# Core Deformations of Topological Defects

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**Abstract**

We consider phases of spontaneously broken gauge theories in which different types of topological excitations, like fluxtubes, monopoles etc. coexist. We analyse the topological relations which exist between the charges, and give a description of the admissible core deformations in terms of certain cohomology groups that occur in obstruction theory. Simple examples are worked out in detail.

## 1 Introduction

An interesting feature of (spontaneously) broken gauge theories is the existence of topological excitations. One may think of fluxtubes, magnetic monopoles and instantons. The topological charges are labelled by certain homotopy groups of the coset space $G/H$, where $G$ is the gauge group and $H$ the residual symmetry group of the groundstate [1, 2].

Most of the research on these topological excitations has focussed on a situation where only a single type of excitation is present in the theory, basically because usually only one of the relevant homotopy groups is non-trivial. In this paper however we will investigate the situation where different types of topological excitations can coexist. The theories in which this occurs, i.e. where the coset space $G/H$ has more than one non-trivial homotopy group, may exhibit novel features which have to do with a non-trivial coupling between different topological quantumnumbers. In simple models we are used to think of the topological singularity, where the topological charge is located, to be point-like in the appropriate dimension (i. e. $d = 2, 3, 4$ for fluxtubes, monopoles and instantons successively), but in models with a coupling between the different quantumnumbers we will encounter situations where the topological singularity may be extended, for example to a ring or a higher dimensional closed surface. This means that the analysis of topological stability becomes more involved. In the conventional cases the stability derives from the asymptotic (large r) behaviour of the fields, which determined the element of the relevant homotopy group $\pi_n(G/H)$. The core was always assumed to be point-like and the question of "instability" of the core never presented itself. However if several topological quantumnumbers are available, the question of core instability becomes relevant. Such an instability may involve a topological deformation of the core. For example, in certain models a magnetic charge may be located on a ring, rather than in a single point [9]. And it is conceivable that by varying the parameters of the theory the deformation of the point charge to a ring charge will become favoured energetically.

In this paper we present a rather general topological analysis of this problem, and show that the appropriate mathematical setting in which to describe such phenomena is *obstruction theory*.

In section 2 we formulate the problem of connecting the ordinary monopole with monopoles with an *extended* topological singularity. We present a construction which relates the normal asymptotic $\pi_n$ classification of defects to the homotopy classification of the extended singularity. In this construction a mathematical theory, called obstruction theory, is employed which we will discuss in section 3. This theory deals with the question of extending maps defined on a space $A$ into a space $Y$, to a map from $X \supset A$ to $Y$. Obstruction theory gives certain criteria for the spaces and maps by which we can see if the map has an extension over $X$.

The paper closes with a few examples. These show how the construction described in section 2 can be applied. The case of a monopole which has an extended topological singularity is discussed in detail.

# 2 The question of extended topological singularities

As is well known, the magnetic charge of monopoles in a gauge theory with gauge group $G$, spontaneously broken to a subgroup $H$, are in one-to-one correspondence with $\pi_2(G/H)$, the second homotopy group of the vacuum manifold. To have a solution with non-vanishing magnetic charge, the asymptotic higgs field

$$\phi_\infty(\theta, \phi) : S^2 \longrightarrow G/H \tag{1}$$

must belong to a non-trivial element of $\pi_2(G/H)$. If $[\phi_\infty]$ is non-trivial, then it follows from continuity considerations that $\phi(\vec{x})$, $(\vec{x} \in \mathbf{R}^3)$ has to be singular in at least one point. This singularity is characterized by the fact that the value of $\phi$ at that point has a different residual symmetrygroup $H' \supset H$. In the case $G = SU(2)$ this implies that $\Phi$ has to vanish sothat $H' = SU(2)$. We say that the gauge symmetry at the singularity is (partially) restored. For larger groups the situation is slightly more involved but not in an essential way. If $\phi$ is in the adjoint representation it fixes a Cartan-element in the Lie-algebra, this will generically break the symmetry to a product of $rank(G)$ $U(1)$ factors, therefore the magnetic charge has in that case

