectors. We can choose a reference particle in the condensate, and then determine the states of the other particles relatvive to this one. As the confined particle continues it path, condensate particles 'on the other side' get transformed, and this costs a finite amount of energy.

The confined particle is said to draw a string, which can be interpreted as a domain wall, on one side of which background particles are in a different, not symmetry-related, state than the one of those on the other side. This domain wall cannot be removed by symmetry transformations, and unconfined particles braiding trivially with the background particles on one side, might not do so on the other side.

We would like to classify the domain walls, preferably in such a way, that the residual symmetry algebra $\mathscr{T}$ is built up from two parts: the unconfined algebra $\mathscr{U}$, and a part describing the domain walls. The concept of a coset of a centralizer in group theory comes to mind: the particles would consist of a part that braids trivially with the condensate, affording a closed multiplication, which induces an equivalence relation. When divided out, we are left with a set of representatives classifying the domain walls.

It turns out that there is such a structure, but it does not satisfy all of our needs, and lacks some features we might expect it to have at first.

Hopf kernels The map $\Gamma : \mathscr{T} \to \mathscr{U}$ is a morphism, and a natural question to ask is what its kernel is. The kernel is usually the set of elements that get sent to 0, and then two elements of $\mathscr{T}$ which are sent to the same element of $\mathscr{U}$ are additively related by an element of the kernel. But for Hopf algebras, this definition is not appropriate, as the coalgebra structure is ignored. Instead we have left and right *Hopf kernels*, defined by

$$\mathrm{LKer}(\Gamma) = \{t \in \mathscr{T} | (\Gamma \otimes \mathrm{id}) \circ \Delta(t) = 1_{\mathscr{U}} \otimes t\}, \tag{3.33}$$

$$\mathrm{RKer}(\Gamma) = \{t \in \mathscr{T} | (\mathrm{id} \otimes \Gamma) \circ \Delta(t) = t \otimes 1_{\mathscr{U}}\}. \tag{3.34}$$

These constructions possess an algebra structure†, and are related by $S\big(\mathrm{LKer}(\Gamma)\big) = \mathrm{RKer}(\Gamma)$ and vice versa‡, where S is the antipode of $\mathscr{T}$, which applies as the left and right kernel are subsets of $\mathscr{T}$.

Hopf kernel classifies domain walls The conjecture is that the domain walls in a $|\phi\rangle$-condensate are classified by either $\mathrm{LKer}(\Gamma)$ or $\mathrm{RKer}(\Gamma)$ (conjecture 1 in [7, §7.2]). The particular choice is not important, as the kernels are related by the antipode, and when these algebras are semisimple, this relation is an isomorphism. We will now only refer to $\mathrm{LKer}(\Gamma)$.

A desirable property of these Hopf kernels is that when ρ denotes a confined particle and σ an unconfined particle, the restriction to a Hopf kernel of the tensor product $(\rho \otimes \sigma) \circ \Delta_{\mathscr{T}}$ decomposes into a number of copies of the restriction of ρ to the Hopf kernel. In other words, a wall is defined up to fusion with an unconfined particle.

Furthermore, it turns out that, as an algebra, $\mathscr{T}$ is isomorphic to a so-called *crossed product* of $\mathrm{LKer}(\Gamma)$ and $\mathscr{U}$. The crossed product is one of many constructions devised to create algebras out of known other algebras¶. This cross product is described in [7, §7.2], a general treatment can be found in [30]. We shall not repeat it here, as it would add little to the understanding of the material at hand. However, this may be an interesting subject for future research; more understanding of the details of the construction may lead to a proof of the above stated conjecture.

Domain walls for $F(H/K) \tilde{\otimes} \mathbb{C}N$ When $\mathscr{T}$ is of the form $F(H/K) \tilde{\otimes} \mathbb{C}N$ and the algebra of unconfined particles is isomorphic to $D(N/N \cap K)$, the left and right Hopf kernels also acquire a compact form. Define $\overline{N}$ as the subgroup of H/K consisting of cosets nK of elements of N; this subgroup is isomorphic to $N/N \cap K$. Then

$$\mathrm{LKer}(\Gamma) \simeq F(\overline{N} \backslash (H/K)) \otimes \mathbb{C}(N \cap K), \tag{3.35}$$

$$\mathrm{RKer}(\Gamma) \simeq F((H/K)/\overline{N}) \otimes \mathbb{C}(N \cap K). \tag{3.36}$$

Here the symbol $\backslash$ denotes a right coset. This is shown in corollary 3 in [7, §7.3].

†Moreover they are right resp. left *coideal subalgebras*: they are subalgebras of $\mathscr{T}$ and $\Delta\big(\mathrm{LKer}(\Gamma)\big) \subset \mathscr{T} \otimes \mathrm{LKer}(\Gamma)$, $\Delta\big(\mathrm{RKer}(\Gamma)\big) \subset \mathrm{RKer}(\Gamma) \otimes \mathscr{T}$.

‡For semisimple algebras, the "vice versa"-statement is trivial by $S^2 = \mathrm{id}$ (lemma A.3)

¶For an overview of such constructions for Hopf algebras, see [2, §2]

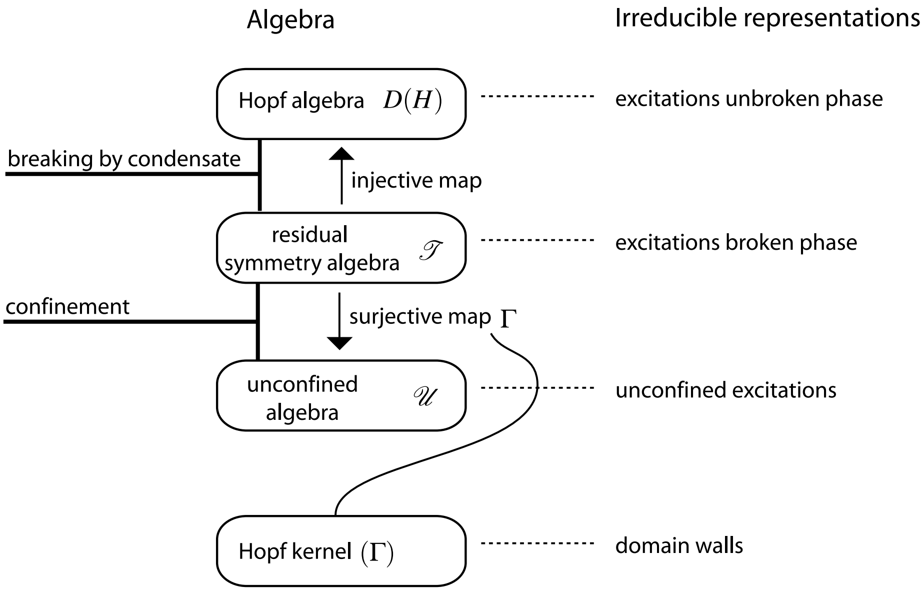


Figure 3.1: Overview of Hopf symmetry breaking

3.4 Summary

In this chapter we have seen how to describe symmetry breaking in a system featuring Hopf symmetry. To a large extent, this can be carried out for any Hopf algebra $\mathscr{A}$.

First, when a condensate in a certain state $|\phi\rangle$ is formed, the Hopf algebra $\mathscr{A}$ is broken down to a sub-Hopf algebra $\mathscr{T}$ of elements that leave the condensate invariant in the sense that their action on condensate particles is the same as on 'vacuum particles' ε. The particles of the original Hopf algebra decompose into irreducible representations of the residual symmetry algebra.

Next, making use of the braiding prescription of the original Hopf algebra, we are able to determine which of the possible excitations in the condensate braid trivially with condensate particles, and will therefore be unconfined. These unconfined particles are representations of yet another Hopf algebra $\mathscr{U}$, which is a Hopf quotient of the residual symmetry algebra $\mathscr{T}$ through the surjective Hopf map Γ. The unconfined algebra $\mathscr{U}$ is again (at least in most cases) quasi-triangular, so that there is a prescription for braiding of unconfined particles.

Furthermore, we can define the Hopf analogue of a kernel of this map, the left or right Hopf kernel of Γ, a subalgebra of $\mathscr{T}$. The excitations in $\mathscr{T}$ then decompose into irreducible representations of this Hopf kernel, in such a way that these representations label the inequivalent domain walls that may form in the condensate: the process of an unconfined particle fusing with such a wall does not alter that wall, and all unconfined excitations decompose into the 'trivial' domain wall.

This is all pictured schematically in figure 3.1.

For the case of the quantum double of a finite group, explicit formulae where found for a large class of condensates.

Finite group G	Hopf algebra $\mathscr{A}$
Group multiplication gg'	Algebra multiplication aa'
Irreducible representations	
$\pi : G \to GL(V)$	$\Pi : \mathscr{A} \to GL(V)$
Every representation of a finite group is completely reducible	$\mathscr{A}$ semisimple $\Rightarrow$ all representations are completely reducible; $D(H)$ is semisimple when $\mathbb{C}H$, $\mathbb{C}H^*$ are semisimple
Tensor product representation	
$(\pi \otimes \pi')(g) = \pi(g) \otimes \pi'(g)$	$(\Pi \otimes \Pi')(a) = (\Pi \otimes \Pi') \circ \Delta(a)$
Anti-particle	
$\overline{\pi}(g) = \pi^t(g^{-1})$	$\overline{\Pi} = \Pi^t \circ S$
Symmetry breaking	
Residual symmetry group $\{g \in G \vert g \rightharpoonup \vert\phi\rangle = \vert\phi\rangle\}$	Residual symmetry algebra $\{a \in \mathscr{A} \vert a \rightharpoonup \vert\phi\rangle = \varepsilon(a)\vert\phi\rangle\}$

Table 3.1: Comparison of group and Hopf algebra concepts

Comparison to group symmetry It may be enlightening to compare the familiar concepts of group theory, applied to symmetries in physics, to the quantum double symmetries we have developed here. In both cases elements of the group or algebra represent symmetry transformations, which for our purposes include projections P_h, being the result of a sequence of symmetry transformations, see §2.1.1. These transformations affect particles by action on the particle state by the representation value of such elements.

Symmetry breaking by some mechanism leaves the groundstate invariant under a subgroup H of the original symmetry group G. The invariance demand is $g \rightharpoonup |\phi\rangle = |\phi\rangle$. In Hopf symmetry we define invariance as "acting in the same way as on the vacuum", that is $a \rightharpoonup |\phi\rangle = \varepsilon(a)|\phi\rangle$.

An overview of corresponding concepts is listed in table 3.1.

The other feautures of the Hopf algebra symmetry breaking scheme do not translate as nicely. For instance confinement is a consequence of the braid properties of the excitations in the condensate, which has no equivalent in ordinary group symmetry breaking.

Chapter 4

Quantum double symmetry of even dihedral groups

We have now built up a giant toolbox which allows us to calculate the particle 'spectrum' (the spectrum of *internal* quantum numbers) of a system featuring quantum double symmetry, as well as all possible condensates with their unconfined particles and domain walls. We can now turn to actually performing those calculations, which constitute an extensive part of the work done for this thesis.

Finite subgroups of $SO(3)$ In chapter 1, we discussed how we could get quantum double symmetries when a continuous symmetry group was spontaneouly broken down to a finite subgroup. A logical choice for such a group to look at would be the group of rotations in three dimensions $SO(3)$, as many systems possess such symmetry. However, $SO(3)$ is not simply connected, which leads to some subtleties in the arguments concerning homotopy theory. These problems can be evaded by proceeding to the *universal covering group*, which is always simply connected (see [33, §5],[52, §1.4.1],[31, §2.3.1]). For example, the universal covering group of $SO(3)$ is $SU(2)$.

Still, the finite subgroups of $SO(3)$ are interesting examples to look at, not in the least because they comprise some of the simplest groups. The finite subgroups of $SO(3)$ are the cyclic groups of order n, $\mathbb{Z}_n$, the dihedral groups of order $2n$, D_n, the tetrahedral group T of order 24, the octahedral group O of order 48, and the icosahedral group I of order 60.

The cases of $D(H)$ with H Abelian, and of $D(D_n)$ for n odd were studied in [7]. Some cases of T, O and I were studied in [5]. We will now treat the dihedral groups with n even.

4.1 The even dihedral groups

The group $D_n \simeq \mathbb{Z}_n \rtimes \mathbb{Z}_2$ is given by the generators r en s as follows:

$$D_n = \{s^m r^k \mid s^2 = r^n = 1,\ sr^k = r^{n-k}s\} \quad m = 0,1;\ k = 0,\dots,n-1 \tag{4.1}$$

Note that $r^{n-k} = (r^k)^{-1} \equiv r^{-k}$ and $sr^k = r^{n-k}s = (sr^k)^{-1}$. This group has $2n$ elements and $e = s^0 r^0$. The main difference between the odd and even dihedral groups comes from the fact that $r^{n/2} = r^{-n/2}$ for n even, so that this is a central element.

We will take n to be even from now on. The structure of D_n, n odd, is given in the appendix A.4.1.

The $\frac{n}{2}+3$ conjugacy classes of D_n are

$$\begin{aligned}&[e]=\{e\}; \quad [r^{n/2}]=\{r^{n/2}\}; \quad [r^k]=\{r^k,r^{-k}\};\\ &[s]=\{s,sr^2,\ldots,sr^{n-2}\}; \quad [sr]=\{sr,sr^3,\ldots,sr^{n-1}\}.\end{aligned} \tag{4.2}$$

The centralizers of the elements of D_n are

$$\begin{aligned}N_e &= D_n,\\ N_{r^{n/2}} &= D_n,\\ N_{r^k} &= \{r^j \mid j=0,\ldots,n-1\} \simeq \mathbb{Z}_n,\\ N_{sr^{2i}} &= \{e,r^{n/2},sr^{2i},sr^{2i+\frac{n}{2}}\} \simeq D_2 \qquad && i=0,\ldots,\frac{n}{2}-1,\\ N_{sr^{2i+1}} &= \{e,r^{n/2},sr^{2i+1},sr^{2i+1+\frac{n}{2}}\} \simeq D_2 \qquad && i=0,\ldots,\frac{n}{2}-1.\end{aligned} \tag{4.3}$$

The fact that the last two centralizers are isomorphic to D_2 can be seen from the multiplication tables

N_{sr^k}	e	$r^{n/2}$	sr^k	$sr^{k+n/2}$
e	e	$r^{n/2}$	sr^k	$sr^{k+n/2}$
$r^{n/2}$	$r^{n/2}$	e	$sr^{k+n/2}$	sr^k
sr^k	sr^k	$sr^{k+n/2}$	e	$r^{n/2}$
$sr^{k+n/2}$	$sr^{k+n/2}$	sr^k	$r^{n/2}$	e

D_2	e	r	s	sr
e	e	r	s	sr
r	r	e	sr	s
s	s	sr	e	r
sr	sr	s	r	e

The centralizer of an element in a conjugacy class is isomorphic to those of every element in that class (lemma A.7). We can therefore define the centralizer of a class by the centralizer of the distinguished element.

4.1.1 Irreducible representations of D_n

The even dihedral group D_n has four one-dimensional representations

$$\begin{aligned}J_0(d\in D_n)&=1,\\ J_1(e,r^k)&=1,\ J_1(sr^k)=-1,\\ J_2(e)=1,\ J_2(r^k)&=(-1)^k,\ J_2(sr^k)=(-1)^k,\\ J_3(e)=1,\ J_3(r^k)&=(-1)^k,\ J_3(sr^k)=-(-1)^k.\end{aligned} \tag{4.4}$$

The other $\frac{n}{2}-1$ irreducible representations α_j are two-dimensional, and when we denote the first root of unity by $q\equiv \mathrm{e}^{\imath\frac{2\pi}{n}}$, they are given by†

$$\alpha_j(r^k)=\begin{pmatrix} q^{jk} & 0\\ 0 & q^{-jk}\end{pmatrix}, \quad \alpha_j(sr^k)=\begin{pmatrix} 0 & q^{-jk}\\ q^{jk} & 0\end{pmatrix}. \tag{4.5}$$

The character table of D_n is given by

†We have here chosen a particular basis, actually the one in which the matrices take the simplest form. We will always work in this basis without further comment. Also note that these are the complex representations. It is also possible to give real representations, as was done for example in [7, §5.1]. But as we will always have Hilbert spaces as representation spaces, there is no need to restrict ourselves in this manner.

D_n	$[e]$	$[r^{n/2}]$	$[r^k]$	$[s]$	$[sr]$
J_0	1	1	1	1	1
J_1	1	1	1	-1	-1
J_2	1	$(-1)^{n/2}$	$(-1)^k$	1	-1
J_3	1	$(-1)^{n/2}$	$(-1)^k$	-1	1
α_j	2	$2(-1)^j$	$q^{jk}+q^{-jk}$	0	0

We will also need the irreducible representations of $\mathbb{Z}_n$, as some centralizers are of that form. The group is given by one generator r and one relation $r^n = e$. The n irreducible representations β_j are defined by

$$\beta_j(r^k) = q^{jk}. \tag{4.6}$$

4.1.2 Irreducible representations of $D(D_n)$

Now we turn to the quantum double of D_n. It is a $4n^2$-dimensional Hopf algebra, and its irreducible representations are labelled by conjugacy classes of D_n, and irreducible representations of the centralizers of those classes (see §2.1.3).

We will label the irreducible representations of the centralizers as follows:

$$\begin{aligned} N_{[e]}, N_{[r^{n/2}]} &\simeq D_n \text{ by } J_i, \alpha_j, \\ N_{[r^k]} &\simeq \mathbb{Z}_n \text{ by } \beta_l, \\ N_{[s]}, N_{[sr]} &\simeq D_2 \text{ by } J_i. \end{aligned} \tag{4.7}$$

Note that although we have given them the same labels, the representations J_i are not isomorphic: for $[e]$ and $[r^{n/2}]$ they are the one-dimensional representations of D_n, while the D_2-representations of $N_{[s]}$ and $N_{[sr]}$ act on $\{e, r^{n/2}, s, sr^{n/2}\}$ and $\{e, r^{n/2}, ss, sr^{n/2+1}\}$ respectively. For example $J_2(r^{n/2}) = 1$ for $[e]$ and $[r^{n/2}]$ when $\frac{n}{2}$ is even, whereas $J_1(r^{n/2}) = -1$ for $[s]$ and $[sr]$.

There are $\frac{1}{2}n^2 + 14$ irreducible representations. They are

$$\left.\begin{array}{l} \Pi^e_{J_i}\ \Pi^e_{\alpha_j}\ \Pi^{r^{n/2}}_{J_i}\ \Pi^{r^{n/2}}_{\alpha_j} \\ \Pi^{r^k}_{\beta_l}\ \Pi^s_{J_i}\ \Pi^{sr}_{J_i} \end{array}\right\} \begin{array}{l} i = 0,1,2,3;\ j = 0,\dots,\frac{n}{2}-1; \\ k = 0,\dots,\frac{n}{2}-1;\ l = 0,\dots,n-1. \end{array} \tag{4.8}$$

The dimension d^A_α of an irreducible representation Π^A_α of $D(H)$ is given in (2.8) by the product of the number of elements in A and the dimension of α.

Recall from §2.5 that the spin s^A_α of a particle Π^A_α is determined by the representation value on the ribbon element $c = \sum_h (P_h, h)$, which amounts to $\Pi^A_\alpha(c) = \alpha(a)$, where a is the distinguished element of A. This value is a scalar times the unit matrix, because c is central in $D(H)$, and we identified this value, a root of unity because Π^A_α is unitary, with the spin of the particle.

The dimensions and spin values of the irreducible representations of $D(H)$ are listed in table 4.1.

Symmetry breaking Now we proceed to examine the possible condensates that can form in $D(D_n)$-theories, together with a treatment of the unconfined particles and domain walls.

$D(D_n)$	$\Pi^e_{J_i}$	$\Pi^e_{\alpha_j}$	$\Pi^{r^{n/2}}_{J_{0,1}}$	$\Pi^{r^{n/2}}_{J_{2,3}}$	$\Pi^{r^{n/2}}_{\alpha_j}$	$\Pi^{r^k}_{\beta_l}$	$\Pi^s_{J_{0,2}}$	$\Pi^s_{J_{1,3}}$	$\Pi^{sr}_{J_{0,3}}$	$\Pi^{sr}_{J_{1,2}}$
d^A_α	1	2	1	1	2	2	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{n}{2}$
s^A_α	1	1	1	$(-1)^{n/2}$	$(-1)^j$	q^{-lk}	1	-1	1	-1

Table 4.1: Dimensions and spin of irreducible $D(D_n)$-representations

We will treat consecutively condensates in the electric, the magnetic and the dyonic sector (see p.17). We start each section by exploring a possible general structure of these condensates. When this fails, the condensates are calculated by hand, that is by checking condition (3.8) for all irreducible representations of $D(D_n)^*$, and conditions (3.25) and (3.26) for irreducible representations of $\mathscr{T}$.

We will be able to calculate most condensates for all n, but for the dyonic condensates $|\phi\rangle \in \Pi^{r^{n/2}}_{J_i}$, $i = 1,2,3$, this is not as easy. However, it turns out to be a very interesting case, which we treat extensively in chapter 5 for $\Pi^{r^2}_{J_1}$ in a $D(D_4)$-theory.

4.2 Electric condensates

By electric condensates we mean condensates of states $|\phi\rangle$ in the representation space of a representation carrying trivial flux, so when $A = [e]$. Because these particles do not cause topological interactions on other particles entering the condensate, we expect that the theory should reduce to the regular breaking of the gauge group; that is, the magnetic part $F(H)$ of the symmetry algebra $D(H)$ should not be affected.

Trivial self-braiding and spin Particles Π^e_α have trivial self-braiding:

$$\begin{aligned}
\tau\circ(\Pi^e_\alpha\otimes\Pi^e_\alpha)(R)(|\phi\rangle\otimes|\phi\rangle) &= \tau\circ(\Pi^e_\alpha\otimes\Pi^e_\alpha)\big(\sum_h (P_h,e)\otimes(1,h)(|\phi\rangle\otimes|\phi\rangle)\big)\\
&= \tau\circ\sum_h \Pi^e_\alpha(P_h,e)\otimes\Pi^e_\alpha(1,h)(|\phi\rangle\otimes|\phi\rangle)\\
&= \tau\circ\sum_h \delta_{e,h}\otimes\Pi^e_\alpha(1,h)(|\phi\rangle\otimes|\phi\rangle)\\
&= \tau\circ 1\otimes\Pi^e_\alpha(1,e)(|\phi\rangle\otimes|\phi\rangle)\\
&= \tau(|\phi\rangle\otimes|\phi\rangle) = |\phi\rangle\otimes|\phi\rangle
\end{aligned}\tag{4.9}$$

Furthermore, all electric condensates have trivial spin factor ($\exp(\imath 2\pi s)$ with $s = 1$), as $\alpha(e) = 1$ because it is a group representation. Therefore all irreducible representations are allowed candidates for condensates.

Residual symmetry algebra Choose a vector $|\phi\rangle \in V^A_\alpha$. We determine $\mathscr{T}_{|\phi\rangle}$ by proposition 3.2. The irreducible representations of $D(D_n)^*$ are just the tensor product of the irreducible representations of $\mathbb{C}D_n$ and those of $F(D_n)$, which we label by $\rho_i\otimes E_g$ (cf. (3.5)) or (ρ_i, E_g). They must satisfy (3.8), in this case

$$\Pi^e_\alpha(1,g^{-1})|\phi\rangle = \frac{\chi_{\rho_i}(e)}{d_{\rho_i}}|\phi\rangle \tag{4.10}$$

Now $\chi_{\rho_i}(e) = d_{\rho_i}\quad \forall i$, because it is a group character. There is no restriction on ρ_i, so the magnetic part of $D(D_n)$ is unbroken.

Since the centralizer of e is all of D_n, $\Pi^e_\alpha(1,g)$ is just the group action of g on $|\phi\rangle$. We see that this equation is satisfied for all g for which $\alpha(g^{-1}) = 1$. These elements constitute a subgroup of D_n, which is called the *stabilizer* $N_{|\phi\rangle}$ of $|\phi\rangle$.

Therefore we find

$$\mathscr{T}_{|\phi\rangle} \simeq F(D_n)\tilde{\otimes}\mathbb{C}N_{|\phi\rangle}. \tag{4.11}$$

We have the following choices for α: J_1, J_2, J_3, α_j. For the one-dimensional representations J_i, we have no choice (up to a scalar factor) for the state $|\phi\rangle$, and the stabilizer is just the set of elements g for which $J_i(g) = 1$. From the defintions (4.4), we see that $N_{J_1} = \langle r\rangle \simeq \mathbb{Z}_n$, the subgroup generated by r; $N_{J_2} = \langle r^2\rangle \cup s\langle r^2\rangle \simeq D_{n/2}$; $N_{J_3} = \langle r^2\rangle \cup sr\langle r^2\rangle \simeq D_{n/2}$.

For α_j, the stabilizer may depend upon $|\phi\rangle$. We write out the action of the elements r^k.

$$\alpha_j(r^k)\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix} = \begin{pmatrix} q^{jk} & 0\\ 0 & q^{-jk}\end{pmatrix}\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix} = \begin{pmatrix} q^{jk}\lambda_1\\ q^{-jk}\lambda_2\end{pmatrix}. \tag{4.12}$$

This can only be satisfied by elements r^k for which $q^{jk} = 1 \wedge q^{-jk} = 1$, which is equivalent to the demand $q^{jk} = 1 \Rightarrow jk = 0 \bmod n$. The smallest k to satisfy this demand is $k = \frac{n}{\gcd(n,j)}$, where gcd denotes the greatest common divisor of two integers. All multiples of this k will also satisfy the demand, so we have a group $\{e, r^x, r^{2x}, \ldots, r^{\gcd(n,j)}\} \simeq \mathbb{Z}_{\gcd(n,j)}$, where $x \equiv \frac{n}{\gcd(n,j)}$.

This result does not depend upon the particular state $|\phi\rangle$. But let's now look at the representation values of the elements sr^k:

$$\alpha_j(sr^k)\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix} = \begin{pmatrix} 0 & q^{-jk}\\ q^{jk} & 0\end{pmatrix}\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix} = \begin{pmatrix} q^{-jk}\lambda_2\\ q^{jk}\lambda_1\end{pmatrix}. \tag{4.13}$$

We already demanded that $q^{jk} = q^{-jk} = 1$, but the vector can only be invariant under this action when $\lambda_1 = \lambda_2$. If this is the case, then all elements of the form sr^{mx}, with m an integer and x as above, are also in the stabilizer of $|\phi\rangle$, which is then isomorphic to $D_{\gcd(n,j)}$.

Thus we see that the residual algebra $\mathscr{T}_{|\phi\rangle}$ is either of the form $F(D_n)\tilde{\otimes}\mathbb{C}\mathbb{Z}_{\gcd(n,j)}$ or of the form $F(D_n)\tilde{\otimes}\mathbb{C}D_{\gcd(n,j)}$, the latter case arises when $|\phi\rangle$ is of the form $(\lambda\ \lambda)^T$ for some $\lambda \in \mathbb{C}$.

Particles in the condensates The residual symmetry algebra $\mathscr{T}_{|\phi\rangle}$ is of the form $F(D_n/K)\tilde{\otimes}\mathbb{C}N$, with $K = \{e\}$ and $N = N_{|\phi\rangle}$. Then the irreducible representations of $F(D_n)\tilde{\otimes}\mathbb{C}N_{|\phi\rangle}$ are labelled by orbits B of D_n of the N-action and irreducible representations β of the stabilizer of a distinguished element of that orbit, where the stabilizer should now be a subgroup of $N_{|\phi\rangle}$.

We denote these particles by Ω^B_β.

Confinement for electric condensates We examine which of these representations Ω^B_β are not confined. Such particles should satisfy the conditions of proposition 3.5. Because the distinguished element of Π^e_α is e, (3.28) is always satisfied. Because the magnetic part is unbroken $K = \{e\}$, and (3.27) reduces to

$$\Pi^A_\alpha(1,\eta)|\phi\rangle = |\phi\rangle \quad \forall \eta \in B. \tag{4.14}$$

This amounts to $\alpha(\eta) = 1 \quad \forall\eta \in B$. This holds when all elements of B are contained in the stabilizer $N_{|\phi\rangle}$, so when B is a subgroup of $N_{|\phi\rangle}$. In §3.3.3 we mentioned that all elements of the algebra $D\big(N_{|\phi\rangle}/(K \cap N_{|\phi\rangle})\big)$ satisfy said conditions, and we showed here that there are no others.

We have found

$$\mathscr{U}_{|\phi\rangle} \simeq D(N_{|\phi\rangle}). \tag{4.15}$$

The representations of $\mathscr{T}_{|\phi\rangle}$ for which the orbit B is contained in the stabilizer $N_{|\phi\rangle}$ are unconfined, and are the irreducible representations of $\mathscr{U}_{|\phi\rangle}$ as well; the other representations are confined.

Domain walls The confined particles give rise to domain walls as in §3.3.4. They are given by the irreducible representations of the left or right Hopf kernel of the map $\Gamma : \mathscr{T} \to \mathscr{U}$. From (3.36) we see that the right Hopf kernel is isomorphic to

$$\mathrm{RKer}(\Gamma) \simeq F((H/K)/(N/N \cap K)) \otimes \mathbb{C}(N \cap K) = F(D_n/N_{|\phi\rangle}) \otimes \mathbb{C}e. \tag{4.16}$$

The walls are therefore classified by functions on the $N_{|\phi\rangle}$-cosets. The irreducible representations are labelled by these cosets, and given by

$$E_{hN_{|\phi\rangle}}(f,e) = f(h). \tag{4.17}$$

There are $|D_n|/|N_{|\phi\rangle}|$ distinguishable domain walls, which are all one-dimensional; this is obvious, as $F(D_n/N_{|\phi\rangle}) \otimes \mathbb{C}e$ is Abelian.

Each particle Ω^B_β restricts to a domain wall, and all unconfined particles restrict to the trivial element $h = e$. Let's calculate the representation values of Ω^B_β on the Hopf kernel $F(D_n/N_{|\phi\rangle}) \otimes \mathbb{C}e$:

$$\Omega^B_\beta(f,e)|b_p w_j\rangle = f(b_p)|b_p w_j\rangle. \tag{4.18}$$

Comparing this with (4.17), we see that each basis vector $|b_p w_j\rangle$ corresponds to one representation of the coset $b_p N_{|\phi\rangle}$. So for each $b_p \in B$, there are d_β copies of the representation $E_{b_p N_{|\phi\rangle}}$. Some of these cosets may coincide. We also see that when $B \subset N_{|\phi\rangle}$, this representation does indeed restrict only to copies of $E_{eN_{|\phi\rangle}}$, all other Ω^B_β are confined.

4.2.1 Particles in the J_1-condensate

For $|\phi\rangle \in V^e_{J_1}$, the stabilizer $N_{|\phi\rangle}$ is the subgroup of D_n generated by r. The residual symmetry algebra is $\mathscr{T}_{|\phi\rangle} = F(D_n)\tilde{\otimes}\mathbb{C}\mathbb{Z}_n$. The orbits of D_n by the action of $N_{|\phi\rangle}$ are

$$[r^k] = \{r^k\};\ [s] = \{s, sr^2, \ldots, sr^{n-2}\};\ [sr] = \{sr, sr^3, \ldots, sr^{n-1}\};\quad k = 0, \ldots, n-1. \tag{4.19}$$

The stabilizers in $N_{|\phi\rangle}$ of (distinguished elements) of these orbits are

$$N_{r^k} = \mathbb{Z}_n;\ N_s = \{e, r^{n/2}\} \simeq \mathbb{Z}_2;\ N_{sr} = \{e, r^{n/2}\}. \tag{4.20}$$

The particles in this condensate are then labelled by (dimensions within parentheses).

$$\Omega^{r^k}_{\beta_l}\ (1),\ \Omega^{s}_{\gamma_{0,1}}\ (\tfrac{n}{2}),\ \Omega^{sr}_{\gamma_{0,1}}\ (\tfrac{n}{2}).\quad k,l = 0, \ldots, n-1 \tag{4.21}$$

Here $\gamma_{0,1}$ denote the two irreducible $\mathbb{Z}_2$-representations.

We can determine the branching rules for the decomposition of irreducible $D(D_n)$-representations as $\mathcal{T}$-representation into irreducible $\mathcal{T}$-representation by using the characters (see §3.2.4). We then find

$$\Pi^e_{J_{0,1}} \simeq \Omega^e_{\beta_0} \qquad \Pi^e_{J_{2,3}} \simeq \Omega^e_{\beta_{n/2}} \qquad \Pi^e_{\alpha_j} \simeq \Omega^e_{\beta_j} \oplus \Omega^e_{\beta_{-j}} \tag{4.22}$$

$$\Pi^{r^{n/2}}_{J'_{0,1}} \simeq \Omega^{r^{n/2}}_{\beta_0} \qquad \Pi^{r^{n/2}}_{J'_{2,3}} \simeq \Omega^{r^{n/2}}_{\beta_{n/2}} \qquad \Pi^{r^{n/2}}_{\alpha'_j} \simeq \Omega^{r^{n/2}}_{\beta_j} \oplus \Omega^{r^{n/2}}_{\beta_{-j}} \tag{4.23}$$

$$\Pi^{r^k}_{\beta_l} \simeq \Omega^{r^k}_{\beta_l} \oplus \Omega^{r^{-k}}_{\beta_{-l}} \tag{4.24}$$

$$\Pi^s_{J_{0,2}} \simeq \Omega^s_{\gamma_0} \qquad \Pi^s_{J_{1,3}} \simeq \Omega^s_{\gamma_1} \tag{4.25}$$

$$\Pi^{sr}_{J_{0,2}} \simeq \Omega^{sr}_{\gamma_0} \qquad \Pi^{sr}_{J_{1,3}} \simeq \Omega^{sr}_{\gamma_1}. \tag{4.26}$$

We have included one calculation, the branching of $\Pi^{r^k}_{\beta_l}$ in the appendix §A.4.2, to show how this works. We will state only the results of such calculations throughout this chapter.

The particles Ω^B_β which are unconfined have an orbit B that lies within the stabilizer $N_{|\phi\rangle} = \langle r\rangle \simeq \mathbb{Z}_n$. The domain walls are characterized by the Hopf kernel algebra $F(\mathbb{Z}_2) \otimes \mathbb{C}e$, and its irreducible representations are labelled by the cosets of $N_{|\phi\rangle}$ in D_n, of which there are two, represented by e and s.

Then the particles $\Omega^{r^k}_{\beta_l}$ are unconfined, and are representations of $\mathcal{U}_{|\phi\rangle} \simeq D(\mathbb{Z}_n)$. They branch to E_e. The representations $\Omega^{s,sr}_{\gamma_{0,1}}$ are confined, and must necessarily branch to E_s.

4.2.2 Particles in the J_2-condensate

For $|\phi\rangle \in V^e_{J_2}$, the stabilizer $N_{|\phi\rangle}$ is the subgroup of D_n given by $\{s^q r^{2p} | q = 0,1;\ p = 0,\dots,\frac{n}{2}-1\} \simeq D_{n/2}$. The residual symmetry algebra is $\mathcal{T}_{|\phi\rangle} = F(D_n)\,\tilde{\otimes}\,\mathbb{C}D_{n/2}$. We now have to distinguish between the cases $n/2$ even or odd. If $n/2$ odd, the stabilizer is not a group of the form described in §4.1, but rather an odd dihedral group given in §A.4.1.

Case $n/2$ even The stabilizer $N_{|\phi\rangle}$ is isomorphic to an even dihedral group, and now the elements sr^k of D_n will split up into three orbits, because the action of $r^{2k} \in D_{n/2}$ on $s \in D_n$ now only reaches the elements $\{s, sr^4, \dots, sr^{-4}\}$, because $n \bmod 4 = 0$. From sr, however, the elements sr^{-1+4k} can be reached, by the action the elements sr^{2k}. We therefore have the orbits

$$\begin{gathered}[e] = \{e\};\ [r^{n/2}] = \{r^{n/2}\};\ [r^k] = \{r^k, r^{-k}\}; \quad k = 1,\dots,\frac{n}{2}-1.\\ [s] = \{s, sr^4, \dots, sr^{n-4}\};\ [sr^2] = \{sr^2, sr^6, \dots, sr^{n-2}\};\\ [sr] = \{sr, sr^3, \dots, sr^{n-1}\}.\end{gathered} \tag{4.27}$$

The stabilizers of these orbits in $N_{|\phi\rangle}$ are

$$\begin{gathered}N_e = D_{n/2};\ N_{r^{n/2}} = D_{n/2};\ N_{r^k} = \langle r^2\rangle \simeq \mathbb{Z}_{n/2};\\ N_{sr^{2k}} = \{e, r^{n/2}, sr^{2k}, sr^{2k+n/2}\} \simeq D_2;\ N_{sr} = \{e, r^{n/2}\} \simeq \mathbb{Z}_2.\end{gathered} \tag{4.28}$$

The particles in this condensate are then labelled by (dimensions within parentheses).

$$\begin{aligned}
&\Omega^{e}_{J_i}(1),\ \Omega^{e}_{\alpha_j}(2),\ \Omega^{r^{n/2}}_{J_i}(1),\ \Omega^{r^{n/2}}_{\alpha_j}(2), && i=0,1,2,3;\ j=1,\ldots,\tfrac{n}{4}-1,\\
&\Omega^{r^k}_{\beta_l}(2),\ \Omega^{s,sr^2}_{J_i},(\tfrac{n}{4})\ \Omega^{sr}_{\gamma_{0,1}}(\tfrac{n}{2}), && k=1,\ldots,\tfrac{n}{2}-1;\ l=0,\ldots,\tfrac{n}{2}-1. \qquad (4.29)
\end{aligned}$$

Now we determine the branching rules from $D(D_n)$ to $\mathscr{T}_{|\phi\rangle}$. In making the calculation, we have to be very careful to apply the right reprensentation. For example, $\Pi^{e}_{J_2}(r^2)=1$, but $\Omega^{e}_{J_2}(r^2)=-1$.

$$\Pi^{e}_{J_{0,2}} \simeq \Omega^{e}_{J_0} \qquad \Pi^{e}_{J_{1,3}} \simeq \Omega^{e}_{J_1} \qquad (4.30)$$

$$\Pi^{e}_{\alpha_j} \simeq \Omega^{e}_{\alpha_j},\ j<\tfrac{n}{4} \qquad \Pi^{e}_{\alpha_j} \simeq \Omega^{e}_{\alpha_{-j}},\ j>\tfrac{n}{4} \qquad \Pi^{e}_{\alpha_{n/4}} \simeq \Omega^{e}_{J_2}\oplus\Omega^{e}_{J_3} \qquad (4.31)$$

$$\Pi^{r^{n/2}}_{J_{0,2}} \simeq \Omega^{r^{n/2}}_{J_0} \qquad \Pi^{r^{n/2}}_{J_{1,3}} \simeq \Omega^{r^{n/2}}_{J_1} \qquad (4.32)$$

$$\Pi^{r^{n/2}}_{\alpha_j} \simeq \Omega^{r^{n/2}}_{\alpha_j},\ j<\tfrac{n}{4} \qquad \Pi^{r^{n/2}}_{\alpha_j} \simeq \Omega^{r^{n/2}}_{\alpha_{-j}},\ j>\tfrac{n}{4} \qquad \Pi^{r^{n/2}}_{\alpha_{n/4}} \simeq \Omega^{r^{n/2}}_{J_2}\oplus\Omega^{r^{n/2}}_{J_3} \qquad (4.33)$$

$$\Pi^{r^k}_{\beta_l} \simeq \Omega^{r^k}_{\beta_l},\ l<\tfrac{n}{2} \qquad \Pi^{r^k}_{\beta_l} \simeq \Omega^{r^k}_{\beta_{l-n/2}},\ l>=\tfrac{n}{2} \qquad (4.34)$$

$$\Pi^{s}_{J_i} \simeq \Omega^{s}_{J_i}\oplus\Omega^{sr^2}_{J_i} \qquad \Pi^{sr}_{J_{0,1}} \simeq \Omega^{sr}_{\gamma_0} \qquad \Pi^{sr}_{J_{2,3}} \simeq \Omega^{sr}_{\gamma_1} \qquad (4.35)$$

The unconfined algebra is isomorphic to $D(D_{n/2})$. The particles Ω^{B}_{β} which are unconfined have an orbit B that lies within the stabilizer $N_{|\phi\rangle} \simeq D(D_{n/2})$. The domain walls are characterized by the Hopf kernel algebra $F(\mathbb{Z}_2)\otimes\mathbb{C}e$, and its irreducible representations are labelled by the cosets of $N_{|\phi\rangle}$ in D_n, of which there are two, represented by e and r.