$rank(G)$ components which are topologically conserved. The most degenerate case that can occur in that situation is that the residual symmetrygroup is of the form $U(1) \bigotimes K$ where $K$ is some semisimple subgroup of G, in that case there is only one component of the magnetic charge conserved topologically. In a general group, where the magnetic charge generically has more then a single component which is conserved topologically, there are ofcourse many possible types of topological singularities. These correspond to changes in the value of $\phi$ where the number of conserved components changes some way. If we think of $\phi$ as a vector in the Cartan-subalgebra it means that at a topological singularity one or more components of that vector go to zero in a suitable basis. And at the singularity the gaugesymmetry need only be partially restored These complications are not essential to what we want to discuss in this paper. For reasons of simplicity we stick therefore to the case $G = SU(2)$ where a topological singularity is the subset $\mathcal{T} \subset R^3$ where the Higgsfield $\phi$ vanishes. We remark that in an actual monopole solution the topological singularity is just a point, at the other hand the fields $\phi$ and $A$ are in fact completely regular and the energy of such a configuration is finite.

We should be aware of the fact that the topological singularity is point-like does however not follow from topology, but rather from energy considerations. It is conceivable that instead of in a point, $\phi$ vanishes on some closed compact submanifold of $\mathbf{R}^3$, which is topologically non-trivial. In the remainder of this section we will consider this possibility of an extended topological singularity and relate this to homotopy theory. The topological singularity is enclosed by the so-called monopole core. Inside this core, at the singularity, the symmetry remains unbroken. Outside of the core the fields rapidly approach their asymptotic value, which is governed by the broken symmetry. The core size is roughly the inverse of the mass associated with the symmetry breaking scale. It is the boundary of the core which will play an important role in our considerations.

We mentioned already that the classification of monopoles with point-like singularity in $\mathbf{R}^{k+1}$, is given by $[\phi] \in \pi_k(G/H)$, where we consider $\phi$ now as a map of the boundary of the core ($\simeq S^k$) to $G/H$. In the following we shall consider a field on a smooth and closed $k$-dimensional manifold $A$ in $\mathbf{R}^{k+1}$, where $A$ is the boundary of a core which contains some extended topological singularity. The question we want to adress in general is whether the point-like monopole can smoothly be deformed ("decay") into an object with the extended charge. We may think of this deformation as a result of slowly changing the parameters in the model. So we want to establish a relation between possible field configurations on $S^k$, which are characterized by elements of the $n$-th homotopy group, and possible field configurations on $A$, which are characterized by the homotopy classes of $[A;Y]$. In order to answer this question, we start by recalling the notion of *cobordism*. A cobordism between two $n$-dimensional manifolds $C_1$ and $C_2$ is defined as a $(n+1)$-dimensional manifold $D$ such that $\partial D = C_1 \cup C_2$, where $C_1$ and $C_2$ are disjunct. If such a manifold exists then $C_1$ and $C_2$ are called cobordant.

Let us assume that $A$ and $S^k$ are cobordant. For the situation we are interested

Figure 1: Schematical representation of the "cobordism"-construction.

in, this is not a very restrictive condition. We can always put the manifold $A$ inside the sphere. The cobordism $B$ is the space between $A$ and $S^k$ with the latter as boundaries. This cobordism is well defined because $A$ and $S^k$ are closed compact manifolds without boundary in $\mathbf{R}^{k+1}$. It is obvious that $\partial B = A \cup S^k$.

The procedure to establish the relation between the configurations on $A$ and $S^k$ starts with a field configuration (map) on $S^k$, $f : S^k \to Y$. This map is labelled by the homotopy class of f, i.e. $[f] \in \pi_k(Y)$. We try to extend this map over the cobordism $B$. The central question is whether this can be done and if so, for which elements of $\pi_k(Y)$.

If it is possible to extend $f$ over $B$, the next question is, which configurations we can obtain on $A$ this way. In other words, we want to determine the homotopy classes of the extension restricted to $A$. Furthermore, we want to know which elements of $[A; Y]$ we may reach by different extensions. To answer this last question we use the same arguments in reverse order: we start with a configuration $h : A \to Y$ and then look whether it is possible to extend this map over $B$. We restrict this extension to $S^k$ and determine which elements of $\pi_k(Y)$ we can get. Comparing the results of the two constructions described above, we can decide which homotopy classes $[A; Y]$ and $\pi_k(Y)$ correspond to each other, i.e.. which topological defects can be smoothly deformed into each other by changing their core. An interesting possibility arises if more than one class $[A; Y]$ corresponds to a single element of $\pi_k(Y)$. Then one may expect that by varying the parameters in the potential, the solution with a point-like singularity changes smoothly into one with an extended topogical singularity.

In figure 1 we have sketched schematically the cobordism and the manifolds $S^k$ and $A$. The construction amounts to trying to extend $f$ and $g$ over the cobordism $B$ and then to determine the map at the other boundary.

So the things we need to know to make a classification of the allowed monopole core deformation are the structure of $\pi_k(Y)$, of $[A; Y]$, and in which cases there exists an extension. This latter, purely mathematical question can be answered in a very general way by means of obstruction theory, which is the subject of the next section.

# 3 Obstruction theory

In this section we will give a brief account of obstruction theory as it may be found in standard text books about algebraic topology. There are various approaches known in literature, such as in [3] or [4]. Here we will mainly follow the approach used in [5], which is the most accesible.

Obstruction theory tells us under which conditions a function, which maps a space $A$ to a space $Y$, can be extended continously to a map from a space $X$, with $A \subset X$, into the space $Y$. This extension problem can be diagrammatically represented as in figure 2.

From the physical point of view we are faced with the question of extending maps,

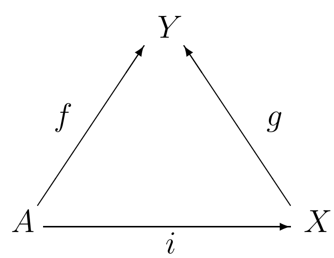


Figure 2: The extension problem

where $A$, $X$ and $Y$ are smooth manifolds. However, the natural setting in which obstruction theory is defined, is that of cell complexes. So we will restrict ourselves to maps of finite cell complexes into a pathwise connected space $Y$. Furthermore, we will mostly think in terms of a special kind of cell complex, namely the simplicial complex. To settle the notation we will start by giving definitions of simplicial complexes, chain complexes etc. , and other mathematical notions appearing in obstruction theory.

## 3.1 Basic definitions

All of the definitions given in this section can be found in the mathematical textbooks [3, 6]. Alternatively one may consult the book of Nash & Sen [7].

An $m$-dimensional *simplex* ($m$-simplex) $\sigma^m$ is defined as the set of points $x$ in

$\mathbf{R}^n$ given by

$$\sigma^m = \{x = \sum_{i=1}^{m+1} \lambda_i x_i | \lambda_i \geq 0, \sum_{i=1}^{m+1} \lambda_i = 1\}, \tag{2}$$

where $x_i, \ldots, x_{m+1}$ are independent.

A *simplicial complex* $K$ of dimension $k$ is a finite collection of simplexes in $\mathbf{R}^n$, with dimension less or equal than $k$, satisfying

1. if $\sigma^p \in K$, then all faces of $\sigma^p$ belong to $K$,
2. if $\sigma^p$,$\sigma^q \in K$ then either $\sigma^p \cap \sigma^q = \emptyset$ or $\sigma^p \cap \sigma^q$ is a common face of $\sigma^p$ and $\sigma^q$

A simplicial complex is a special kind of cell complex, the cells being simplexes.

The union of members of $K$ with the Euclidean subspace topology is called the *polyhedron* associated with $K$. The $m$-skeleton of a simplicial complex consists of all simplexes with dimension less or equal than $m$.

A simplicial complex is closely related to the familiar concept of triangulation, namely, a toplogical space $X$ which is homeomorphic to a polyhedron $K$ is said to be triangulable and the polyhedron $K$ is called a triangulation of $X$.

By introducing the concept of orientation on simplexes it is possible to make out of a simplicial complex another complex, the chain complex.

A *chain complex* $C = \{C_n, \partial_n\}$ is a sequence of abelian groups $C_n$ , $n = 0, \pm 1, \pm 2$ , . . ., and a sequence of homomorphisms (boundary operators) $\partial_n : C_n \rightarrow C_{n-1}$ which are required to satisfy the condition

$$\partial_{n-1} \circ \partial_n = 0 \tag{3}$$

for all $n$.

The relation between simplicial- and chain complexes can be phrased exactly by saying that there exists a functor from the category of simplicial complexes to the category of chain complexes. This relation makes it possible to introduce homology on simplicial complexes in the usual way.