The particles which are unconfined are

$$\Omega^{e,r^{n/2}}_{J_i},\ \Omega^{e,r^{n/2}}_{\alpha_j},\ \Omega^{r^{2k}}_{\beta_l},\ \Omega^{s,sr^2}_{J_i} \quad i=0,1,2,3;\ j,k=1,\ldots,\tfrac{n}{4}-1;\ l=0,\ldots,\tfrac{n}{2}-1. \qquad (4.36)$$

One can check that the squares of the dimensions of these representations add up to $n^2=\dim D(D_{n/2})$. These representations restrict to the trivial wall E_e, all others are confined and restrict to E_r.

Case $n/2$ odd When $n/2$ is odd, the stabilizer $N_{|\phi\rangle}$ has the structure of an odd dihedral group, described in §A.4.1. In particular, there no longer is a non-trivial element $r^{n/4}$ in $N_{|\phi\rangle}$.

The action of the element $r^2 \in N_{|\phi\rangle}$ on $s\in D_n$ still yields sr^4, but because $n/2$ is odd, $r^{n/2+1}\in N_{|\phi\rangle}$, and $r^{-n/2-1}sr^{n/2+1}=sr^2$. Therefore the $N_{|\phi\rangle}$-orbits in D_n are

$$\begin{aligned}
&[e]=\{e\};\ [r^{n/2}]=\{r^{n/2}\};\ [r^k]=\{r^k,r^{-k}\}; \quad k=1,\ldots,\frac{n}{2}-1.\\
&[s]=\{s,sr^2,\ldots,sr^{n-2}\};\ [sr]=\{sr,sr^3,\ldots,sr^{n-1}\}. \qquad (4.37)
\end{aligned}$$

The stabilizers in $N_{|\phi\rangle}$ of these orbits are

$$\begin{aligned}
&N_e=D_{n/2};\ N_{r^{n/2}}=D_{n/2};\ N_{r^k}=\langle r^2\rangle\simeq\mathbb{Z}_{n/2};\\
&N_s=\{e,s\}\simeq\mathbb{Z}_2;\ N_{sr}=\{e,sr^{1+n/2}\}\simeq\mathbb{Z}_2. \qquad (4.38)
\end{aligned}$$

The particles in this condensate are then given by (dimensions within parentheses).

$$\begin{aligned}&\Omega^{e}_{J_i}(1),\ \Omega^{e}_{\alpha_j}(2),\ \Omega^{r^{n/2}}_{J_i}(1),\ \Omega^{r^{n/2}}_{\alpha_j}(2),\quad i=0,1;\ j=1,\dots,\tfrac{n/2-1}{2},\\ &\Omega^{r^k}_{\beta_l}(2),\ \Omega^{s,sr}_{\gamma_{0,1}}(\tfrac{n}{2}),\quad k=1,\dots,\tfrac{n}{2}-1;\ l=0,\dots,\tfrac{n}{2}-1.\end{aligned} \tag{4.39}$$

The branching rules from $D(D_n)$ to $\mathscr{T}_{|\phi\rangle}$ are given by

$$\begin{aligned}&\Pi^{e,r^{n/2}}_{J_{0,2}} \simeq \Omega^{e,r^{n/2}}_{J_0} && \Pi^{e,r^{n/2}}_{J_{1,3}} \simeq \Omega^{e,r^{n/2}}_{J_1}\\ &\Pi^{e,r^{n/2}}_{\alpha_j} \simeq \Omega^{e,r^{n/2}}_{\alpha_j},\quad j<\tfrac{n}{4} && \Pi^{e,r^{n/2}}_{\alpha_j} \simeq \Omega^{e,r^{n/2}}_{\alpha_{-j}},\quad j>\tfrac{n}{4}\\ &\Pi^{r^k}_{\beta_l} \simeq \Omega^{r^k}_{\beta_l},\quad l<\tfrac{n}{2} && \Pi^{r^k}_{\beta_l} \simeq \Omega^{r^k}_{\beta_{l-n/2}},\quad l>=\tfrac{n}{2}\\ &\Pi^{s}_{J_{0,2}} \simeq \Omega^{s}_{\gamma_0} && \Pi^{s}_{J_{1,3}} \simeq \Omega^{s}_{\gamma_1}\\ &\Pi^{sr}_{J_{0,3}} \simeq \Omega^{sr}_{\gamma_0} && \Pi^{sr}_{J_{1,2}} \simeq \Omega^{sr}_{\gamma_1}.\end{aligned} \tag{4.40}$$

Particles Ω^{B}_{β} of which the orbit B is not contained in the stabilizer $N_{|\phi\rangle}$ are confined. The unconfined algebra $\mathscr{U}$ is again isomorphic to $D(D_{n/2})$, and the Hopf kernel of Γ is $F(\mathbb{Z}_2)\otimes\mathbb{C}e$, with irreducible representations E_e and E_r. The unconfined particles are

$$\Omega^{e}_{J_{0,1}},\ \Omega^{e}_{\alpha_j},\ \Omega^{r^{2k}}_{\beta_l},\ \Omega^{s}_{\gamma_{0,1}}\quad j,k=1,\dots,\tfrac{n}{4}-\tfrac{1}{2};\ l=0,\dots,\tfrac{n}{2}-1. \tag{4.41}$$

4.2.3 Particles in the J_3-condensate

For $|\phi\rangle\in V^{e}_{J_3}$, the stabilizer $N_{|\phi\rangle}$ is the subgroup of D_n given by $\{(sr)^q r^{2p}|q=0,1;\ p=0,\dots,\tfrac{n}{2}-1\}\simeq D_{n/2}$. The residual symmetry algebra is $\mathscr{T}_{|\phi\rangle}=F(D_n)\tilde{\otimes}\mathbb{C}D_{n/2}$. The calculations are all very similar to those for $|\phi\rangle\in V^{e}_{J_2}$. We again have to distinguish between the cases $n/2$ even or odd.

Case $n/2$ even The stabilizer $N_{|\phi\rangle}$ is isomorphic to an even dihedral group; the orbits are now

$$\begin{aligned}&[e]=\{e\};\ [r^{n/2}]=\{r^{n/2}\};\ [r^k]=\{r^k,r^{-k}\};\quad k=1,\dots,\frac{n}{2}-1.\\ &[sr]=\{sr,sr^5,\dots,sr^{n-3}\};\ [sr^3]=\{sr^3,sr^7,\dots,sr^{n-1}\};\\ &[s]=\{s,sr^2,\dots,sr^{n-2}\}.\end{aligned} \tag{4.42}$$

The stabilizers of these orbits in $N_{|\phi\rangle}$ are

$$\begin{aligned}&N_e=D_{n/2};\ N_{r^{n/2}}=D_{n/2};\ N_{r^k}=\langle r^2\rangle\simeq\mathbb{Z}_{n/2};\\ &N_{sr^{2k+1}}=\{e,r^{n/2},sr^{2k+1},sr^{2k+1+n/2}\}\simeq D_2;\ N_s=\{e,r^{n/2}\}\simeq\mathbb{Z}_2.\end{aligned} \tag{4.43}$$

The particles in this condensate are then labelled by (dimensions within parentheses).

$$\begin{aligned}&\Omega^{e}_{J_i}(1),\ \Omega^{e}_{\alpha_j}(2),\ \Omega^{r^{n/2}}_{J_i}(1),\ \Omega^{r^{n/2}}_{\alpha_j}(2),\quad i=0,1,2,3;\ j=1,\dots,\tfrac{n}{4}-1,\\ &\Omega^{r^k}_{\beta_l}(2),\ \Omega^{sr,sr^3}_{J_i}(\tfrac{n}{4}),\ \Omega^{s}_{\gamma_{0,1}}(\tfrac{n}{2}),\quad k=1,\dots,\tfrac{n}{2}-1;\ l=0,\dots,\tfrac{n}{2}-1.\end{aligned} \tag{4.44}$$

The branching rules are similar to those for $|\phi\rangle \in V^e_{J_2}$. To see that the restriction of $\Pi^e_{\alpha_j}$, $\;j < \frac{n}{4}$ is equivalent to $\Omega^e_{\alpha_j}$, one should check that

$$\begin{pmatrix} 1 & \\ & q_n^{-j} \end{pmatrix} \begin{pmatrix} & q_n^{-j(2p+1)} \\ q_n^{j(2p+1)} & \end{pmatrix} \begin{pmatrix} 1 & \\ & q_n^{j} \end{pmatrix} = \begin{pmatrix} & q_n^{-2jp} \\ q_n^{2jp} & \end{pmatrix} = \begin{pmatrix} & q_{n/2}^{-jp} \\ q_{n/2}^{jp}, & \end{pmatrix} \tag{4.45}$$

where $q_n \equiv e^{\imath \frac{2\pi}{n}}$ and $q_{n/2} \equiv e^{\imath \frac{2\pi}{n/2}}$, so that the values on sr^{2p+1}, $\;p \in \mathbb{Z}$ of $\Pi^e_{\alpha_j}$ and $\Omega^e_{\alpha_j}$ are related by this similarity transformation (cf. §5.1.2).

$$\Pi^e_{J_{0,3}} \simeq \Omega^e_{J_0} \qquad \Pi^e_{J_{1,2}} \simeq \Omega^e_{J_1} \tag{4.46}$$

$$\Pi^e_{\alpha_j} \simeq \Omega^e_{\alpha_j},\ j < \tfrac{n}{4} \qquad \Pi^e_{\alpha_j} \simeq \Omega^e_{\alpha_{-j}},\ j > \tfrac{n}{4} \qquad \Pi^e_{\alpha_{n/4}} \simeq \Omega^e_{J_2} \oplus \Omega^e_{J_3} \tag{4.47}$$

$$\Pi^{r^{n/2}}_{J_{0,3}} \simeq \Omega^{r^{n/2}}_{J_0} \qquad \Pi^{r^{n/2}}_{J_{1,2}} \simeq \Omega^{r^{n/2}}_{J_1} \tag{4.48}$$

$$\Pi^{r^{n/2}}_{\alpha_j} \simeq \Omega^{r^{n/2}}_{\alpha_j}\ j < \tfrac{n}{4} \qquad \Pi^{r^{n/2}}_{\alpha_j} \simeq \Omega^{r^{n/2}}_{\alpha_{-j}},\ j > \tfrac{n}{4} \qquad \Pi^{r^{n/2}}_{\alpha_{n/4}} \simeq \Omega^{r^{n/2}}_{J_2} \oplus \Omega^{r^{n/2}}_{J_3} \tag{4.49}$$

$$\Pi^{r^k}_{\beta_l} \simeq \Omega^{r^k}_{\beta_l},\ l < \tfrac{n}{2} \qquad \Pi^{r^k}_{\beta_l} \simeq \Omega^{r^k}_{\beta_{l-n/2}},\ l >= \tfrac{n}{2} \tag{4.50}$$

$$\Pi^s_{J_{0,1}} \simeq \Omega^s_{\gamma_0} \qquad \Pi^s_{J_{2,3}} \simeq \Omega^s_{\gamma_1} \qquad \Pi^{sr}_{J_i} \simeq \Omega^{sr}_{J_i} \oplus \Omega^{sr^3}_{J_i} \tag{4.51}$$

The unconfined algebra is isomorphic to $D(D_{n/2})$. The domain walls are characterized by the Hopf kernel algebra $F(\mathbb{Z}_2) \otimes \mathbb{C}e$, and its irreducible representations are labelled by the cosets of $N_{|\phi\rangle}$ in D_n, of which there are two, represented by e and r.

The particles which are unconfined are

$$\Omega^{e,r^{n/2}}_{J_i},\ \Omega^{e,r^{n/2}}_{\alpha_j},\ \Omega^{r^{2k}}_{\beta_l},\ \Omega^{sr,sr^3}_{J_i} \qquad i = 0,1,2,3;\ j,k = 1,\ldots,\tfrac{n}{4}-1;\ l = 0,\ldots,\tfrac{n}{2}-1. \tag{4.52}$$

These representations restrict to the trivial wall E_e, all others are confined and restrict to E_r.

Case $n/2$ odd When $n/2$ is odd, the stabilizer $N_{|\phi\rangle}$ has the structure of an odd dihedral group, described in §A.4.1. In particular, there no longer is a non-trivial element $r^{n/4}$ in $N_{|\phi\rangle}$.

The action of the element $r^2 \in N_{|\phi\rangle}$ on $s \in D_n$ still yields sr^4, but because $n/2$ is odd, $r^{n/2+1} \in N_{|\phi\rangle}$, and $r^{-n/2-1}sr^{n/2+1} = sr^2$. Therefore the $N_{|\phi\rangle}$-orbits in D_n are

$$\begin{aligned} &[e] = \{e\};\ [r^{n/2}] = \{r^{n/2}\};\ [r^k] = \{r^k, r^{-k}\}; \quad k = 1,\ldots,\frac{n}{2}-1. \\ &[s] = \{s, sr^2,\ldots,sr^{n-2}\};\ [sr] = \{sr, sr^3,\ldots,sr^{n-1}\}. \end{aligned} \tag{4.53}$$

The stabilizers in $N_{|\phi\rangle}$ of these orbits are

$$\begin{aligned} &N_e = D_{n/2};\ N_{r^{n/2}} = D_{n/2};\ N_{r^k} = \langle r^2\rangle \simeq \mathbb{Z}_{n/2}; \\ &N_s = \{e, sr^{n/2}\} \simeq \mathbb{Z}_2;\ N_{sr} = \{e, sr\} \simeq \mathbb{Z}_2. \end{aligned} \tag{4.54}$$

The particles in this condensate are then given by (dimensions within parentheses).

$$\begin{aligned} &\Omega^e_{J_i}(1),\ \Omega^e_{\alpha_j}(2),\ \Omega^{r^{n/2}}_{J_i}(1),\ \Omega^{r^{n/2}}_{\alpha_j}(2), \quad i = 0,1;\ j = 1,\ldots,\tfrac{n/2-1}{2}, \\ &\Omega^{r^k}_{\beta_l}(2),\ \Omega^{s,sr}_{\gamma_{0,1}}(\tfrac{n}{2}), \quad k = 1,\ldots,\tfrac{n}{2}-1;\ l = 0,\ldots,\tfrac{n}{2}-1. \end{aligned} \tag{4.55}$$

The branching rules from $D(D_n)$ to $\mathscr{T}_{|\phi\rangle}$ are given by

$$\begin{aligned}
&\Pi^{e,r^{n/2}}_{J_{0,3}} \simeq \Omega^{e,r^{n/2}}_{J_0} && \Pi^{e,r^{n/2}}_{J_{1,2}} \simeq \Omega^{e,r^{n/2}}_{J_1} \\
&\Pi^{e,r^{n/2}}_{\alpha_j} \simeq \Omega^{e,r^{n/2}}_{\alpha_j}, \quad j < \tfrac{n}{4} && \Pi^{e,r^{n/2}}_{\alpha_j} \simeq \Omega^{e,r^{n/2}}_{\alpha_{-j}}, \quad j > \tfrac{n}{4} \\
&\Pi^{r^k}_{\beta_l} \simeq \Omega^{r^k}_{\beta_l}, \quad l < \tfrac{n}{2} && \Pi^{r^k}_{\beta_l} \simeq \Omega^{r^k}_{\beta_{l-n/2}}, \quad l >= \tfrac{n}{2} \\
&\Pi^{s}_{J_{0,3}} \simeq \Omega^{s}_{\gamma_0} && \Pi^{s}_{J_{1,2}} \simeq \Omega^{s}_{\gamma_1} \\
&\Pi^{sr}_{J_{0,2}} \simeq \Omega^{sr}_{\gamma_0} && \Pi^{sr}_{J_{1,3}} \simeq \Omega^{sr}_{\gamma_1}.
\end{aligned} \tag{4.56}$$

The unconfined algebra is $\mathscr{U} \simeq D(D_{n/2})$, and $\mathrm{RKer}(\Gamma) \simeq F(\mathbb{Z}_2) \otimes \mathbb{C}e$. The following particles are unconfined:

$$\Omega^{e}_{J_{0,1}},\ \Omega^{e}_{\alpha_j},\ \Omega^{r^{2k}}_{\beta_l},\ \Omega^{sr}_{\gamma_{0,1}} \qquad j,k = 1,\ldots,\tfrac{n}{4}-\tfrac{1}{2};\ l = 0,\ldots,\tfrac{n}{2}-1. \tag{4.57}$$

4.2.4 Particles in α_j-condensates

Take $|\phi\rangle \in V^{e}_{\alpha_j}$, which is two-dimensional. Condition (3.8) gives no restriction on the ρ-part of the matrix elements (ρ_{ab}, E_g) of representations of $D(D_4)^*$, and boils down to

$$\alpha_j(s^q r^p)|\phi\rangle = |\phi\rangle. \tag{4.58}$$

When $q = 0$, so for elements E_{r^p} this is equivalent to

$$\alpha_j(r^p) = \begin{pmatrix} q^{jp} & \\ & q^{-jp} \end{pmatrix} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}. \tag{4.59}$$

The smallest p that satisfies this equation is $p = x \equiv \frac{n}{\gcd(n,j)}$, and the subgroup $\langle r^x \rangle \simeq \mathbb{Z}_{\gcd(n,j)}$ leaves any $|\phi\rangle$ invariant.

For elements E_{sr^p}, the condition becomes

$$\alpha_j(sr^p)\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} & q^{-jp} \\ q^{jp} & \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} q^{-jp}\lambda_2 \\ q^{jp}\lambda_1 \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}. \tag{4.60}$$

For the element E_s, so when $p = 0$, we see that this condition is satisfied only for $|\phi\rangle$ for which $\lambda_2 = \lambda_1$. Because α_j is a representation, all elements $s\langle r^x \rangle$ will also leave this $|\phi\rangle$ invariant, and it is not hard to see that no other sr^p have this property. We find that for this state vector, the stabilizer $N_{|\phi\rangle}$ is isomorphic to $D_{\gcd(n,j)}$.

Alternatively, we could have $\lambda_2 = q^{jy}\lambda_1$, for some y so that $jy \in \{1,\ldots,x-1\}$. In that case the elements $sr^y\langle r^x \rangle$ leave the condensate invariant, but the stabilizer will still be isomorphic to $D_{\gcd(n,j)}$. It is only of importance in which of the two classes $[s]$ or $[sr]$ the element sr^y lies, since we can always perform some gauge transformation which carries sr^y over to the distinguished element, s or sr, of that class (this gauge transformation corresponds to a change of coordinates so that the values $\lambda_{1,2}$ change in a way that leads to an invariant action of the distinguished element).

We treat the cases of these different stabilizers separately.

Case $N_{|\phi\rangle} \simeq \mathbb{Z}_{\gcd(n,j)}$ The residual symmetry algebra $\mathscr{T} \simeq F(D_n) \tilde{\otimes} \mathbb{C}\mathbb{Z}_{\gcd(n,j)}$ has dimension $2n.\gcd(n,j)$. The irreducible representations Ω^B_β are given by orbits B in D_n and stabilizers in $\mathbb{C}\mathbb{Z}_{\gcd(n,j)}$ of distinguished elements in these orbits.

The action of r^{zx}, $z \in 0,\ldots,\gcd(n,j)-1$ on an element r^k yields only r^k, which therefore constitutes an entire orbit. Its stablizer is the entire group $\mathbb{Z}_{\gcd(n,j)}$.

The action of r^{zx} on sr^k gives sr^{k+2zx}. If j is even then $\gcd(n,j)$ is also even because n is even, and $\gcd(n,j)/2 \in \{0,\ldots,\gcd(n,j)\}$. The action of all r^{zx} then gives us two times every element of $sr^k\langle r^{2x}\rangle$, and there are $2x$ orbits labelled by $[sr^k]$, $k = 0,\ldots,2x-1$, the stabilizer of each of which is $\{e, r^{n/2}\}$ as $r^{n/2} \in \mathbb{Z}_{\gcd(n,j)}$ because $\gcd(n,j)$ is even.

If j is odd then $\gcd(n,j)$ is odd, so $\gcd(n,j)/2+\frac{1}{2} \in \{0,\ldots,\gcd(n,j)\}$, and the action of $r^{(\gcd(n,j)/2+\frac{1}{2})x}$ on sr^k gives sr^{k+x}. The orbits are $[sr^k] = sr^k\langle r^x\rangle$, $k = 0,\ldots,x-1$, with stabilizer just $\{e\}$, as $r^{n/2} \notin \mathbb{Z}_{\gcd(n,j)}$.

The irreducible representations of $\mathscr{T}$ are:

$$
\begin{array}{ll}
\Omega^{r^k}_{\beta_l}(1), & k=0,\ldots,n-1,\ l=0,\ldots,\gcd(n,j)-1 \\
\begin{cases} \Omega^{sr^k}_{\gamma_{0,1}}\ (\frac{\gcd(n,j)}{2}) \\ \Omega^{sr^k}_{\gamma_0}\ (\gcd(n,j)) \end{cases} & \begin{array}{ll} k=0,\ldots,2x-1, & j \text{ even} \\ k=0,\ldots,\ x-1, & j \text{ odd} \end{array}
\end{array}
\tag{4.61}
$$

In both cases, the dimensions correctly add up to $2n.\gcd(n,j)$. The branching rules of irreducible $D(D_n)$-representations are not hard to calculate:

$$
\begin{array}{l}
\Pi^e_{J_{0,1}} \simeq \Omega^e_{\beta_0} \quad \Pi^e_{J_{2,3}} \simeq \Omega^e_{\beta_0}\ (x \text{ even}) \quad \Pi^e_{J_{2,3}} \simeq \Omega^e_{\beta_{\gcd(n,j)/2}}\ (x \text{ odd}) \\
\Pi^{r^{n/2}}_{J_{0,1}} \simeq \Omega^{r^{n/2}}_{\beta_0} \quad \Pi^{r^{n/2}}_{J_{2,3}} \simeq \Omega^{r^{n/2}}_{\beta_0}\ (x \text{ even}) \quad \Pi^{r^{n/2}}_{J_{2,3}} \simeq \Omega^{r^{n/2}}_{\beta_{\gcd(n,j)/2}}\ (x \text{ odd}) \\
\Pi^e_{\alpha_j} \simeq \Omega^e_{\beta_{j \bmod \gcd(n,j)}} \oplus \Omega^e_{\beta_{-j \bmod \gcd(n.j)}} \\
\Pi^{r^{n/2}}_{\alpha_j} \simeq \Omega^{r^{n/2}}_{\beta_{j \bmod \gcd(n,j)}} \oplus \Omega^{r^{n/2}}_{\beta_{-j \bmod \gcd(n.j)}} \\
\Pi^{r^k}_{\beta_l} \simeq \Omega^{r^k}_{\beta_{l \bmod \gcd(n,j)}} \oplus \Omega^{r^{-k}}_{\beta_{l \bmod \gcd(n,j)}} \\
\left.\begin{array}{l}
\Pi^s_{J_{0,1}} \simeq \Omega^s_{\gamma_0} \oplus \Omega^{sr^2}_{\gamma_0} \oplus \ldots \oplus \Omega^{sr^{2x-2}}_{\gamma_0} \\
\Pi^s_{J_{2,3}} \simeq \Omega^s_{\gamma_1} \oplus \Omega^{sr^2}_{\gamma_1} \oplus \ldots \oplus \Omega^{sr^{2x-2}}_{\gamma_1} \\
\Pi^{sr}_{J_{0,1}} \simeq \Omega^{sr}_{\gamma_0} \oplus \Omega^{sr^3}_{\gamma_0} \oplus \ldots \oplus \Omega^{sr^{2x-1}}_{\gamma_0} \\
\Pi^{sr}_{J_{2,3}} \simeq \Omega^{sr}_{\gamma_1} \oplus \Omega^{sr^3}_{\gamma_1} \oplus \ldots \oplus \Omega^{sr^{2x-1}}_{\gamma_1}
\end{array}\right\} j \text{ even} \\
\left.\begin{array}{l}
\Pi^s_{J_i} \simeq \Omega^s_{\gamma_0} \oplus \Omega^{sr^2}_{\gamma_0} \oplus \ldots \oplus \Omega^{sr^{x-2}}_{\gamma_0} \\
\Pi^{sr}_{J_i} \simeq \Omega^{sr}_{\gamma_0} \oplus \Omega^{sr^3}_{\gamma_0} \oplus \ldots \oplus \Omega^{sr^{x-1}}_{\gamma_0}
\end{array}\right\} j \text{ odd}
\end{array}
\tag{4.62}
$$

The particles of which the orbit B is contained within the stabilizer $N_{|\phi\rangle} = \langle r^x\rangle$ are unconfined, which leaves only $\Omega^{r^{zx}}_{\beta_l}$, $z,l = 0,\ldots,\gcd(n,j)-1$, and $\mathscr{U} \simeq D(\mathbb{Z}_{\gcd(n,j)})$. From (4.16) the Hopf kernel is $F(D_n/\mathbb{Z}_{\gcd(n,j)}) \otimes \mathbb{C}e \simeq F(D_x) \otimes \mathbb{C}e$. The restriction of $\mathscr{T}$-representations is:

$$
\Omega^{r^k}_{\beta_l} \simeq E_{r^k \bmod x}, \quad \Omega^{sr^k}_{\gamma_{0,1}} \simeq E_{sr^k \bmod x}. \tag{4.63}
$$

Here the index k runs over all valid values according to (4.61), and we see that indeed only the representations $\Omega^{r^{zx}}_{\beta_l}$ restrict to E_e and are (therefore) unconfined.

Case $N_{|\phi\rangle} \simeq D_{\gcd(n,j)}$ The residual symmetry algebra $\mathscr{T} \simeq F(D_n) \tilde{\otimes} \mathbb{C}D_{\gcd(n,j)}$ is $2n.2\gcd(n,j)$-dimensional. As mentioned above, we only have to consider the cases $N_{|\phi\rangle} = \langle r^x\rangle \cup s\langle r^x\rangle$ and $N_{|\phi\rangle} = \langle r^x\rangle \cup sr\langle r^x\rangle$. We will denote these two cases by $D^s_{\gcd(n,j)}$ and $D^{sr}_{\gcd(n,j)}$.

We determine the $D_{\gcd(n,j)}$-orbits in $F(D_n)$. Analogous to the case of $N_{|\phi\rangle} \simeq \mathbb{Z}_{\gcd(n,j)}$, we have to distinguish between j even and odd: when j is odd, then $\gcd(n,j)$ is odd and x is even; when j is even, then $\gcd(n,j)$ is even and x may be odd.

The orbit of r^k now includes r^{-k}. The stabilizer is $\langle r^x\rangle \simeq \mathbb{Z}_{\gcd(n,j)}$. The stabilizer of e and $r^{n/2}$ is the entire group $N_{|\phi\rangle} \simeq D_{\gcd(n,j)}$.

For elements sr^k, the action of $\langle r^x\rangle$ gives the set $sr^k\langle r^{2x}\rangle$, which reduces to $sr^k\langle r^x\rangle$ when j is odd. The action of $s\langle r^x\rangle$ on sr^k adds to this the elements $sr^{-k}\langle r^{2x}\rangle$ ($sr^{-k}\langle r^x\rangle$ when j is odd).

The sets $sr^k\langle r^{2x}\rangle$ and $sr^{-k}\langle r^{2x}\rangle$ coincide when $k=0$ or $k=x$. The sets $sr^k\langle r^x\rangle$ and $sr^{-k}\langle r^x\rangle$ coincide when $k=0$ or $k=\frac{x}{2}$. This leads to the orbits $[s]$ and $[sr^x]$ (resp. $[sr^{x/2}]$) of $\frac{\gcd(n,j)}{2}$ (resp. $\gcd(n,j)$) elements and the orbits $[sr^k]$, $k = 1,\ldots,x-1$ (resp. $\frac{x}{2}-1$) of $\gcd(n,j)$ (resp. $2\gcd(n,j)$) elements.

When j is even, $\gcd(n,j)$ is even and $r^{n/2} \in N_{|\phi\rangle}$. Therefore the stabilizer of s is $\{e, r^{n/2}, s, sr^{n/2}\}$, and the stabilizer of sr^x is $\{e, r^{n/2}, sr^x, sr^{x+n/2}\}$. For the other orbits $[sr^k]$, no element in $s\langle r^x\rangle$ leaves the distinguished elements invariant, and the stabilizers are $\{e, r^{n/2}\}$.

When j is odd, $\gcd(n,j)$ is also odd and $r^{n/2} \notin N_{|\phi\rangle}$. The stabilizer of s is $\{e,s\}$. The elements $sr^{x/2}$ is left invariant by $sr^{x/2+n/2}$, which is contained in $N_{|\phi\rangle}$ because $\frac{x}{2}+\frac{n}{2} = (\frac{1}{2}+\frac{\gcd(n,j)}{2})x$ is a multiple of x. The stabilizer of $sr^{x/2}$ therefore is $\{e, sr^{x/2+n/2}\}$. The stabilizers of the other orbits $[sr^k]$ are just the trivial group.

When we are dealing with $D^{sr}_{\gcd(n,j)}$ these orbits 'shift by r': relabelling every orbit by substituting $sr^p \mapsto sr^{p+1}$ gives the correct results. In particular, the orbits $[sr]$ and (for j even) $[sr^{x+1}]$ now have half as much elements as the other orbits $[sr^k]$ etc.

Summarizing, we have for j even

$$\begin{aligned}
&\Omega^{e,r^{n/2}}_{J_i}(1),\ \Omega^{e,r^{n/2}}_{\alpha_j}(2), && i=0,1,2,3,\ j=1,\ldots,\tfrac{\gcd(n,j)}{2}-1\\
&\Omega^{r^k}_{\beta_l}(2), && k=1,\ldots,\tfrac{n}{2}-1,\ l=0,\ldots,\gcd(n,j)-1\\
&\Omega^{s,sr^x}_{J_i}\big(\gcd(n,j)/2\big),\ \Omega^{sr^k}_{\gamma_{0,1}}\big(\gcd(n,j)\big) && i=0,1,2,3,\ k=1,\ldots,x-1
\end{aligned}\tag{4.64}$$

and for j odd

$$\begin{aligned}
&\Omega^{e,r^{n/2}}_{J_{0,1}}(1),\ \Omega^{e,r^{n/2}}_{\alpha_j}(2), && j=1,\ldots,\tfrac{\gcd(n,j)-1}{2}\\
&\Omega^{r^k}_{\beta_l}(2), && k=1,\ldots,\tfrac{n}{2}-1,\ l=0,\ldots,\gcd(n,j)-1\\
&\Omega^{s,sr^{x/2}}_{\gamma_{0,1}}\big(\gcd(n,j)\big),\ \Omega^{sr^k}\big(2\gcd(n,j)\big) && k=1,\ldots,\tfrac{x}{2}-1
\end{aligned}\tag{4.65}$$

The branching rules give the isomorphisms between restrictions of irreducible $D(D_n)$-representations and irreducible $\mathscr{T}$-representations:

For j even:

$$
\begin{aligned}
&\Pi^{e,r^{n/2}}_{J_{0,2}} \simeq \Omega^{e,r^{n/2}}_{J_0},\ \Pi^{e,r^{n/2}}_{J_{1,3}} \simeq \Omega^{e,r^{n/2}}_{J_1},\\
&\Pi^{e,r^{n/2}}_{\alpha_j} \simeq \begin{cases} \Omega^{e,r^{n/2}}_{\alpha_{j \bmod \gcd(n,j)}} & 1 < j \bmod \gcd(n,j) < \frac{\gcd(n,j)}{2} \\ \Omega^{e,r^{n/2}}_{J_2} \oplus \Omega^{e,r^{n/2}}_{J_3} & j = \frac{\gcd(n,j)}{2} \bmod \gcd(n,j) \\ \Omega^{e,r^{n/2}}_{\alpha_{-j \bmod \gcd(n,j)}} & \frac{\gcd(n,j)}{2} < j \bmod \gcd(n,j) < \gcd(n,j) \\ \Omega^{e,r^{n/2}}_{J_0} \oplus \Omega^{e,r^{n/2}}_{J_1} & j = 0 \bmod \gcd(n,j) \end{cases}\\
&\Pi^{r^k}_{\beta_l} \simeq \Omega^{r^k}_{\beta_{l \bmod \gcd(n,j)}}, \quad k = 1, \dots, \tfrac{n}{2} - 1\\
&\Pi^{s}_{J_{0,1}} \simeq \begin{cases} \Omega^{s}_{J_{0,1}} \oplus \Omega^{sr^x}_{J_{0,1}} \bigoplus_{k \text{ even}} \Omega^{sr^k}_{\gamma_0} & x \text{ even} \\ \Omega^{s}_{J_{0,1}} \bigoplus_{k \text{ even}} \Omega^{sr^k}_{\gamma_0} & x \text{ odd} \end{cases}\\
&\Pi^{s}_{J_{2,3}} \simeq \begin{cases} \Omega^{s}_{J_{2,3}} \oplus \Omega^{sr^x}_{J_{2,3}} \bigoplus_{k \text{ even}} \Omega^{sr^k}_{\gamma_1} & x \text{ even} \\ \Omega^{s}_{J_{2,3}} \bigoplus_{k \text{ even}} \Omega^{sr^k}_{\gamma_1} & x \text{ odd} \end{cases}\\
&\Pi^{sr}_{J_{0,1}} \simeq \begin{cases} \Omega^{sr^x}_{J_{0,1}} \bigoplus_{k \text{ odd}} \Omega^{sr^k}_{\gamma_0} & x \text{ odd} \\ \bigoplus_{k \text{ odd}} \Omega^{sr^k}_{\gamma_0} & x \text{ even} \end{cases}\\
&\Pi^{sr}_{J_{2,3}} \simeq \begin{cases} \Omega^{sr^x}_{J_{2,3}} \bigoplus_{k \text{ odd}} \Omega^{sr^k}_{\gamma_1} & x \text{ odd} \\ \bigoplus_{k \text{ odd}} \Omega^{sr^k}_{\gamma_1} & x \text{ even} \end{cases}
\end{aligned} \tag{4.66}
$$

For j odd:

$$
\begin{aligned}
&\Pi^{e,r^{n/2}}_{J_{0,2}} \simeq \Omega^{e,r^{n/2}}_{J_0},\ \Pi^{e,r^{n/2}}_{J_{1,3}} \simeq \Omega^{e,r^{n/2}}_{J_1},\\
&\Pi^{e,r^{n/2}}_{\alpha_j} \simeq \begin{cases} \Omega^{e,r^{n/2}}_{\alpha_{j \bmod \gcd(n,j)}} & 1 < j \bmod \gcd(n,j) < \frac{\gcd(n,j)}{2} \\ \Omega^{e,r^{n/2}}_{\alpha_{-j \bmod \gcd(n,j)}} & \frac{\gcd(n,j)}{2} < j \bmod \gcd(n,j) < \gcd(n,j) \end{cases}\\
&\Pi^{r^k}_{\beta_l} \simeq \Omega^{r^k}_{\beta_{l \bmod \gcd(n,j)}}, \quad k = 1, \dots, \tfrac{n-1}{2}\\
&\Pi^{s}_{J_{0,2}} \simeq \begin{cases} \Omega^{s}_{\gamma_0} \oplus \Omega^{sr^x}_{\gamma_0} \bigoplus_{k \text{ even}} \Omega^{sr^k} & x \text{ even} \\ \Omega^{s}_{\gamma_0} \bigoplus_{k \text{ even}} \Omega^{sr^k} & x \text{ odd} \end{cases}\\
&\Pi^{s}_{J_{1,3}} \simeq \begin{cases} \Omega^{s}_{\gamma_1} \oplus \Omega^{sr^x}_{\gamma_1} \bigoplus_{k \text{ even}} \Omega^{sr^k} & x \text{ even} \\ \Omega^{s}_{\gamma_1} \bigoplus_{k \text{ even}} \Omega^{sr^k} & x \text{ odd} \end{cases}\\
&\Pi^{sr}_{J_{0,3}} \simeq \begin{cases} \Omega^{sr^{x/2}}_{\gamma_0} \bigoplus_{k \text{ odd}} \Omega^{sr^k} & \frac{x}{2} \text{ odd} \\ \bigoplus_{k \text{ odd}} \Omega^{sr^k} & \frac{x}{2} \text{even} \end{cases}\\
&\Pi^{sr}_{J_{1,2}} \simeq \begin{cases} \Omega^{sr^{x/2}}_{\gamma_1} \bigoplus_{k \text{ odd}} \Omega^{sr^k} & \frac{x}{2} \text{ odd} \\ \bigoplus_{k \text{ odd}} \Omega^{sr^k} & \frac{x}{2} \text{ even} \end{cases}
\end{aligned} \tag{4.67}
$$

When we have $N_{|\phi\rangle} \simeq D^{sr}_{\gcd(n,j)}$ instead of $D^{s}_{\gcd(n,j)}$, the branching rules of $\Pi^{s}_{J_i}$ and $\Pi^{sr}_{J_i}$ are 'reversed'; for example $\Pi^{sr}_{J_1} \simeq \Omega^{sr}_{\gamma_1} \bigoplus_{k \text{ odd}} \Omega^{sr^k}$ for j odd and $\frac{x}{2}$ even.

The representations carrying flux contained in $N_{|\phi\rangle}$ are unconfined, which are $\Omega^{r^{zx}}$ and Ω^{s,sr^x} for j even and $\Omega^{r^{zx}}$, and Ω^{s} for j odd. Note that $\Omega^{r^{n/2}}$ is only unconfined when j is even. The unconfined algebra is $\mathcal{U} \simeq D(D_{\gcd(n,j)})$ and the Hopf kernel is $\mathrm{RKer}(\Gamma) \simeq F(\mathbb{Z}_{\gcd(n,j)}) \otimes \mathbb{C}e$ with irreducible representations E_{r^k}, $\;k = 0,\ldots,x-1$.

The restriction of irreducible $\mathcal{T}$-representations to $\mathrm{RKer}(\Gamma)$ gives

$$\Omega_\beta^{r^k} \simeq E_{r^k \bmod x}, \quad \Omega_\beta^{sr^k} \simeq E_{r^{-k} \bmod x}. \tag{4.68}$$

This concludes the treatment of electric condensates in $D(D_n)$-theories.

4.3 Gauge-invariant magnetic condensates

Now we turn to representations carrying flux but trivial electric charge, denoted by Π_1^A with $A \neq [e]$. We call these *magnetic* representations, and we are interested in the formation of condensates in this sector.