If we have a chain complex $C$ and a module $G$, we can make a cochain complex (the same definition as a chain complex only the homomorphisms (coboundary operators) map into an abelian group with index one higher, instead of one lower) by using the functor Hom, which assigns to every chain complex $C$ and module $G$ the cochain complex $Hom(C, G) = \{Hom(C_q, G), \delta^q\}$. The coboundary operator is defined by

$$(\delta^q f)(c) = f(\partial_{q+1} c) \, , \qquad\qquad c \in C_{q+1}, \; f \in Hom(C_q, G).$$

Just as on chain complexes one can introduce homology on cochain complexes, now called cohomology.

The link between chain- and cochain complexes enables us to derive properties of cohomology from these of homology. We will however not pursue this link and give an axiomatic description of the properties of cohomology in the appendix. This description suffices to do the calculations needed for the determination of cohomology groups appearing in this paper. There is however another reason for emphasizing this link as will become apparent in the following: it leads to a description of obstruction in terms of cohomology.

## 3.2 Measuring the obstruction

We shall use the notation

$$\bar{X}^n = A \cup X^n,$$

with $X$ a cell complex and $A$ a subcomplex of $X$. The obstruction method consists of trying to extend the map $f : A \to Y$ step-by-step over the subcomplexes [5]

$$\bar{X}^n,\ n = 0, 1, 2, \cdots.$$

We will do so untill we meet an obstruction. We illustrate the subcomplex $\bar{X}^n$ for various $n$ in figure 3 where we take $X$ to be a triangulation of $S^2$ and as subcomplex we take the boundary of a face.

Figure 3: The subcomplexes $\bar{X}^n$ where $X$ is the triangulation of $S^2$

The obstruction method starts with a map on $A$. Then you try to extend this map over $\bar{X}^0$, so over the vertices outside $A$. This can always be done arbitrarily. Then you proceed with extending the map on $\bar{X}^0$ over $\bar{X}^1$, which are the lines outside $A$. This can always be done if $Y$ is pathwise connected. The next step is trying to extend over $\bar{X}^2$ etc. untill you reach $\bar{X}^m$, where $m$ is the dimension of $X$. Obstruction theory tells you when an extension is possible or not. If you run into trouble during the stepwise extension this is called an *obstruction*. We will call a map $n$-*extendable* over $X$ if it is extendable to $\bar{X}^n$.

Let $n$ be a given positive integer and assume that $Y$ is $n$-simple (a space $Y$ is $n$-simple if the fundamental group of $Y$ acts trivially on the homotopy groups $\pi_p(Y)$ for all $p \leq n$). If this is the case, every map of any oriented $n$-sphere into $Y$ determines a unique element of the homotopy group $\pi_n(Y)$.

The idea behind obstuction theory is that, if you have an extension over an $n$-skeleton, the obstruction to extending the map over the $(n+1)$-skeleton has something to do with the homotopy group $\pi_n(Y)$. This arises because the $(n+1)$-skeleton is built up out of the $n$-skeleton by filling in the $n$-cells. Extending over an $(n+1)$-cell, if the map is already defined on the boundary of the cell, is only possible if the element of the homotopy group $\pi_n(Y)$, which is associated with the boundary, is trivial. Of course this has to be true for all $(n+1)$-cells. This basic idea will be worked out in the remainder of this section.

Let us consider a given map

$$g : \bar{X}^n \to Y. \tag{4}$$

This map $g$ determines an $(n+1)$-cochain $c^{n+1}(g)$ of $X$ with coefficients in the homotopy group $\pi_n(Y)$ as follows. Let $\sigma$ be any $(n+1)$-cell of $X$. Then the set theoretic boundary $\partial\sigma$ of $\sigma$ is an orientated $n$-sphere. Since $\partial\sigma \subset X^n$, the partial map $g_\sigma = g|\partial\sigma$ determines an element $[g_\sigma]$ of $\pi_n(Y)$. So for every $(n+1)$-cell of $X$, which determines an element of the chain group $C_{n+1}(K)$, we can associate an element of $\pi_n(Y)$. This element is given by the homotopy class of $g_\sigma$. The association establishes a homomorphism $C_{n+1}(X) \to \pi_n(Y)$ for every $(n+1)$-cell of $X$. Since the cochain group $C^{n+1}(X; \pi_n(Y))$ of $X$ is defined as the set of homomorphisms from $C_{n+1}(X)$ to $\pi_n(Y)$, we get an element of this cochain group. This cochain is defined by taking

$$[c^{n+1}(g)](\sigma) = [g_\sigma] \in \pi_n(Y) \tag{5}$$

for every $(n+1)$-cell of $X$. This $(n+1)$-cochain $c^{n+1}(g)$ of $X$ is called the *obstruction* of the map $g$.

With all $(n+1)$-cells of $A$, there is also associated an element of $\pi_n(Y)$. But, since $g$ is defined on the whole of $A$, $g_\sigma$ has an extension. Hence $[g_\sigma]$ is the zero element of $\pi_n(Y)$ for every cell $\sigma$ of $A$. So $c^{n+1}(g)$ is not only an element of $C^{n+1}(X; \pi_n(Y))$ but in particular an element of $C^{n+1}(X, A; \pi_n(Y))$.

We will show that $c^{n+1}(g)$ is not only a cochain, but also a cocycle: $c^{n+1}(g) \in Z^{n+1}(X, A; \pi_n(Y))$. So it determines a cohomology class

$$\gamma^{n+1}(g) \in H^{n+1}(X, A; \pi_n(Y)) \tag{6}$$

represented by $c^{n+1}(g)$.

Let $\sigma$ be any $(n+2)$-cell. It is sufficient to show that $[\delta c^{n+1}(g)](\sigma) = 0$. Let $W$ be the boundary of $\sigma$ and $W^n$ be the $n$-skeleton of $W$. Then we have the following sequence of groups and homomorphisms:

$$C_{n+1}(W) \xrightarrow{\partial} Z_n(W) = Z_n(W^n) = H_n(W^n) \xleftarrow{h} \pi_n(W^n) \xrightarrow{k_*} \pi_n(Y), \qquad (7)$$

where $h$ is the Hurewicz homomorphism and $k_*$ is the homomorphism induced by the partial map $g_\sigma$. Because $W^n$ is $(n-1)$-connected, $h$ is an isomorphism for $n > 1$ and an epimorphism for $n = 1$. So we have a well defined homomorphism

$$k_* h^{-1} : Z_n(W) \longrightarrow \pi_n(Y) \qquad (8)$$

and we can extend this homomorphism to

$$d : C_{n+1}(W) \longrightarrow \pi_n(Y). \qquad (9)$$

Since $[c^{n+1}(g)](\tau)$, with $\tau$ an $(n+1)$-cell, is represented by the partial map $k|\partial\tau$, we have

$$[c^{n+1}(g)](\tau) = k_* h^{-1}(\partial\tau) = d(\partial\tau) = (\delta d)(\tau). \qquad (10)$$

And so

$$[c^{n+1}(g)](\partial\sigma) = \delta d(\partial\sigma) = (\delta\delta d)(\sigma) = 0. \qquad (11)$$

So $c^{n+1}(g)$ is a cocycle. The obstruction cocycle $c^{n+1}(g)$ does not depend on homotopy, so homotopic maps give the same obstruction.

We now come to the very important *Eilenberg extension theorem*, which tells us when a function can be extended. It says that $\gamma^{n+1}(g) = 0$ if and only if there exists a map $h^* : \bar{X}^{n+1} \to Y$ such that $h^*|\bar{X}^{n-1} = g|\bar{X}^{n-1}$. The theorem says that if $c^{n+1}(g) \sim 0$, then we can modify the open cells in $X/A$ such that an extension of $g$ over $\bar{X}^{n+1}$ exists.

We are going to define the *obstruction set* of $f : A \to Y$. This $(n+1)$-dimensional obstruction set is a subset of the $(n+1)$-dimensional cohomology group

$$O^{n+1}(f) \subset H^{n+1}(X, A; \pi_n(Y)).$$

If $f$ is not $n$-extendable over $X$, we define $O^{n+1}(f)$ as the vacuous set. Now, suppose that $f$ is $n$-extenbable over $X$. Then there exists an extension $g : \bar{X}^n \to Y$ of $f$. The cohomology class $\gamma^{n+1}(g)$ in $H^{n+1}(X, A; \pi_n(Y))$ is called an $(n+1)$-dimensional obstruction element of $f$. Then $O^{n+1}(f)$ is defined as the set of all $(n+1)$-obstruction elements of $f$. Two immediate consequences of this definition are that the map $f : L \to Y$ is $n$-extensible over $X$ if and only if $O^{n+1}(f)$ is non-empty and the map $f$ is $(n+1)$-extendable over $X$ if and only if $O^{n+1}(f)$ contains the zero element of $H^{n+1}(X, A; \pi_n(Y))$. This last statement follows from Eilenbergs theorem. It is also important to know that this obstruction set does not depend on the triangulation of $A$, so we can really speak of *the* obstruction set.