Trivial self-braiding Let us first look under which circumstances we have condensate particles which braid trivially amongst themselves. The general form (2.8) of the vector $|\phi\rangle$ reduces to $\sum_{j\in A} \lambda_j |a_j\rangle$, because the electric part of the representation space is trivial. We then find

$$\begin{aligned}
\tau \circ (\Pi_1^A \otimes \Pi_1^A)(R)(|\phi\rangle \otimes |\phi\rangle) &= \tau \circ (\Pi_1^A \otimes \Pi_1^A)\big(\sum_h (P_h, e) \otimes (1,h)\big)(|\phi\rangle \otimes |\phi\rangle) \\
&= \tau \circ \sum_h \Pi_A^1(P_h, e) \otimes \Pi_A^1(1,h)(\sum_j \lambda_j |a_j\rangle \otimes \sum_{j'} \lambda_{j'} |a_{j'}\rangle) \\
&= \tau \circ \sum_h \sum_j \delta_{a_j,h} \lambda_j |a_j\rangle \otimes \sum_{j'} \lambda_{j'} |h a_{j'} h^{-1}\rangle \\
&= \tau \circ \sum_{j,j'} (\lambda_j |a_j\rangle \otimes \lambda_{j'} |a_j a_{j'} a_j^{-1}\rangle) \\
&= \sum_{j,j'} \lambda_{j'} |a_j a_{j'} a_j^{-1}\rangle \otimes \lambda_j |a_j\rangle.
\end{aligned} \tag{4.69}$$

In order that this be equal to $|\phi\rangle \otimes |\phi\rangle = \sum_{j,j'} \lambda_{j'} |a_{j'}\rangle \otimes \sum_j \lambda_j |a_j\rangle$, we must demand that $\lambda_{a_j^{-1}} \lambda_{a_{j'}} \lambda_{a_j} = \lambda_{a_{j'}} \quad \forall j, j'$. This holds for at least two general gauge orbits: the gauge-invariant orbit where $\lambda_j = 1 \;\; \forall j$: $\sum_j |a_j\rangle$, and the *pure fluxes* $\lambda_j = \delta_{j,j'}$ for some j': $|a_{j'}\rangle$. The rest of this section describes the gauge-invariant magnetic condensates, in the next we look at condensates of pure flux. They may be other states that satisfy the condition of trivial self-braiding, see e.g. §4.4.4.

The condition for a trivial spin factor is automatically satisfied for all magnetic condensates.

Residual symmetry algebra The residual symmetry algebra $\mathcal{T}$ is spanned by matrix elements of those irreducible representations of $D(D_n)^*$ that satisfy (3.8), for our choice of the condensate vector: $|\phi\rangle = \sum_{j\in A} |a_j\rangle$. The demand

$$\Pi_1^A(1, E_{g^{-1}})|\phi\rangle \otimes \frac{\chi_\rho(a)}{d_\rho}|\phi\rangle \tag{4.70}$$

reduces to

$$\chi_\rho(a) = d_\rho \Rightarrow \rho(a) = 1, \tag{4.71}$$

because $\sum_j |g^{-1}a_j g\rangle = \sum_j |a_j\rangle$.

Since ρ is a representation of D_n, any element a_j in the class A will also have $\rho(a_j) = 1$, and furthermore the entire subgroup K_A generated by elements of A has this property. Because A is a conjugacy class, this subgroup is normal in D_n.

So we have a set of irreducible representations of which the value is trivial on a normal subgroup K_A. They are then equivalent to the irreducible representations of the group with this normal subgroup divided out. The matrix elements of these representations span the space $F(D_n/K_A)$. This can be seen as a subalgra of $F(D_n)$ by identifying the element $P_h \in F(D_n/K_A)$ (so hK_A is a K_A-coset in D_n) with $\sum_{\kappa\in K_A} P_{h\kappa} \in F(D_n)$.

Because there was no restriction on the elements E_g, the residual symmetry algebra is given by

$$\mathscr{T} \simeq F(D_n/K_A) \tilde{\otimes} \mathbb{C}D_n, \tag{4.72}$$

which is of the form described in corollary 3.3. The particles in the condensate are given by Ω^B_β with B a D_n-orbit in D_n/K_A and β an irreducible representation of the stabilizer in D_n of the distinguished element of B.

The character (3.18) of such a representation is given by

$$\chi^B_\beta(P_h, g) = 1_{N_h}(g) 1_B(h) \chi_\beta(k_h^{-1} g k_h), \quad h \in D_n/K_A,\ g \in D_n. \tag{4.73}$$

The restriction of a $D(D_n)$-representation $\Pi^{A'}_\alpha$ has character

$$\chi^{A'}_\alpha|_{\mathscr{T}}(P_h, g) = \sum_{\kappa\in K_A} 1_{N_h\kappa}(g) 1_{A'}(h\kappa) \chi_\alpha(k_{h\kappa}^{-1} g k_{h\kappa}). \tag{4.74}$$

We can calculate the branching of $\Pi^{A'}_\alpha$ into irreducible $\mathscr{T}$-representations Ω^B_β by using the orthogonality relation (3.19). We see that we get non-zero values only when the elements of the K_A-cosets that constitute B comprise the elements of A'. Because K_A is normal in D_n and A' is a conjugacy class, there is only one B that will satisfy this condition.

Confinement The Ω^B_β that satisfy (3.27) and (3.28) are unconfined. The first of these relations is always satisfied because of the form of $|\phi\rangle$. The second condition states that β must be trivial on all of K_A. As above, such representations are equivalent to irreducible representations on the quotient group D_n/K_A. The unconfined algebra is therefore given by $\mathscr{U} \simeq D(D_n/K_A)$.

One can now guess (correctly) that the domain walls should be characterized by elements of K_A. From (3.36) we find

$$\mathrm{RKer}(\Gamma) = F\big((D_n/K_A)/(D_n/K_A)\big) \tilde{\otimes} \mathbb{C}(D_n \cap K_A) = F(e) \tilde{\otimes} \mathbb{C}(K_A). \tag{4.75}$$

4.3.1 Particles in the gauge-invariant $r^{n/2}$-condensate

The subgroup generated by $r^{n/2}$ is $\{e, r^{n/2}\} \simeq \mathbb{Z}_2$, so the residual symmetry algebra is $\mathscr{T} \simeq F(D_n/\langle r^{n/2}\rangle) \tilde{\otimes} \mathbb{C}D_n \simeq F(D_{n/2}) \tilde{\otimes} \mathbb{C}D_n$, which is $2n^2$-dimensional.

The algebra $F(D_{n/2})$ has the form of functions on a dihedral group, where this group consists of $\{e, r^{n/2}\}$-cosets. It is a subalgebra of $F(D_n)$ by condidering the element $P_e \in F(D_{n/2})$ as the element $P_e + P_{r^{n/2}} \in F(D_n)$. Elements of the form $P_g - P_{gr^{n/2}}$ are projected onto $0 \in F(D_{n/2})$.

In order to determine the other properties of this symmetry breaking, we have to distinguish between the cases where $\frac{n}{2}$ is even or odd.

Case $\frac{n}{2}$ even When $r^{n/2}$ is even, the element $r^{n/4}$ exists in $F(D_{n/2})$ and forms an entire D_n-orbit, so its stabilizer is the full group D_n. The other orbits are $[e]$, $[r^k] = \{r^k, r^{-k}\}$, $[s] = \{s, sr^2, \ldots, sr^{\frac{n}{2}-2}\}$ and $[sr] = \{sr, sr^3, \ldots, sr^{\frac{n}{2}-1}\}$. The D_n-stabilizers of $[e]$ and $[r^k]$ are just D_n and $\langle r\rangle \simeq \mathbb{Z}_n$, but the elements that leave $sr^k \in D_{n/2}$ invariant are $\{e, r^{n/4}, r^{n/2}, r^{3n/4}, sr^k, sr^{k+n/4}, sr^{k+n/2}, sr^{k+3n/4}\} \simeq D_4$. We then find the following irreducible representations for $\mathscr{T}$ (dimensions within parentheses):

$$\begin{array}{lll} \Omega^{e,r^{n/4}}_{J_i}(1),\ \Omega^{e,r^{n/4}}_{\alpha_j}(2), & i=0,1,2,3,\ j=1,\ldots,\frac{n}{2}-1 & \\ \Omega^{r^k}_{\beta_l}(2), & k=1,\ldots,\frac{n}{4}-1,\ l=0,\ldots,n-1 & \\ \Omega^{s,sr}_{J_i}(\frac{n}{4}),\ \Omega^{s,sr}_{\alpha}(\frac{n}{2}) & i=0,1,2,3. & (4.76) \end{array}$$

The squares of the dimensions correctly add up to $2n^2$.

We calculate the branching rules. The orbit B of the Ω^B_β should contain the conjugacy class A of Π^A_α. The determination of β is sometimes a lot trickier. For example, the representations $\Pi^{r^{n/4}}_{\beta_l}$ are two-dimensional, because $[r^{n/4}] = \{r^{n/4}, r^{3n/4}\}$; the $\Omega^{r^{n/4}}_{J_i}$ are one-dimensional. We therefore get a direct sum of two of those representations, which should correspond properly to the action on the coset space represented by $\{e,s\}$.

Worse still, the orbit $[s]$ of $\Pi^s_{J_i}$ is twice as large as the orbit $[s]$ of Ω^s_α. However, the D_4-representation α is two-dimensional. We should find a basis of the representation space of the restriction $\Pi^A_\alpha|_{\mathscr{T}}$ so that the action on the coset space corresponds to the action of α for Ω^s_α. An example of this is the branching of $\Pi^s_{J_2}$, which is worked out in the appendix, §A.4.2.

The branching rules are given by

$$\begin{aligned}
\Pi^{e,r^{n/2}}_{J_i} &\simeq \Omega^e_{J_i}; \quad \Pi^{e,r^{n/2}}_{\alpha_j} \simeq \Omega^e_{\alpha_j} \\
\Pi^{r^{n/4}}_{\beta_l} &\simeq \begin{cases} \Omega^{r^{n/4}}_{J_0} \oplus \Omega^{r^{n/4}}_{J_1} & l=0 \\ \Omega^{r^{n/4}}_{J_2} \oplus \Omega^{r^{n/4}}_{J_3} & l=\frac{n}{2} \\ \Omega^{r^{n/4}}_{\alpha_l} & 0<l<\frac{n}{2} \\ \Omega^{r^{n/4}}_{\alpha_{l-\frac{n}{2}}} & \frac{n}{2}<l<n \end{cases} \\
\Pi^{r^k}_{\beta_l} &\simeq \begin{cases} \Omega^{r^k}_{\beta_l} & 0<k<\frac{n}{4} \\ \Omega^{r^{-k+n/2}}_{\beta_l} & \frac{n}{4}<k<\frac{n}{2} \end{cases} \\
\Pi^{s,sr}_{J_0} &\simeq \Omega^{s,sr}_{J_0} \oplus \Omega^{s,sr}_{J_2} \\
\Pi^{s,sr}_{J_1} &\simeq \Omega^{s,sr}_{J_1} \oplus \Omega^{s,sr}_{J_3} \\
\Pi^{s,sr}_{J_{2,3}} &\simeq \Omega^{s,sr}_{\alpha}
\end{aligned} \tag{4.77}$$

In particular the purely magnetic particles $\Pi^{s,sr}_{J_0}$ may branch to dyonic particles in the residual symmetry algebra, so the electric symmetry is then also broken. This will happen often in magnetic and dyonic condensates, and forms an interesting feature in these theories.

The condensate particles Ω^B_β for which β is trivial on the normal subgroup $K_A = \langle r^{n/2}\rangle$ are unconfined, and they are in one-to-one correpondence with irreducible representations of $\mathscr{U} \simeq D(D_n/\langle r^{n/2}\rangle) \simeq D(D_{n/2})$. This holds for the following representations:

$$\Omega^{e,r^{n/4}}_{J_i},\ \Omega^{e,r^{n/4}}_{\alpha_{2j}},\ \Omega^{r^k}_{\beta_{2l}},\ \Omega^{s,sr}_{J_i}. \tag{4.78}$$

The sum of the squares of the dimensions adds up to $\frac{1}{2}n^2$, which equals $(\dim D_{n/2})^2$.

The Hopf kernel of the surjection from $\mathscr{T}$ onto $\mathscr{U}$ is isomorphic to just $\mathbb{C}K_A \simeq \mathbb{C}\langle r^{n/2}\rangle \simeq \mathbb{C}\mathbb{Z}_2$, which has two irreducible representations. The unconfined particles restrict to the trivial domain wall, the other particles to the other one.

Case $\frac{n}{2}$ odd When $\frac{n}{2}$ is odd, there is no element $r^{n/4}$, and $D_{n/2}$ is an odd dihedral group (§A.4.1). The D_n-orbits in $D_{n/2}$ are now $[e]$, $[r^k] = \{r^k, r^{-k}\}$, $k = 1,\ldots,\frac{n/2-1}{2}$ and $[s] = \{s, sr, \ldots, sr^{n/2-1}\}$. The D_n-stabilizers are D_n for e, $\langle r\rangle \simeq \mathbb{Z}_n$ for r^k and $\{e, r^{n/2}, sr^k, sr^{k+n/2}\} \simeq D_2$ for sr^k. We find the find the following irreducible representations for the residual symmetry algebra $\mathscr{T} \simeq F(D_{n/2}) \tilde{\otimes} \mathbb{C}D_{n/2}$:

$$\begin{array}{ll}
\Omega^e_{J_i}(1),\ \Omega^e_{\alpha_j}(2), & i = 0,1,2,3,\ j = 1,\ldots,\frac{n}{2}-1 \\
\Omega^{r^k}_{\beta_l}(2), & k = 1,\ldots,\frac{n/2-1}{2},\ l = 0,\ldots,n-1 \\
\Omega^s_{J_i}(\frac{n}{2}) & i = 0,1,2,3.
\end{array} \tag{4.79}$$

The squares of the dimensions correctly add up to $2n^2$.

The branching rules are given by

$$\begin{aligned}
\Pi^{e,r^{n/2}}_{J_i} \simeq \Omega^e_{J_i}; \quad \Pi^{e,r^{n/2}}_{\alpha_j} &\simeq \Omega^e_{\alpha_j} \\
\Pi^{r^k}_{\beta_l} &\simeq \begin{cases} \Omega^{r^k}_{\beta_l} & 0 < k < \frac{n}{4} \\ \Omega^{r^{-k+n/2}}_{\beta_l} & \frac{n}{4} < k < \frac{n}{2} \end{cases} \\
\Pi^{s,sr}_{J_i} &\simeq \Omega^s_{J_i}
\end{aligned} \tag{4.80}$$

The particles Ω^B_β for which β is trivial on $K_A = \{e, r^{n/2}\}$ are unconfined. This holds for

$$\Omega^e_{J_{0,1}},\ \Omega^e_{\alpha_{2j}}, \Omega^{r^k}_{\beta_{2l}},\ \Omega^s_{J_{0,1}}, \quad j,k = 1,\ldots,\tfrac{n/2-1}{2},\ l = 0,\ldots,\tfrac{n}{2}. \tag{4.81}$$

The unconfined algebra $\mathscr{U} \simeq D(D_{n/2})$ is of dimension n^2. The Hopf kernel is given by $\mathbb{C}K_A = \mathbb{C}\langle r^{n/2}\rangle \simeq \mathbb{C}\mathbb{Z}_2$. The unconfined particles restrict to the trivial representation and the confined particles restrict to the non-trivial $\mathbb{Z}_2$-representation.

4.3.2 Particles in gauge-invariant r^k-condensates

The orbit $[r^k]$ in D_n is $\{r^k, r^{-k}\}$, where $k = 1,\ldots,\frac{n}{2}-1$. The condition for the residual symmetry algebra (4.71) states that the matrix elements of any $D(D_n)^*$-representation (ρ, E_g) for which $\rho(r^k) = 1$ leave the condensate invariant. $J_{0,1}$ are always trivial on r^k, $J_{2,3}$ when k is even, and $\alpha_j(r^k) = 1$ when $j = z\gcd(n,k)$, $z \in \mathbb{N}$. It can be seen that the residual symmetry algebra is then given by $\mathscr{T} = F(D_{\gcd(n,k)}) \tilde{\otimes} \mathbb{C}D_n$.

This agrees with the general description $F(D_n/K_{[r^k]})\,\tilde{\otimes}\,\mathbb{C}D_n$, because the smallest subgroup of D_n which contains $[r^k]$ is $\langle r^{\gcd(n,k)}\rangle \simeq \mathbb{Z}_x$ where $x \equiv \frac{n}{\gcd(n,k)}$, and $D_n/\mathbb{Z}_x = D_{\gcd(n,k)}$.

The D_n-orbits in $D_{\gcd(n,k)}$ are the regular conjugacy classes for the (odd or even) dihedral group. For example $r^{p'} \rightharpoonup sr^p = sr^{p-2p' \bmod \gcd(n,k)}$. The D_n-stabilizers for $[e], [r^{n/2}]$ and $[r^{k'}]$ are just D_n, D_n and $\mathbb{Z}_n$, but the stabilizer of $sr^{k'}$ is now given by $D_{2x} = \langle r^{\gcd(n,k)/2}\rangle \cup sr^{k'}\langle r^{\gcd(n,k)/2}\rangle$ when $\gcd(n,k)$ is even, and $D_x = \langle r^{\gcd(n,k)}\rangle \cup sr^{k'}\langle r^{\gcd(n,k)}\rangle$ when $\gcd(n,k)$ is odd.

Case $\gcd(n,k)$ **even** The irreducible representations of $\mathscr{T}$ are (dimensions within parentheses):

$$
\begin{array}{ll}
\Omega_{J_i}^{e,r^{\gcd(n,k)/2}}(1), \Omega_{\alpha_j}^{e,r^{\gcd(n,k)/2}}(2) & i = 0,1,2,3;\ j = 1,\ldots,\frac{n}{2}-1 \\
\Omega_{\beta_l}^{r^{k'}}(2) & k' = 1,\ldots,\frac{\gcd(n,k)}{2}-1;\ l = 0,\ldots,n-1 \\
\Omega_{J_i}^{s,sr}(\frac{\gcd(n,k)}{2}), \Omega_{\alpha_j}^{s,sr}\big(\gcd(n,k)\big) & i = 0,1,2,3;\ j = 1,\ldots,x-1
\end{array}
\tag{4.82}
$$

The squares of the dimensions correctly add up to $4n\gcd(n,k)$.

To calculate the branching rules of Π_α^A, we know that we only have to consider those Ω_β^B for which A is completely contained in $\bigcup bK_{[r^k]},\quad b \in B$. Because $\gcd(n,k)$ is even, $\Pi^{r^{n/2}}$ will always restrict to Ω^e. Furthermore $\Pi^{r^{k'}} \simeq \Omega^{r^{k' \bmod \gcd(n,k)}}$, and $\Pi^{s,sr} \simeq \Omega^{s,sr}$.

Special care has to be taken when $r^{k' \bmod \gcd(n,j)} = r^{\gcd(n,k)/2}$.

$$
\begin{aligned}
&\Pi^{e}_{J_i} \simeq \Omega^{e}_{J_i};\ \Pi^{e}_{\alpha_j} \simeq \Omega^{e}_{\alpha_j}\\
&\Pi^{r^{n/2}}_{J_i} \simeq \begin{cases} \Omega^{e}_{J_i} & x \text{ even}\\ \Omega^{r^{\gcd(n,k)/2}}_{J_i} & x \text{ odd}\end{cases}\\
&\Pi^{r^{n/2}}_{\alpha_j} \simeq \begin{cases} \Omega^{e}_{\alpha_j} & x \text{ even}\\ \Omega^{r^{\gcd(n,k)/2}}_{\alpha_j} & x \text{ odd}\end{cases}\\
&\Pi^{r^{k'}}_{\beta_l} \simeq \begin{cases} \Omega^{r^{\gcd(n,k)/2}}_{J_0} \oplus \Omega^{r^{\gcd(n,k)/2}}_{J_1} & l=0\\ \Omega^{r^{\gcd(n,k)/2}}_{J_2} \oplus \Omega^{r^{\gcd(n,k)/2}}_{J_3} & l=\frac{n}{2}\\ \Omega^{r^{\gcd(n,k)/2}}_{\alpha_l} & 0<l<\frac{n}{2}\\ \Omega^{r^{\gcd(n,k)/2}}_{\alpha_{l-\frac{n}{2}}} & \frac{n}{2}<l<n \end{cases} k' \bmod \gcd(n,k) = \tfrac{\gcd(n,k)}{2}\\
&\Pi^{r^{k'}}_{\beta_l} \simeq \begin{cases} \Omega^{r^{k' \bmod \gcd(n,k)}}_{\beta_l} & 0<k' \bmod \gcd(n,k) < \frac{\gcd(n,k)}{2}\\ \Omega^{r^{-k' \bmod \gcd(n,k)}}_{\beta_l} & \frac{\gcd(n,k)}{2} < k' \bmod \gcd(n,k) < \gcd(n,k)\end{cases}\\
&\Pi^{s,sr}_{J_0} \simeq \begin{cases} \Omega^{s,sr}_{J_0} \oplus \Omega^{s,sr}_{J_2} \bigoplus_{j \text{ even}} \Omega^{s,sr}_{\alpha_j} & x \text{ even}\\ \Omega^{s,sr}_{J_0} \bigoplus_{j \text{ even}} \Omega^{s,sr}_{\alpha_j} & x \text{ odd}\end{cases}\\
&\Pi^{s,sr}_{J_1} \simeq \begin{cases} \Omega^{s,sr}_{J_1} \oplus \Omega^{s,sr}_{J_3} \bigoplus_{j \text{ even}} \Omega^{s,sr}_{\alpha_j} & x \text{ even}\\ \Omega^{s,sr}_{J_1} \bigoplus_{j \text{ even}} \Omega^{s,sr}_{\alpha_j} & x \text{ odd}\end{cases}\\
&\Pi^{s,sr}_{J_2} \simeq \begin{cases} \Omega^{s,sr}_{J_2} \bigoplus_{j \text{ odd}} \Omega^{s,sr}_{\alpha_j} & x \text{ odd}\\ \bigoplus_{j \text{ odd}} \Omega^{s,sr}_{\alpha_j} & x \text{ even}\end{cases}\\
&\Pi^{s,sr}_{J_3} \simeq \begin{cases} \Omega^{s,sr}_{J_3} \bigoplus_{j \text{ odd}} \Omega^{s,sr}_{\alpha_j} & x \text{ odd}\\ \bigoplus_{j \text{ odd}} \Omega^{s,sr}_{\alpha_j} & x \text{ even}\end{cases}
\end{aligned} \tag{4.83}
$$

One may verify that the dimensions of the representations are the same on both sides. We also see that this description just generalizes the case of $|\phi\rangle \in V^{r^{n/2}}_{J_0}$; this happens because we divide out the entire subgroup K_A, leading to the same expressions all along.

The particles for which β is trivial on $\langle r^{\gcd(n,k)}\rangle$ are unconfined. This applies to

$$
\Omega^{e,r^{\gcd(n,k)/2}}_{J_i},\ \Omega^{e,r^{\gcd(n,k)/2}}_{\alpha_{zx}},\ \Omega^{r^{k'}}_{\beta_{z'x}},\ \Omega^{s,sr}_{J_i},\quad z=1,\ldots,\tfrac{\gcd(n,k)}{2}-1;\ z'=0,\ldots,\gcd(n,k)-1 \tag{4.84}
$$

The unconfined algebra is $\mathscr{U} \simeq D(D_n/K_{[r^k]}) \simeq D(D_{\gcd(n,k)})$. The Hopf kernel of the map from $\mathscr{T}$ to $\mathscr{U}$ is $\mathbb{C}\langle r^k\rangle \simeq \mathbb{C}\mathbb{Z}_x$. Denote its irreducible representations by $\rho_l,\quad l=0,\ldots,x-1$. The restrictions of the Ω^B_β to this Hopf kernel are given by

$$
\begin{aligned}
&\Omega^{e,r^{\gcd(n,k)/2}}_{J_i} \simeq \rho_0 && \Omega^{s,sr}_{J_i} \simeq \tfrac{\gcd(n,k)}{2}\rho_0\\
&\Omega^{e,r^{\gcd(n,k)/2}}_{\alpha_j} \simeq \rho_{j \bmod x} \oplus \rho_{-j \bmod x} && \Omega^{s,sr}_{\alpha_j} \simeq \tfrac{\gcd(n,k)}{2}\rho_{j \bmod x} \oplus \rho_{-j \bmod x}\\
&\Omega^{r^{k'}}_{\beta_l} \simeq \rho_{j \bmod x} \oplus \rho_{-j \bmod x}
\end{aligned} \tag{4.85}
$$

and indeed the particles we claimed to be unconfined restrict to the trivial wall ρ_0.

Case $\gcd(n,k)$ **odd** In this case the left tensorand of $\mathscr{T} \simeq F(D_{\gcd(n,k)}) \tilde{\otimes} \mathbb{C}D_n$ is the function algebra of an odd dihedral group. Its D_n-orbits are the regular orbits for dihedral groups: $[e]$, $[r^k] = \{r^k, r^{-k}\}$ and $[s] = \{s, \ldots, sr^{\gcd(n,k)-1}\}$, with D_n-stabilizers D_n, $\mathbb{Z}_n$ and $\langle r^{\gcd(n,k)}\rangle \cup s\langle r^{\gcd(n,k)}\rangle \simeq D_x$. Recall that when $\gcd(n,k)$ is odd, $x = \frac{n}{\gcd(n,k)}$ is even, and therefore $r^{n/2} \in \langle r^{\gcd(n,k)}\rangle$.

The irreducible representations of $\mathscr{T}$ are (dimensions within parentheses):

$$\begin{aligned} &\Omega^e_{J_i}(1), \Omega^e_{\alpha_j}(2) && i = 0,1,2,3;\ j = 1, \ldots, \tfrac{n}{2} - 1 \\ &\Omega^{r^{k'}}_{\beta_l}(2) && k' = 1, \ldots, \tfrac{\gcd(n,k)-1}{2};\ l = 0, \ldots, n-1 \\ &\Omega^s_{J_i}\big(\gcd(n,k)\big), \Omega^s_{\alpha_j}\big(2\gcd(n,k)\big) && i = 0,1,2,3;\ j = 1, \ldots, \tfrac{x}{2} - 1 \end{aligned} \tag{4.86}$$

The branching rules of $D(D_n)$-representations as $\mathscr{T}$-representations are

$$\begin{aligned} &\Pi^{e,r^{n/2}}_{J_i} \simeq \Omega^e_{J_i};\ \Pi^{e,r^{n/2}}_{\alpha_j} \simeq \Omega^e_{\alpha_j} \\ &\Pi^{r^{k'}}_{\beta_l} \simeq \begin{cases} \Omega^{r^{k' \bmod \gcd(n,k)}}_{\beta_l} & 0 < k' \bmod \gcd(n,k) < \frac{\gcd(n,k)}{2} \\ \Omega^{r^{-k' \bmod \gcd(n,k)}}_{\beta_l} & \frac{\gcd(n,k)}{2} < k' \bmod \gcd(n,k) < \gcd(n,k) \end{cases} \\ &\Pi^{s,sr}_{J_0} \simeq \begin{cases} \Omega^s_{J_0} \oplus \Omega^s_{J_2} \bigoplus_{j \text{ even}} \Omega^s_{\alpha_j} & x \text{ even} \\ \Omega^s_{J_0} \bigoplus_{j \text{ even}} \Omega^s_{\alpha_j} & x \text{ odd} \end{cases} \\ &\Pi^{s,sr}_{J_1} \simeq \begin{cases} \Omega^s_{J_1} \oplus \Omega^s_{J_3} \bigoplus_{j \text{ even}} \Omega^s_{\alpha_j} & x \text{ even} \\ \Omega^s_{J_1} \bigoplus_{j \text{ even}} \Omega^s_{\alpha_j} & x \text{ odd} \end{cases} \\ &\Pi^{s,sr}_{J_2} \simeq \begin{cases} \Omega^s_{J_2} \bigoplus_{j \text{ odd}} \Omega^s_{\alpha_j} & x \text{ odd} \\ \bigoplus_{j \text{ odd}} \Omega^s_{\alpha_j} & x \text{ even} \end{cases} \\ &\Pi^{s,sr}_{J_3} \simeq \begin{cases} \Omega^s_{J_3} \bigoplus_{j \text{ odd}} \Omega^s_{\alpha_j} & x \text{ odd} \\ \bigoplus_{j \text{ odd}} \Omega^s_{\alpha_j} & x \text{ even} \end{cases} \end{aligned} \tag{4.87}$$

The unconfined algebra $\mathscr{U} \simeq D(D_{\gcd(n,j)})$ has irreducible representations given by those Ω^B_β for which $\beta(r^{z\gcd(n,k)}) = 1, \quad \forall z \in \mathbb{Z}$. This holds for

$$\Omega^e_{J_{0,1}},\ \Omega^{r^{k'}}_{\beta_{zx}},\ \Omega^s_{J_{0,1}}, \quad z = 0, \ldots, \gcd(n,k) - 1 \tag{4.88}$$

The restriction of irreducible $\mathscr{T}$-representations to the Hopf kernel $\mathbb{C}K_{[r^k]} \simeq \mathbb{Z}_x$ is similar to the case $\gcd(n,k)$ even, but now x is certain to be even, and the several $J_{2,3}$-representations restrict to $\rho_{n/2}$.

$$\begin{aligned} &\Omega^e_{J_{0,1}} \simeq \rho_0 && \Omega^s_{J_{0,1}} \simeq \gcd(n,k)\rho_0 \\ &\Omega^e_{J_{2,3}} \simeq \rho_{n/2} && \Omega^s_{J_{2,3}} \simeq \gcd(n,k)\rho_{n/2} \\ &\Omega^e_{\alpha_j} \simeq \rho_{j \bmod x} \oplus \rho_{-j \bmod x} && \Omega^s_{\alpha_j} \simeq \gcd(n,k)\rho_{j \bmod x} \oplus \rho_{-j \bmod x} \\ &\Omega^{r^{k'}}_{\beta_l} \simeq \rho_{j \bmod x} \oplus \rho_{-j \bmod x} \end{aligned} \tag{4.89}$$

4.3.3 Particles in gauge-invariant s-condensates

The conjugacy class of s in D_n for n even is $[s] = \{s, sr^2, \ldots, sr^{-2}\}$ (4.2). The smallest (normal) subgroup that contains this orbit is $K_{[s]} = \langle r^2\rangle \cup s\langle r^2\rangle \simeq D_{n/2}$. The residual symmetry algebra $\mathscr{T}$ is isomorphic to $F(D_n/D_{n/2})\tilde{\otimes}\mathbb{C}D_n \simeq F(\mathbb{Z}_2)\tilde{\otimes}\mathbb{C}D_n$, and is $4n$-dimensional. We label the D_n-orbits in $\mathbb{Z}_2$ by e and r, and the stabilizer of both these orbits is the entire group D_n.

The irreducible representations of $\mathscr{T}$ are then (dimensions within parentheses):

$$\Omega^{e,r}_{J_i},\ \Omega^{e,r}_{\alpha_j},\quad i=0,1,2,3;\ j=1,\dots,\tfrac{n}{2}-1 \tag{4.90}$$

The restriction of $D(D_n)$ irreps to $\mathscr{T}$ gives the following isomorphisms:

$$\begin{aligned}
&\Pi^{e,r^{n/2}}_{J_i} \simeq \Omega^{e}_{J_i} \quad \Pi^{e,r^{n/2}}_{\alpha_j} \simeq \Omega^{e}_{\alpha_j}\\
&\Pi^{r^k}_{\beta_l} \simeq \begin{cases} \Omega^{r^{k \bmod 2}}_{J_0} \oplus \Omega^{r^{k \bmod 2}}_{J_1} & l=0\\ \Omega^{r^{k \bmod 2}}_{J_2} \oplus \Omega^{r^{k \bmod 2}}_{J_3} & l=\frac{n}{2}\\ \Omega^{r^{k \bmod 2}}_{\alpha_l} & 0<l<\frac{n}{2}\\ \Omega^{r^{k \bmod 2}}_{\alpha_{l-\frac{n}{2}}} & \frac{n}{2}<l<n \end{cases}\\
&\Pi^{s,sr}_{J_0} \simeq \begin{cases} \Omega^{e,r}_{J_0} \oplus \Omega^{e,r}_{J_2} \bigoplus_{j\text{ even}} \Omega^{e,r}_{\alpha_j} & \frac{n}{2}\text{ even}\\ \Omega^{e,r}_{J_0} \bigoplus_{j\text{ even}} \Omega^{e,r}_{\alpha_j} & \frac{n}{2}\text{ odd} \end{cases}\\
&\Pi^{s,sr}_{J_1} \simeq \begin{cases} \Omega^{e,r}_{J_1} \oplus \Omega^{e,r}_{J_3} \bigoplus_{j\text{ even}} \Omega^{e,r}_{\alpha_j} & \frac{n}{2}\text{ even}\\ \Omega^{e,r}_{J_1} \bigoplus_{j\text{ even}} \Omega^{e,r}_{\alpha_j} & \frac{n}{2}\text{ odd} \end{cases}\\
&\Pi^{s,sr}_{J_2} \simeq \begin{cases} \Omega^{e,r}_{J_2} \bigoplus_{j\text{ odd}} \Omega^{e,r}_{\alpha_j} & \frac{n}{2}\text{ odd}\\ \bigoplus_{j\text{ odd}} \Omega^{e,r}_{\alpha_j} & \frac{n}{2}\text{ even} \end{cases}\\
&\Pi^{s,sr}_{J_3} \simeq \begin{cases} \Omega^{e,r}_{J_3} \bigoplus_{j\text{ odd}} \Omega^{e,r}_{\alpha_j} & \frac{n}{2}\text{ odd}\\ \bigoplus_{j\text{ odd}} \Omega^{e,r}_{\alpha_j} & \frac{n}{2}\text{ even} \end{cases}
\end{aligned} \tag{4.91}$$

When β is trivial on the entire group $\langle r^2\rangle \cup s\langle r^2\rangle$, the particle Ω^B_β is confined. It is easily seen that this only holds for $\beta = J_{0,2}$ and the unconfined particles $\Omega^{e,r}_{J_{0,2}}$ are irreducible representations of the unconfined algebra $\mathscr{U} \simeq F(\mathbb{Z}_2)\otimes\mathbb{C}\mathbb{Z}_2$.

The left and right Hopf kernel of $\Gamma: F(\mathbb{Z}_2)\tilde{\otimes}\mathbb{C}D_n \to F(\mathbb{Z}_2)\otimes\mathbb{C}\mathbb{Z}_2$ are isomorphic to $\mathbb{C}D_{n/2}$. We have to distinguish between the cases where $\frac{n}{2}$ is either even or odd.

When $\frac{n}{2}$ is even, the irreducible representations of $\mathbb{C}D_{n/2}$ are given by $J_i,\ \ i=0,1,2,3$ and $\alpha_j,\ \ j=1,\dots,\frac{n}{4}-1$. The irreducible $\mathscr{T}$-representations restrict to these as:

$$\begin{aligned}
&\Omega^{e,r}_{J_{0,2}} \simeq J_0 \quad \Omega^{e,r}_{J_{1,3}} \simeq J_1 \quad \Omega^{e,r}_{\alpha_{n/4}} \simeq J_2 \oplus J_3\\
&\Omega^{e,r}_{\alpha_j} \simeq \begin{cases} \alpha_j & 0<j<\frac{n}{4}\\ \alpha_{-j \bmod n/2} & \frac{n}{4}<j<\frac{n}{2} \end{cases}
\end{aligned} \tag{4.92}$$

When $\frac{n}{2}$ is odd the irreducible representations of $\mathbb{C}D_{n/2}$ are $J_{0,1}$ and $\alpha_j,\ \ j=1,\dots,\frac{n}{4}-\frac{1}{2}$. The restrictions follow the same prescription (4.92), where the case for $\Omega^{e,r}_{\alpha_{n/4}}$ should be ignored.

4.3.4 Particles in gauge-invariant sr-condensates

The conjugacy class of sr is $[sr]=\{sr, sr^3, \dots, sr^{-1}\}$. The smallest (normal) subgroup that contains this orbit is $K_{[sr]} = \langle r^2\rangle \cup sr\langle r^2\rangle \simeq D_{n/2}$. The residual symmetry algebra

$\mathcal{T}$ is isomorphic to $F(\mathbb{Z}_2)\tilde{\otimes}\mathbb{C}D_n$.

The rest of the calculations are very similar to those for the $\Pi^{[s]}_{J_0}$-condensate. The labels of the irreducible representations of $\mathcal{T}$ are the same:

$$\Omega^{e,r}_{J_i},\ \Omega^{e,r}_{\alpha_j}, \quad i=0,1,2,3;\ j=1,\dots,\tfrac{n}{2}-1 \tag{4.93}$$

The restriction of $D(D_n)$ irreps to $\mathcal{T}$ is also identical to (4.91), except for $\Pi^s \simeq \Omega_r$, $\Pi^{sr} \simeq \Omega_e$.

On the group $\langle r^2\rangle \cup sr\langle r^2\rangle$, only $\beta = J_{0,2}$ are trivial.The unconfined algebra $\mathcal{U}$ is isomorphic to $F(\mathbb{Z}_2)\otimes\mathbb{C}\mathbb{Z}_2$.

The Hopf kernel the map from $\mathcal{T}$ to $\mathcal{U}$ is isomorphic to $\mathbb{C}D_{n/2}$. When $\frac{n}{2}$ is even, the restrictions of Ω^B_β are:

$$\Omega^{e,r}_{J_{0,3}} \simeq J_0 \quad \Omega^{e,r}_{J_{1,2}} \simeq J_1 \quad \Omega^{e,r}_{\alpha_{n/4}} \simeq J_2 \oplus J_3$$
$$\Omega^{e,r}_{\alpha_j} \simeq \begin{cases} \alpha_j & 0<j<\frac{n}{4} \\ \alpha_{-j \bmod n/2} & \frac{n}{4}<j<\frac{n}{2} \end{cases} \tag{4.94}$$

When $\frac{n}{2}$ is odd the irreducible representations of $\mathbb{C}D_{n/2}$ are $J_{0,1}$ and α_j, $\ j=1,\dots,\frac{n}{4}-\frac{1}{2}$. The restrictions are as above, but the representation $\Omega^{e,r}_{\alpha_{n/4}}$ does not exist.

4.4 Pure flux magnetic condensates

Next to the class sum there are other possible magnetic condensates, when the conjugacy class contains more than one element. In the previous section we showed that condensates of pure flux a_j, so when $|\phi\rangle = |a_j\rangle \in V^A_1$ for a single j, always have trivial self-braiding. The residual symmetry breaking is exactly as one would expect: the normal subgroup containing the class A is divided out, and electric symmetry is broken into the transformations that leave the flux under consideration invariant.

Residual symmetry algebra We now show that $\mathcal{T}$ is indeed $F(D_n/K_A)\tilde{\otimes}\mathbb{C}N_{a_j}$. The matrix elements (ρ_{ab},g) of irreducible representations of $D(D_n)^*$ that leave the condensate $|\phi\rangle$ invariant must satisfy (3.8). The action of g^{-1} on $|\phi\rangle$ reduces to a scalar factor only if g^{-1} commutes with the chosen flux, so we demand

$$g \in N_{a_j}. \tag{4.95}$$

When this is the case, the scalar factor is always equal to 1. Then, just as we have seen before, the other demand is

$$\rho(a) = 1, \tag{4.96}$$

and leads to $F(D_n/K_A)$, with just as for gauge-invariant condensates. So we find $\mathcal{T} \simeq F(D_n/K_A)\otimes\mathbb{C}N_{a_j}$, which is of the general form of corollary 3.3. From now on we make the choice $a_j = a_1 = a$, which we can do without loss of generality, because a was chosen in A arbitrarily.

The irreducible representations are again given by Ω^B_β with B an N_a-orbit in D_n/K_A, and β an irreducible N_a-representation. The character of such a representation is

$$\chi^B_\beta(P_h,g) = 1_{N_h}(g)1_B(h)\chi_\beta(k_h^{-1}gk_h), \quad h\in D_n/K_A,\ g\in N_a. \tag{4.97}$$

The restriction of a character $\chi^{A'}_\alpha$ of a $D(D_n)$-representation $\Pi^{A'}_\alpha$ to $\mathcal{T}$ is the same as in (4.74), but with $g\in N_a$, instead of D_n.