We now turn to the case of an $(n-1)$-connected space $X$ for a given positive integer $n$. (A space is a $k$-connected space ($k \geq 0$) if it is pathwise connected and $\pi_i(Y) = 0$ for all $i = 1, \ldots, k$). Then $f$ is always $n$-extendable over $X$. The first obstruction we may meet, is $\gamma^{n+1}(f)$ and is called the primary obstruction. In this case the $(n+1)$-dimensional obstruction set consists of one element and it can be shown that

$$\gamma^{n+1}(f) = (-1)^n \delta^* f^* \iota^n(Y) \in H^{n+1}(X, A; \pi_n(Y)), \tag{12}$$

where $\iota^n(Y) \in H^n(Y; \pi_n(Y))$. The mappings correspond to the diagram

$$H^n(Y; \pi_n(Y)) \xrightarrow{f^*} H^n(A; \pi_n(Y)) \xrightarrow{\delta^*} H^{n+1}(X, A; \pi_n(Y)), \tag{13}$$

where $\delta^*$ is the coboundary operator and $f^*$ is the homomorphism induced by $f : A \to Y$. The condition on $f_n : X_n \to Y$ to be extendable over $X_{n+1}$ is that $\gamma^{n+1}(f) = 0$, which is necessary and sufficient.

Let us suppose that $\gamma^{n+1}(f)$ vanishes. If $\dim(X, A) = n+1$ then we have an extension and the procedure has come to an end. If $\dim(X, A) > n+1$, we reach a secondary obstruction if $H^{r+1}(X, A; \pi_n(Y)) \neq 0$ for some $r > n$. This secondary obstruction is an element of this cohomology group:

$$z^{r+1}(f_u) \in H^{r+1}(X, A; \pi_r(Y)), \tag{14}$$

where $f_u : X_{n+1} \to Y$. The vanishing of $z^{r+1}$ is necessary and sufficient for $f_u$ to be extendable over $X_{r+1}$. (This follows from the Eilenberg extension theorem). Of course, $z^{r+1}$ depends only on the homotopy class of $f_u$.

This is as far as we will go. In the next section we apply this method in monopole theory to the case of extended topological singularities.

# 4 Monopoles and Obstruction Theory

## 4.1 What do we learn from obstruction theory ?

Let us suppose that we have calculated, for manifolds $A$ and $Y$, the possible extensions of $f : S^k \to Y$ to $g : B \to Y$. Following the construction described in section 2 we now can decide which configurations on the sphere, characterized by $\pi_k(Y)$, lead to which configurations on the manifold $A$, characterized by $[A; Y]$.

If we find that each class $[f] \in \pi_k(Y)$ gives us only a single class $[h] \in [A; Y]$, then obviously we have a unique relation between the topological charges of both objects. In practice this means that we do not obtain any additional topological structure by deforming the sphere-like core to the deformed core with boundary $A$.

The configuration on $A$ is essentially the same as the configuration on $S^k$. In this sense this situation is not very interesting.

More interesting is the case where we can reach different classes $[h] \in [A; Y]$ for a single homotopy class $[f] \in \pi_k(Y)$. The configurations on $A$ are different from the configurations on $S^k$ because for one $[f] \in \pi_k(Y)$ we get topologically different elements of $[A; Y]$. There are several different extensions allowed over the cobordism $B$. In this case, we have to do physics to see which configuration on $A$ is favourable in terms of energy.

In the following we will consider two examples to illustrate these ideas. The theories are very similar, but there is one difference, which has to do with the fundamental group. It is very closely related to the difference between ordinary and "alice" electrodynamics. It turns out, that this makes the difference between an interesting - and a non-interesting case explicit.

## 4.2 The Georgi-Glashow model

Consider a $SO(3)$ gauge theory with a Higgs field in the adjoint representation. After symmetry breaking we are left with a $U(1)$ symmetry. $G/H$ is then identified with the 2-sphere $S^2$. This is the theory in which 't Hooft and Polyakov originally found their celebrated magnetic monopole. We will now argue why it is not such an interesting case in the present context.

Let us start with a configuration $f : S^2 \to S^2$ which belongs to a class $[f] \in \pi_2(S^2) = \mathbf{Z}$. The core deformation we are going to look at is the deformation of the singular point to a ring. The manifold which encloses the extended topological singularity is the torus. The cobordism is the manifold $B$, with $\partial B = S^2 \cup T^2$. We try to extend $f$ over the cobordism $B$ to the torus. The homotopy groups which

Figure 4: The cobordism fills the space between the sphere and the torus.

will be needed in obstruction theory are $\pi_1(S^2) \simeq 0$ and $\pi_2(S^2) \simeq \mathbf{Z}$.

Let us try to extend $f$ over $B$. One glance at obstruction theory tells us that this can always be done because all relevant cohomology groups vanish (for the calculation of the cohomology groups, see the appendix)

$$H^2(B, S^2; 0) = 0 \tag{15}$$

$$H^3(B, S^2; \mathbf{Z}) = 0, \tag{16}$$

so we do not bump into any obstructions. Now we start with a map $h : T^2 \to S^2$ which belongs to a class $[h] \in [T^2; S^2] = \mathbf{Z}$. But we see again that the relevant cohomology groups vanish:

$$H^2(B, T^2; 0) = 0 \tag{17}$$

$$H^3(B, T^2; \mathbf{Z}) = 0 \tag{18}$$

so there is no obstruction to extending $h : T^2 \to S^2$ over $B$. This is not a very interesting example in the sense of the previous section. We can always make an extension over the cobordism to the other space, and every element of $\pi_2(S^2)$ gives us exactly one element of $[T^2, S^2]$ and vice-versa. So there is not really a difference between the configuration on the torus and the configuration on the 2-sphere.

The physical interpretation we should give is that in this example the ring singularity enclosed by the torus $T^2$ is not locally stable for a topological reason. It can be pinched of and will most probably shrink to a point singularity, as we know that to be the minimal energy solution. In a realistic model one expects the actual solution to be maximally symmetric compatible with the topological structure.

## 4.3 The Alice-Georgi-Glashow model

Consider an $SO(3)$ gauge theory in the 5-dimensional representation. The 5-dimensional representation corresponds with a symmetric second-rank tensor $\Phi_{ab}$. This tensor can in general be decomposed with respect to as particular $SO(2)$ subgroup of rotations about a particular vector $\eta$ as follows.

$$\Phi_{ab} = \alpha(\hat{\eta}_a\hat{\eta}_b - \frac{1}{3}\delta_{ab}) + \beta(\hat{\mu}_a\hat{\lambda}_b + \hat{\lambda}_a\hat{\mu}_b) \tag{19}$$

where $\hat{\eta}$, $\hat{\mu}$ and $\hat{\lambda}$ form an orthonormal basis and $\alpha$ and $\beta$ are constants. From this decomposition one easily sees what the possible orbits and stability groups are [8]. The generic case with $\beta \neq 0$ and $\alpha/\beta \neq \frac{1}{2}$ the residual symmetrygroup is the dihedral group $D_2$, with four elements. Because this group is discrete one has that $\pi_2(G/H) \simeq 0$ and $\pi_1(G/H) \simeq D_2$. Hence in this phase the model only supports (non-abelian) fluxtubes. In the present context we are more interested in the case where the potential for $\Phi$ is so arranged that its minimum is achieved for values $\beta = 0$ or $\alpha/\beta = \frac{1}{2}$. In that case the residual symmetrygroup is $H = \mathcal{N}(U(1)$, i.e. the normalizer of $U(1)$ in $SO(3)$. That is to say, $U(1)$ plus rotations of $\pm\pi$ around the axis perpendicular to the axis around which $U(1)$ works. (See [9]). $G/H$ is then identified with $P^2$, the real projective plane. The real projective plane $P^2$ is topologically equivalent to $S^2$ with opposite points identified. As $S^2$ is the double

Figure 5: The deformation of $(B, S^2)$ into $(S^2 \cup L^1, S^2)$.

covering of $P^2$, we have that $\pi_2(P^2) \simeq \pi_2(S^2) \simeq \mathbf{Z}$. However, the fundamental group $\pi_1(P^2) \simeq \mathbf{Z}_2$, the integers modulo 2.