Confinement Now we want to find out which of these Ω^B_β are confined. We can use the conditions (3.27) and (3.28). The first reduces to

$$\sum_{k\in K_A}(1,\eta k)\rightharpoonup|\phi\rangle=\sum_{k\in K_A}(1,k)\rightharpoonup|\phi\rangle. \tag{4.98}$$

Because K_A is a normal group, we have $hK_Ah^{-1}=K_A \quad \forall h\in H$. With this we find that

$$\sum_{k\in K_A}(1,\eta k)\rightharpoonup|\phi\rangle=\sum_{k\in K_A}(1,\eta k\eta^{-1})\rightharpoonup(\eta\rightharpoonup|\phi\rangle) \tag{4.99}$$

can only be equal to the right-hand side of (4.98) when $\eta\rightharpoonup|\phi\rangle=|\phi\rangle$. Now $|\phi\rangle=|a\rangle$, so $\eta\rightharpoonup|\phi\rangle=\eta\rightharpoonup|a\rangle=|\eta a\eta^{-1}\rangle$. This can only be equal to $|\phi\rangle$ when η commutes with a.

Considering that η was chosen arbitrarily in the coset ηK_A, we see that the only unconfined representations have an orbit B that consists of elements of the form $nK_A, \quad n\in N_a$.

Because $a\in N_a$, the other condition reduces to

$$\beta(x_\eta^{-1}k_pak_p^{-1}x_\eta)=1 \quad \forall\eta\in B, \tag{4.100}$$

where the N_a-coset representative k_p can now only be e, because $|\phi\rangle=|a\rangle$. But as we have just shown that $\eta=nK_A$ for some $n\in N_a$, $x_\eta\in N_a$, and we then only demand

$$\beta(a)=1. \tag{4.101}$$

These conditions do not always lead to a general form as they did for electric and gauge-invariant condensates. In §4.4.2 we will see that sometimes the set of representations that satisfy these relations are the irreducible representations of a Hopf algebra for which the element given by (3.21) does not constitute a valid universal R-matrix. Because we want to be able to define braiding for unconfined particles, this full set of solutions must be restricted so that the remaining solutions do allow a universal R-matrix. In [7, §11.2] it was conjectured that in that case the unconfined algebra would be given by

$$\mathscr{U}\simeq D\big(N_a/(K_A\cap N_a)\big). \tag{4.102}$$

4.4.1 Particles in pure r^k-condensates

The conjugacy class $[r^k]$ has two elements: r^k and r^{-k}. We choose r^k as our distinguished element. The minimal normal subgroup $K_{[r^k]}$ that contains $[r^k]$ is the subgroup generated by r^k, which was shown to be equivalent to $\langle r^{\gcd(n,k)}\rangle\simeq\mathbb{Z}_x$ in §4.3.2.

The stabilizer of r^k is $\langle r\rangle\simeq\mathbb{Z}_n$, so that $\mathscr{T}\simeq F(D_{\gcd(n,k)})\,\tilde{\otimes}\,\mathbb{C}\mathbb{Z}_n$, a $2n\gcd(n,k)$-dimensional Hopf algebra. We must distinguish between the cases in which $\gcd(n,k)$ is even or odd.

Case $\gcd(n,k)$ **even** The orbits in $D_{\gcd(n,k)}$ of the $\langle r\rangle$-action are $\{r^{k'}\}, \quad 0\le k'<\gcd(n,k)$ and $[s],[sr]$ with $\frac{\gcd(n,k)}{2}$ elements each. Their $\langle r\rangle$-stabilizers are $\langle r\rangle$ for $[r^{k'}]$ and $\langle r^{\gcd(n,k)/2}\rangle\simeq\mathbb{Z}_{2x}$ for $[s],[sr]$. The irreducible representations of $\mathscr{T}$ are then (dimensions within parentheses)

$$\begin{array}{ll}\Omega^{r^{k'}}_{\beta_l}(1) & k'=0,\dots,\gcd(n,k)-1;\ l=0,\dots,n-1\\ \Omega^{s,sr}_{\beta_{l'}}(\frac{\gcd(n,k)}{2}) & l'=0,\dots,2x-1\end{array} \tag{4.103}$$

The dimensions correctly add up to $2n\gcd(n,k)$. The branching rules for restrictions of $D(D_n)$-representations Π^A_α to $\mathscr{T}$ are

$$\begin{aligned}
\Pi^{e,r^{n/2}}_{J_{0,1}} &\simeq \Omega^{e,r^{n/2}}_{\beta_0} & \Pi^{e,r^{n/2}}_{J_{2,3}} &\simeq \Omega^{e,r^{n/2}}_{\beta_{n/2}} \\
\Pi^{e,r^{n/2}}_{\alpha_j} &\simeq \Omega^{e,r^{n/2}}_{\beta_j} \oplus \Omega^{e,r^{n/2}}_{\beta_{-j}} & \Pi^{r^k}_{\beta_l} &\simeq \Omega^{r^{k \bmod \gcd(n,k)}}_{\beta_l} \oplus \Omega^{r^{-k \bmod \gcd(n,k)}}_{\beta_{-l}} \\
\Pi^{s,sr}_{J_{0,1}} &\simeq \bigoplus_{l' \text{ even}} \Omega^{s,sr}_{\beta_{l'}} & \Pi^{s,sr}_{J_{2,3}} &\simeq \bigoplus_{l' \text{ odd}} \Omega^{s,sr}_{\beta_{l'}}
\end{aligned} \tag{4.104}$$

We have two conditions for the irreducible representations Ω^B_β to be unconfined. The first says that the orbit B must consist of elements of the form $n\langle r^{\gcd(n,k)}\rangle, n \in \langle r\rangle$, the other states that $\beta(r^k) = 1$. From the first condition we see that the representations $\Omega^{s,sr}$ are confined, and the second allows only those β_l for which l is a multiple of x. What is left are representations of an algebra $F(\mathbb{Z}_{\gcd(n,k)}) \otimes \mathbb{C}\mathbb{Z}_{\gcd(n,k)} \simeq D(\mathbb{Z}_{\gcd(n,k)})$. This is indeed of the general form $D(N_a/K_A \cap N_a) = D(\mathbb{Z}_n/\mathbb{Z}_x)$.

The right Hopf kernel is given by (3.36), which amounts to

$$\begin{aligned}
\mathrm{RKer}(\Gamma) &\simeq F\big((D_n/\langle r^{\gcd(n,k)}\rangle)/(\langle r\rangle/\langle r^{\gcd(n,k)}\rangle)\big) \tilde{\otimes} \mathbb{C}(\langle r\rangle \cap \langle r^{\gcd(n,k)}\rangle) \\
&\simeq F(D_n/\mathbb{Z}_n) \tilde{\otimes} \mathbb{C}\langle r^{\gcd(n,k)}\rangle \simeq F(\mathbb{Z}_2) \tilde{\otimes} \mathbb{C}\mathbb{Z}_x.
\end{aligned} \tag{4.105}$$

We label its representations by $(E_{e,s}, \rho_l)$, $\quad l = 0,\ldots,x-1$. The restriction of $\mathscr{T}$-representations is then given by

$$\Omega^{r^{k'}}_{\beta_l} \simeq (E_e, \rho_{l \bmod x}), \quad \Omega^{s,sr}_{\beta_{l'}} \simeq \tfrac{\gcd(n,k)}{2}(E_s, \rho_{l' \bmod x}). \tag{4.106}$$

Case $\gcd(n,k)$ **odd** In this case the $\langle r\rangle$-orbits in $D_{\gcd(n,k)}$ are $\{r^{k'}\}$, $\quad k' = 0,\ldots,n-1$ and $[s] = \{s,\ldots,sr^{-1}\}$, with stabilizers $\langle r\rangle$ and $\langle r^{\gcd(n,k)}\rangle$. The irreducible representations are (dimensions within parentheses):

$$\begin{aligned}
&\Omega^{r^{k'}}_{\beta_l}(1) && k' = 0,\ldots,\gcd(n,k)-1;\ l = 0,\ldots,n-1 \\
&\Omega^{s}_{\beta_{l'}}(\gcd(n,k)) && l' = 0,\ldots,x-1
\end{aligned} \tag{4.107}$$

The branching rules are given by:

$$\begin{aligned}
\Pi^{e,r^{n/2}}_{J_{0,1}} &\simeq \Omega^{e,r^{n/2}}_{\beta_0} & \Pi^{e,r^{n/2}}_{J_{2,3}} &\simeq \Omega^{e,r^{n/2}}_{\beta_{n/2}} \\
\Pi^{e,r^{n/2}}_{\alpha_j} &\simeq \Omega^{e,r^{n/2}}_{\beta_j} \oplus \Omega^{e,r^{n/2}}_{\beta_{-j}} & \Pi^{r^k}_{\beta_l} &\simeq \Omega^{r^{k \bmod \gcd(n,k)}}_{\beta_l} \oplus \Omega^{r^{-k \bmod \gcd(n,k)}}_{\beta_{-l}} \\
\Pi^{s,sr}_{J_{0,1}} &\simeq \bigoplus_{l' \text{ even}} \Omega^{s}_{\beta_{l'}} & \Pi^{s,sr}_{J_{2,3}} &\simeq \bigoplus_{l' \text{ odd}} \Omega^{s}_{\beta_{l'}}
\end{aligned} \tag{4.108}$$

Following the same reasoning as for $\gcd(n,k)$ even, only $\Omega^{r^k}_{\beta_{zx}}$, $\quad z = 0,\ldots,\gcd(n,k)-1$ are unconfined, so that $\mathscr{U} \simeq D(\mathbb{Z}_{\gcd(n,k)})$. The Hopf kernel again is isomorphic to $F(\mathbb{Z}_2) \tilde{\otimes} \mathbb{C}\mathbb{Z}_x$, and the restrictions of $\mathscr{T}$-representations are

$$\Omega^{r^{k'}}_{\beta_l} \simeq (E_e, \rho_{l \bmod x}), \quad \Omega^{s}_{\beta_{l'}} \simeq (E_s, \rho_{l'}). \tag{4.109}$$

4.4.2 Particles in pure s-condensates

When forming a condensate of particles in the state $|\phi\rangle = |s\rangle$, the residual symmetry algebra is $\mathscr{T} \simeq F(\mathbb{Z}_2) \otimes \mathbb{C}D_2$. The smallest normal subgroup $K_{[s]}$ of D_n that contains s is $\langle r^2\rangle \cup s\langle r^2\rangle$. The stabilizer N_s of the element s is $\{e, r^{n/2}, s, sr^{n/2}\}$. Its irreducible representations are

$$\Omega^e_{J_i}, \quad \Omega^r_{J_i} \quad i = 0,1,2,3 \tag{4.110}$$

and are all of dimension one. The branching of the restriction of representations Π^A_α of $D(D_n)$ to $\mathscr{T}$ is given by

$$
\begin{cases}
\Pi^{e,r^{n/2}}_{J_{0,2}} \simeq \Omega^{e}_{J_0}, \quad \Pi^{e,r^{n/2}}_{J_{1,3}} \simeq \Omega^{e}_{J_1} & \frac{n}{2} \text{ even} \\
\Pi^{e}_{J_i} \simeq \Omega^{e}_{J_i}, \qquad \Pi^{r^{n/2}}_{J_i} \simeq \Omega^{r}_{J_i} & \frac{n}{2} \text{ odd}
\end{cases}
$$

$$
\Pi^{e}_{\alpha_j} \simeq \begin{cases}
\Omega^{e}_{J_0} \oplus \Omega^{e}_{J_1} & j \text{ even} \\
\Omega^{e}_{J_2} \oplus \Omega^{e}_{J_3} & j \text{ odd}
\end{cases}
$$

$$
\Pi^{r^{n/2}}_{\alpha_j} \simeq \begin{cases}
\Omega^{e}_{J_0} \oplus \Omega^{e}_{J_1} & j \text{ even}, \frac{n}{2} \text{ even} \\
\Omega^{e}_{J_2} \oplus \Omega^{e}_{J_3} & j \text{ odd}, \frac{n}{2} \text{ even} \\
\Omega^{r}_{J_0} \oplus \Omega^{r}_{J_1} & j \text{ even}, \frac{n}{2} \text{ odd} \\
\Omega^{r}_{J_2} \oplus \Omega^{r}_{J_3} & j \text{ odd}, \frac{n}{2} \text{ odd}
\end{cases}
$$

$$
\Pi^{r^k}_{\beta_l} \simeq \begin{cases}
\Omega^{e}_{J_0} \oplus \Omega^{e}_{J_1} & l \text{ even}, k \text{ even} \\
\Omega^{e}_{J_2} \oplus \Omega^{e}_{J_3} & l \text{ odd}, k \text{ even} \\
\Omega^{r}_{J_0} \oplus \Omega^{r}_{J_1} & l \text{ even}, k \text{ odd} \\
\Omega^{r}_{J_2} \oplus \Omega^{r}_{J_3} & l \text{ odd}, k \text{ odd}
\end{cases}
$$

$$
\Pi^{s}_{J_i} \simeq \Omega^{e}_{J_i}, \quad \Pi^{sr}_{J_i} \simeq \Omega^{r}_{J_i} \tag{4.111}
$$

The particles Ω^{B}_{β} that are unconfined have the properties that the orbit B consists of elements $nK_{[s]}$ with n in the s-stabilizer N_s, and that $\beta(s) = 1$. This last requirement holds for $\beta = J_0, J_2$; for the first we see that rK_s is the same orbit as $r^{n/2}K_{[s]}$ when $\frac{n}{2}$ is odd. We distinguish the following cases:

Case $\frac{n}{2}$ even The only representations which are unconfined are $\Omega^{e}_{J_{0,2}}$. These two are irreducible representations of a Hopf algebra $F(e) \otimes \mathbb{C}\mathbb{Z}_2$. However, when calculating the universal R-matrix according to (3.21), we are left with

$$
\mathscr{R} = (1,e) \otimes \big(1,(e+s)\big), \tag{4.112}
$$

where we have labelled the basis vectors of $F(e) \otimes \mathbb{C}\mathbb{Z}_2$ by $(1,e)$ and $(1,s)$. This not a valid universal R-matrix, it is not even invertible. So here we have a concrete example of a set of solutions for which the braiding is not well-defined, as mentioned earlier. The suggested solution was to define the unconfined algebra by (4.102), which in this case would leave only $D(e)$, the one-dimensional, trivial Hopf algebra.

In this case we could also define $\mathscr{R} = (1,e) \otimes (1,e)$, because $F(e) \otimes \mathbb{C}\mathbb{Z}_2$ is cocommutative; it is then unclear, however, how this braiding would relate to the braiding of $D(D_n)$ itself.

When we choose $\mathscr{U} \simeq D(e)$, all representations but $\Omega^{e}_{J_0}$ are confined; the Hopf kernel is isomorphic to $\mathscr{T}$ itself, and the domain walls are characterized by the labels B, β as well.

Case $\frac{n}{2}$ odd Now the orbit $[r]$ can be unconfined as well: the unconfined particles are $\Omega^{e,r}_{J_{0,2}}$, and $\mathscr{U} \simeq F(\mathbb{Z}_2) \otimes \mathbb{Z}_2 \simeq D(D_2)$. This agrees with (4.102), because $K_{[s]} \cap N_s = \{e,s\}$, so $\mathscr{U} \simeq D(D_2/\mathbb{Z}_2) \simeq D(\mathbb{Z}_2)$. The Hopf kernel is given by $F(e) \otimes \mathbb{C}\mathbb{Z}_2$, and the confined particles $\Omega^{e,r}_{J_{1,3}}$ restrict to the non-trivial representation of this algebra.

We see that in this case, the description of confinement follows the general rules.

4.4.3 Particles in pure *sr*-condensates

The calculation for pure *sr*-condensates is similar to that for pure *s*-condensates. We have $K_{[sr]} = \langle r^2 \rangle \cup sr\langle r^2 \rangle \simeq D_{n/2}$ and $N_{sr} = \{e, r^{n/2}, sr, sr^{n/2+1}\} \simeq D_2$. The residual symmetry algebra $\mathscr{T}$ is isomorphic to $F(\mathbb{Z}_2) \otimes \mathbb{C}D_2$. Its irreducible representations are

$$\Omega^e_{J_i}, \quad \Omega^r_{J_i}. \quad i = 0,1,2,3 \tag{4.113}$$

The branching rules for restrictions of $D(D_n)$-representations are identical to (4.111), except for

$$\begin{cases} \Pi^{e,r^{n/2}}_{J_{0,3}} \simeq \Omega^e_{J_0}, \quad \Pi^{e,r^{n/2}}_{J_{1,2}} \simeq \Omega^e_{J_1} & \frac{n}{2} \text{ even} \\ \Pi^e_{J_{0,1}} \simeq \Omega^e_{J_{0,1}}, \quad \Pi^e_{J_{2,3}} \simeq \Omega^e_{J_{3,2}}, \quad \Pi^{r^{n/2}}_{J_{0,1}} \simeq \Omega^r_{J_{0,1}}, \quad \Pi^{r^{n/2}}_{J_{2,3}} \simeq \Omega^r_{J_{3,2}} & \frac{n}{2} \text{ odd} \end{cases}$$
$$\Pi^s_{J_i} \simeq \Omega^r_{J_i}, \quad \Pi^{sr}_{J_i} \simeq \Omega^e_{J_i} \tag{4.114}$$

For confinement, we again distinguish between the cases whether $\frac{n}{2}$ is even or odd.

Case $\frac{n}{2}$ even The representations $\Omega^e_{J_{0,2}}$ satisfy the unconfining relations (4.100) and (4.101), which would lead to $\mathscr{U} \simeq F(e) \otimes \mathbb{C}\mathbb{Z}_2$. Again, this cannot be made into a braided Hopf algebra. When we then restrict ourselves to $\mathscr{U} \simeq D(e)$, only the trivial representation is unconfined, the rest classify inequivalent domain walls.

Case $\frac{n}{2}$ odd The representations $\Omega^{e,r}_{J_{0,2}}$ are unconfined, so that $\mathscr{U} \simeq F(\mathbb{Z}_2) \otimes \mathbb{C}\mathbb{Z}_2 \simeq D(\mathbb{Z}_2)$. This follows the general description $\mathscr{U} \simeq D\big(N_{sr}/(K_{[sr]} \cap N_{sr})\big)$. The Hopf kernel is isomorphic to $F(e) \otimes \mathbb{C}\mathbb{Z}_2$, and the unconfined representations restrict to the trivial, the confined representations to the non-trivial representation of this Hopf algebra.

4.4.4 Another case: a two-flux sum condensate

We were able to show that in general gauge-invariant and pure flux magnetic condensate obey trivial self-braiding. This does not exclude, however, that in special cases there are other states, other superpositions of pure flux basis vectors, that have trivial self-braiding as well. We treat one such example here.

We take $n/2$ even. In that case the element $sr^{n/2}$ is in the same conjugacy class as s. We assume a condensate forms in the state $|\phi\rangle = |s\rangle + |sr^{n/2}\rangle \in V^s_1$, a superposition of two out of $n/2$ possible basis vectors. For $n = 4$ this is identical to the class sum (gauge-invariant state) and our results should reduce those found in §4.3.3.

Trivial self-braiding and spin Particles represented by a magnetic representation always have trivial spin factor, due to the trivial electric representation.

The demand for trivial self-braiding is:

$$\Pi^s_1 \otimes \Pi^s_1(R)|\phi\rangle \otimes |\phi\rangle = |\phi\rangle \otimes |\phi\rangle. \tag{4.115}$$

Writing out the left-hand side:

$$\begin{aligned}
\Pi_1^s \otimes \Pi_1^s(R)|\phi\rangle \otimes |\phi\rangle &= \sum_{h \in D_n} \Pi_1^s(P_h, e) \otimes \Pi_1^s(1,h)(|\phi\rangle \otimes |\phi\rangle) \\
&= \sum_{h \in D_n} \Pi_1^s(P_h, e)(|s\rangle + |sr^{n/2}\rangle) \otimes \Pi_1^s(1,h)(|s\rangle + |sr^{n/2}\rangle) \\
&= |s\rangle \otimes \Pi_1^s(1,s)(|s\rangle + |sr^{n/2}\rangle) + \\
&\quad |sr^{n/2}\rangle \otimes \Pi_1^s(1,sr^{n/2})(|s\rangle + |sr^{n/2}\rangle) \\
&= |s\rangle \otimes (|s\rangle + |sr^{n/2}\rangle) + |sr^{n^2}\rangle \otimes (|s\rangle + |sr^{n/2}\rangle) \\
&= |\phi\rangle \otimes |\phi\rangle.
\end{aligned} \tag{4.116}$$

Therefore this constitutes a valid condensate.

Residual symmetry algebra The residual symmetry algebra is spanned by matrix elements (ρ_{ab}, g) of representations of $D(D_n)^*$, that satisfy the relation (3.8). Now

$$(1, r^k) \rightharpoonup |s\rangle + |sr^{n/2}\rangle = |sr^{-2k}\rangle + |sr^{n/2-2k}\rangle. \tag{4.117}$$

The right-hand side of this equation will be equal to the condensate vector itself when $k = 0 \bmod \frac{n}{4}$ (recall that $\frac{n}{2}$ is even). One can readily see that in those cases, the elements sr^k for the same k will leave the condensate invariant as well. This gives the set $\langle r^{n/4}\rangle \cup s\langle r^{n/4}\rangle \simeq D_4$, which all act on the condensate vector by a scalar factor 1.

Next, we wish to know which representations ρ of D_n are equal to the identity matrix on the distinguished element s. This is only so for the representations J_0 and J_2, which together span $F(\mathbb{Z}_2)$.

So we have found that $\mathscr{T} \simeq F(\mathbb{Z}_2) \tilde{\otimes} \mathbb{C}D_4$. We see that this is indeed different from pure s-condensates (for which $\mathscr{T} \simeq F(\mathbb{Z}_2) \tilde{\otimes} \mathbb{C}D_2$) and that it is equal to the gauge-invariant s-condensate for $n = 4$.

The representations of $\mathscr{T}$ are $\Omega^{e,r}_{J_i}$ (one-dimensional) and $\Omega^{e,r}_{\alpha}$ (two-dimensional).

The branching rules from restrictions of $D(D_n)$-representations Π^A_α to these $\mathscr{T}$-representations are given by:

$$\begin{aligned}
\Pi^{e,r^{n/2}}_{J_i} &\simeq \Omega^e_{J_i} \\
\Pi^{e,r^{n/2}}_{\alpha_j} &\simeq \begin{cases} \Omega^e_\alpha & j \text{ odd} \\ \Omega^e_{J_0} \oplus \Omega^e_{J_1} & j \bmod 4 = 0 \\ \Omega^e_{J_2} \oplus \Omega^e_{J_3} & j \bmod 4 = 2 \end{cases} \\
\Pi^{r^k}_{\beta_l} &\simeq \begin{cases} \Omega^{r^k \bmod 2}_\alpha & l \text{ odd} \\ \Omega^{r^k \bmod 2}_{J_0} \oplus \Omega^e_{J_1} & l \bmod 4 = 0 \\ \Omega^{r^k \bmod 2}_{J_2} \oplus \Omega^e_{J_3} & l \bmod 4 = 2 \end{cases} \\
\Pi^{s,sr}_{J_0} &\simeq \tfrac{n}{4}\Omega^{e,r}_{J_0} \oplus \Omega^{e,r}_{J_2} \\
\Pi^{s,sr}_{J_1} &\simeq \tfrac{n}{4}\Omega^{e,r}_{J_1} \oplus \Omega^{e,r}_{J_3} \\
\Pi^{s,sr}_{J_{2,3}} &\simeq \tfrac{n}{4}\Omega^{e,r}_{\alpha}
\end{aligned} \tag{4.118}$$

Confinement Because $\mathscr{T}$ is of the general form $F(H/K)\tilde{\otimes}\mathbb{C}N$, with $H = D_n$, $K = D_{n/2}$ and $N = D_4$, we can use proposition 3.5 to determine the unconfined algebra $\mathscr{U}$. The two demands on the representations Ω^B_β are

$$\sum_{k \text{ even}} (1,\eta r^k) \rightharpoonup |\phi\rangle = \sum_{k \text{ even}} (1,r^k) \rightharpoonup |\phi\rangle \quad \forall \eta \in B, \tag{4.119}$$

$$\beta(s) = \beta(sr^{n/2}) = 1. \tag{4.120}$$

The first relation is trivially satisfied when $B = e$. When $B = r$, the relation holds only when $n/4$ is odd. The second relation holds only when β is either J_0 or J_2. So we find

$$\mathscr{U} \simeq \begin{cases} F(\mathbb{Z}_2)\tilde{\otimes}\mathbb{C}\mathbb{Z}_2 & \frac{n}{4} \text{ odd} \\ F(e)\tilde{\otimes}\mathbb{C}\mathbb{Z}_2 & \frac{n}{4} \text{ even} \end{cases} \tag{4.121}$$

So when $n/4$ is odd, this conforms to the general form $\mathscr{U} \simeq D\big(N/(K\cap N)\big)$, because then $K\cap N = D_{n/2}\cap D_4 = D_2$. However, when $n/4$ is even, we get the same form as we did for pure s-condensates in §4.4.2. Because now $K\cap N = D_{n/2}\cap D_4 = D_4$, we may decide to choose $\mathscr{U} \simeq D(e)$, in which case we can define a braiding for the unconfined algebra.

The rest of the treatment is actually analogous to both cases ($n/2$ even or odd) of pure s-condensates, and we do not repeat that here. Do note, that when $n = 4$, $n/4 = 1$ is odd, and the unconfined algebra $D(\mathbb{Z}_2)$ we found above corresponds correctly to the gauge-invariant condensate.

We have now seen one example of another type of valid magnetic condensate, which is still described by the general formalism of chapter 3. The residual symmetry algebra and the unconfined algebra may be larger than that of the corresponding pure-flux condensates.

4.5 Dyonic condensates

Dyonic condensates consist of particles represented by state vectors in the representation space of representations of the form Π^A_α, where $A \neq [e]$ and α is not the trivial representation J_0 or β_0.

The dyonic condensates are the most interesting objects in our discussion. As we will see, some dyonic condensates are not of the form $F(H/K)\tilde{\otimes}\mathbb{C}N$; then the representation structure may not be directly deducible, and confinement can no longer be described by the reduced conditions (3.27) and (3.28).

Trivial spin and self-braiding Let us first look at which irreducible representations of $D(D_n)$ allow condensate vectors which have trivial spin factor and trivial self-braiding. By proposition A.8 some condensate of every quantum double of a group with non-trivial center will satisfy the conditions, so we expect at least one condensate of representations with conjugacy class $[r^{n/2}]$ to be a valid condensate.

If we look at table 4.1, we see that $\Pi^{r^{n/2}}_{J_1}$, $\Pi^{r^{n/2}}_{\alpha_j}$ (j even), $\Pi^{r^k}_{\beta_l}$ ($lk \bmod n = 0$), $\Pi^{s}_{J_2}$ and $\Pi^{sr}_{J_3}$ always have trivial spin; $\Pi^{r^{n/2}}_{J_2}$ and $\Pi^{r^{n/2}}_{J_3}$ will have as well if $n/2$ is even.

The representation space of Π^A_α is spanned by basis vectors $|a_i, v_j\rangle$ as in (2.8). A general vector will then be $\sum_i \lambda_i |a_i, \mathbf{v}\rangle$, with $\lambda_i \in \mathbb{C}$, and we have left the state of the

$\mathbb{C}H$-representation implicit. The trivial self-braiding condition (3.20) is then written as

$$\begin{aligned}\sum_i \lambda_i |a_i, \mathbf{v}\rangle \otimes \sum_j \lambda_j |a_j, \mathbf{v}\rangle &= \tau \circ \sum_h \left(\Pi^A_\alpha(P_h, e)(\Sigma_i \lambda_i |a_i, \mathbf{v}\rangle) \otimes \Pi^A_\alpha(1,h)(\Sigma_j \lambda_j |a_j, \mathbf{v}\rangle)\right) \\ &= \tau \circ \sum_i \left(\lambda_i |a_i, \mathbf{v}\rangle \otimes \Pi^A_\alpha(a_i)(\Sigma_j \lambda_j |a_j, \mathbf{v}\rangle)\right) \\ &= \tau \circ \sum_i \left(\lambda_i |a_i, \mathbf{v}\rangle \otimes (\Sigma_j \lambda_j |a_i a_j a_i^{-1}, \alpha(k_x^{-1} a_i k_j)\mathbf{v}\rangle)\right)\end{aligned} \tag{4.122}$$

If we want this equation to be satisfied, then $\alpha(k_x^{-1} a_i k_j)$ must be the identity matrix for all i and j, and moreover, $\forall i, j \quad \lambda_j = \lambda_{i \rightharpoonup j}$, where $\lambda_{i \rightharpoonup j}$ denotes the coefficient of the coset to which $a_i a_j a_i^{-1}$ belongs.

These seem perhaps to be very strong demands, but when all the λ_i are equal, or when the elements a_i of the conjugacy class commute, this will boil down to just the demand on α. However, in each specific case there may be other linear combinations of basis vectors which have trivial self-braiding as well.

For $D(D_n)$, we find that the trivial spin factor condition is equal to that of trivial self-braiding for all vectors in all representations except for those in $\Pi^s_{J_2}$ and $\Pi^{sr}_{J_3}$. The conjugacy classes $[s]$ and $[sr]$ are large and its elements do not all commute, so that we may only expect certain state vectors to comply.

For example, when $A = [s] = \{s, sr^2, \ldots, sr^{-2}\}$, we have $sr^{2p}sr^{2q}sr^{2p} = sr^{2(2p-q)}$. If $n/2$ is odd, we will reach all sr^{2q} again and the only allowed vector is the one with identical coefficients. But if $n/2$ is even we will have 'steps of four'. We can then devide $[s]$ into two equivalence classes, labelled by s and sr^2, the elements of which can be reached by conjugation by some sr^{2p}. Then the coefficient of a certain basis vector need only be equal for all elements in its class, but can differ from those of the other class. In other words, the vector

$$\sum_{i=0}^{n/4-1} |sr^{4i}, \mathbf{v}\rangle + \lambda |sr^{4i+2}, \mathbf{v}\rangle \tag{4.123}$$

will have trivial self-braiding if the condition on α is satisfied for any $\lambda \in \mathbb{C}$. For $\Pi^{sr}_{J_3}$ there is a similar calculation.

Condensates of state vectors $|\phi\rangle$ for which the flux-part is one-dimensional, like for pure flux-condensates, can have trivial self-braiding as well, and in specific cases there may be other vectors still. We will only treat class sum dyonic condensates, for which $\lambda_i = \lambda \quad \forall i$.

Particles in the condensates We calculate the residual symmetry algebras of these condensates by looking at which representations (ρ, E_g) of $D(D_n)^*$ satisfy (3.8). When all representations for a certain condensate of Π^A_α have $\chi_\rho(a)/d_\rho = 1$, then $\mathscr{T}$ is of the form $F(D_n/K) \tilde{\otimes} \mathbb{C}N$, and we will give the algebra in question.

Many condensates are not of this form. If they are not, the residual symmetry algebra will be larger than one would expect, that is with $F(H/K)$ where K the group generated by the conjugacy class A, and with $\mathbb{C}N$ where N the intersection of K and $N_{|\phi\rangle}$, the stabilizer of $|\phi\rangle$. The algebra $F(H/K) \tilde{\otimes} \mathbb{C}N$ will be then be a subalgebra of $\mathscr{T}$ (cf. [7, prop 9 in §7.3]). We have assumed† that $\mathscr{T}$ is a Hopf algebra, and then $\dim \mathscr{T}$ divides $\dim D(D_n)$, and $\dim F(H/K) \tilde{\otimes} \mathbb{C}N$ divides $\dim \mathscr{T}$ by lemma A.5.

† See the remark on p.28.

Up until now, no general description for these cases has been found. Of course, manual calculations can be performed, but this can get quite complicated. In the next chapter, we fully describe a condensate of the representation $\Pi^{r^2}_{J_1}$ in a $D(D_4)$-theory. Other condensates will not be discussed, except for some crude indications of the form of the residual symmetry algebra below.

4.5.1 Particles in dyonic $r^{n/2}$-condensates

The conjugacy class $[r^{n/2}]$ contains but one element, because $r^{n/2}$ is central in D_n. We determine the value of the character on this element for all representations of D_n:

$$J_0 : 1, \quad J_1 : 1, \quad J_2 : (-1)^{n/2}, \quad J_3 : (-1)^{n/2}, \quad \alpha_m : (-1)^m \quad m = 1, \ldots, \frac{n}{2} - 1. \tag{4.124}$$

From corrolary 3.3 we know that we have a residual algebra of the form $F(H/K) \tilde{\otimes} \mathbb{C}N$ only when for the representations $(\rho, g) \equiv (\rho, E_g)$ of $D(D_n)^*$ satisfying (3.8) this value is 1. With the values given above, we expect that this does not hold in many cases.

The $\Pi^{r^{n/2}}_{J_1}$-condensate We calculate $\Pi^{r^{n/2}}_{J_1}(1, g^{-1})|\phi\rangle$ for all $g \in D_n$. Because $r^{n/2}$ is central in D_n, $|\phi\rangle$ is equivalent to just the vector in the representation space of J_1. From (4.4):

$$J_1(r^{-k}) = 1, \quad J_1(sr^k) = -1. \tag{4.125}$$

Then the following $D(D_n)^*$-representations (ρ, g) leave any condensate in $V^{r^{n/2}}_{J_1}$ invariant:

$$(J_j, r^k),\ (\alpha_m, r^k)\ m \text{ even},\ (\alpha_m, sr^k)\ m \text{ odd} \qquad \frac{n}{2} \text{ even}, \tag{4.126}$$

$$(J_{0,1}, r^k),\ (J_{2,3}, sr^k),\ (\alpha_m, r^k)\ m \text{ even},\ (\alpha_m, sr^k)\ m \text{ odd} \qquad \frac{n}{2} \text{ odd}. \tag{4.127}$$

Because $n > 2$, we always have at least (α_1, sr^k) as $D(D_n)^*$-representations satisfying (3.8), and tr $\alpha_1(r^{n/2}) = -1$, so there is no residual algebra of the form $F(H/K) \tilde{\otimes} \mathbb{C}N$.

The case for $n = 4$ is treated in the next chapter. It will turn out that the residual symmetry algebra $\mathscr{T}$ will consist of two copies of $F(D_2) \tilde{\otimes} \mathbb{C}\mathbb{Z}_4$, which are connected through the action of one special element $(\alpha_{11} + \alpha_{21}, s)$. The unconfined algebra $\mathscr{U}$ will also consist of two copies of $D(\mathbb{Z}_2)$. This explicitly shows that the residual symmetry in these Hopf-symmetric theories may be larger than one would naively expect.

The $\Pi^{r^{n/2}}_{J_{2,3}}$-condensates These condensates only have trivial spin and self-braiding if $n/2$ is even.

We have

$$J_{2,3}(r^{-k}) = (-1)^k, \quad J_2(sr^k) = (-1)^k, \quad J_3(sr^k) = (-1)^{k+1}. \tag{4.128}$$

Then the following $D(D_n)^*$-representations (ρ, g) leave any condensate in $V^{r^{n/2}}_{J_2}$ invariant:

$$(J_j, r^k),\ (J_j, sr^k) \quad k \text{ even},\ (\alpha_m, r^k), (\alpha_m, sr^k) \quad \text{sgn}(mk) = +1. \tag{4.129}$$

The following (ρ, g) leave any condensate in $V_{J_3}^{r^{n/2}}$ invariant:

$$(J_j, r^k)\ k \text{ even},\ (J_j, sr^k)\ k \text{ odd},\ (\alpha_m, r^k)\ \mathrm{sgn}(mk) = +1,\ (\alpha_m, sr^k)\ \mathrm{sgn}(mk) = -1. \tag{4.130}$$

We have at least (α_1, sr) for J_2 and (α_1, s) for J_3 satisfying (3.8), so there is no residual algebra of the form $F(H/K)\tilde{\otimes}\mathbb{C}N$.

The $\Pi_{\alpha_j}^{r^{n/2}}$-condensates To satisfy trivial spin and self-braiding, j has to be even.

According to (4.124), the value of $\chi_\rho(r^{n/2})/d_\rho$ can be either $+1$ or -1. For those ρ for which this expression equals 1, we find the subgroup $\langle r^x\rangle$ of D_n with the property $g^{-1} \rightharpoonup |\phi\rangle = 1|\phi\rangle$, analogous to the derivation of §4.2.4. In that section it is also explained that some $|\phi\rangle$ allow the elements $s\langle r^x\rangle$ as well.

Now when x is even, $j\frac{x}{2} = \frac{j}{\gcd(n,j)}\frac{n}{2} = \frac{n}{2} \bmod n$, because in this case $\frac{j}{\gcd(n,j)}$ is odd. The value of α_j on the elements $r^{x/2+zx}$, $z = 0, \ldots, \gcd(n, j)-1$ is then -1 times the unit matrix. These elements will be coupled to those representations of D_n for which the representation value on $r^{n/2}$ is -1. When $\frac{n}{2}$ is odd, x can never be even, because j is even.

When $|\phi\rangle$ is of the appropriate form, all these elements multiplied (from the right) by (J_0, s) will leave the condensate invariant as well. We find that the following elements of $D(D_n)$ span the residual symmetry algebra $\mathscr{T}$:

$$\begin{cases} (J_i, r^{zx}), & (\alpha_{2m}, r^{zx}), \quad (\alpha_{2m+1}, r^{x/2+zx}), \quad \frac{n}{2} \text{ even}, x \text{ even} \\ (J_i, r^{zx}), & (\alpha_{2m}, r^{zx}), \qquad\qquad\qquad\qquad\quad\ \frac{n}{2} \text{ even}, x \text{ odd} \\ (J_{0,1}, r^{zx}), & (\alpha_{2m}, r^{zx}), \qquad\qquad\qquad\qquad\quad\ \frac{n}{2} \text{ odd} \end{cases} \tag{4.131}$$

accompanied by the same number of basis vectors obtained by right multiplication of the above by (J_0, s), for some $|\phi\rangle$.

We see that when x is odd, all the basis vectors have the property $\chi_\rho(r^{n/2})/d_\rho = 1$, and the residual symmetry algebra is therefore of the form $F(H/K)\tilde{\otimes}\mathbb{C}N$ by corollary 3.3. This amounts to

$$\mathscr{T} \simeq F(D_{n/2})\tilde{\otimes}\mathbb{C}\mathbb{Z}_{\gcd(n,j)} \text{ or } F(D_{n/2})\tilde{\otimes}\mathbb{C}D_{\gcd(n,j)}. \tag{4.132}$$

When adhering to the general form $\mathscr{U} \simeq D\big(N_{r^{n/2}}/(K_{r^{n/2}} \cap N_{r^{n/2}})\big)$ for the unconfined algebra, we find $\mathscr{U} \simeq D(\mathbb{Z}_{\gcd(n,j)})$ or $D(D_{\gcd(n,j)})$, because $\gcd(n, j)$ is always even, as both j and $\frac{n}{2}$ are.

4.5.2 Particles in dyonic r^k-condensates

Recall that the only $\Pi_{\beta_l}^{r^k}$-condensates with trivial spin and self-braiding are those with $kl = 0 \bmod n$.

The conjugacy class $[r^k] = \{r^k, r^{-k}\}$ for each k. The values of the characters on the distinguished element r^k are

$$J_0 : 1, \quad J_1 : 1, \quad J_2 : (-1)^k, \quad J_3 : (-1)^k, \quad \alpha_m : q^{mk} + q^{-mk} \quad m = 1, \ldots, \frac{n}{2} - 1. \tag{4.133}$$

Now $\chi_{\alpha_m}(r^{n/2})/d_{\alpha_m} = \frac{1}{2}(q^{mk} + q^{-mk})$ can only be a root of unity if $mk = -mk \bmod n$ (cf. proof of proposition 3.2), so when $mk = 0 \bmod \frac{n}{2}$, i.e. $m = z\frac{n/2}{\gcd(n/2,k)}$ $\ z \in \mathbb{Z}$; if

$\gcd(n/2,k)=1$, this can never be satisfied, as $m\in\{1,\dots,\frac{n}{2}-1\}$; also if $n/2$ is odd and k is even, q^{mk} can never be -1. If it is not a root of unity, the matrix elements of such representations cannot leave the condensate invariant by corrolary 3.3.

The representation space $V^{r^k}_{\beta_l}$ is two-dimensional, and the two basis vectors are labelled by the elements of $[r^k]$, namely r^k and r^{-k}; the coset representatives are $k_{r^k}=e$ and $k_{r^{-k}}=s$. A general vector is given by a linear combination of these with coefficients λ_{r^k} and $\lambda_{r^{-k}}$. The action of $\Pi^{r^k}_{\beta_l}(1,g^{-1})$ is then given by

$$\Pi^{r^k}_{\beta_l}(1,r^{-p})\begin{pmatrix}\lambda_{r^k}\\ \lambda_{r^{-k}}\end{pmatrix}=\begin{pmatrix}\beta_l(r^{-p})\lambda_{r^k}\\ \beta_l(r^{p})\lambda_{r^{-k}}\end{pmatrix}=\begin{pmatrix}q^{-lp}\lambda_{r^k}\\ q^{lp}\lambda_{r^{-k}}\end{pmatrix}, \tag{4.134}$$

$$\Pi^{r^k}_{\beta_l}(1,sr^{p})\begin{pmatrix}\lambda_{r^k}\\ \lambda_{r^{-k}}\end{pmatrix}=\begin{pmatrix}\beta_l(r^{-p})\lambda_{r^{-k}}\\ \beta_l(r^{p})\lambda_{r^{k}}\end{pmatrix}=\begin{pmatrix}q^{-lp}\lambda_{r^{-k}}\\ q^{lp}\lambda_{r^{k}}\end{pmatrix}. \tag{4.135}$$

We need the vectors to be eigenvectors of these actions with eigenvalues either 1 or -1, according to (4.133). For the elements $(1,r^p)$, this leads to the condition $lp=0 \bmod n$ or $lp=\frac{n}{2} \bmod n$, independent of the values of $\lambda_{r^{\pm k}}$. We denote $x\equiv\frac{n}{\gcd(n,l)}$ and $\tilde{x}=\frac{n/2}{\gcd(n/2,l)}$. The smallest p satisfying this equation is x resp. $x+\tilde{x}$, and the subgroups $\{r^{zx}\}$ resp. $\{r^{zx+\tilde{x}}\}$, $z\in\mathbb{Z}$, which are both isomorphic to $\mathbb{Z}_{\gcd(n,l)}$ leave the condensate invariant. For $n/2$ odd, lp can only be -1, when l is odd as well, so then we have an extra condition, and fewer elements may be left.

For the elements $(1,sr^p)$, the outcome does depend on the coefficients. We have two equations leading both to the same condition:

$$\lambda_{r^{-k}}=\pm q^{lp}\lambda_{r^k}. \tag{4.136}$$

If the values of $\lambda_{r^{\pm k}}$ allow this equation to hold for some p, then the elements sr^{zx+p} or $sr^{zx+\frac{n/2}{\gcd(n/2,l)}+p}$ also leave the condensate invariant for any $z\in\mathbb{Z}$.

When k is even, we find the following (ρ,g) to satisfy (3.8), where for $\frac{n}{2}$ odd, the elements containing $\tilde{x}$ only occur when l is odd:

$$\begin{gathered}(J_j,r^{zx}),\ (\alpha_m,r^{zx}),\ (\alpha_{m'},r^{zx+\tilde{x}})\\ \text{if (4.136) holds also: } (J_j,sr^{zx+p}),\ (\alpha_m,sr^{zx+p}),\ (\alpha_{m'},sr^{zx+\tilde{x}+p})\\ z\in 0,\dots,\gcd(n,l),\ m=z'\tfrac{n}{\gcd(n,k)},\ m'=z'\tfrac{n}{\gcd(n,k)}+\tfrac{n/2}{\gcd(n/2,k)},\ z'\in 0,\dots,\gcd(n,k).\end{gathered} \tag{4.137}$$

When k is odd, $\rho=J_{2,3}(r^k)=-1$ and they are then coupled to the $F(D_n)$-representations E_g with $g=(s)r^{zx+\tilde{x}}$.

There are some cases where the only $D(D_n)^*$-representations found in this manner all have $\chi_\rho(r^k)/d_\rho=1$. This occurs when k is even and $n/2$ is odd, or when k is odd, $n/2$ is odd and l is even. In these cases the residual symmetry algebra $\mathscr{T}$ is of the form $F(D_{\gcd(n,k)})\tilde{\otimes}\mathbb{C}\mathbb{Z}_{\gcd(n,l)}$, or $F(D_{\gcd(n,k)})\tilde{\otimes}\mathbb{C}D_{\gcd(n,l)}$ when we can satisfy (4.136).

The particles in the condensate are irreducible representations of this algebra, and the branching rules are calculated in the same way as we did for electric and magnetic condensates.

The unconfined algebra will be given by $\mathscr{U}\simeq D(\mathbb{Z}_w)$ or $\mathscr{U}\simeq D(D_w)$, where w is determined by

$$\gcd(n,k)\gcd(n,l)=wn. \tag{4.138}$$

This can be seen by the following argument [7, §12.2]: we know that $kl = 0 \bmod n$, and therefore $\gcd(n,l)\gcd(n,k) = 0 \bmod n$, which gives us (4.138). Now $K = \langle r^{\gcd(n,k)} \rangle$ and $N = \langle r^x \rangle$. Then $K \subset N$, because $\gcd(n,k) = w\frac{n}{\gcd(n,l)} = wx$. The unconfined algebra is shown to be $\mathscr{U} \simeq D(N/K \cap N)$, which now amounts to $D(N/K) = D(\langle r^x \rangle / \langle r^{wx} \rangle) \simeq D(\mathbb{Z}_w)$. When $N \simeq D_{\gcd(n,l)}$ the same argument leads to $\mathscr{U} \simeq D(D_w)$.

4.5.3 Particles in dyonic s- and sr-condensates

We have already seen that only $\Pi^s_{J_2}$ and $\Pi^{sr}_{J_3}$ allow trivial self-braiding, and again we look at the class sum state vector only. In that case very few elements $(1,g)$ leave the condensate vector invariant: take $|\phi\rangle = \sum_{k=0}^{n/2-1} |sr^{2k}\mathbf{v}_{sr^{2k}}\rangle \in V^s_{J_2}$. The action of $(1,r^p)$ on $|\phi\rangle$ is

$$(1,r^p) \rightharpoonup \sum_{k=0}^{n/2-1} |sr^{2k}\mathbf{v}_{sr^{2k}}\rangle = \sum_k J_2(k^{-1}_{sr^{2k-2p}} r^p k_{sr^{2k}}) sr^{2k-2p}. \tag{4.139}$$

Now the representation values of J_2 differ in general depending on the value of k. In fact, one can calculate that the argument of J_2 will be e when $p = 0 \vee p \geq \frac{n}{2} - k$ and will be $r^{n/2}$ when $p \neq 0 \wedge p < \frac{n}{2} - k$, for $p = 1, \ldots, \frac{n}{2} - 1$. Therefore only when $p = 0, \frac{n}{2}$ all values of J_2 will be equal, and $|\phi\rangle$ is left invariant. Similarly, the elements $(1, sr^p)$ never leave $|\phi\rangle$ invariant, and there is an analogous argument for $|\phi\rangle \in V^{sr}_{J_3}$.

Particles in dyonic $\Pi^s_{J_2}$-condensates We consider which matrix elements (ρ_{ab}, E_g) (3.8) can be satisfied. Above, we argued that only $g = e, r^{n/2}$ can be allowed. The value of $\chi_\rho(s)/d_\rho$ is 1 for $\rho = J_{0,2}$ and -1 for $\rho = J_{1,3}$. Then the matrix elements which span $\mathscr{T}$ are

$$(J_{0,2}, E_e) \text{ and } (J_{1,3}, E_{r^{n/2}}). \tag{4.140}$$

This Hopf algebra $\mathscr{T}$ is not of the form $F(H/K_{[s]}) \tilde{\otimes} \mathbb{C}N$, and we cannot say any more without turning to explicit calculation.

Particles in dyonic $\Pi^{sr}_{J_3}$-condensates Following the same reasoning, we find that the matrix elements

$$(J_{0,3}, E_e) \text{ and } (J_{1,2}, E_{r^{n/2}}). \tag{4.141}$$

span the residual symmetry algebra, which is again not of the form $F(H/K_{[sr]}) \tilde{\otimes} \mathbb{C}N$.

With this we have mentioned all dyonic condensates for which the state vector is the sum of all possible flux-components. There are other possibilities, as mentioned at the beginning of this section, which we are not considering, but which may be calculated in the same way we have done throughout this chapter.

4.6 Summary

Using the machinery developed in chapter 3, we have calculated the residual symmetry algebra, the spectrum of possible excitations in the condensate, the branching rules of the particles in the unbroken theory, the unconfined particles and the unconfined algebra, the possible domain walls and the branching of condensate excitations to these walls for many possible condensates in theories with the symmerty of quantum doubles of even dihedral groups.

Which condensates have we treated exactly to this extent? All electric condensates, all gauge-invariant and all pure flux magnetic condensates. For all possible (showing trivial self-braiding and spin factor) class-sum dyonic condensates, we have calculated at least the form of the residual symmetry algebra and of the unconfined algebra for those cases for which the residual symmetry algebra is of the general form $F(H/K)\tilde{\otimes}\mathbb{C}N$.

So the cases that are left out are several magnetic condensates for which the condensate vector is some superposition of pure flux basis vectors and not the entire class sum, and which show trivial self-braiding. One example of these is mentioned. The dyonic condensates which are not of the general form must be calculated by hand, which is done for one case in the next chapter.

Some the peculiarities of Hopf symmetry breaking have come forward in the calculations of this chapter. One of those is that in a purely magnetic condensate, even electric degrees of freedom may be broken. Also, we have seen explicitly in §4.4.2 that the relations determining the unconfined algebra $\mathscr{U}$ may be insufficient to provide a braiding derived from the original algebra.

Chapter 5

The $\Pi^{r^2}_{J_1}$-condensate in $D(D_4)$

In this chapter we fully describe the residual symmetry algebra and confinement in a $D(D_4)$ theory when a condensate is formed with condensate vector in $V^{r^2}_{J_1}$. We have already seen that the residual symmetry algebra $\mathscr{T}$ is not just $F(D_4/\langle r^2\rangle)\,\tilde{\otimes}\,\mathbb{C}\langle r\rangle \simeq F(D_2)\,\tilde{\otimes}\,\mathbb{C}\mathbb{Z}_4$, as one might expect, but will be a larger Hopf algebra with this one as a sub-Hopf algebra. The fact that this example somehow falls outside of our general scheme makes it of course of special interest. That is why we want to analyze this case in detail.

It turns out that there is a basis in $\mathscr{T}$ so that it can be written in a convenient form. It even turns out that this basis has the nice property that the multiplication of two basis vector always gives another basis vector, as if it were a group algebra. However, the coproduct and braid matrix are still non-trivial, so it is not isomorphic to a group algebra as a Hopf algebra.

This basis can be extended to the whole algebra $D(D_4)$. Then it may be hoped that we can choose such a basis for any quantum double, or perhaps just for $D(D_n)$ for any even n. Unfortunately, this turns out not to be the case. We can construct a Hopf algebra from a group algebra so that $D(D_n)$ is a Hopf quotient, but this does not add much structure to what is already known of the quantum doubles.

5.1 Residual symmetry algebra

The group D_4 has five conjugacy classes $\{e\}$, $\{r^2\}$, $\{r^k, r^{-k}\}$, $\{s, sr^2\}$, $\{sr, sr^3\}$, and five irreducible representations J_i and $\alpha = \alpha_1$. We are interested in a condensate of the dyonic representation $\Pi^{r^2}_{J_1}$, which is one-dimensional, so the result will be the same for all choices of condensate vectors $|\phi\rangle$.

We start with determining the residual symmetry algebra. We know from §4.5.1 that this will leave us with an algebra which is not of the form $F(D_n/K)\,\tilde{\otimes}\,\mathbb{C}N$. From (4.126), we see that the matrix elements of the following representations of $D(D_4)^*$ leave $|\phi\rangle$ invariant, and which therefore span $\mathscr{T}$:

$$(J_j, E_{r^k}),\ (\alpha, E_{sr^k}) \quad j,k = 0,1,2,3. \tag{5.1}$$

There are $4\times 4 = 16$ elements (J_j, E_{r^k}) and also 16 matrix elements (α_{ab}, E_{sr^k}), because α is two-dimensional. This gives us an Hopf algebra $\mathscr{T}$ of 32 dimensions, whereas $D(D_4)$ is 64-dimensional.

5.1.1 Hopf algebra structure

We know from proposition 3.1 that this should indeed be a sub-Hopf algebra of $D(D_n)$ if these matrix elements close under tensor products. Proposition 3.2 asserts that this is the case, but we calculate it anyway, as it will also provide the multiplication of these elements.

Tensor products of $D(D_4)^*$-representations The tensor product of representations of $D(D_4)^*$ is defined by the comultiplication in $D(D_4)^*$. Recall from (3.4) that this comultiplication is not just the tensor product of the comultiplication of $\mathbb{C}D_4$ and that of $F(D_4)$, but rather a twisted tensor product, coming from the multiplication (2.4) of $D(D_4)$. For $D(D_n)^*$, n even, we find

$$\begin{aligned}
\Delta^*(r^p, P_{r^l}) &= \sum_{l'} (r^p, P_{r^{l'}}) \otimes (r^p, P_{r^{l-l'}}) + (r^p, P_{sr^{l'}}) \otimes (r^{-p}, P_{sr^{l+l'}}), \\
\Delta^*(sr^p, P_{r^l}) &= \sum_{l'} (sr^p, P_{r^{l'}}) \otimes (sr^{p+2l'}, P_{r^{l-l'}}) + (sr^p, P_{sr^{l'}}) \otimes (sr^{2l'-p}, P_{sr^{l+l'}}), \\
\Delta^*(r^p, P_{sr^l}) &= \sum_{l'} (r^p, P_{r^{l'}}) \otimes (r^p, P_{sr^{l+l'}}) + (r^p, P_{sr^{l'}}) \otimes (r^{-p}, P_{r^{l-l'}}), \\
\Delta^*(sr^p, P_{sr^l}) &= \sum_{l'} (sr^p, P_{r^{l'}}) \otimes (sr^{p+2l'}, P_{sr^{l+l'}}) + (sr^p, P_{sr^{l'}}) \otimes (sr^{2l'-p}, P_{r^{l-l'}}),
\end{aligned} \tag{5.2}$$

or, when all put into one equation:

$$\begin{aligned}
\Delta^*(s^q r^p, P_{s^i r^l}) = \sum_{l'} & (s^q r^p, P_{r^{l'}}) \otimes (s^q r^{p+2ql'}, P_{s^i r^{l-(-1)^i l'}}) + \\
& (s^q r^p, P_{sr^{l'}}) \otimes (s^q r^{-p+2ql'}, P_{s^{i+1} r^{l+(-1)^i l'}}).
\end{aligned} \tag{5.3}$$

Tensor products of (J_j, E_{r^k}) For the tensor products of the irreducible representations (J_j, E_{r^k}) this gives us

$$\begin{aligned}
\big((J_j, E_{r^k}) \otimes (J_{j'}, E_{r^{k'}})\big)\Delta^*(s^q r^p, P_{r^l}) &= \sum_{l'} J_j(s^q r^p)\delta_{k,l'} \otimes J_{j'}(s^q r^{p+2ql'})\delta_{k',l-l'} && (5.4) \\
&= \delta_{k+k',l} J_j(s^q r^p) J_{j'}(s^q r^p), && (5.5)
\end{aligned}$$

where the second equality holds because $J_{j'}(s^q r^{p+2z}) = J_{j'}(s^q r^p) \;\; \forall z \in \mathbb{Z}, \; j = 0,1,2,3$. These representations always give zero on elements with P_{sr^l}. So we see that for (J_j, E_{r^k}) the tensor product is identical to the regular tensor product. The J_j then obey the regular tensor product 'decomposition'—they are one-dimensional and decompose into just one irreducible representation—for irreducible D_2-representations. Written as a multiplication table, this gives us

	J_0	J_1	J_2	J_3
J_0	J_0	J_1	J_2	J_3
J_1	J_1	J_0	J_3	J_2
J_2	J_2	J_3	J_0	J_1
J_3	J_3	J_2	J_1	J_0

From this we see that these representations form a group which is isomorphic to D_2. We could have known this beforehand, because these are the representations of $F(D_2)$, the ring of which forms a group isomorphic to D_2 itself because D_2 is Abelian (Pontryagin duality).

The part with E_{r^k} just forms the group $\mathbb{Z}_4$, because as we can read off from (5.4) $E_{r^k}E_{r^{k'}} = E_{r^{k+k'}}$. Therefore the representations (J_j, E_{r^k}), seen as elements of $D(D_4)$ span the subalgebra $F(D_2)\tilde{\otimes}\mathbb{C}\mathbb{Z}_4$, which one would expect from just electric and magnetic symmetry breaking. Note that this algebra is isomorphic to the regular tensor product $F(D_2)\otimes\mathbb{C}\mathbb{Z}_4$, because the action of $\mathbb{C}\mathbb{Z}_4$ is trivial on $F(D_2)$. Consequently, this algebra is commutative, because $\mathbb{Z}_4$ is Abelian; moreover it is a group algebra because $F(D_2) \simeq \mathbb{C}D_2$.

Tensor products of (α, E_{sr^k}) Next, we look at the multiplication of the elements (α_{ab}, E_{sr^k}) amongst each other. The D_4-representation α is defined by

$$\alpha(r^p) = \begin{pmatrix} \imath^p & 0 \\ 0 & (-\imath)^p \end{pmatrix},\ \alpha(sr^p) = \begin{pmatrix} 0 & (-\imath)^p \\ \imath^p & 0 \end{pmatrix}. \tag{5.6}$$

The matrix elements α_{ab} are therefore given by

$$\alpha_{11}(s^i r^p) = \delta_{i,0}\imath^p,\ \alpha_{22}(s^i r^p) = \delta_{i,0}(-\imath)^p, \alpha_{12}(s^i r^p) = \delta_{i,1}(-\imath)^p,\ \alpha_{21}(s^i r^p) = \delta_{i,1}\imath^p. \tag{5.7}$$

For the tensor product we find

$$\begin{aligned}(\alpha, E_{sr^k})\otimes(\alpha, E_{sr^{k'}})\Delta^*(s^q r^p, P_{r^l}) &= \sum_{l'} \alpha(s^q r^p)\delta_{k,l'}\otimes\alpha(s^q r^{-p+2ql'})\delta k', l+l' \\ &= \delta_{k'-k,l}\,\alpha(s^q r^p)\otimes\alpha(s^q r^{-p+2qk}).\end{aligned} \tag{5.8}$$

The action on elements with P_{sr^l} is always zero. For the E_{sr^k} we again find the regular tensor product, extending to the multiplication of a group: $E_{sr^k}E_{sr^{k'}} = E_{r^{k'-k}}$. For the other part, we write the tensor product $(\alpha, E_{sr^k})\otimes(\alpha, E_{sr^{k'}})$ as matrices:

$$\text{on } r^p: \begin{pmatrix} 1 & & & \\ & (-1)^p & & \\ & & (-1)^p & \\ & & & 1 \end{pmatrix}, \quad \text{on } sr^p: \begin{pmatrix} & & & (-1)^k \\ & & (-1)^{-p+k} & \\ & (-1)^{p-k} & & \\ (-1)^k & & & \end{pmatrix}. \tag{5.9}$$

By comparing these matrices to the representations J_j in (4.4), we see that the matrix elements are either zero or a linear combination of certain J_j:

$$(\alpha, E_{sr^k})\otimes(\alpha, E_{sr^{k'}}) = \begin{cases} \frac{1}{2}\begin{pmatrix} J_0+J_1 & & & J_0-J_1 \\ & J_2+J_3 & J_2-J_3 & \\ & J_2-J_3 & J_2+J_3 & \\ J_0-J_1 & & & J_0+J_1 \end{pmatrix}, & k \text{ even} \\ \frac{1}{2}\begin{pmatrix} J_0+J_1 & & & J_1-J_0 \\ & J_2+J_3 & J_3-J_2 & \\ & J_3-J_2 & J_2+J_3 & \\ J_1-J_0 & & & J_0+J_1 \end{pmatrix}, & k \text{ odd} \end{cases} \tag{5.10}$$

Mixed tensor products In the same way, we can calculate the 'mixed' tensor products $(\alpha_{ab}, E_{sr^k}) \otimes (J_j, E_{r^{k'}})$ and $(J_j, E_{r^k}) \otimes (\alpha_{ab}, E_{sr^{k'}})$. We find

$$(\alpha_{ab}, E_{sr^k}) \otimes (J_0, E_{r^{k'}}) = (\alpha_{ab}, E_{sr^{k+k'}}) \tag{5.11}$$

$$(\alpha_{ab}, E_{sr^k}) \otimes (J_1, E_{r^{k'}}) = (-1)^{(1-\delta_{a,b})}(\alpha_{ab}, E_{sr^{k+k'}}) \tag{5.12}$$

$$(\alpha_{ab}, E_{sr^k}) \otimes (J_2, E_{r^{k'}}) = (\alpha_{\overline{a}\overline{b}}, E_{sr^{k+k'}}) \tag{5.13}$$

$$(\alpha_{ab}, E_{sr^k}) \otimes (J_3, E_{r^{k'}}) = (-1)^{(1-\delta_{a,b})}(\alpha_{\overline{a}\overline{b}}, E_{sr^{k+k'}}), \tag{5.14}$$

$$(J_0, E_{r^k}) \otimes (\alpha_{ab}, E_{sr^{k'}}) = (-1)^{k(1-\delta_{a,b})}(\alpha_{ab}, E_{sr^{k'-k}}) \tag{5.15}$$

$$(J_1, E_{r^k}) \otimes (\alpha_{ab}, E_{sr^{k'}}) = (-1)^{(k+1)(1-\delta_{a,b})}(\alpha_{ab}, E_{sr^{k'-k}}) \tag{5.16}$$

$$(J_2, E_{r^k}) \otimes (\alpha_{ab}, E_{sr^{k'}}) = (-1)^{k(1-\delta_{a,b})}(\alpha_{\overline{a}\overline{b}}, E_{sr^{k'-k}}) \tag{5.17}$$

$$(J_3, E_{r^k}) \otimes (\alpha_{ab}, E_{sr^{k'}}) = (-1)^{(k+1)(1-\delta_{a,b})}(\alpha_{\overline{a}\overline{b}}, E_{sr^{k'-k}}) \tag{5.18}$$

Here $a, b = 1, 2$ and $\overline{1} = 2$, $\overline{2} = 1$.

Unit By examining the multiplication rules of the previous paragraph, one sees that the element (J_0, E_e) is a unit for this multiplication.

Comultiplication All in all, we find that this subset does indeed close under the algebra multiplication, and is therefore a subalgebra of $D(D_4)$. We also want it to be closed under comultiplication.

The comultiplication in $D(D_4) \simeq D(D_4)^{**}$ is related to the multiplication of $D(D_4)^*$ by $\Delta(\rho_{ab}, E_g) = (\rho_{ab}, E_g) \circ \mu^*$ when regarded as functions on $D(D_4)^{*\dagger}$.

For (J_j, E_{r^k}) we find

$$\begin{aligned}\Delta(J_j, E_{r^k})\big((g, P_h) \otimes (g', P_{h'})\big) &= (J_j, E_{r^k})(gg', \delta_{h,h'}P_h) = J_j(gg')\delta_{hh'}E_{r^k}(P_h) \\ &= J_j(g)J_j(g')P_h(r^k)P_{h'}(r^k),\end{aligned} \tag{5.19}$$

where the last equality holds because J_j is a representation. So we see that $\Delta(J_j, E_{r^k}) = (J_j, E_{r^k}) \otimes (J_j, E_{r^k})$, the trivial coproduct. We could have expected this, as we noted before that the (J_j, E_{r^k}) form a group algebra.

For the elements (α_{ab}, E_{sr^k}) it is a little more complicated. The E_{sr^k}-part will still give $E_{sr^k} \otimes E_{sr^k}$, but we have to calculate the other part. Let's do this for α_{11}. First note that this function on D_4 is only non-zero for elements in $\langle r \rangle$. Then we have

$$\alpha_{11}(r^p r^{p'}) = \imath^{p+p'} = \imath^p \imath^{p'} \quad \text{and} \quad \alpha_{11}(sr^p sr^{p'}) = \imath^{p'-p} = \imath^{-p}\imath^{p'}. \tag{5.20}$$

Now $\imath^p$ corresponds to α_{11} when the argument is r^p. The function that gives $\imath^{-p}$ for the elements sr^p is α_{12}, and $\imath^p$ corresponds to α_{21}. This leads to

$$\Delta(\alpha_{11}, E_{sr^k}) = (\alpha_{11}, E_{sr^k}) \otimes (\alpha_{11}, E_{sr^k}) + (\alpha_{12}, E_{sr^k}) \otimes (\alpha_{21}, E_{sr^k}). \tag{5.21}$$

Because α_{ab} is only non-zero on elements $s^{1-\delta_{a,b}}r^p$, the first terms only gives values for elements of the form $r^p \otimes r^{p'}$ and the second only for $sr^p \otimes sr^{p'}$. Mixed terms as $r^p \otimes sr^{p'}$ always gives zero, as they should.

†We could also write out the matrix elements of the representations of $D(D_4)^*$ on the basis $\{(P_h, g)\}$ of $D(D_4)$, but because the multiplication in $D(D_4)^*$ is that of the regular tensor product, this method is easier.

In the same way one can calculate that

$$\Delta(\alpha_{22},E_{sr^k}) = (\alpha_{22},E_{sr^k})\otimes(\alpha_{22},E_{sr^k}) + (\alpha_{21},E_{sr^k})\otimes(\alpha_{12},E_{sr^k}); \tag{5.22}$$
$$\Delta(\alpha_{12},E_{sr^k}) = (\alpha_{11},E_{sr^k})\otimes(\alpha_{12},E_{sr^k}) + (\alpha_{12},E_{sr^k})\otimes(\alpha_{22},E_{sr^k}); \tag{5.23}$$
$$\Delta(\alpha_{21},E_{sr^k}) = (\alpha_{22},E_{sr^k})\otimes(\alpha_{21},E_{sr^k}) + (\alpha_{21},E_{sr^k})\otimes(\alpha_{11},E_{sr^k}). \tag{5.24}$$

So we see that $\mathscr{T}$ is also closed under comultiplication. Also note that the particular value of E_{sr^k} has no influence on the α-part of these tensor products, a property that we already know from §2.2.1.

Counit The counit of $D(D_4) \simeq D(D_4)^{**}$ is given by the unit map η^* of $D(D_4)^*$. So

$$\varepsilon(\rho_{ab},E_g) = \sum_{h\in H}(e,P_h)(\rho_{ab},E_g) = \rho_{ab}(e)\sum_h P_h(g) = \delta_{a,b}. \tag{5.25}$$

Antipode From both proposition 3.1 and lemma A.4 we know that $\mathscr{T}$ should also close under the antipode map of $D(D_4)$. But let's check it anyway.

The antipode S of $D(D_4) \simeq D(D_4)^{**}$ is given by

$$S(\rho_{ab},E_g) = (\rho_{ab},g)\circ S^*. \tag{5.26}$$

The antipode of the dual quantum double is given by (A.31); for $D(D_4)^*$:

$$S^*(s^q r^p, P_{s^{q'}r^{p'}}) = (s^q r^{(-1)^{q'}(p-2qp')}, P_{s^{q'}r^{p'(-1)^{q'}}}). \tag{5.27}$$

With this we find

$$S(J_j,E_{r^k}) = (J_j,E_{r^k})\circ S^* = (J_j,E_{r^{-k}}), \tag{5.28}$$
$$S(\alpha_{ab},E_{sr^k}) = (\alpha_{ab},E_{sr^k})\circ S^* = (-1)^{k(1-\delta_{a,b})}(\alpha_{\bar{a}\bar{b}},E_{sr^k}). \tag{5.29}$$

So $\mathscr{T}$ is indeed closed under the antipode, and it is a sub-Hopf algebra of $D(D_4)$.

5.1.2 Irreducible representations

We want to know what the irreducible representations of $\mathscr{T}$ are: if it were an algebra of the form $F(H/K)\tilde{\otimes}\mathbb{C}N$, the representations would be labelled by N-orbits in H/K and irreducible representations of the stabilizers in N of the distinguished elements of these orbits. Now however, all we know is that it is a finite-dimensional Hopf algebra, and such algebras are just starting to get classified (see for instance the introduction of [3]), and even then, its representation structure may not be fully known.

Finding irreducible representations However, we do know that $\mathscr{T}$ is a sub-Hopf algebra of $D(D_4)$, so the (irreducible) representations of $D(D_4)$ are also representations of $D(D_4)$ of $\mathscr{T}$ by restriction, and should decompose into irreducible representations. For $D(D_4)$, all irreducible representations are either one- or two-dimensional, and it is not a hopeless task to try and examine all representations.

The two things that we should look for are equivalent representations, and reducible representations. Two representations π and π' of an algebra $\mathscr{A}$ are equivalent if $\pi'(a) = M\pi(a)M^{-1}\ \ \forall a\in A$, for some matrix M, which does not depend on the algebra element a. We denote this by $\pi' \sim \pi$, and the transformation by M is called a *similarity transformation*.

A representation is (completely) reducible if it is equivalent to a direct sum of irreducible representations. In that case, we can write it as

$$\pi = \begin{pmatrix} \pi_1 & & \\ & \ddots & \\ & & \pi_k \end{pmatrix}, \tag{5.30}$$

where the π_k are irreducible representations.

For our case of $D(D_4)$, we should therefore choose convenient bases for the irreducible representations, determine whether the two-dimensional repesentations are equivalent to the direct sum of two one-dimensional representations, and then look for equivalences.

So we are trying to find the irreducible rerpesentations of $\mathscr{T}$ by the branching of the representations of $D(D_4)$, as opposed to the procedure we followed in chapter 4 where we first determined the representations of the residual symmetry algebra and then calculated the branching rules.

What can we expect? Because we have condensed a vector in $V^{r^2}_{J_1}$, we expect the representations that differ from others only by the 'properties' of r^2 or J_1 to be equivalent to those other representations. For example, we expect representations $\Pi^{r^2}_{J_j}$ to be equivalent to $\Pi^e_{J_j} \otimes \Pi^e_{J_1}$. The representations $\Pi^r_{\beta_l}$ are two-dimensional because of the conjugacy class with two elements r and r^3, and should reduce to the direct sum of two one-dimensional representations whenever $\beta_l(r) = \beta_l(r^3)$.

Irreducible $\mathscr{T}$-representations The determination of the equivalences is handwork, and we therefore do not give calculations. It can be checked directly that the results are correct. We have

$$\Pi^{r^2}_{J_0} \sim \Pi^e_{J_1}; \qquad \Pi^{r^2}_{J_1} \sim \Pi^e_{J_0}; \qquad \Pi^{r^2}_{J_2} \sim \Pi^e_{J_3}; \qquad \Pi^{r^2}_{J_3} \sim \Pi^e_{J_2}; \tag{5.31}$$

$$\Pi^{r^2}_{\alpha} \sim \Pi^e_{\alpha}; \tag{5.32}$$

$$\Pi^r_{\beta_0} \simeq \Omega^r_{J_0} \oplus \Omega^r_{J_1}; \qquad \Pi^r_{\beta_2} \simeq \Omega^r_{J_2} \oplus \Omega^r_{J_3}; \qquad \Pi^r_{\beta_1} \sim \Pi^r_{\beta_3}; \tag{5.33}$$

$$\Pi^s_{J_1} \sim \Pi^s_{J_0}; \qquad \Pi^s_{J_3} \sim \Pi^s_{J_2}; \qquad \Pi^{sr}_{J_1} \sim \Pi^{sr}_{J_0}; \qquad \Pi^{sr}_{J_3} \sim \Pi^{sr}_{J_2}. \tag{5.34}$$

Here the one-dimensional representations $\Omega^r_{J_j}$ are defined by

$$\Omega^r_{J_j}(f, s^q r^k) = f(r) J_j(s^q r^k), \tag{5.35}$$

where $f \in F(D_4)$, but $(f, s^q r^k) \in \mathscr{T}$ and J_j are the regular one-dimensional D_4-representations.

We just relabel the other representations to now denote irreducible $\mathscr{T}$-representations, and call $\Omega^r_\beta = \Pi^r_{\beta_1}|_{\mathscr{T}}$. This gives us (dimensions in parentheses):

$$\Omega^e_{J_j}\ (1),\ \Omega^e_\alpha\ (2),\ \Omega^r_{J_j}\ (1),\ \Omega^r_\beta\ (2),\ \Omega^s_{J_{0,2}}\ (2),\ \Omega^{sr}_{J_{0,2}}\ (2). \tag{5.36}$$

The squares of the dimensions correctly add up to 32.

5.1.3 Probing the structure of the residual symmetry algebra

Up until now, we have regarded $\mathscr{T}$ as just a set of basis vectors of $D(D_4)$ that happen to span a sub-Hopf algebra. We have not discerned a particular nice structure from which it is clear why it is this sub-Hopf algebra that leaves $|\phi\rangle \in V_{J_1}^{r^2}$ invariant.

We have seen that the (J_j, E_{r^k}) span the sub-Hopf algebra $F(D_2)\,\tilde{\otimes}\,\mathbb{C}\mathbb{Z}_4$. It turns out that $\mathscr{T}$ is somehow isomorphic to two copies of this Hopf algebra, accompanied by an element taking each element from one copy to the other. This element is $(\aleph, s) \equiv (\aleph, E_s) \equiv (\alpha_{11}+\alpha_{12}, E_s)$. Indeed we see from (5.11)-(5.14) that

$$(\alpha_{11}+\alpha_{21}, E_s)(J_0, E_{r^k}) = (\alpha_{11}+\alpha_{21}, E_{sr^k}), \tag{5.37}$$

$$(\alpha_{11}+\alpha_{21}, E_s)(J_1, E_{r^k}) = (\alpha_{11}-\alpha_{21}, E_{sr^k}), \tag{5.38}$$

$$(\alpha_{11}+\alpha_{21}, E_s)(J_2, E_{r^k}) = (\alpha_{22}+\alpha_{12}, E_{sr^k}), \tag{5.39}$$

$$(\alpha_{11}+\alpha_{21}, E_s)(J_3, E_{r^k}) = (\alpha_{22}-\alpha_{12}, E_{sr^k}). \tag{5.40}$$

The elements on the right hand side span the same space as the elements $\{(\alpha_{ab}, E_{sr^k})\}$, which together with (J_j, E_{r^k}) span $\mathscr{T}$. We have already seen that this is a subalgebra, so the multiplication of these elements works out correctly. One can also check that $(\aleph, s)^2 = (J_0, e) = 1_{D(D_4)}$.

If we agree to always write the elements of $\mathscr{T}$ in the standard from

$$(\aleph, s)^i (J_j, E_{r^k}) \quad i = 0,1;\ j,k = 0,1,2,3; \tag{5.41}$$

so with $(\aleph, s)$ on the left—the multiplication in *not* commutative, as one can read off from (5.11)-(5.18)—we only need the indices i, j and k to describe a basis vector of $\mathscr{T}$.

In fact, the whole Hopf algebra $D(D_4)$ can be written as elements of $\{(J_j, E_{r^k})\}$ multiplied by $(\aleph, e)$, (J_0, E_s), or both, or neither. The Hopf algebra $D(D_4)$ can then be thought of as four blocks, pictured by

$$\begin{array}{c|c} (J_0, E_e) & (J_0, E_s) \\ \hline (\aleph, e) & (\aleph, s) \end{array}$$

and $\{(J_j, E_{r^k})\}$ within each block.

In fact, we see that these basis vectors form a group: each multiplication of two such vectors gives exactly one other basis vector, with scalar factor 1. We explore this further in chapter 6.

5.2 Confinement

The next step is finding out which of the $\mathscr{T}$-representations are confined. We could use the conditions (3.25) and (3.26) on matrix elements of representations of $\mathscr{T}$, which span a sub-Hopf algebra $\mathscr{U}^*$ of $\mathscr{T}^*$, which is then dual to the unconfined algebra $\mathscr{U}$.

We are going to use a slightly different approach: we will check these conditions for all irreducible representations of $\mathscr{T}$, obtaining the unconfined representations directly, which must be isomorphic to the irreducible representations of $\mathscr{U}$ (cf. §3.3.3). We will then 'guess' the algebra structure of $\mathscr{U}$, but we will see that it follows quite naturally.

5.2.1 Braiding condensate particles

Projection of the R-matrix The first thing we need is the restriction to $\mathscr{T}$ of the left and right tensorand of the braid matrix $R_{\mathscr{A}}$. Now R is defined in §2.3 on the basis $\{(P_h, g)\}$ of $D(D_4)$. We therefore need to write our elements $(\aleph, s)^i(J_j, E_{r^k})$ out on this basis.

From the definitions of J_j and α, (4.4) and (4.5), we see

$$
\begin{aligned}
P_e &= \tfrac{1}{8}(1+\ \aleph)(J_0+J_1+J_2+J_3), &(5.42)\\
P_r &= \tfrac{1}{8}(1-\imath\aleph)(J_0+J_1-J_2-J_3), &(5.43)\\
P_{r^2} &= \tfrac{1}{8}(1-\ \aleph)(J_0+J_1+J_2+J_3), &(5.44)\\
P_{r^3} &= \tfrac{1}{8}(1+\imath\aleph)(J_0+J_1-J_2-J_3), &(5.45)\\
P_s &= \tfrac{1}{8}(1+\ \aleph)(J_0-J_1+J_2-J_3), &(5.46)\\
P_{sr} &= \tfrac{1}{8}(1-\imath\aleph)(J_0-J_1-J_2+J_3), &(5.47)\\
P_{sr^2} &= \tfrac{1}{8}(1-\ \aleph)(J_0-J_1+J_2-J_3), &(5.48)\\
P_{sr^3} &= \tfrac{1}{8}(1+\imath\aleph)(J_0-J_1-J_2+J_3). &(5.49)
\end{aligned}
$$

With this and $R_{\mathscr{A}} = \sum_h (P_h, e)\otimes(J_0, h)$ we find

$$
(P\otimes \mathrm{id})(R_{\mathscr{A}}) = \tfrac{1}{8}\sum_{\substack{i=0,1\\k=0,1,2,3}} \big((J_0+(-1)^iJ_1+(-1)^kJ_2+(-1)^{i+k}J_3), e\big)\otimes(J_0, s^ir^k), \tag{5.50}
$$

$$
(\mathrm{id}\otimes P)(R_{\mathscr{A}}) = \tfrac{1}{8}\sum_{k=0,1,2,3}(1+\imath^k\aleph)\big((J_0+J_1+(-1)^kJ_2+(-1)^kJ_3), e\big)\otimes(J_0, r^k). \tag{5.51}
$$

We calculate the values of the counit on the left and right projected tensorands:

$$
\big(1(\varepsilon\circ P)\otimes\mathrm{id}\big)(R_{\mathscr{A}}) = \tfrac{1}{8}4\otimes\big(J_0, (e+r^2)\big), \tag{5.52}
$$

$$
\big(\mathrm{id}\otimes 1(\varepsilon\circ P)\big)(R_{\mathscr{A}}) = \tfrac{1}{8}\sum_{k=0,1,2,3}(1+\imath^k\aleph)\big((J_0+J_1+(-1)^kJ_2+(-1)^kJ_3), e\big)\otimes 1. \tag{5.53}
$$

Imposing the confinement conditions Now we can check the conditions for unconfined particles, (3.22), which reduce to (using $\Pi^{r^2}_{J_1}(s^ir^k) = (-1)^i$):

$$
\tfrac{1}{8}\sum_{\substack{i=0,1\\k=0,1,2,3}} \Omega\big((J_0+(-1)^iJ_1+(-1)^kJ_2+(-1)^{i+k}J_3), e\big)\otimes(-1)^i|\phi\rangle = \Omega(1)\otimes|\phi\rangle, \tag{5.54}
$$

$$
|\phi\rangle\otimes\Omega(J_0, r^2) = |\phi\rangle\otimes\Omega(1). \tag{5.55}
$$

From the second condition, we see that any Ω^B_β that doesn't have $\beta(r^2)=1$ is confined. This is true for Ω^e_α and Ω^r_β. The first condition imposed on $\Omega^{s,sr}_{J_0,J_2}$ leads to a difference of a minus sign between right and left hand sides, so these particles are confined as well.

This leaves only the $\Omega^e_{J_j}$ and $\Omega^r_{J_j}$ as unconfined particles. These eight representations are all one-dimensional, so $\mathscr{U}$ should be eight-dimensional.

Hopf algebra structure Let's now look at the values of these representations on the basis vectors of $\mathscr{T}$. Because representations satisfy the algebra multiplication rule, we can distinguish between those elements, as mentioned before, by elements that do or do not contain a factor $(\aleph, s)$. Ignoring the representation values on that element for a while, we have

$$\Omega^e_{J'_j}(J_j, E_{r^k}) = J_{j'}(r^k); \quad \Omega^r_{J_{j'}}(J_j, E_{r^k}) = J_j(r) J_{j'}(r^k). \tag{5.56}$$

We immediately see that, fixing k, the elements (J_0, E_{r^k}) and (J_1, E_{r^k}) give the same representation values for every representation Ω^B_β. Similary (J_j, E_e) and (J_j, E_{r^2}) for fixed j give identical representation values. We then define the surjective Hopf morphism $\Gamma : \mathscr{T} \to \mathscr{U}$ of §3.3.3 by

$$\Gamma : (J_{0,1}, E_{r^k}) \mapsto (J_0, E_{r^k \bmod 2}), \; (J_{2,3}, E_{r^k}) \mapsto (J_2), E_{r^k \bmod 2}). \tag{5.57}$$

It is not hard to check that this is indeed a morphism. Then the unconfined algebra is spanned by

$$\mathscr{U} = \{(\aleph, s)^i (J_j, E_{r^k})\} \quad i, k = 0, 1; \; j = 0, 2; \tag{5.58}$$

where the elements (J_j, r^k) represent equivalence classes

$$\{(J_j, E_{r^k}), (J_{j+1}, E_{r^k}), (J_j, E_{r^{k+2}}), (J_{j+1}, E_{r^{k+2}})\}, \tag{5.59}$$

in agreement with (5.57).

5.2.2 Braiding

The Hopf algebra $\mathscr{U}$ is obviously not isomorphic to a quantum double, because its dimension is not the square of an integer. However, we do want to have a description of braiding of unconfined particles, so we need to obtain the universal R-matrix by some other method.

The braiding in $\mathscr{U}$ should correspond to the braiding in $\mathscr{A}$, and we mentioned earlier (§3.3.2), that we should use the projection of $\mathscr{A}$ onto $\mathscr{T}$ and then use Γ to go to $\mathscr{U}$.

We then find

$$R_{\mathscr{U}} = \tfrac{1}{2}\big((J_0 + J_2, E_e) \otimes (J_0, E_e) + (J_0 - J_2, E_e) \otimes (J_0, E_r)\big). \tag{5.60}$$

Comultiplication in $\mathscr{U}$ To check whether this element indeed constitutes a valid universal R-matrix, we need the comultiplication on $\mathscr{U}$, which we can determine by using $\Gamma \otimes \Gamma$ on the comultiplication of $\mathscr{T}$ on an element of $\mathscr{U}$, which we consider as an element of $\mathscr{T}$ by choosing a representative of each equivalence class (5.59) by the obvious choice

$$(J_j, E^k_r) \in \mathscr{U} \rightsquigarrow (J_j, E^k_r) \in \mathscr{T}. \tag{5.61}$$

For (J_j, E_{r^k}) the comultiplication is trivial (5.19), and this is carried over to $\mathscr{U}$. For $(\aleph, s)$ we find from (5.21) and (5.23):

$$\begin{aligned}
\Delta_{\mathscr{T}}(\aleph, s) =& \Delta(\alpha_{11}, E_s) + \Delta(\alpha_{22}, E_s) \\
=& (\alpha_{11}, E_s) \otimes (\alpha_{11}, E_s) + \big(\alpha_{12}, E_s) \otimes (\alpha_{21}, E_s) + \\
& (\alpha_{22}, E_s) \otimes (\alpha_{21}, E_s) + (\alpha_{21}, E_s) \otimes (\alpha_{11}, E_s) \\
=& (\aleph, s) \otimes (\aleph, s) \tfrac{1}{2}(J_0 + J_1, E_e) + (\aleph, s)(J_2, E_e) \otimes (\aleph, s) \tfrac{1}{2}(J_0 - J_1, E_e).
\end{aligned} \tag{5.62}$$

Now, when going over to $\mathscr{U}$ by Γ the second term drops out, which gives us

$$\Delta_{\mathscr{U}}(\aleph,s) = (\aleph,s)\otimes(\aleph,s). \tag{5.63}$$

So the comultiplication is trivial for all elements (recall from §2.2.4 that the comultiplication in a bialgebra is an algebra morphism).

Non-trivial braiding despite cocommutativity Before we check that the expression in (5.60) is indeed a universal R-matrix, we note that when this is the case, we have found a non-trivial braiding for a Hopf algebra with a trivial coproduct. It is known that the universal R-matrix does not have to be unique, even when the Hopf algebra is cocommutative. For example the universal R-matrices for the group algebra a finite Abelian group were classified† in [42], and for cocommutative Hopf algebras (so when the comultiplication is trivial) such structures are explored in [14].

Quasi-triangularity conditions The element R of (5.60) is its own inverse. We calculate the quasi-cocommutativity condition for $\Delta(J_j,E_{r^k})$:

$$\begin{aligned} R\Delta(J_j,E_{r^k})R^{-1} =& \tfrac{1}{2}\big((J_0+J_2,E_e)\otimes(J_0,E_e)+(J_0-J_2,E_e)\otimes(J_0,E_r)\big)\times \\ &\big((J_j,E_{r^k})\otimes(J_j,E_{r^k})\big)R^{-1} \\ =&\Delta(J_j,E_{r^k})RR^{-1} = \Delta^{\mathrm{op}}(J_j,E_{r^k}). \end{aligned} \tag{5.64}$$

For the calculation for $\Delta(\aleph,s)$ we use that the product $(J_0,r)(\aleph,s) = (\alpha_{11},sr^3) - (\alpha_{12},sr^3) = (\aleph,sr^3)(J_1,e)$ in $\mathscr{T}$ reduces to $(\aleph,s)(J_0,r)$ in $\mathscr{U}$. With this one can see that R commutes with $(\aleph,s)$ and obeys quasi-commutativity.

The other quasi-triangularity conditions are also satisfied:

$$\begin{aligned} (\Delta\otimes\mathrm{id})(R) =& \tfrac{1}{2}(J_0+J_2,E_e)\otimes(J_0+J_2,E_e)\otimes(J_0,E_e)+ \\ & \tfrac{1}{2}(J_0-J_2,E_e)\otimes(J_0-J_2,E_e)\otimes(J_0,E_r). \end{aligned} \tag{5.65}$$

$$\begin{aligned} R_{13}R_{23} =& \tfrac{1}{4}\big((J_0+J_2,E_e)\otimes(J_0,E_e)\otimes(J_0,E_e)+(J_0-J_2,E_e)\otimes(J_0,E_e)\otimes(J_0,E_r)\big) \\ & \times\big((J_0,E_e)\otimes(J_0+J_2,E_e)\otimes(J_0,E_e)+(J_0,E_e)\otimes(J_0-J_2,E_e)\otimes(J_0,E_r)\big) \\ =& \tfrac{1}{4}\big((J_0+J_2,E_e)\otimes(J_0+J_2,E_e)+(J_0-J_2,E_e)\otimes(J_0-J_2,E_e)\big)\otimes(J_0,E_e) \\ & +\big((J_0+J_2,E_e)\otimes(J_0-J_2,E_e)+(J_0-J_2,E_e)\otimes(J_0+J_2,E_e)\big)\otimes(J_0,E_r) \\ =& \tfrac{1}{2}(J_0+J_2,E_e)\otimes(J_0+J_2,E_e)\otimes(J_0,E_2)+ \\ & \tfrac{1}{2}(J_0-J_2,E_e)\otimes(J_0-J_2,E_e)\otimes(J_0,E_r). \end{aligned} \tag{5.66}$$

The calculation for $(\mathrm{id}\otimes\Delta)(R) = R_{13}R_{12}$ is similar.

Obtaining $R_{\mathscr{U}}$ from $R_{\mathscr{A}}$ At this point, we should remark that the universal R-matrix obtained here is $(\Gamma\otimes\Gamma)\circ(P\otimes P)(R_{\mathscr{A}})$. If we had chosen the orthogonal projection of R from $\mathscr{A}$ on $\mathscr{U}$, by considering $\mathscr{U}$ as a subset according to (5.61), then we would have found

$$(P_{\mathscr{U}}\otimes P_{\mathscr{U}})(R_{\mathscr{A}}) = \frac{1}{8}\big((J_0+J_2,E_e)\otimes(J_0,E_e)+(J_0-J_2,E_e)\otimes(J_0,E_r)\big), \tag{5.67}$$

†The author of this article does not, however, prove that every R-matrix for such an algebra falls within this classification.

which is $\frac{1}{4}$ times the R obtained by Γ. Then the quasi-triangularity conditions $(\Delta \otimes \mathrm{id})(R) = R_{13}R_{23}$ and $(\mathrm{id} \otimes \Delta)(R) = R_{13}R_{12}$ are no longer satisfied.

This can be fixed, however, by replacing (5.61) by

$$(J_j, E_r^k) \in \mathscr{U} \rightsquigarrow \tfrac{1}{4}(J_j, E_r^k) \in \mathscr{T}. \tag{5.68}$$

So the matter whether we should use the Hopf map Γ or some projection to obtain the universal R-matrix for $\mathscr{U}$ is still unresolved, although Γ seems to automatically give the right solution, whereas for the projection a specific choice is required.

5.2.3 Domain walls

The domain walls are characterized by a Hopf kernel of Γ. We calculate the right Hopf kernel, defined by (3.34):

$$(\mathrm{id} \otimes \Gamma)\Delta(J_j, E_{r^k}) = (\mathrm{id} \otimes \Gamma)\big((J_j, E_{r^k}) \otimes (J_j, E_{r^k})\big). \tag{5.69}$$

Using (5.57), this is only identical to $(J_j, E_{r^k}) \otimes 1 = (J_0, E_e)$ if $j = 0, 1;\quad r = 0, 2$.

$$(\mathrm{id} \otimes \Gamma)\Delta(\aleph, s) = (\mathrm{id} \otimes \Gamma)\big((\aleph, s) \otimes (\aleph, s)\big). \tag{5.70}$$

This can never be equal to $(\aleph, s) \otimes (J_0, E_e)$.

So we find that $\mathrm{RKer}(\Gamma)$ is spanned by $\{J_{0,1}E_{r^{0,2}}\}$, and is therefore isomorphic (as an algebra) to $F(\mathbb{Z}_2) \otimes \mathbb{C}\mathbb{Z}_2$. Its four one-dimensional representations are labelled by $(J_{0,1}, \gamma_{0,1})$.

Restrictions of $\mathscr{T}$-representations We can determine to which of these representations the particles in the condensate correspond, by looking at their values on the restriction of $\mathscr{T}$ to $\mathrm{RKer}(\Gamma)$. The unconfined particles of course correspond to (J_0, γ_0), and for the others we find:

$$\begin{aligned} \Omega^e_\alpha &\mapsto (J_0, \gamma_1) & \qquad \Omega^r_\beta &\mapsto (J_0, \gamma_1) \\ \Omega^{s,sr}_{J_0} &\mapsto (J_1, \gamma_0) & \qquad \Omega^{s,sr}_{J_2} &\mapsto (J_1, \gamma_1). \end{aligned} \tag{5.71}$$

5.3 Summary

In this chapter we calculated the structure of the residual symmetry algebra $\mathscr{T}$ and the unconfined algebra $\mathscr{U}$, along with their representations and branching rules thereof, for one particular condensate of one particular quantum-double-symmetric theory. Still, this calculation was quite elaborate, and does not generalize easily to other condensates or other quantum doubles.

It is however a good example of the richness of the residual symmetry, even though we started out with the group D_4, which has only eight elements. It also illustrates nicely that dyonic condensates may allow for larger residual symmetry than one might naively expect. In this case, $\mathscr{T}$ was twice as large as just the tensor product of the unbroken magnetic and electric parts, see the first paragraph of this chapter and §5.1.3.

We have also seen that the issue raised in §3.3.3 of how to define a proper braid matrix for the unconfined algebra, has been clarified to some extent. It is most likely that first projecting the universal R-matrix of $\mathscr{A}$ to $\mathscr{T}$ and then carrying over to $\mathscr{U}$ by Γ will give correct results. However, this is still open to verification or proof.

Chapter 6

Group algebra structure

As we have already mentioned, by choosing the basis $\{(\rho_{ab}, E_g)\}$ for $D(D_4)$, the multiplication of two basis vectors gives exactly one other basis vector, and they therefore have the structure of a *semigroup*. The basis vector (J_0, E_e) is a left and right identity for this multiplication, turning it into a *monoid*. Furthermore, each element (J_j, E_{r^k}) has a left and right inverse element by $(J_j, E_{r^{-k}})$, (J_0, E_s) is its own inverse and $(\aleph, e)^4 = (J_0, E_e)$.

Therefore this basis has the structure of a finite group. Another way of saying the same thing is stating that $D(D_4)$ is isomorphic to the group algebra of some finite group as an algebra, but certainly not as a Hopf algebra; for instance, $D(D_4)$ is not cocommutative.

We immediately face two questions: do other quantum doubles possess such a nice basis? And does this lead to new physical insights for this model? Unfortunately, both of these questions have to be answered mainly negatively. We will explain this a little further in the next few pages. But let us first look more closely at the group structure of this basis.

6.1 $D(D_4)$ as a group algebra

The multiplication of $D(D_4)$ is of course given by construction, and the multiplication on the basis $\{(\rho_{ab}, E_g)\}$ was determined in §5.1.1-5.1.3. We will now write it in a more suggestive notation to obtain a group.

Rewriting elements of $D(D_4)$ Firstly, we see that the matrix elements of the representations of $\mathbb{C}D_4$ correspond to:

$$J_0 = \sum_{p=0}^{3} (P_{r^p} + P_{sr^p}) \qquad J_1 = \sum_{p=0}^{3} (P_{r^p} - P_{sr^p})$$

$$J_2 = \sum_{p=0}^{3} (-1)^p (P_{r^p} + P_{sr^p}) \qquad J_3 = \sum_{p=0}^{3} (-1)^p (P_{r^p} - P_{sr^p})$$

$$\aleph = \alpha_{11} + \alpha_{21} = \sum_{p=0}^{3} \iota^p (P_{r^p} + P_{sr^p}). \tag{6.1}$$

We can ignore the case of J_3 by writing it as the product J_1J_2.

We must also deal with the twisted multiplication of the quantum double (2.4). For $D(D_4)$, this amounts to the following interchanging rules:

$$(J_0,E_s)(\aleph,e) = (\aleph,e)(J_2,E_e)(J_0,E_s), \tag{6.2}$$
$$(J_0,E_r)(\aleph,e) = (\aleph,e)(J_1,E_e)(J_0,E_r). \tag{6.3}$$

Other combinations can either be obtained from these or are commutative.

Finally, note that $(J_2,E_e) = (\aleph,e)^2$ which one can confirm by writing this out on the (P_h,g) basis:

$$(\aleph,e)^2 = \Big(\sum_{p=0}^{3} q^p(P_{r^p}+P_{sr^p}),E_e\Big)^2 = \Big(\sum_{p=0}^{3}(\imath^2)^p(P_{r^p}+P_{sr^p}),E_e\Big) = (J_2,E_e). \tag{6.4}$$

The group G_4 This leads us to the following construction. Let G_4 be the group with four generators r,s,a and d and the relations (the unit element is denoted by 1)

$$r^4=1 \quad s^2=1 \quad a^4=1 \quad d^2=1, \tag{6.5}$$
$$rs = sr^{-1}, \tag{6.6}$$
$$ra = dar, \tag{6.7}$$
$$rd = dr, \tag{6.8}$$
$$sa = a^{-1}s, \tag{6.9}$$
$$sd = ds, \tag{6.10}$$
$$ad = da. \tag{6.11}$$

Then every element can be written in the form

$$d^ia^jr^ks^l \quad j,k=0,1,2,3;\ i,l=0,1. \tag{6.12}$$

and G_4 has 64 elements. This group does not fall in a general class of groups, but can be seen as an extension of D_4: it is a semidirect product $(\mathbb{Z}_4\times\mathbb{Z}_2)\rtimes D_4$ (cf. §6.3.1).

Correspondence between $\mathbb{C}G_4$ and $D(D_4)$ The group algebra $\mathbb{C}G_4$ is isomorphic as an algebra to $D(D_4)$ by the following correpondences

$$\begin{aligned} r^k &\mapsto (J_0,E_{r^k}),\\ s &\mapsto (J_0,E_s),\\ a^j &\mapsto \Big(\sum_{p=0}^{3} q^{jk}(P_{r^p}+P_{sr^p}),E_e\Big),\\ d &\mapsto (J_1,E_e). \end{aligned} \tag{6.13}$$

We see that a^2 corresponds (J_2,E_e) and a to $(\aleph,E_e)$. The elements r and s generate D_4, corresponding to the $\mathbb{C}D_4$-part of $D(D_4)$. The 'non-trivial' relations (6.7) and (6.9) correspond to (6.3) and (6.2).

This group has a very large center, given by the elements $\{1,r^2,a^2,d\}$ and all their products. This corresponds to saying that all elements (J_j,E_{e,r^2}) are central.

Hopf algebra structure We can now bring over all other structure from $D(D_4)$ to $\mathbb{C}G_4$:

$$\Delta(r) = r \otimes r \tag{6.14}$$
$$\Delta(s) = s \otimes s \tag{6.15}$$
$$\Delta(a^i) = \tfrac{1}{2}\big(a^i \otimes a^i(1+d) + a^{-i} \otimes a^i(1-d)\big) \tag{6.16}$$
$$\Delta(d) = d \otimes d, \tag{6.17}$$

$$\varepsilon(x) = 1 \quad \forall \text{ basis vectors of } \mathbb{C}G_4, \tag{6.18}$$

$$S(r^i) = r^{-i}, \tag{6.19}$$
$$S(s) = s, \tag{6.20}$$
$$S(a^i) = a^{-i}(1+d) + a^i(1-d), \tag{6.21}$$
$$S(d) = d. \tag{6.22}$$

Using

$$P_{s^q r^p} = \big(1 + (-1)^q d\big) \sum_k (-\imath)^{pk} a^k \tag{6.23}$$

the universal R-matrix $\sum_h (P_h, e) \otimes (1, h)$ can be rewritten:

$$R = \sum_{p,k} (-\imath)^{pk} \big((1+d)a^k \otimes r^p + (1-d)a^k \otimes sr^p\big). \tag{6.24}$$

So we see that this group algebra can be equipped with non-trivial coproduct, antipode and braiding, instead of the trivial structures as mentioned in chapter 2. This raises of course many questions, which we are mostly unable to answer. In the literature, we have not found any other cases comparable to this occurence. In fact, mathematicians prefer to invert the reasoning, saying that this non-trivial Hopf algebra has a special basis, so that the basis vectors possess a group structure. We shall also see that this does not generalize to $D(D_n)$ with $n > 4$.

Furthermore, it is not clear whether this leads to special properties of the physical model it describes.

6.2 Correspondence between G_n and $D(D_n)$

In the previous section we found that we could find a basis on $D(D_n)$ such that the basis vectors form a group G_n. Now we are going to look if we can do the same thing for larger n, n still even.

Let's take another look at the twisted multiplication, which is after all the most important and defining property of the quantum double. For general n, the commutation relation between $a = \big(\sum_{p=0}^n q^k (P_{r^p} + P_{sr^p}), E_e\big)$ and $r = (J_0, E_r)$ is given by

$$\begin{aligned}(J_0, E_r)\big(\sum_{p=0}^n q^k (P_{r^p} + P_{sr^p}), E_e\big) &= \big(\sum_{p=0}^n q^k (P_{r^p} + P_{sr^{p-2}}), E_e\big)(J_0, E_r) \\ &= \big(\sum_{p=0}^n (P_{r^p} + q^2 P_{sr^p}), E_e\big)\big(\sum_{p=0}^n q^k (P_{r^p} + P_{sr^p}), E_e\big)(J_0, E_r),\end{aligned} \tag{6.25}$$

so the interchange of a and r introduces a central element $\left(\sum_{p=0}^{n}(P_{r^p}+q^2P_{sr^p}),E_e\right)$. When $n=4$, $q^2=-1$ and this element takes a particularly nice form, which enabled us to identify it with J_1 in (6.13). For general n this can only be done in the following way: define the element of $D(D_n)$

$$\delta \equiv \big(\sum_{p=1}^{n}(P_{r^p}+qP_{sr^p}),E_e\big). \tag{6.26}$$

Here as before $q\equiv e^{2\pi\imath/n}$. We define the group G_n of dimension $2n^3$ by the generators r,s,a,δ and relations of (6.5)-(6.11), but now with $\delta^n=1$ and (6.7) replaced by

$$ra=\delta^2 ar. \tag{6.27}$$

The element $d=\sum_p P_{r^p}-P_{sr^p}$ now corresponds to the element $\delta^{n/2}$, which is why we had to define δ with the value q rather than q^2 as we did for d in (6.13): $n/2$ may be odd. Please note that the group G_4 of the previous section is isomorphic to the group G_4 defined here with the subgroup $\{1,\delta\}$ divided out.

Now one may hope that, when turning this group into an algebra $\mathbb{C}G_n$ it will be isomorphic to $D(D_n)$. However, this is immediately contradicted by looking at the dimensions, $2n^3$ for G_n and $4n^2$ for $D(D_n)^\dagger$.

We can, however, provide a surjective homomorphism $f_n:\mathbb{C}G_n\to D(D_n)$ by

$$\begin{aligned} r &\mapsto r,\\ s &\mapsto s,\\ a &\mapsto a,\\ \delta^i &\mapsto \tfrac{1}{2}(1+q^i+d-q^i d), \end{aligned} \tag{6.28}$$

where the elements of the right hand side correspond to the definitions of (6.13). The kernel of this map is

$$\begin{aligned} &\{(r^x s^y a^z)\big((\delta^k-1)(d+1)\big)\}\cup\{(r^x s^y a^z)\big((\delta^k-q^k)(d-1)\big)\}\\ &k=1,\ldots,\tfrac{n}{2}-1\ ;\ x,z=0,\ldots,n-1\ ;\ y=0,1 \end{aligned} \tag{6.29}$$

which has dimension $2n^2(n-2)$, so that $\dim(\mathbb{C}G_n)=\dim\big(D(D_n)\big)+\dim(\ker f_n)$.

In the next section we determine the irreducible representations of G_n, and it is then also shown that the ones that factor over this morphism are in one-to-one correspondence with the irreducible representations of $D(D_n)$.

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†Actually, these dimensions are equal for $n=2$, but D_2 is Abelian, and it is already known that the quantum double of every finite Abelian group is isomorphic as an algebra to a group algebra. This description does not give this group algebra, however, as the element δ is central, and does not show up in any group relation.

The rest of the braided Hopf algebra structure is given by

$$\begin{aligned}
\Delta(r) &= r \otimes r \\
\Delta(s) &= s \otimes s \\
\Delta(a^i) &= \frac{1}{2}\big(a^i \otimes a^i(1+d) + a^{-i} \otimes a^i(1-d)\big) \\
\Delta(\delta^i) &= \frac{1}{2}\big(d\delta^i \otimes \delta^i + (-1)^{i+1}\delta^{-i}) + \delta^i \otimes (\delta^i + (-1)^i\delta^{-i})\big)
\end{aligned}$$

$$\varepsilon(x) = 1 \quad \forall \text{ basisvectors of } \mathbb{C}G_n$$

$$\begin{aligned}
S(r^i) &= r^{-i} \\
S(s) &= s^{-1} \\
S(a^i) &= a^{-i}(1+d) + a^i(1-d) \\
S(\delta^i) &= \delta^i.
\end{aligned}$$

Analogy for n odd The group G_n is well-defined for any positive integer n, but the surjection onto $D(D_n)$ makes explicit use of the element $d = \delta^{n/2}$. This element does not exist, of course, when n is odd.

We can, however, formulate the surjection in another way, by making use of the fact that the sum of the n-roots of unity add up to zero†. Recall that the element d represented the $D(D_n)$ element $\sum_p \big((P_{r^p} - P_{sr^p}), e\big)$. We calculate

$$\sum_{i=0}^{n-1} \delta_i \mapsto \sum_{p,i} \big((P_{r^p} + q^{ip} P_{sr^p}), e\big) = \sum_p n(P_{r^p}, e). \tag{6.30}$$

With $(1,e) = \sum_p \big((P_{r^p} + P_{sr^p}), e\big)$ we find

$$d - 1 \equiv \frac{2}{n}\sum_{i=0}^{n-1} \delta_i - 1 \mapsto \frac{2}{n} n \sum_p (P_{r^p}, e) - \sum_{p'} \big((P_{r^{p'}} + P_{sr^{p'}}), e\big) = \sum_p \big((P_{r^p} - P_{sr^p}), e\big). \tag{6.31}$$

So with this new definition of d all equations that were introduced above are valid also for odd values of n. Note that d does reduce to $\delta^{n/2}$ when n is even, because $q^0 + q^{n/2} = 0$, so the sum of all roots of unity these two must be zero as well.

$D(D_n)$ is not isomorphic to a group algebra We return to the question posed at the beginning of this chapter: are there quantum doubles that are isomorphic to a group algebra? Although we have found some relation to a group algebra in what is discussed above, it has not added much information or insight into the quantum double structure. As n is chosen larger, the difference in dimensions between $\mathbb{C}G_n$ and $D(D_n)$ increases linearly with n. A surjection onto an algebra from a much larger algebra is in most cases not very useful to explore that algebra's structure.

† This can be easily seen by multiplying $\sum_i q^i$ by the non-zero factor $(q-1)$: $\sum_i (q^{i+1} - q^i) = \sum_i (q^i - q^i) = 0$.

However, through this construction we are able to define a non-trivial comultiplication and non-trivial braiding for a large family of non-Abelian groups. Were we to find a physical model exhibiting symmetry according to such a group, then perhaps it would show interesting braid statistics as well. Still, these groups, although having connections to the dihedral groups, seem up until now mathematical possibilities with no direct physical consequences.

It does provide a way, however, to determine the irreducible representations of $D(D_n)$ by other means (§6.3.2), but as the original representation theory is not very hard, this is of no practical use. All in all the description with the group G_n seems to fail to expand the knowledge of $D(D_n)$.

It is desirable to know whether $D(D_n)$ for other values of n permit a basis that forms a group at all. It is not a straightforward task to find out what that basis should look like. We can make use, though, of the fact that the irreducible representations of a group algebra are given by the irreducible representations of that group. These representations can be calculated on a computer, and we can compare some of their properties to those of irreducible representations of $D(D_n)$. In particular, we can look at the number of representations and their dimensions.

Using the computer program GAP with the code listed in appendix B, we have calculated the dimensions of the irreducible representations of all groups of dimension $4n^2$ for $n = 5, \dots, 20$, with the exception of $n = 8, 16$, because the number of groups of dimensions 256 and 1024 is very large. It turns out that for these n, there are no groups with the right number of representations of the right dimensions, so we cannot hope to find any group algebra that is isomorphic (as an algebra) to $D(D_n)$.

6.3 Irreducible representations of $\mathbb{C}G_n$

In this section, we will construct the irreducible (algebra) representations of our group algebra $\mathbb{C}G_n$. This is done by the use of induced representations, through a method by Serre [43, §8.2] which is laid out in the appendix §A.5.

We then use this mechanism to work out the details for our group G_n, for n even. Subsequently we show how the irreducible representations of $D(D_n)$ are related to those of G_n.

6.3.1 Irreducible representations of G_n

We are going to apply the method developed in §A.5 to our group G_n. Firstly, we must show that this group is a semidirect product by an Abelian group. Recall that our group is given by the elements $\{d^i a^j r^k s^l \mid i, j, k = 0, \dots, n-1\ ;\ l = 0, 1\}$. Using the relations (6.5)-(6.11) and (6.27), we see that the subgroup generated by d and a form the Abelian group $\mathbb{Z}_n \times \mathbb{Z}_n$, and the subgroup generated by r and s form the group D_n. We show that $G_n \simeq (\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes D_n$.

$$\begin{aligned}
(d^i a^j, r^k s^l) \cdot (d^p a^q, r^x s^y) &= d^i a^j r^k s^l d^p a^q r^x s^y \\
&= d^i a^j d^p r^k s^l a^q r^x s^y \\
&= d^i a^j d^p r^k a^{q-2ql} s^l r^x s^y \\
&= d^i a^j d^{p+2k(q-2ql)} a^{q-2ql} r^k s^l r^x s^y \\
&= (d^i a^j d^{p+2k(q-2ql)} a^{q-2ql}, r^k s^l r^x s^y)
\end{aligned}$$

We see that the action of D_n on $\mathbb{Z}_n \times \mathbb{Z}_n$ is given by

$$r^k s^l \rightharpoonup d^p a^q = d^{p+2k(q-2ql)} a^{q-2ql} \ \in \mathbb{Z}_n \times \mathbb{Z}_n,$$

which is equal to $r^k s^l d^p a^q (r^k s^l)^{-1}$, so this is a regular semidirect product.

We can now use the machinery of §A.5 by identifying $\mathbb{Z}_n \times \mathbb{Z}_n$ to A and D_n to H. The characters of $\mathbb{Z}_n \times \mathbb{Z}_n$ are given by

$$\chi_{y,z} : d^p a^q \mapsto e^{\frac{i2\pi}{n}(yp+zq)} \quad y,z,p,q = 0,\ldots,n-1. \tag{6.32}$$

The action of D_n on a character $\chi_{y,z}$ is given by

$$\begin{aligned}(r^k s^l \rightharpoonup \chi_{y,z})(d^p a^q) &= \chi_{y,z}\big((r^k s^l)^{-1} d^p a^q r^k s^l\big) \\ &= \chi_{y,z}(d^{p-2qk} a^{q-2lq}) \\ &= e^{\frac{i2\pi}{n}\big(y(p-2qk)+z(q-2lq)\big)} \\ &= e^{\frac{i2\pi}{n}\big(py+q(z-2lz-2ky)\big)} \\ &= \chi_{y,z-2lz-2ky}(d^p a^q)\end{aligned}$$

We want to know which characters χ can be reached through the action of D_n for fixed y and z. This is an equivalence relation, and we write $\chi \sim \chi_{y,z}$, and $[\chi_{y,z}]$ for the corresponding equivalence class, which is the D_n-orbit of the character group X for the element $\chi_{y,z}$. We see that

$$\chi_{y,z} \sim \chi_{y,\pm z+2my \mod n} \quad m = 0,\ldots,n-1.$$

Set $x \equiv \frac{n/2}{\gcd(\frac{n}{2},y)}$. We propose that for fixed y there are $\gcd(\frac{n}{2},y)+1$ orbits represented by $\chi_{y,0},\ldots,\chi_{y,\gcd(\frac{n}{2},y)}$, having x elements if $z \in \{0, \gcd(\frac{n}{2},y)\}$ and $2x$ elements otherwise.

Proof. Disregarding the action of s, all elements in the orbit $[\chi_{y,z}]$ are of the form $\chi_{y,z+2my}$. As we increase m, the smallest m which does not give a new character in this orbit is $m = x$. Then each orbit has x elements, evenly spaced with an interval $I = 2\gcd(\frac{n}{2},y)$, so that $xI = n$.

Because of these regular intervals, the orbits $[\chi_{y,0}],\ldots,[\chi_{y,I-1}]$ are distinct. Now we include the action of s. The orbits then also comprise all elements of the form $\chi_{y,-z+2my}$, so $\chi_{y,I-z} \sim \chi_{y,z}$. We see at once that we are left with the orbits $[\chi_{y,0}],\ldots,[\chi_{y,\gcd(\frac{n}{2},y)}]$, of which the first and last still have x elements, and the others now have $2x$ elements. □

Next, we wish to determine the subgroup $(D_n)_{y,z}$ of D_n, which leaves the character $\chi_{y,z}$ invariant. The action of r^k will leave the character invariant if $2ky = 0 \mod n$, so if k is a multiple of x. Since x always divides n ($n/x = 2\gcd(\frac{n}{2},y)$) the subgroup $\mathbb{Z}_{2\gcd(\frac{n}{2},y)} = \{r^{\lambda x} | \lambda = 0,\ldots,2\gcd(\frac{n}{2},y)-1\}$ leaves $\chi_{y,z}$ invariant.

Including the action of s gives $r^k s \rightharpoonup \chi_{y,z} = \chi_{y,-z+2ky}$. This is only equal to $\chi_{y,z}$ if $2ky = 2z \mod n$, so when $yk = z \mod \frac{n}{2}$. This will only occur if $z = 0 \mod \gcd(\frac{n}{2},y)$. In that case, the elements $\{r^{z/\gcd(\frac{n}{2},y)+\lambda x} s\}$ will also leave $\chi_{y,z}$ invariant, and the subgroup will be $D_{2\gcd(\frac{n}{2},y)}$. Summarizing:

$$(D_n)_{y,z} = \begin{cases} D_{2\gcd(\frac{n}{2},y)} & \text{if } z = 0 \mod \gcd(\frac{n}{2},y) \\ \mathbb{Z}_{2\gcd(\frac{n}{2},y)} & \text{otherwise} \end{cases} \tag{6.33}$$

Denote the irreducible representations of these groups by ρ_α. From §A.5, we know that the irreducible representations of $(\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes (D_n)_{y,z}$ are given by $\chi_{y,z} \otimes \rho_\alpha$. These representations can be induced to G_n; the resulting representations θ_{yz,ρ_α} are irreducible by proposition A.9, and give all irreducible representations of G_n. The dimension of θ_{yz,ρ_α} is the number of cosets $G_n/\big((\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes (D_n)_{y,z}\big)$ times the dimension of ρ_α.

6.3.2 Connection with $D(D_n)$

We now wish to link the representations θ_{yz,ρ_α} of $\mathbb{C}G_n$ to those of our original theory of $D(D_n)$. To do this, we have to realize the following: as $\mathbb{C}G_n$ has a larger dimension than $D(D_n)$, it will have more irreducible representations, or of higher dimension, or a combination of these. Some of these representations will not have any connection to those of the quotient. The appropriate requirement is that, using the morphism $f_n : \mathbb{C}G_n \to D(D_n)$ of (6.28), the representations $\theta : \mathbb{C}G_n \to GL(V)$ *factor over* f_n. By this we mean that there is a map $\Pi : D(D_n) \to GL(V)$, so that (cf. §3.3.3)

$$\theta = \Pi \circ f_n \tag{6.34}$$

We will now show that this is the case if θ is zero on the kernel of f_n, written as $\ker f_n$, which is the subspace of $\mathbb{C}G_n$ that is mapped to $0 \in D(D_n)$ by f_n.

We firstly remark that $\mathbb{C}G_n/\ker f_n \simeq D(D_n)$, because no two distinct non-zero elements $g_1, g_2 \in \mathbb{C}G_n$ can map to the same element $h \neq 0$ of $D(D_n)$:

$$\begin{aligned} f_n(0_{\mathbb{C}G_n}) = 0_{D(D_n)} &= h + (-h) = f_n(g_1) + \big(-f_n(g_2)\big) \\ &= f_n(g_1) + f_n(-g_2) = f_n(g_1 - g_2) \\ &\Rightarrow g_1 - g_2 = 0 \;\Rightarrow\; g_1 = g_2. \end{aligned}$$

If $\theta(g) = 0\ \forall g \in \ker f_n$, it directly defines a representation Π of $D(D_n)$ by considering g as a representative of $\mathbb{C}G_n/\ker f_n$, so $\Pi\big(f_n(g)\big) = \theta(g)\ \forall g$. If, on the other hand, there is an element $g \in \ker f_n$ for which $\theta(g) \neq 0$, and another element $g' \in \mathbb{C}G_n$, then $\theta(g) \neq \theta(g + g')$ whereas $\Pi(f_n(g + g')) = \Pi(f_n(g))$, and the above identification of representations no longer holds.

One can also see, that through this morphism $D(D_n)$ will be semisimple if $\mathbb{C}G_n$ is, which it is, and also that an irreducible represenation of $\mathbb{C}G_n$ will be sent to an irreducible representation of $D(D_n)$.

Representations that factor over f_n We have determined the kernel of the morphism $f_n : \mathbb{C}G_n \to D(D_n)$ in (6.29):

$$\begin{aligned} &\{(r^k s^l a^j)\big((d^i - 1)(d^{\frac{n}{2}} + 1)\big)\} \cup \{(r^k s^l a^j)\big((d^i - q^i)(d^{\frac{n}{2}} - 1)\big)\} \\ &i = 1, \ldots, \frac{n}{2} - 1 \;;\; k, j = 0, \ldots, n-1 \;;\; l = 0, 1 \end{aligned}$$

Because we know that for a representation θ, $\theta(g_1 g_2) = \theta(g_1)\theta(g_2)$, we now only consider the values of the θ_{yz,ρ_α} on the parts $(d^i - 1)(d^{\frac{n}{2}} + 1)$ and $(d^i - q^i)(d^{\frac{n}{2}} - 1)$: if these are zero, products including these factors will also be zero. Furthermore, the values of the representations will never be zero on the basisvectors r, s, a, d, as these form a group; in other words, we can only get zero on linear combinations. So in considering just the above factors, we will get all representations that are zero on the kernel.

$$\begin{aligned}
\theta_{yz,\rho_\alpha}\big((d^i-1)(d^{\frac{n}{2}}+1)\big) &= \big(\theta_{yz,\rho_\alpha}(d^i)-\theta_{yz,\rho_\alpha}(1)\big)\big(\theta_{yz,\rho_\alpha}(d^{\frac{n}{2}})+\theta_{yz,\rho_\alpha}(1)\big)\\
&= \big(\chi_{y,z}(d^i)-1\big)\big(\chi_{y,z}(d^{\frac{n}{2}})+1\big)\\
&= (q^{yi}-1)(q^{y\frac{n}{2}}+1) \qquad (6.35)\\
\theta_{yz,\rho_\alpha}\big((d^i-q^i)(d^{\frac{n}{2}}-1)\big) &= \big(\theta_{yz,\rho_\alpha}(d^i)-q^i\theta_{yz,\rho_\alpha}(1)\big)\big(\theta_{yz,\rho_\alpha}(d^{\frac{n}{2}})-\theta_{yz,\rho_\alpha}(1)\big)\\
&= \big(\chi_{y,z}(d^i)-q^i\big)\big(\chi_{y,z}(d^{\frac{n}{2}})-1\big)\\
&= (q^{yi}-q^i)(q^{y\frac{n}{2}}-1) \qquad (6.36)
\end{aligned}$$

According to the statement above, the θ_{yz,ρ_α} must give zero on *all* these elements. From (6.35) and (6.36) we then get the conditions

$$\begin{aligned}
yi = 0 \mod n\ \forall i \ \vee\ y\frac{n}{2} = \frac{n}{2} \mod n \Rightarrow y=0 \ \vee\ y \text{ odd} \quad \text{and}\\
yi = i \mod n\ \forall i \ \vee\ y\frac{n}{2} = 0 \mod n \Rightarrow y=1 \ \vee\ y \text{ even}
\end{aligned}$$

Combining these, we see that the only representations that factor over f_n are those for which $y=0$ or $y=1$.

Correspondence with $D(D_n)$-representations We have now found that the representations θ_{yz,ρ_α} with $y=0,1$; $z=0,\dots,\gcd(\frac{n}{2},y)$ factor over f_n. For these specific cases, they can be given in more detail by using $\gcd(\frac{n}{2},0)=n/2$ and $\gcd(\frac{n}{2},1)=1$. Then from (6.33) we see that

$$(D_n)_{0,z} = D_n \quad z=0,\frac{n}{2} \qquad (6.37)$$

$$(D_n)_{0,z} = \mathbb{Z}_n \quad z=1,\dots,\frac{n}{2}-1 \qquad (6.38)$$

$$(D_n)_{1,z} = D_2 \quad z=0,1 \qquad (6.39)$$

Furthermore, we can determine the cosets $D_n/(D_n)_{y,z}$:

$$\begin{array}{lcll}
D_n/(D_n)_{0,z} & : & [e]=D_n & z=0,\frac{n}{2}\\
D_n/(D_n)_{0,z} & : & [e]=\langle r\rangle,\ [s]=s\langle r\rangle & z=1,\dots,\frac{n}{2}-1\\
D_n/(D_n)_{1,0} & : & [r^k]=\{r^k,r^{k+\frac{n}{2}},sr^k,sr^{k+\frac{n}{2}}\} & k=0,\dots,\frac{n}{2}-1\\
D_n/(D_n)_{1,1} & : & [r^k]=\{r^k,r^{k+\frac{n}{2}},sr^{k-1},sr^{k-1+\frac{n}{2}}\} & k=0,\dots,\frac{n}{2}-1
\end{array} \qquad (6.40)$$

We can already see that these subgroups correspond to the centralizers of elements in D_n, and the cosets correspond to those of these centralizers, which are used in the determination of the irreducible representations of $D(D_n)$. In particular, there is a direct isomorphism between the representations spaces of our $\theta_{yz,\rho}$ and Π^A_α. We only need to compare the actions of the elements.

We immediately note that the action of the elements $r^k s^l$ for $\theta_{yz,\rho}$ is identical to that for Π^A_α, as the '$\chi_{y,z}$-part' is not affected by these elements, and the others parts correspond directly.

Let's now look at the subspace of $\mathbb{C}G_n$ generated by a and d. The action of these elements is given by

$$\begin{aligned}
d^i a^j \rightharpoonup_{\theta_{yz,\rho_\alpha}} r^u s^w \otimes \mathbf{e}\otimes \mathbf{v}_l &= r^u s^w \otimes \chi_{y,z}(d^{i-2uj}a^{j(1-2w)})\mathbf{e}\otimes\mathbf{v}_z\\
&= r^u s^w \otimes q^{y(i-2uj)+zj(1-2w)}\mathbf{e}\otimes\mathbf{v}_z \qquad (6.41)
\end{aligned}$$

$$\begin{aligned}
d_i a_j \rightharpoonup_{\Pi^A_\alpha} r^u s^w \otimes \mathbf{v}_z &= \sum_{p,q=0}^{n-1} (\delta_{r^p} + q^i \delta_{sr^p})\big(q^j q(\delta_{r^q} + \delta_{sr^q})\big) \rightharpoonup r^u s^w \otimes \mathbf{v}_z \\
&= \sum_{p=0}^{n-1} (q^{jp}\delta_{r^p} + q^{i+jp}\delta_{sr^p}) \rightharpoonup r^u s^w \otimes \mathbf{v}_z \\
&= \sum_{p=0}^{n-1} (q^{jp}\delta_{r^p, s^w r^u g^A r^{-u} s^w} + q^{i+jp}\delta_{sr^p, s^w r^u g^A r^{-u} s^w}) r^u s^w \otimes \mathbf{v}_z \\
&= q^{j\gamma(1-2w)+\beta\left(i-2ju(1-2w)\right)} r^u s^w \otimes \mathbf{v}_z \quad g^A = s^\beta r^\gamma \qquad (6.42)
\end{aligned}$$

Now, if $\beta = 1$, then A is either the conjugacy class $[s]$ or $[sr]$, and the coset representatives of the centralizers of these are the r^u, so $w = 0$. Then we see the following correspondence by comparing the values of θ_{yz,ρ_α} and Π^A_α:

$$\begin{aligned}
\theta_{00,\rho_\alpha} &\leftrightarrow \Pi^e_\alpha && (6.43)\\
\theta_{0\frac{n}{2},\rho_\alpha} &\leftrightarrow \Pi^{r^{\frac{n}{2}}}_\alpha && (6.44)\\
\theta_{0z,\rho_\alpha} &\leftrightarrow \Pi^{r^z}_\alpha \quad z = 1,\ldots,\frac{n}{2}-1 && (6.45)\\
\theta_{10,\rho_\alpha} &\leftrightarrow \Pi^s_\alpha && (6.46)\\
\theta_{11,\rho_\alpha} &\leftrightarrow \Pi^{sr}_\alpha && (6.47)\\
& && (6.48)
\end{aligned}$$

6.4 Summary

In this section we showed that the quantum double of D_4 allows for a basis so that the basis vector form a group under the algebra multiplication. This is not true, however for many, and probably all, other dihedral groups (except for the Abelian groups D_1 and D_2).

These calculations did lead to a family of groups, the groups algebras of which can be equipped with a non-trivial braided Hopf algebra structure. This is certainly interesting, but does not immediately yield physical insight. But perhaps future investigations may show that other, more physical, group algebras have non-trivial comultiplication as well.

Because this family of groups was constructed from the group algebra of the dihedral groups, we were able to determine its irreducible representations by an induction method. Perhaps such properties should be taken into consideration when looking for non-trivial comultiplication on group algebras.

Chapter 7

Conclusions

The formalism developed in [7] to describe the symmetry breaking of theories with Hopf symmetry has been applied to the quantum double of even dihedral groups. Together with the work of cited paper and [5], most of the condensates of quantum doubles of finite subgroups of $SO(3)$ have been worked out.

As one would have expected, the most interesting or new phenomena arise in situations with magnetic and dyonic condensates, as those are not treated to such extent by other theories. Because the Hopf symmetry description does not really distinguish between topological and fundamental charges, the procedure for calculating the particle spectrum after condensation and subsequent confinement is the same for all types of condensates. The electric and magnetic sector do provide, however, some simplifying general properties.

In particular, there are many options for taking magnetic condensates, as the magnetic sector is organized in conjugacy classes. The class sum provides a gauge invariant solution, but we may also choose pure flux condensates, or certain sums of fluxes, as long as they satisfy the trivial self-braiding condition (§4.4.4).

When the consendate leaves a residual symmetry of the form $F(H/K)\tilde{\otimes}\mathbb{C}N$, the calculations, although requiring ample bookkeeping, are straightforward. This gives hope that generalizations to for instance quantum doubles of continuous groups are also within reach.

If the condensate does not reduce to this form, which happens for some dyonic condensates, such as the one worked out in chapter 5, there is no general way to determine condensates and confinement, although they can be calculated by hand. In any case the residual symmetry algebra always has $F(H/K)\tilde{\otimes}\mathbb{C}N$ as a sub-Hopf algebra, but may be larger. Perhaps the special properties of Hopf algebras, such as lemma A.5 will lead to a general description for all types of condensates.

It is remarkable that D_4, a group of only eight elements, has such a rich structure when the quantum double construction followed by symmmetry breaking is applied.

Throughout this work, some observations have been made. Firstly, on the basis of the experience of §§3.3.2,5.2.2, we would like to conjecture that in general the prescription for braiding of unconfined particles (representations of $\mathscr{U}$) should be given by

$$R_{\mathscr{U}} = (\Gamma\otimes\Gamma)\circ(P_{\mathscr{T}}\otimes P_{\mathscr{T}})R_{\mathscr{A}}, \tag{7.1}$$

the universal R-matrix obtained by projecting the universal R-matrix of the original algebra onto the residual symmetry algebra, and then carrying it over to the unconfined algebra by the Hopf map Γ.

Secondly, the $D(D_4)$-theory is of a particular interesting form, because its basis can be chosen in such a way that it forms a group. The braiding is non-trivial, however, and remains so even in the unconfined algebra of the $\Pi'^{r^2}_{J_1}$-condensate. Unfortunately, this group structure is not present in most quantum doubles, nor does it seem to have a direct physical consequence, apart from the non-trivial braiding given by the quantum double description. We showed more generally that all quantum doubles of even dihedral groups are Hopf quotients of a certain group algebra, so that its irreducible representations can be determined in this way.

Outlook There is still a lot of unexplored territory concerning these Hopf symmetric models. For quantum doubles of finite groups, such as treated in this thesis, the main open issue is finding a general form for the residual symmetry algebra and the confined algebra. The rapid development of the theory of Hopf algebras may be of use; for instance, there is much recent work on the classification of finite-dimensional Hopf algebras.

Another important question is whether the unconfined algebra always possesses an R-matrix related to the one of the original symmetry algebra, and whether it is of the form (7.1). It is our belief that in order to give an answer to this, one first has to know more of the structure of the residual symmetry algebra.

Apart from these unresolved issues, one can try to generalize this formalism to related structures. The particle spectrum, tensor products and braiding have been worked out for quasi-quantum doubles, abandoning coassociativity (see p.103), in [51, §§2.5,3.1], and for quantum doubles of locally compact (infinite) groups in [34]. Conformal field theory has connections with quantum groups (Hopf algebras) other than quantum doubles, and the way in which these describe braiding and fusion properties for quantum Hall systems has been treated in [44].

The aspect of symmetry breaking has not been touched by these papers, and in order to do so, one has to take precautions to circumvent problems when dropping for example coassociativity, the closedness of the coproduct when taking a subalgebra or finite-dimensionality, from the definition of the residual symmetry algebra of §3.2.1.

And then we are left with the question: "Now that we have all these Hopf symmetric models, where can they be applied?" Although discrete gauge theories originated from high-energy physics, condensed matter systems may be more suitable for this description, as the vacua or groundstates, which might be described as resulting from spontaneous symmetry breaking, found in that area are far more diverse. For example the disclinations and dislocations in crystals and liquid crystals are the kind of topological defects that lend themselves to be treated by the Hopf symmetry description [5, 32].

There is much recent work concerning fractional statistics (representations of a truncated braid group) in condensed matter systems. Claims have been made that fractional (Abelian) statictics of Laughlin quasiparticles in a fractional quantum Hall fluid has been directly observed [10]. A proposal on how to detect the suspected non-Abelian statistics in the $\nu = \frac{5}{2}$ fractional quantum Hall state is posed in [8]. Furthermore, there are many proposals for models showing non-Abelian statistics (e.g. [18, 41]).

Hopf symmetry is to be found as well in quantum liquid crystals. I will be working with professor Jan Zaanen in Leiden to look at symmetry breaking in the models developed in [54] (see also [32]).

Appendix A

Background material

A.1 Mathematical definitions

In this section we define some of the mathematical constructs used in this thesis, with the purpose as a quick reference for physicists not very familiar with these terms. It is not at all comprehesive. For more information on quantum groups one can for instance consult [20], or for a physical point of view [30].

Homomorphism Considering two instances of a mathematical construct, we can define a map between them, which is called a (homo)morphism if it respects all their structure. For example a linear map between vector spaces is a vector space morphism, as it respects addition, distributive scalar multiplication et cetera. It is usually clear what structure is referred to, but we will define all relevant morphisms.

Algebra An algebra A is a vector space over a field k, for which we shall always take $\mathbb{C}$, with a bilinear multiplication $\mu : A \otimes A \rightarrow A$. We usually denote $ab \equiv \mu(a,b)$.

A unit 1 is an element for which $1a = a1 = a \quad \forall a \in A$. This is equivalent to introducing a map $\eta : k \rightarrow A$, with $\eta(\lambda)$ commuting with every element in A.

The multiplication is called associative if $a(bc) = (ab)c \quad \forall a,b,c \in A$. We always say *algebra* for an associative unital algebra.

Ideals and simplicity A left ideal I of an algebra A is a subset of A such that $aI \in I \quad \forall a \in A$. A right ideal I' satisfies $I'a \in I' \quad \forall a \in A$. A two-sided ideal is both a left and a right ideal.

An algebra A is called simple if it has no two-sided ideals other than $\{0\}$ and A itself.

An algebra A is called semisimple if it has no two-sided nilpotent ideals other than $\{0\}$. A semisimple algebra can be written as a direct sum of simple algebras.

Group algebra The group algebra $\mathbb{C}G$ of a group g is an algebra in which all basis vectors are labeled by group elements, and the multiplication is given by the group multiplication.

Function algebra With the name function algebra, we denote the algebra $F(G)$ of linear functions on the group G into $\mathbb{C}$. A convenient basis is the set $\{P_g \mid g \in G\}$ for which $P_g(h) = \delta_{g,h} \quad \forall g,h \in G$. The multiplication is given by $P_g P'_g = \delta_{g,g'} P_g$. The unit is $\sum_g P_g$.

The function algebra is dual to the group algebra, in the sense that its vector space is the dual vector space, its multiplication is derived from the comultiplication of the group algebra, and vice versa.

Algebra morphism An algebra morphism is a map $\phi : A \to B$, where A and B are algebras, for which $\phi(aa') = \phi(a)\phi(a')$ and $\phi(a+a') = \phi(a)+\phi(a') \quad \forall a,a' \in A$. This implies $1_B = \phi(1_A)$.

Module A module comprises what physicists commonly refer to as a representation. Let A be an algebra. A left A-module M is a vector space[†] and a bilinear map $A \otimes M \to M$, called the action, denoted by $(a \otimes m) \mapsto a \rightharpoonup m$ such that

$$a \rightharpoonup (a' \rightharpoonup m) = (aa') \rightharpoonup m \quad \forall a,a' \in A,\ m \in M. \tag{A.1}$$

A right A-module is defined analogously, but now with a map $M \otimes A \to M$. In the text we often say "module" when we mean "left module", adhering to the physicist's convention of acting to the right by a group.

We write M to mean both the vector space and the map. We introduce this as it is sometimes more convenient to speak of "the module" instead of "the represenation space of a representation".

A submodule S is a subspace of an A-module such that $aS \in S \quad \forall a \in A$. This corresponds to an invariant subspace of a representation.

A simple module is a module having no other submodules than $\{0\}$ and the module itself. This corresponds to an irreducible representation.

A semisimple module is a module that is a direct sum of simple modules. This corresponds to a completely reducible representation. It can be shown that every module of a semisimple algebra is semisimple.

Coalgebra A coalgebra C is a vector space over a field k with two linear maps called the comultiplication or coproduct $\Delta : C \to C \otimes C$ and counit $\varepsilon : C \to k$. We demand coassociativity:

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta, \tag{A.2}$$

and counitality:

$$(\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon) \circ \Delta. \tag{A.3}$$

We can regard a coalgebra as a structure dual to an algebra, where Δ is dual to μ, and ε dual to η. See, however, the definition of a bialgebra.

Sweedler notation The coproduct of a vector will in general be a linear combination of tensor products of vectors. We write this compactly as

$$\Delta(c) = \sum_{(c)} c' \otimes c''. \tag{A.4}$$

It is assumed that this stands for a combination with as few terms as possible.

[†] Some mathematics textbooks begin by defining a module over a ring and then define a vector space as a module over a field. As most physicists are more familiar with vector spaces, we do it the other way around.

Cocommutativity A coalgebra C is called cocommutative if

$$\Delta(c) = \Delta^{\text{op}}(c) \equiv (\tau \circ \Delta)(c) \quad \forall c \in C, \tag{A.5}$$

where τ is the flip, interchanging the two tensorands in a tensor product.

Coalgebra morphism A coalgebra morphism is a map $\phi : C \to D$, where C and D are coalgebras, for which

$$(\phi \otimes \phi)\big(\Delta_C(c)\big) = \Delta_D\big(\phi(c)\big) \quad \text{and} \quad \varepsilon_C(c) = \varepsilon_D\big(\phi(c)\big), \tag{A.6}$$
$$\Delta(c + c') = \Delta(c) + \Delta(c') \quad \text{and} \quad \varepsilon(c + c') = \varepsilon(c) + \varepsilon(c'). \tag{A.7}$$

Please note that if $\Delta(c) = c \otimes c$, then $\Delta(c + c') \neq (c + c') \otimes (c + c')$, but $c \otimes c + c' \otimes c'$.

Bialgebra A bialgebra A is a vector space with compatible algebra and coalgebra structure. This means that μ and η are coalgebra morphisms, or equivalently Δ and ε are algebra morphisms. This implies

$$\Delta(ab) = \Delta(a)\Delta(b) \quad \text{and} \quad \varepsilon(ab) = \varepsilon(a)\varepsilon(b), \tag{A.8}$$
$$\Delta(1) = 1 \otimes 1 \quad \text{and} \quad \varepsilon(1_A) = 1_k. \tag{A.9}$$

Bialgebra morphism A bialgebra morphism is a map $\phi : A \to B$, where A and B are bialgebras, where ϕ is an algebra morphism and a coalgebra morphism for its respective structures.

Hopf algebra A Hopf algebra H is a bialgebra with a map $S : H \to H$, called the antipode, for which, using the Sweedler notation,

$$\sum_{(a)} a' S(a'') = \sum_{(a)} S(a') a'' = \varepsilon(a) 1 \tag{A.10}$$

The antipode is an anti-algebra morphism and an anti-coalgebra morphism, i.e.

$$S(ab) = S(b)S(a) \quad \text{and} \quad (S \otimes S)\big(\Delta(a)\big) = \Delta^{\text{op}}\big(S(a)\big) \tag{A.11}$$

Furthermore $S(1) = 1$ and $\varepsilon \circ S = \varepsilon$.

Hopf algebra morphism A Hopf (algebra) morphism is a bialgebra morphism $\phi : A \to B$, for which $\phi \circ S_A = S_B \circ \phi$.

Quasi-cocommutativity A bialgebra A is called quasi-cocommutative if there exists an invertible element $R \in A \otimes A$, called the *universal R-matrix* such that

$$\Delta^{\text{op}}(a) = R\Delta(a)R^{-1} \quad \forall a \in A. \tag{A.12}$$

One could say that the degree of non-cocommutativity of A is determined by the manner in which R deviates from $1 \otimes 1$.

Quasi-triangularity, braided Let $\{A,R\}$ be a quasi-commutative bialgebra. If we write $R=\sum_k R_l^k \otimes R_r^k$ then by R_{ij}, $i,j=1,2,3$ we mean a tensor product of three factors, where the R_l^k are placed at the i-th position, the R_r^k at the j-th position and the other factor is always 1. For example $R_{32}=\sum_k 1\otimes R_r^k \otimes R_l^k$.

Now $\{A,R\}$ is called quasi-triangular or braided if

$$(\Delta\otimes \mathrm{id})(R)=R_{13}R_{23} \tag{A.13}$$
$$(\mathrm{id}\otimes\Delta)(R)=R_{13}R_{12} \tag{A.14}$$

A cocommutative bialgebra is braided with $R=1\otimes 1$.

Yang–Baxter equation An isomorphism of vector spaces $\phi: V\otimes V\to V\otimes V$ is called an R-matrix if it satifies the Yang–Baxter equation

$$(\phi\otimes\mathrm{id})(\mathrm{id}\otimes\phi)(\phi\otimes\mathrm{id})=(\mathrm{id}\otimes\phi)(\phi\otimes\mathrm{id})(\mathrm{id}\otimes\phi) \tag{A.15}$$

The universal R-matrix of a quasi-triangular bialgebra can be shown to satisfy

$$R_{12}R_{13}R_{23}=R_{23}R_{13}R_{12}. \tag{A.16}$$

This property leads to solutions of the Yang–Baxter equation for every module of this bialgebra [20, §VIII.3].

Ribbon algebra A ribbon algebra is a braided Hopf algebra (H,R) with an element $c\in H$, which satistfies (τ is the flip, see (A.5))

$$\Delta(c)=\big(\tau(R)R\big)^{-1}(c\otimes c) \tag{A.17}$$
$$\varepsilon(c)=1 \tag{A.18}$$
$$S(c)=c \tag{A.19}$$

Quantum double The quantum double is a construction by Drinfeld [17], which yields a non-commutative, non-cocommutative braided Hopf algebra out of any Hopf algebra. We give the definition of the quantum double of a finite-dimensional group algebra.

Let H be a finite group with unit e. Then $D(H)\equiv D(\mathbb{C}H)$ is called the quantum double of H. As a vector space it is isomorphic to $F(H)\otimes\mathbb{C}H$, with basis $\{(P_h,g)\mid g,h\in H\}$. Its ribbon algebra structure is given by

$$(P_h,g)(P_h',g')=\delta_{h,gh'g^{-1}}(P_h,gg') \tag{A.20}$$
$$1=\sum_{h\in H}(P_h,e) \tag{A.21}$$
$$\Delta(P_h,g)=\sum_{h'\in H}(P_{h'},g)\otimes(P_{h'^{-1}h},g) \tag{A.22}$$
$$\varepsilon(P_h,g)=\delta_{h,e} \tag{A.23}$$
$$S(P_h,g)=(P_{g^{-1}h^{-1}g},g^{-1}) \tag{A.24}$$
$$R=\sum_{g\in H}(P_g,e)\otimes(1_{F(H)},g) \tag{A.25}$$
$$c=\sum_{g\in H}(P_g,g). \tag{A.26}$$

Dual quantum double The dual $D(H)^*$ of the quantum double $D(H)$ is the vector space $\big(F(H)\otimes\mathbb{C}H\big)^*\simeq\big(F(H)\big)^*\otimes\big(\mathbb{C}H\big)^*\simeq\mathbb{C}H\otimes F(H)$ together with Hopf algebra structure

$$(g,P_h)(g',P'_h)=\delta_{h,h'}(gg',P_h) \tag{A.27}$$
$$1=\sum_{h\in H}(e,P_h) \tag{A.28}$$
$$\Delta(g,P_h)=\sum_{h'\in H}(g,P_{h'})\otimes(h'^{-1}gh',P_{h'^{-1}h}) \tag{A.29}$$
$$\varepsilon(g,P_h)=\delta_{h,e} \tag{A.30}$$
$$S(g,P_h)=(h^{-1}g^{-1}h,P_{h^{-1}}). \tag{A.31}$$

The dual quantum double is not necessarily braided; however it is *cobraided* [20, §VIII.5].

A.2 Some properties of Hopf algebras

The definition of an algebra is so general, that it can usually only be made interesting by imposing additional structure. We will always assume associativity and unitality (p.102).

A Hopf algebra, on the other hand, has so much structure that it has many desirable properties. We will list some of those here as lemmas, which may be useful to the reader.

Lemma A.1. *Let A be a Hopf algebra. If A is semisimple, then A is finite-dimensional.*

Proof. See corollary 2.7 in [45]. ☐

Lemma A.2. *Let A be a Hopf algebra, and B be a sub-Hopf algebra of A. If A is semisimple then B is semisimple.*

Proof. See corollary 2.5 in [26]. ☐

Lemma A.3. *Let A be a semisimple Hopf algebra over a field of characteristic 0, and let S be its antipode. Then* $S^2=\mathrm{id}$, *so the antipode is its own inverse.*

Proof. See theorem 4 in [25]. ☐

Lemma A.4. *Let A be a Hopf algebra, and B be a finite-dimensional subbialgebra of A. Then B is a sub-Hopf algebra of A.*

Proof. See lemma 1 in [39]. ☐

Lemma A.5. *Let A be a finite-dimensional Hopf algebra, and B be a sub-Hopf algebra of A. Then* $\dim B$ *divides* $\dim A$.

Proof. See corrolary 1.6 in [26]. ☐

Lemma A.6. *Let A and B be a Hopf algebras and let* $\Gamma : A\to B$ *be a surjective Hopf morphism. If A is braided with R-matrix* R_A *then B is braided by* $R_B=(\Gamma\otimes\Gamma)(R_A)$.

Proof. First, we note that $R_B^{-1} = (\Gamma \otimes \Gamma)(R_A^{-1})$, because

$$\begin{aligned}1_B \otimes 1_B &= (\Gamma \otimes \Gamma)(1_A \otimes 1_A) = (\Gamma \otimes \Gamma)(R_A R_A^{-1}) = (\Gamma \otimes \Gamma)(R_A) \cdot (\Gamma \otimes \Gamma)(R_A^{-1}) \\ &= R_B \cdot (\Gamma \otimes \Gamma)(R_A^{-1}),\end{aligned}$$

where we made use of the fact that Γ is a Hopf morphism several times.

We check all conditions for braided Hopf algebras, namely (A.12), (A.13) and (A.14). Because Γ is surjective, all elements $b \in B$ can be written as $\Gamma(a)$ for some $a \in A$.

$$\begin{aligned}\Delta_B^{\mathrm{op}}(b) &= \tau \circ \Delta_B(\Gamma(a)) = \tau \circ (\Gamma \otimes \Gamma)\Delta_A(a) = (\Gamma \otimes \Gamma)(\tau \circ \Delta_A)(a) \\ &= (\Gamma \otimes \Gamma)\big(R_A \Delta_A(a) R_A^{-1}\big) = R_B\big((\Gamma \otimes \Gamma)\Delta_A(a)\big)R_B^{-1} \\ &= R_B \Delta_B(b) R_B^{-1}.\end{aligned}$$

Here we used that Γ is a Hopf morphism in the second and the last two equalities.

$$\begin{aligned}(\Delta_B \otimes \mathrm{id}_B)(R_B) &= (\Delta_B \otimes \mathrm{id}_B)\big((\Gamma \otimes \Gamma)(R_A)\big) = (\Gamma \otimes \Gamma \otimes \Gamma)\big((\Delta_A \otimes \mathrm{id}_A)(R_A)\big) \\ &= (\Gamma \otimes \Gamma \otimes \Gamma)\big((R_A)_{13}(R_A)_{23}\big) \\ &= (\Gamma \otimes \Gamma \otimes \Gamma)(R_A)_{13} \cdot (\Gamma \otimes \Gamma \otimes \Gamma)(R_A)_{23} \\ &= (R_B)_{13}(R_B)_{23}.\end{aligned}$$

Here, in the last equality, one has to recall that $(R_A)_{ij}$ is a linear combination of elements of A, and because Γ is a linear map, each of those elements will be sent to the corresponding element in B. In other words:

$$R_B = (\Gamma \otimes \Gamma)(R_A) = (\Gamma \otimes \Gamma)\Big(\sum_k (R_A)_l^k \otimes (R_A)_r^k\Big) = \sum_k \Gamma\big((R_A)_l^k\big) \otimes \Gamma\big((R_A)_r^k\big).$$

Condition (A.14) is checked in the same way. □

A.3 Additional proofs

Lemma A.7. *Let H be a group, let A be a conjugacy class of that group. The centralizers of two elements of A are isomorphic.*

Proof. From the definition of a conjugacy class $hah^{-1} \in A \quad \forall a \in A,\ h \in H$, and furthermore, every $a \in A$ can be written as at least one combination $ha'h^{-1}, \quad h \in H$ for every $a' \in A$.

Now take the centralizer $N_a = \{n \in H | nan^{-1} = 1\}$ of a, so. Take $h \in H - N_a$ and $a' \in A$ so that $ha'h = a \neq a'$. With this

$$\begin{aligned}nha'h^{-1} &= ha'hn; \\ h^{-1}nha'h^{-1}h &= h^{-1}ha'h^{-1}nh; \\ h^{-1}nha' &= a'h^{-1}nh \quad \forall n \in N_a.\end{aligned}$$

Then the set $N_{a'} = \{n' = h^{-1}nh \mid n \in N_a\}$ commutes with a'.

We now show that the map $N_a \to N_{a'}$ by this $h \in H$ is an injection: take $n_1, n_2 \in N_a$. Then

$$\begin{aligned}n_1' = h^{-1}n_1h &= h^{-1}n_2h = n_2'; \\ hh^{-1}n_2hh^{-1} &= hh^{-1}n_2hh^{-1}; \\ n_1 &= n_2.\end{aligned}$$

In the same way the map $N_{a'} \to N_a$ defined by $n' \mapsto hnh^{-1}$ is injective. Then h defines a bijection, which also respects the group multiplication:

$$n_1'n_2' = hn_1h^{-1}hn_2h^{-1} = hn_1n_2h^{-1} = (n_1n_2)'.$$

□

Proposition A.8. *Let H be a finite non-Abelian group with non-trivial center. Then there is at least one dyonic representation of $D(H)$ which allows a condensate vector with trivial spin and self-braiding.*

Proof. Let Z be the center of H. It is a normal subgroup, so the quotient H/Z is a group. Each element $h \in H$ can be written as an, where $a \in H/Z$ seen as Z-coset representative and $n \in Z$. H/Z must be non-Abelian, because if $h_1, h_2 \in H$ do not commute we have

$$a_1n_1a_2n_2 = a_1a_2n_1n_2 \neq a_2n_2a_1n_1 = a_2a_1n_1n_2 \Rightarrow a_1a_2 \neq a_2a_1. \tag{A.32}$$

Because H/Z is non-Abelian, it must have at least one non-trivial one-dimensional (irreducible) representation, which we denote by J. Then $J' : an \mapsto J(a)$ is a non-trivial one-dimensional irreducible representation for H. Indeed we have

$$J'(a_1n_1a_2n_2) = J'(a_1a_2n_1n_2) = J(a_1a_2) = J(a_1)J(a_2) = J'(a_1n_1)J'(a_2n_2). \tag{A.33}$$

The first equality holds because n_1 is central in H, the second because H/Z is a group so $a_1a_2 \in H/Z$, and the last by the definition of J'.

By the same argument, higher-dimensional irreducible representations of H/Z will induce representations of H, but those can then be reducible.

Take an element $n \in Z$; then $\Pi^n_{J'}$ is an irreducible dyonic representation of $D(H)$. Because it is one-dimensional and n is central in H, the trivial self-braiding condition (3.20) reduces to a simple form:

$$\begin{aligned}
|\phi\rangle \otimes |\phi\rangle &= \tau \circ (\Pi^n_{J'} \otimes \Pi^n_{J'})(R)(|\phi\rangle \otimes |\phi\rangle) \\
&= \tau \circ \sum_{h \in H} \big(\Pi^n_{J'}(P_h, e) \otimes \Pi^n_{J'}(1, e)\big)(|\phi\rangle \otimes |\phi\rangle) \\
&= \tau\big(|\phi\rangle \otimes \Pi^n_{J'}(1, n)|\phi\rangle\big) = J'(n)|\phi\rangle \otimes |\phi\rangle.
\end{aligned}$$

So the condition is $J'(n) = 1$, which is true by our definition of J'. Furthermore, the condition of trivial spin is also $J'(n) = 1$, which is then immediately satisfied as well.

□

A.4 Additional calculations

A.4.1 The odd dihedral groups

The odd dihedral groups D_n, n odd, have the same group definition as the even dihedral groups,

$$D_n = \{s^m r^k \mid s^2 = r^n = 1,\ sr^k = r^{-k}s\} \quad m = 0, 1;\ k = 1, \ldots, n-1. \tag{A.34}$$

but the have a different general structure, because there is no non-trivial central element. Its conjugacy classes are The $\frac{n-1}{2} + 2$ conjugacy classes of D_n are

$$[e] = \{e\}; \quad [r^k] = \{r^k, r^{-k}\}; \quad [s] \quad = \{s, sr, \ldots, sr^{n-1}\}; \quad k = 0, \ldots, \frac{n-1}{2}. \tag{A.35}$$

The centralizers of the elements of D_n are

$$\begin{aligned} N_e &= D_n, \\ N_{r^k} &= \{r^j \mid j = 0, \ldots, n-1\} \simeq \mathbb{Z}_n, \\ N_{sr^i} &= \{e, sr^i\} \simeq \mathbb{Z}_2 \; i = 0, \ldots, n-1. \end{aligned} \tag{A.36}$$

(A.37)

There are two one-dimensional representations given by

$$J_0(s^m r^k) = 1; \quad J_1(s^m r^k) = (-1)^m. \tag{A.38}$$

There are $\frac{n-1}{2}$ two-dimensional irreducible representations α_j, given by the same definition as (4.5):

$$\alpha_j(r^k) = \begin{pmatrix} q^{jk} & 0 \\ 0 & q^{-jk} \end{pmatrix}, \quad \alpha_j(sr^k) = \begin{pmatrix} 0 & q^{-jk} \\ q^{jk} & 0 \end{pmatrix}. \tag{A.39}$$

The character table of D_n, n odd, is

D_n	$[e]$	$[r^k]$	$[s]$
J_0	1	1	1
J_1	1	1	-1
α_j	2	$q^{jk}+q^{-jk}$	0

The irreducible representations of the quantum double of $D(D_n)$, n odd are (see also [7, §5.2]).

$D(D_n)$	$\Pi^e_{J_{0,1}}$	$\Pi^e_{\alpha_j}$	$\Pi^{r^k}_{\beta_l}$	$\Pi^s_{\gamma_0}$	$\Pi^s_{\gamma_1}$
d^A_α	1	2	1	n	n
s^A_α	1	1	q^{-lk}	1	-1

A.4.2 Branching of representations to the residual symmetry algebra

We give some examples of how to calculate the tensor product decomposition of the restriction of an irreducible $D(D_n)$-representation to the residual symmetry algebra $\mathscr{T}$.

Branching of $\Pi^{r^k}_{\beta_l}$ in a $\Pi^e_{J_1}$-condensate For this example we take the condensate vector $|\phi\rangle$ to lie in the representation space of $\Pi^e_{J_1}$, so that $\mathscr{T} \simeq F(D_n)\tilde{\otimes}\mathbb{C}\mathbb{Z}_n$, and $\Pi^{r^k}_{\beta_l}$ as irreducible $D(D_n)$-representation (see (4.24)).

The character for Π^A_α is just the restriction of its character to $\mathscr{T}$. The formula for the characters is

$$\sum_{\substack{h\in D_n \\ g\in\langle r\rangle}} \chi^B_\beta(P_h,g)\chi^A_\alpha(P_h,g)^* = \sum_{h,g} 1_B(h)1_{N_h}(g)\chi_\beta(k_h^{-1}gk_h)1_A(h)1_{N_h}(g)\chi_\alpha(k'^{-1}_h gk'_h)^*$$

From this we immediatly see that this will always be zero when B is not contained in A. For our Π^A_α, $A = \{r^k, r^{-k}\}$, so the only candidates are $\Omega^{r^k}_{\beta_{l'}}$ and $\Omega^{r^{-k}}_{\beta_{l'}}$. We then see that the stabilizer of these orbits in $N_{|\phi\rangle} = \langle r\rangle$ is the whole group. We calculate further:

$$\sum_{\substack{h\in\{r^k,r^{-k}\}\\ r^p}} \chi_{\beta_{l'}}(k_h^{-1} r^p k_h)\chi_{\beta_l}(k'^{-1}_h r^p k'_h)^*.$$

Next, note that the orbits B consist of just one element, either r^k or r^{-k}, but that the orbit (conjugacy class) $A = \{r^k, r^{-k}\}$. The cosets representatives are $k_{r^k} = e$ and $k_{r^{-k}} = s$. We then find

$$\sum_p \chi_{\beta_{l'}}(r^p)\chi_{\beta_l}(r^p)^*, \quad B = \{r^k\},$$
$$\sum_p \chi_{\beta_{l'}}(r^p)\chi_{\beta_l}(r^{-p})^*, \quad B = \{r^{-k}\}.$$

Now we wish to know for which $\beta_{l'}$ these equations are satisfied. The representations β_l and $\beta_{l'}$ are one-dimensional, and are therefore isomorphic to their characters. They are given by $\beta_l(r^p) = q^{lp}$, where $q \equiv e^{\imath\frac{2\pi}{n}}$. This gives us

$$\sum_p q^{l'p}q^{-lp} = q^{(l'-l)p}, \quad B = \{r^k\},$$
$$\sum_p q^{l'p}q^{lp} = q^{(l'+l)p}, \quad B = \{r^{-k}\}.$$

Because the sum of all roots of unity is zero, these equations can only be non-zero when $(l'-l)$ resp. $(l'+l)$ are zero. So we find the decomposition

$$\Pi^{r^k}_{\beta_l}|_{\mathscr{T}} \simeq \Omega^{r^k}_{\beta_l} \oplus \Omega^{r^{-k}}_{\beta_{-l}}.$$

Branching of $\Pi^s_{J_2}$ in a gauge-invariant $\Pi^{r^{n/2}}_{J_0}$-condensate Normally one would use characters to determine the branching rules. In this example, as an illustration we do an explicit calculation.

For this example we take $\frac{n}{2}$ to be even, so that the D_n-orbit $[s] = \{s, sr^2, \ldots, sr^{n/2-2}\}$ in $D_{n/2}$. The D_n-stabilizer of s is $\{e, r^{n/4}, r^{n/2}, r^{3n/4}, sr^k, sr^{k+n/4}, sr^{k+n/2}, sr^{k+3n/4}\} \simeq D_4$, so for this orbit we have irreducible $\mathscr{T}$-representations $\Omega^s_{J_i}$, $\ i = 0,1,2,3$ and Ω^s_α.

For $\Pi^s_{J_2}$, the representation space is spanned by $|sr^{2k}\rangle$, $\ k = 0,\ldots,\frac{n}{2}-1$, for which $(P_h, e) \rightharpoonup |sr^{2k}\rangle = \delta_{h,sr^{2k}}|sr^{2k}\rangle$. When restricting to $\mathscr{T} \simeq F(D_{n/2}) \tilde{\otimes} \mathbb{C}D_n$, we consider only functions of the form $P_h + P_{hr^{n/2}}$. We can perform a basis transformation on the representation space $V^s_{J_2}$, so that the basis vectors are now

$$|sr^{2k} \pm \imath sr^{2k+n/2}\rangle \equiv |sr^{2k}\rangle \pm \imath|sr^{sk+n/2}\rangle, \quad k = 0,\ldots,\tfrac{n}{4}. \tag{A.40}$$

This transformation will enable us to identify this representation space as a representation space of an irreducible $\mathscr{T}$-representation. Let's calculate the representation values of the restriction of $\Pi^s_{J_2}$ to $\mathscr{T}$ on this basis. The action of the $F(D_{n/2})$-part is obvious, and we suppress it in the following equations.

$$\begin{aligned}
\Pi^s_{J_2}(r^y)|sr^{2k} \pm \imath sr^{2k+n/2}\rangle &= \begin{cases} |J_2(r^{n/2})sr^{2(k-y)} \pm \imath J_2(e)sr^{2(k-y)+n/2}\rangle & -\frac{n}{2} < k-y \le 0 \\ |J_2(e)sr^{2(k-y)} \pm \imath J_2(r^{n/2})sr^{2(k-y)+n/2}\rangle & 0 < k-y \le \frac{n}{2} \end{cases} \\
\Pi^s_{J_2}(sr^y)|sr^{2k} \pm \imath sr^{2k+n/2}\rangle &= \begin{cases} |J_2(s)sr^{2y-2k} \pm \imath J_2(sr^{n/2})sr^{2y-2k+n/2}\rangle & -\frac{n}{2} \le k-y < 0 \\ |J_2(sr^{n/2})sr^{2y-2k} \pm \imath J_2(s)sr^{2y-2k+n/2}\rangle & 0 \le k-y < \frac{n}{2} \end{cases}
\end{aligned} \tag{A.41}$$

The action of $r^{n/2}$ will just give $J_2(r^{n/2})$ as it commutes with the entire orbit. This should be calculated separately from the formulae above.

We similarly calculate the representation values of Ω^s_α. Its representation space is labelled by and element sr^{2k} of the orbit $\{s, sr^2, \dots, sr^{n/2-2}\}$, and some vector $\mathbf{v}$ in the representations space of the D_4-representation α.

It is now easiest to split off an element of the stabilizer of s, and let this act first; so we write $r^y = r^{y'} r^{zn/4}$ with $y \in \{0, \dots, \frac{n}{4} - 1\}$ and $z \in \{0, 1, 2, 3\}$. This element will end up in the argument of α, prehaps multiplied with an additional factor of $r^{n/4}$ depending on the values of y' and k.

$$\begin{aligned}
\Omega^s_\alpha(r^y)|sr^{2k}, \mathbf{v}\rangle &= r^{y'} \rightharpoonup |sr^{2k}, \alpha(r^{zn/4})\mathbf{v}\rangle \\
&= \begin{cases} |sr^{2k-2y'}, \alpha(r^{zn/4})\mathbf{v}\rangle & k - y' > 0 \\ |sr^{2k-2y'}, \alpha(r^{(z+1)n/4})\mathbf{v}\rangle & k - y' < 0 \end{cases} \\
\Omega^s_\alpha(sr^y)|sr^{2k}, \mathbf{v}\rangle &= sr^{y'} \rightharpoonup |sr^{2k}, \alpha(r^{zn/4})\mathbf{v}\rangle \\
&= \begin{cases} |sr^{-2k+2y'}, \alpha(r^{(z+1)n/4})\mathbf{v}\rangle & k - y' > 0 \\ |sr^{-2k+2y'}, \alpha(r^{zn/4})\mathbf{v}\rangle & k - y' < 0 \end{cases}
\end{aligned} \tag{A.42}$$

Let's now look at the specific representation values for the actions of $r^{n/4}$ and s:

$$\Pi^s_{J_2}(r^{n/4})|sr^{2k} \pm \imath sr^{2k+n/2}\rangle = \pm\imath|\pm\imath sr^{sk+n/2} + sr^{2k}\rangle \tag{A.43}$$

$$\Omega^s_\alpha(r^{n/4})|sr^{2k}, \mathbf{v}\rangle = |sr^{2k}, \alpha(r^{n/4})\mathbf{v}\rangle = |sr^{2k}, \begin{pmatrix} \imath & \\ & -\imath \end{pmatrix}\mathbf{v}\rangle \tag{A.44}$$

$$\Pi^s_{J_2}(s)|sr^{2k} \pm \imath sr^{2k+n/2}\rangle = \mp\imath|sr^{n-2k+n/2} \mp sr^{n-2k}\rangle \tag{A.45}$$

$$\Omega^s_\alpha(s)|sr^{2k}, \mathbf{v}\rangle = |sr^{n/2-2k}, \alpha(sr^{n/4})\mathbf{v}\rangle = |sr^{n/2-2k}, \begin{pmatrix} & -\imath \\ \imath & \end{pmatrix}\mathbf{v}\rangle \tag{A.46}$$

If we now make the identification

$$\begin{aligned}
|sr^{2k} + \imath sr^{2k+n/2}\rangle &= |sr^{2k} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\rangle \\
|sr^{2k} - \imath sr^{2k+n/2}\rangle &= |sr^{2k} \begin{pmatrix} 0 \\ 1 \end{pmatrix}\rangle
\end{aligned} \tag{A.47}$$

we see that the two representations are indeed equivalent: $\Pi^s_{J_2}|_{\mathscr{T}} \simeq \Omega^s_\alpha$. Similarly $\Pi^s_{J_3}|_{\mathscr{T}} \simeq \Omega^s_\alpha$, because the only difference is $J_3(s) = -J_2(s)$, but

$$\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\begin{pmatrix} & -\imath \\ \imath & \end{pmatrix}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} = \begin{pmatrix} & \imath \\ -\imath & \end{pmatrix}, \tag{A.48}$$

so it is still equivalent to $\alpha(s)$.

A.5 Induced representations for a semidirect product by an Abelian group

This section describes the construction of the irreducible group-representations of the semidirect product of a group (with known irreducible represenations) by an Abelian group. This formalism is used in §6.3 to construct the irreducible representations of the group G_n.

We follow Serre [43, §8.2], but the derivation is presented more elaborately here.

A.5.1 Extending representations of the Abelian subgroup

We take a group G and subgroups A and H, such that A is Abelian and normal, and G is a semidirect product of H by A, denoted as $G = A \rtimes H$. This means that every element of G is written as ah with $a \in A,\ h \in H$, and the multiplication is given by

$$(ah)(a'h') = a(ha'h^{-1})hh'.^{\dagger} \tag{A.49}$$

Because A is Abelian, all its irreducible representations are one-dimensional, and the characters are equal to these representations and form a group of group morphisms from A to the non-zero complex numbers:

$$X = \mathrm{Hom}(A, \mathbb{C}^*) \qquad \mathbb{C}^* = \mathbb{C} - \{0\}. \tag{A.50}$$

There are $|A|$ characters, which are orthogonal by the orthogonality theorem for finite groups. The multiplication is given by $\chi.\chi'(a) = \chi(a)\chi'(a)$. Then $\chi(a)$ is never zero, as there is some integer m such that a^m is the unit element, and $1 = \chi(a^m) = \chi(a)^m$. The unit of the group is of course the character of the trivial representation $\chi : a \in A \mapsto 1$.

We can define an action of the group G on X by

$$(g \rightharpoonup \chi)(a) = \chi(g^{-1}ag) \qquad g \in G, \chi \in X, a \in A. \tag{A.51}$$

This is a well-defined group action because

$$\begin{aligned}(g \rightharpoonup \chi)(a_1a_2) &= \chi(g^{-1}a_1a_2g) = \chi(g^{-1}a_1gg^{-1}a_2g) = \chi(g^{-1}a_1g)\chi(g^{-1}a_2g) \\ &= \big((g \rightharpoonup \chi)(a_1)\big)\big((g \rightharpoonup \chi)(a_1)\big)\end{aligned}$$

Because every element g of G can be written as a_gh_g, we have

$$g^{-1}ag = h_g^{-1}a_g^{-1}aa_gh_g = h_g^{-1}ah_g, \tag{A.52}$$

so the action of g on X depends only on the class of $G/A = H$ to which g belongs.

Next we look at the orbits of H in X: we take a certain $\chi \in X$, and we let all the elements of H work on this character by the above defined action. This will give us a certain subset of X, called the orbit of H for χ. Because H is a group, all the characters in a certain orbit will be sent to each other. We will denote an orbit by a representative χ_i:

$$\chi_i = \{\chi \in X | \exists h \in H : h \rightharpoonup \chi_i = \chi\}. \tag{A.53}$$

Now we define a subgroup $H_i \subset H$ of all elements that leave the representative χ_i invariant:

$$H_i = \{h \in H | h \rightharpoonup \chi_i = \chi_i\}. \tag{A.54}$$

Set $G_i = A \rtimes H_i \subset G$ and extend $\chi_i : A \to \mathbb{C}^*$ to $\chi_i : G_i \to \mathbb{C}^*$ by

$$\chi_i(ah) = \chi(a) \qquad \forall a \in A, h \in H_i. \tag{A.55}$$

We will now show that this is a character of degree 1, i.e. $\chi_i(e) = 1$, for G_i. For clarity denote the respective maps by $\chi_i^{G_i}$ and χ_i^A. If $\chi_i^{G_i}$ is to be a character of degree

† There is a more general definition of a semidirect product: take two groups K and H, and $\varphi \in \mathrm{Hom}_{\mathrm{Group}}\big(H, \mathrm{Aut}(K)\big)$. Then $K \rtimes H$ is the group of pairs kh, with multiplication $(kh)(k'h') = k(\varphi(h) \rightharpoonup k')hh'$. In the above case, $\varphi(h)$ is the action of conjugation by h.

one, it has to be equal to a one-dimensional representation of G_i. Take two elements $g_j = a_{g_j} h_{g_j},\ j = 1,2$ of G_i. By definition we have:

$$\begin{aligned}
\chi_i^{G_i}(g_1 g_2) &= \chi_i^{G_i}(a_{g_1} h_{g_1} a_{g_2} h_{g_2}) = h_{g_1} \rightharpoonup \chi_i^{G_i}(a_{g_1} h_{g_1} a_{g_2} h_{g_2}) \\
&= \chi_i^{G_i}(\underbrace{h_{g_1}^{-1} a_{g_1} h_{g_1}}_{\in A \text{ as } A \text{ normal}} a_{g_2} \underbrace{h_{g_2} h_{g_1}}_{\in H}) = \chi_i^{A}(h_{g_1}^{-1} a_{g_1} h_{g_1} a_{g_2}) \\
&= \chi_i^{A}(h_{g_1}^{-1} a_{g_1} h_{g_1}) \chi_i^{A}(a_{g_2}) = \big(h_{g_1} \rightharpoonup \chi_i^{G_i}(a_{g_1})\big) \chi_i^{g_i}(a_{g_2}) \\
&= \chi_i^{G_i}(a_{g_1}) \chi_i^{G_i}(a_{g_2}) = \chi_i^{G_i}(a_{g_1} h_{g_1}) \chi_i^{G_i}(a_{g_2} h_{g_2}) \\
&= \chi_i^{G_i}(g_1) \chi_i^{G_i}(g_2).
\end{aligned}$$

Together with $\chi_i^{G_i}(e) = \chi_i^{A}(e) = 1$ we see that indeed $\chi_i^{G_i}$ is a one-dimensional representation of G_i, and we will revert to the notation χ_i for both G_i and A.

Let ρ^{H_i} denote an irreducible represenation of H_i, then define a irreducible representation of G_i by $\rho^{G_i} : g = ah \mapsto \rho^{H_i}(h)\ \forall a \in A,\ h \in H_i$. We see that ρ^{G_i} is irreducible by noting that ρ^{H_i} and ρ^{G_i} have the same representation space, and that if that space does not have an invariant subspace under the action of H_i, it certainly is not going to have an invariant subspace under the action of G_i, of which H_i is a subgroup.

By defining $\chi_i \otimes \rho^{G_i} : g \in G \mapsto \chi_i(g) \otimes \rho^{G_i}(g)$ we obtain another irreducible representation of G_i: it is clearly a representation by the above definitions; if we denote by V_ρ the representation space of ρ^{G_i}, then the representation space of $\chi_i \otimes \rho^{G_i}$ is $\mathbb{C}^* \otimes V_\rho$, and we have an injection $V_\rho \to \mathbb{C}^* \otimes V_\rho,\ \mathbf{v} \mapsto 1 \otimes \mathbf{v}$. Through this injection, we see that if V_ρ has no invariant subspace under the action of G_i, then $\mathbb{C}^* \otimes V_\rho$ does not either: $\chi_i \otimes \rho^{G_i}$ is an irreducible representation.

A.5.2 Induced representations

As our next step we are going to induce this representation of G_i to G. There is much to say about induced representations, and we will provide only what is needed for what follows. More can be found in for example [13, 24, 43].

Let $\mathbb{C}E$ be a group algebra and $\mathbb{C}F$ the group algebra of a subgroup, and φ a representation of $\mathbb{C}F$, with representation space V_φ. We can then contruct a representation of $\mathbb{C}E$ by considering $\mathbb{C}E \otimes_{\mathbb{C}F} V_\varphi$ as a $\mathbb{C}E$-module. In the language of represenations this means that the representation space is $\mathbb{C}E/\mathbb{C}F \otimes V_\varphi$; after choosing a basis $e_k \otimes \mathbf{v}_l$, where e_k is considered as a coset representative, the action of $\mathbb{C}E$ is given by

$$\begin{aligned}
p \in \mathbb{C}E \rightharpoonup e_k \otimes \mathbf{v}_l &= p e_k \otimes \mathbf{v}_l \\
&= e_l^p \otimes \varphi(n_l^p) \mathbf{v}_l,
\end{aligned} \tag{A.56}$$

where $p e_k = e_l^p n_l^p,\ n_l^p \in F$. This is a representation, because multiplication in E is associative and φ is a representation. We can work this out by taking two elements $p, q \in \mathbb{C}E$, then

$$e_k^{pq} n_k^{pq} = (pq)e_k = p(qe_k) = (pe_k^q) n_k^q = (e_k^q)^p (n_k^q)^p n_k^q. \tag{A.57}$$

Thus $e_k^{pq} = (e_k^q)^p$ and $n_k^{pq} = (n_k^q)^p n_k^q$. Then we can write

$$\begin{aligned} (pq) \rightharpoonup e_k \otimes \mathbf{v}_l &= e_k^{pq} \otimes \varphi(n_k^{pq})\mathbf{v}_l \\ &= (e_k^q)^p \otimes \varphi\big((n_k^q)^p\big)\varphi(n_k^q)\mathbf{v}_l \\ &= p \rightharpoonup e_k^q \otimes \varphi(n_k^q)\mathbf{v}_l \\ &= p \rightharpoonup \big(q \rightharpoonup e_k \otimes \mathbf{v}_l\big). \end{aligned}$$

Now, returning to our previous case, we induce the irreducible representation $\chi_i \otimes \rho$ from G_i to G and denote the induced representation by $\theta_{i,\rho}$. A basis for its representation space is given by $g_k \otimes \mathbf{e} \otimes \mathbf{v}_l$, where the g_k are coset representatives of $G/G_i \simeq H/H_i$[†], $\mathbf{e}$ spans the one-dimensional representation space of χ_i, and the $\mathbf{v}_l$ form a basis of the representation space of ρ.

Proposition A.9.

(i) $\theta_{i,\rho}$ *is irreducible*

(ii) $\theta_{i,\rho} \simeq \theta_{i',\rho'} \Rightarrow i = i'$ *and* $\rho \simeq \rho'$

(iii) every irreducible representation of G is isomorphic to one of the $\theta_{i,\rho}$.

Proof. (i) We are going to use a theorem by Mackey [29, th.6], which is referred to in [43] as *Mackey's irreducibility criterion*: take a group E, a subgroup F, an irreducible representation ρ of F. For all elements $s \in E - F$, define the subgroup F_s by $sFs^{-1} \cap F$. Then the induced representation of ρ to G is irreducible if and only if for all $s \in E - F$, the two representations of F_s, $\rho^s : f \mapsto \rho(sfs^{-1})$ and $\rho|_{F_s} : f \mapsto \rho(f)$ are disjoint, i.e. have no common irreducible components.

We will show that this applies to our group $G_i = A \rtimes H_i$. So now $s \in G - G_i$, and for $g \in (G_i)_s$ we have $(\chi_i \otimes \rho)^s : g \mapsto \chi(sgs^{-1}) \otimes \rho(sgs^{-1})$, $(\chi \otimes \rho)|_{(G_i)_s} : g \mapsto \chi(g) \otimes \rho(g)$. It is enough to show that these representations are disjoint for the subgroup A of G_i. This can be seen by the following argument: to be disjoint representations, the decomposition of these representations into irreducible components may not contain any factor in common. If there were such a common factor, then considered as A-representations, these components would be isomorphic. But if there are no isomorphic components under the action of the elements of A, there certainly will be no such components under the action of the larger group G_i.

Note that the restriction of $\chi_i \otimes \rho$ to A is $\chi_i \otimes 1$, and all its decomposition factors are isomorphic to χ_i. This also gives $(\chi \otimes \rho)^s|_A = (s \rightharpoonup \chi_i^A) \otimes 1$. Next, we recall that

$$\begin{aligned} (s \rightharpoonup \chi_i^A)(a) &= (a_s h_s \rightharpoonup \chi_i^A)(a) = \chi_i^A(h_s^{-1} a_s^{-1} a a_s h_s) \\ &= \chi_i^A(h_s^{-1} a h_s) = (h_s \rightharpoonup \chi_i^A)(a). \end{aligned}$$

But as $s \notin A \rtimes H_i$, we know that $h_s \rightharpoonup \chi_i^A \neq \chi_i^A$. Therefore $(\chi_i \otimes \rho)^s|_A$ and $(\chi_i \otimes \rho)|_A$ are disjoint, and $\chi_i \otimes \rho$ is an irreducible G-representation.

[†]This isomorphism of equivalence classes holds by $h = x\tilde{h},\ x \in H/H_i,\ \tilde{h} \in H_i \Rightarrow g \in G_i = ah = ax\tilde{h} = xx^{-1}ax\tilde{h} = x\tilde{g},\ \tilde{g} \in G_i$, because $a \in A$ and A normal.

(ii) Let's restrict $\theta_{i,\rho}$ to A. The action of an element $a \in A$ gives

$$\begin{aligned} a \rightharpoonup g_k \otimes \mathbf{e} \otimes \mathbf{v}_l &= (ag_k) \otimes_{\mathbb{C}G_i} \mathbf{e} \otimes \mathbf{v}_l = (g_k \underbrace{g_k^{-1} a g_k}_{\in A}) \otimes_{\mathbb{C}G_i} \mathbf{e} \otimes \mathbf{v}_l \\ &= g_k \otimes \chi_i(g_k^{-1} a g_k) \mathbf{e} \otimes \mathbf{v}_l = \\ &= g_k \otimes (g_k \rightharpoonup \chi_i)(a) \mathbf{e} \otimes \mathbf{v}_l \end{aligned} \tag{A.58}$$

As we see, we will only get characters χ in the orbit $H\chi_i$, so that the label i in $\theta_{i,\rho}$ determines χ_i. This still holds for the action of other elements of G, so those ah for which $h \neq e$, as $\chi_i^{G_i}(ah) = \chi_i^A(a)\ \forall h \in H_i$, and the most left tensor product denotes $\mathbb{C}G_i$-linearity, in other words, it 'lets through' elements of $\mathbb{C}G_i$.

For the second part, of the represenation space W of $\theta_{i,\rho}$ take the subspace $W_i = \{\mathbf{w} \in W | \theta_{i,\rho}(a)\mathbf{w} = \chi_i(a)\mathbf{w}\ \forall a \in A\}$. Writing this out on a basis element:

$$\begin{aligned} a \rightharpoonup (g_k \otimes \mathbf{e} \otimes \mathbf{v}_l) &= g_k g_k^{-1} a g_k \otimes \mathbf{e} \otimes \mathbf{v}_l \\ &= g_k \otimes \chi_i(g_k^{-1} a g_k) \mathbf{e} \otimes \rho^{G_i}(g_k^{-1} a g_k) \mathbf{v}_l \\ &= g_k \otimes (g_k \rightharpoonup \chi_i)(a) \mathbf{e} \otimes \rho^{H_i}(e) \mathbf{v}_l, \end{aligned}$$

so that in order for this to be equal to $g_k \otimes \chi_i(a)\mathbf{e} \otimes \mathbf{v}_l$, we demand $g_k \rightharpoonup \chi_i = \chi_i$. But then $g_k \in H_i$, and since $g_k \in H/H_i$, we must have $g_k = e$. Now, take an element $h \in H_i$, then

$$\begin{aligned} h \rightharpoonup (e \otimes \mathbf{e} \otimes \mathbf{v}_l) &= e \otimes \chi_i^{G_i}(h) \mathbf{e} \otimes \rho(h) \mathbf{v}_l \\ &= e \otimes \chi_i^A(e) \mathbf{e} \otimes \rho(h) \mathbf{v}_l \end{aligned}$$

and we see that this representation of H_i on W_i is isomorphic to ρ. So the label ρ in $\theta_{i,\rho}$ determines the representation ρ.

We conclude that for two G-representations $\theta_{i,\rho}, \theta_{i',\rho'}$ to be isomorphic, we must have $i = i'$ and $\rho \simeq \rho'$.

(iii) The proof can be found in [43, §8.2] and requires some additional preliminary knowledge; I may decide to include it here at a later stage.

So by applying this construction, we are able to collect all irreducible representations of G. □

Appendix B

GAP[†] code comparing group representations to $D(D_n)$

```
#GAP instructions that searches all groups of order 4n^2 for
#those of which the dimensions of the irreducible representations
#correspond to those of the quantum double D(D_n) of the dihedral
#group D_n of order 2n.

searchDDnIrreps := function ( n )
  local  d, i, k, l, m, o, p, q, t, dimlist, nlist;

  #t is the number of irreducible representations of D(D_n)
  if n mod 2 = 0  then
    t := 1 / 2 * n * n + 14;
  else
    t := 1 / 2 * n * n + 7 / 2;
  fi;
  o := 4 * n * n; # o is the dimension of D(D_n)
  l := NrSmallGroups( o );
  Print( "There are ", l, " groups of order ", o, "\n" );
  i := 1; #i is the index looping through the groups of order o
  k := 0; #denotes the number of groups that have the right
          #number of irreducible representations with the right
  #dimensions
  repeat #we loop through all groups of order o, which are
         #retrieved from the GAP library SmallGroups
    Print(i, " " );
    #we first check whether the number of conjugacy classes
    #of this group is equal to the number of irreducible
    #representations of D(D_n). If it is not, we can discard
    #this particular group.
    if NrConjugacyClasses( SmallGroup( o, i ) ) = t  then
      dimlist := [  ]; #dimlist is a list that holds the
                       #different dimensions of the
                       #irreducible representations
      nlist := [  ]; #nlist holds the number of irreducible
                     #representation of dimension corresponding
```

[†]This software is distributed freely at http://www-gap.dcs.st-and.ac.uk/~gap/

```
     #to the same position in dimlist
      q := IrreducibleRepresentations(SmallGroup(o,i));
      for p  in q  do
        d := DimensionOfMatrixGroup( Image( p ) );
        if d in dimlist  then
          nlist[Position( dimlist, d )] :=
            nlist[Position( dimlist, d )] + 1;
        else
          Add( dimlist, d );
          Add( nlist, 1 );
        fi;
      od;
      if n = 4  then
        #case 4 is special, because representations of dimension
        #n/2 and of dimension 2 have the same dimensions
        if Length( dimlist ) = 2 and
          nlist[Position( dimlist, 1 )] = 8 and
          nlist[Position( dimlist, 2 )] = 14
        then
          Print("\nGroup [", o, ",", i, "] complies\n");
          k:=k+1;
        fi;
      elif n mod 2 = 0  then
        if Length( dimlist ) = 3 and
          nlist[Position( dimlist, 1 )] = 8 and
          nlist[Position( dimlist, 2 )] = 1/2*n*n-2 and
          nlist[Position( dimlist, n / 2 )] = 8
        then
          Print("\nGroup [", o, ",", i, "] complies\n");
          k:=k+1;
        fi;
      else
        if Length( dimlist ) = 3 and
          nlist[Position( dimlist, 1 )] = 2 and
          nlist[Position( dimlist, 2 )] = 1/2*n*n-1/2 and
          nlist[Position( dimlist, n / 2 )] = 2
        then
          Print("\nGroup [", o, ",", i, "] complies\n");
          k:=k+1;
        fi;
      fi;
    fi;
    i := i + 1;
    UnloadSmallGroupsData();
  until i > l;
  Print("There are ",k," groups of order ",o," that comply\n");
  return;
end;;
```

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Samenvatting

Dit is de plaats om, in wat in modern Nederlands "Jip-en-Janneke-taal" heet, op te sommen wat vele maanden werk en vele pagina's manuscript nu hebben opgeleverd, zodat familie en vrienden verklaard wordt dat dat toch echt allemaal nodig was. Het gemakkelijkst zou zijn, de welwillende maar ongeïnformeerde lezer te overdonderen met imponerend jargon en termen als "fundamenteel inzicht", "diepe verbanden" en "universele beschrijving". Ik zal dit podium echter gebruiken om, in contrast met de hier en daar droge behandelde stof, wat luchtig door de concepten heen te gaan, in de hoop toch nog het één en ander te kunnen overbrengen.

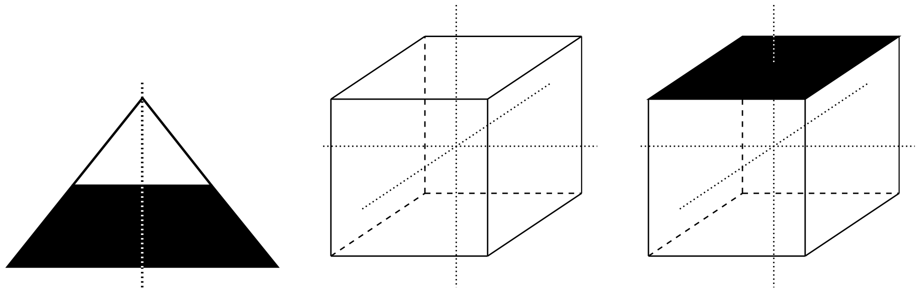
In zeker de helft van de recente natuurkundescripties zal op deze plaats begonnen worden met een frase als: "Symmetrie is uitgegroeid tot het kernbegrip in de natuurkunde, waarmee vele verschijnselen beschreven kunnen worden" (zie ook de eerste zin van de inleiding op pagina vi). Nu, dat is waar, en is ook zeker hier van toepassing. In één zin wordt in deze scriptie een veralgemenisering van een veelgebruikte toepassing van symmetrie in de natuurkunde, genaamd ijksymmetrie, beschreven, waarmee vervolgens een aantal theoretische (hypothetische) modellen wordt doorgerekend. Gelukkig zijn symmetrie en ijksymmetrie goed voorstelbaar en helder uit te leggen; voor mijn eigen werk hierover geldt dat helaas een stuk minder.

Symmetrie in de natuur Een definitie van symmetrie zou kunnen zijn, dat de (natuurkundige) eigenschappen van een systeem voor en na een zekere transformatie hetzelfde zijn. Je kunt je dat als volgt voorstellen: ik stop een systeem in een kamer en laat jou er naar kijken, je mag eraan meten wat je wilt. Vervolgens ga jij de kamer uit, en ik voer er een transformatie op uit: ik draai het rond of iets dergelijks. Daarna laat ik je weer binnen, en je mag weer gaan meten. Als jij het systeem nu niet kunt onderscheiden ten opzichte van de eerdere situatie, is dat systeem *invariant* onder de door mij uitgevoerde *symmetrietransformatie*.

Neem bijvoorbeeld een massief vierkant blok op een tafel. Als ik die over een rechte hoek ronddraai rond zijn middellijn, kan de terugkerende waarnemer niet zeggen of ik die nou rondgedraaid heb of niet. Een rechtopstaande cylinder kan ik zelfs over iedere hoek ronddraaien. Een bol kan ik ook over andere assen draaien.

Een symmetrie hoeft niet noodzakerlijkerwijs praktisch uitvoerbaar te zijn. De driehoek van figuur 1(a) is niet invariant onder draaiingen, maar een spiegeling langs de stippellijn levert wel weer hetzelfde figuur op. Ondanks dat we zo'n spiegeling niet in het echt kunnen bewerkstelligen, verschaft het wel informatie over de kenmerken van het figuur.

Op deze manier zijn er veel vormen van symmetrie: rotatiesymmetrie (draaiingen), spiegelingen in een vlak, lijn of punt, translatiesymmetrie (verschuivingen), en meer abstractere zoals bijvoorbeeld tijdsomkeersymmetrie: de video-opname van de



(a) (b) (c)

Figuur 1: Symmetrische objecten

beweging van een slinger van een staande klok kunnen we niet onderscheiden van die opname die achterstevoren wordt afgespeeld.

Symmetriegroepen Een belangrijke eigenschap van een symmetrietransformatie is dat wanneer we er twee achter elkaar uitvoeren, deze samengestelde transformatie zelf ook een symmetrietransformatie vormt: twee keer draaien over een rechte hoek zal het blok ook ononderscheidbaar achterlaten. We zeggen nu dat alle transformaties die één bepaald systeem invariant laten de wiskundige structuur van een *groep* hebben, afgezien van een paar andere eisen.

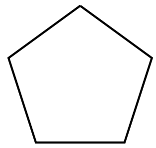
Nu kan een groep verschillende *ondergroepen* hebben, beperkte sets transformaties die samen weer een gesloten groep vormen. Dit zal ik met een voorbeeld duidelijk maken.

Neem een witte kubus, deze is invariant onder draaiingen over rechte hoeken (veelvouden van 90°) over drie assen (figuur 1(b)). Stel dat ik nu één zijde zwart kleur, dan kun jij een draaiing over bijvoorbeeld 90° over twee van de drie assen onderscheiden (figuur 1(c)). Draaiingen over de as die de zwarte zijde doorsnijdt, laten de kubus nog wel invariant. We zeggen nu dat de draaiingen door die as een ondergroep vormen van de draaiingen over drie assen. Verder zeggen we dat door het kleuren van die ene zijde de symmetrie *gebroken* is, en dat de *overgebleven symmetrie* de draaiingen rond één as zijn[†].

Diëdergroepen In mijn scriptie heb ik met name gekeken naar veralgemeniseringen van bepaalde ondergroepen van de groep alle draaiingen van een pijl met vast beginpunt in drie dimensies. Deze ondergroepen heten diëdergroepen, en bestaan uit draaiingen over veelvouden van een vaste hoek èn evenveel spiegelingen. De n-de diëdergroep (genoteerd door D_n) geeft alle symmetrieën van een regelmatige n-hoek. Zo kan een vijfhoek steeds over 72° gedraaid worden, en zijn er vijf spiegellijnen (figuur 2).

Het bijzondere aan draaiingen in drie dimensies is dat twee opeenvolgende draaiingen niet hetzelfde resultaat hoeven te geven als dezelfde twee in omgekeerde volgorde, wat je zelf kunt nagaan met een dobbelsteen. Deze eigenschap van een groep heet

[†]Een interessant geval doet zich voor als we nu drie andere vlakken ook zwart kleuren, zo dat twee tegenover elkaar liggende vlakken wit blijven. Draaiingen over de as door die witte vlakken laten het systeem nu wel weer invariant. Dit heet symmetrieherstel, een verschijnsel dat ik in mijn scriptie echter niet behandel.



Figuur 2: Vijfhoek heeft D_5-symmetrie

niet-commutativiteit. De diëdergroepen (met n groter dan 2) zijn ook niet-commutatief, wat aanleiding geeft tot enkele bijzondere gevolgen.

De modellen die ik in mijn scriptie behandeld heb, zijn alle gebaseerd op diëdergroepen met even n.

IJksymmetrie Symmetrie kan zich, in contrast met de hierboven beschreven gevallen, ook op een andere, meer verhulde manier manifesteren. Een voorbeeld is het meest sprekend: stel dat ik de afstand tussen twee punten wil meten. Dan pak ik een liniaal, lees de twee markeringen ten hoogte van beide punten af, en bereken het verschil. Het maakt nu niet uit, met welke markeringen ik begin: ik kan de liniaal bij het eerste punt op 0 cm leggen, maar net zo goed op 10 cm, het resultaat blijft hetzelfde. Het kiezen van de markeringen, noemen we het kiezen van een *ijk* (*gauge* in het Engels), die voor een zekere *ijkvrijheid* of *ijksymmetrie* zorgt.

In het algemeen duidt een ijkvrijheid op een overcomplete beschrijving van een systeem. Dat wil zeggen dat we de grootheden van dat systeem te specifiek weergeven. In de laatste vijftig jaar is dit concept in de natuurkunde in belang gegroeid, omdat we met ijkvrijheid, die je in eerste instantie als onvolkomenheid in de beschrijving op zou kunnen vatten, juist een heldere formulering van wisselwerking tussen deeltjes verkrijgen. In het bijzonder zijn de deeltjes die geïntroduceerd worden om de ijksymmetrie in bepaalde theorieën te beschrijven, precies de deeltjes die krachten zoals de electromagnetische kracht overdragen.

In deze scriptie worden modellen met ijksymmetrie behandeld die ook nog een andere vorm van wisselwerking vertonen, die hieronder beschreven wordt.

Topologische interacties Als natuurkundigen en ‘gewone mensen’ nadenken over wisselwerking van deeltjes, dan komt al snel het beeld van botsende biljartballen naar voren. Dit is tot op zekere hoogte een goede benadering. Een meer nauwkeurige beschrijving is dat twee deeltjes een ijkdeeltje uitwisselen, zoals hierboven uitgelegd; zo stoten twee electronen elkaar af onder uitwisseling van een foton.

Nu blijken in bijzondere systemen deeltjes ook te kunnen wisselwerken zonder dat er sprake is van de uitwisseling van een krachtdrager. Deze wisselwerking is afhankelijk van bepaalde meetkundige (topologische) eigenschappen van wat we de *configuratieruimte* van die systemen noemen. De configuratieruimte is een manier om alle mogelijke toestanden van alle deeltjes in een systeem te ordenen, en een punt in die ruimte is dan één bepaalde toestand. Voor eenvoudige benaderingen is het bijvoorbeeld voldoende om de plaats en de snelheid van ieder deeltje te weten.

Normaliter zijn de grootheden die het systeem in de configuratieruimte beschrijven *continu*: zij vertonen geen sprongen. Wanneer er zich wel een sprong of *discontinuïteit* voordoet, spreken we van een *topologisch defect*. Deze topologische defecten kunnen nu invloed uitoefenen, wisselwerking aangaan, met gewone deeltjes of met elkaar.

Het bijzondere van dit soort wisselwerkingen, die *topologische interacties* genoemd worden, is dat ze veroorzaakt kunnen worden door aanpassingen aan een systeem, die in eerste instantie geen merkbaar verschil ten gevolg hebben.

Het bekendste voorbeeld is het Aharonov–Bohm effect, schematisch weergegeven in figuur 1.1 op pagina 5. Twee electronenbundels veroorzaken een interferentiepatroon op een scherm. We plaatsen een zeer lange spoel tussen de bundels; door een stroom door de spoel te sturen wordt een magnetisch veld opgewekt, maar dat veld bevindt zich alleen binnen de spoel. Zo'n veld zou invloed hebben op de beweging van de electronen, maar doet dat nu niet, want buiten de spoel is dat veld er niet. Toch zien we dat het interferentiepatroon verschuift, en dat wordt, kort gezegd, veroorzaakt door een overblijfsel van het magnetisch veld, dat zich ook buiten de spoel bevindt, maar dat normaliter geen merkbare invloed op geladen deeltjes heeft.

Een belangrijke eigenschap van deze topologische interacties is dat ze enkel waargenomen kunnen worden, wanneer een topologisch defect geheel omcirkeld wordt. Zo vormen de twee electronenbundels een gesloten lus om de spoel, het topologisch defect hier. Daarom wordt de beschrijving van topologische defecten en interacties wiskundig gegeven door te kijken naar gesloten lussen in de configuratieruimte.

Vlechtstatistiek Een gevolg van topologische interacties is dat de verwisseling van twee deeltjes in een systeem op bijzondere wijze plaatsvindt. De beschrijving van hoe een veel-deeltjes systeem zich gedraagt onder verwisseling van twee of meer deeltjes wordt in de natuurkunde de statistiek van dat systeem genoemd. Tot voor kort waren er twee mogelijkheden bekend, waarbij in beide gevallen het systeem identiek aan zichzelf is na tweemaal verwisseling van dezelfde twee deeltjes.

Wanneer er echter topologische interacties aanwezig zijn, gaat dit niet langer op. We hebben namelijk gezien dat wanneer een deeltje een defect omcirkelt, wat neer komt op het tweemaal verwisselen van de twee, dit deeltje beïnvloed kan worden, zodat het totale systeem veranderd is ten opzichte van de eerdere situatie. Je zou je dit kunnen voorstellen als dat het defect en het deeltje om elkaar heen vlechten, en zo'n vlecht kan niet zomaar worden losgetrokken.

Om deze reden zegt men dat zulke systemen onderhevig zijn aan *vlechtstatistiek* (Engels: *braid statistics*). In bepaalde gevallen levert de verwisseling van twee deeltjes alleen een breuk van gehele getallen, een fractie, op, en er wordt dan gesproken van *fractionele statistiek*, waarnaar op dit moment veel onderzoek plaatsvindt, ook in het lab.

Quantum dubbels Nu is het tijd om al deze zaken aan elkaar te knopen. In deze scriptie is gekeken naar systemen met ijksymmetrie, en eigenschappen onder ijktransformaties classificeren de deeltjes in zo'n systeem. Verder gaan we er vanuit dat er ook topologische defecten zijn die topologische interacties veroorzaken. De defecten vatten we ook als puntdeeltjes op, die eveneens door middel van de ijkgroep geclassificeerd worden. Ten slotte kunnen we ons ook samengestelde deeltjes voorstellen, die zowel *ijklading* als *topologische lading* hebben, ofwel die zich zowel als gewoon deeltje als als defect manifesteren.

Uitgaande van de ijkgroep, kunnen we een ander wiskundig object construeren, de *quantum dubbel* van die ijkgroep. Een quantum dubbel is een speciaal geval van een *Hopf algebra*, en veel van het gepresenteerde werk is van quantum dubbel naar Hopf algebra te generaliseren.

De quantum dubbel verschaft alle informatie over de classificatie van deeltjes, defecten en samengestelde deeltjes. Voorts levert het ook direct een beschrijving voor de vlechtstatistiek van die deeltjes. Anders gezegd zijn de wiskundige eigenschappen van de quantum dubbel precies de veralgemenisering van de eigenschappen van de ijkgroep wanneer we ook topologische interacties in beschouwing nemen.

Verder kunnen we met deze quantum dubbels ook naar eerder genoemde symmetriebreking kijken: er is wederom een natuurlijke manier om symmetriebreking van groepen te veralgemeniseren naar quantum dubbels. Het grootste gedeelte van deze scriptie beslaat het doorrekenen van de symmetriebreking van quantum dubbels van even diëdergroepen.

Wat heb je er nu aan? Samengevat behandel ik dus een reeds opgebouwde theorie die een beschrijving geeft van symmetrie en symmetriebreking van een systeem waarin zowel interacties door middel van uitwisseling van ijkdeeltjes als topologische interacties voorkomen. Verder heb ik een hele familie van mogelijke quantum dubbels doorgerekend.

De grote vraag is natuurlijk: waar vind je zulke systemen dan? Wel, die zijn nog niet waargenomen. Er zijn natuurlijk aanleidingen om deze beschrijving zo op te stellen, maar concrete gevallen zijn nog niet bekend.

De eerste plaats om te zoeken is wat we *gecondenseerde materie* noemen, vaste stoffen en vloeistoffen. Denk aan kristallen en metalen; het onderzoek wordt met name verricht aan hele speciale vormen daarvan. Veel bijzondere eigenschappen zoals supergeleiding zijn daarin waargenomen. Ook zijn er veel voorbeelden van topologische defecten.

Wellicht zal er in de nabije toekomst een materiaal in een bijzondere toestand worden gevonden, waarin deze topologische defecten met andere *excitaties* kunnen wisselwerken op een manier die met een Hopf algebra te beschrijven is. Of dat dan ook de quantum dubbels van diëdergroepen zijn, valt te betwijfelen, maar je weet maar nooit.

Zoals eerder vermeld is een ander *hot topic* de fractionele statistiek. Hier wordt veel experimenteel onderzoek gedaan, en er zijn zeer recent heeft een onderzoeksgroep geclaimd deeltjes die fractionele statistiek vertonen direct gemeten te hebben. De modellen in deze scriptie leveren de fractionele statistiek (en meer exotische vlechtstatistiek) op een natuurlijke manier, en daar ligt misschien ook een toekomst voor hen.

Kort na het afstuderen op deze scriptie zal ik promotieonderzoek gaan verrichten bij professor Jan Zaanen aan de Universiteit Leiden. In dat onderzoek ga ik de Hopf-symmetriebenadering toepassen op zekere modellen analoog aan de beschrijving van vloeibare kristallen, *quantum liquid crystals* genaamd. De hoop is dan een echt verband te vinden tussen deze theoretische modellen en natuurkunde die experimenteel geverifieerd kan worden.

Dankwoord

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