The essential difference with the previous example is that *both* $\pi_1(P^2)$ and $\pi_2(P^2)$ are non-trivial, so that the model supports magnetic charges (monopoles) as well as magnetic fluxtubes. The latter should be compared to those of type-II super conductors, with the property that if two fluxtubes combine than the flux becomes topologically unstable and will most probably decay into gauge- and Higgs particles.

We will work with the same deformation and manifolds as in the previous section, so the cobordism remains the same. Also the relevant cohomology groups are the same, only the coefficient groups differ.

We start again with a map $f : S^2 \to P^2$, and we find with the help of the deformation of $(B, S^2)$ into $(S^2 \cup L^1, S^2)$, see figure 5, that

$$H^2(B, S^2; \mathbf{Z}_2) = 0, \tag{20}$$

$$H^3(B, S^2; \mathbf{Z}) = 0, \tag{21}$$

(see appendix), so this map can always be extended over $B$. But now, the situation is different when we try to extend a map $h : T^2 \to P^2$ over $B$, because $[T^2; P^2]$ $(\simeq (\mathbf{Z}_2 \oplus \mathbf{Z}_2) \otimes \mathbf{Z})$ is different from $[T^2, S^2]$. The participating cohomology groups are

$$H^2(B, T^2; \pi_1(P^2)), \tag{22}$$

$$H^3(B, T^2; \pi_2(P^2)), \tag{23}$$

and notice that they differ with the cohomology groups used in the previous section. This difference lies in the fact that $\pi_1(P^2) \simeq \mathbf{Z}_2$. We calculate these groups with the help of the deformation of $(B, T^2)$ into $(B \cup D^2, T^2)$ as in figure 6. (Notice that during the deformation, $T^2$ remains fixed). So we get (see appendix)

$$H^2(B, T^2; \mathbf{Z}_2) \simeq H^2(T^2 \cup D^2, T^2; \mathbf{Z}_2) \simeq \mathbf{Z}_2. \tag{24}$$

Figure 6: The deformation of $(B, T^2)$ into $(T^2 \cup D^2, T^2)$.

The same procedure holds for the third cohomology group, but with a different result

$$H^3(B, T^2; \mathbf{Z}) \simeq H^3(T^2 \cup D^2, T^2; \mathbf{Z}) = 0. \tag{25}$$

This means that we only have to cope with a primary obstruction. We now turn to the explicit calculation of the primary obstruction.

Bearing in mind the deformation of $(B, T^2)$ we are lead to the conclusion that extending $f : T^2 \to P^2$ is essentially the same as extending $f : T^2 \to P^2$ over the disc $D^2$.

If we restrict $f$ to $\partial D^2$ we get a map

$$f|\partial D^2 : \partial D^2 \longrightarrow P^2 \tag{26}$$

which induces an element of the cohomology group $H^1(\partial D^2; \mathbf{Z}_2)$ which is isomorphic to $\mathbf{Z}_2$. If the homotopy class of $f$ if trivial, then $f$ induces the trivial element of $H^1(\partial D^2; \mathbf{Z}_2)$, but if the homotopy class of $f$ is non-trivial it induces a non-trivial element of this cohomology group. So we know that there is a one-to one correspondence between elements of this cohomology group and elements of the homotopy group $\pi_1(P^2)$. We know $f^*\iota^n(Y)$ and we only have to map this with the coboundary operator into $H^2(D^2, \partial D^2; \mathbf{Z}_2)$. But as we can see in the following exact sequence:

$$H^1(D^2; \mathbf{Z}_2) \simeq 0 \to H^1(\partial D^2; \mathbf{Z}_2) \xrightarrow{\delta^*} H^2(D^2, \partial D^2; \mathbf{Z}_2) \to H^2(D^2; \mathbf{Z}_2) \simeq 0, \tag{27}$$

$\delta^*$ is, in this case, an isomorphism so we come to the conclusion that the conditions for $f : T^2 \to P^2$ to be extendable, depend only on the triviality of curves $\beta$ in figure 6. We could have seen this directly, because we can see (we *have* to see) $\partial D^2$ as the boundary of the 2-simplex $|D^2|$. Extending over the simplex is only possible if and only if the homotopy class associated with $f|\partial D^2$ is trivial. It follows that there is a one-to-one correspondence between $H^2(D^2, \partial D^2; \mathbf{Z}_2)$ and the homotopy group $\pi_1(Y)$.

We conclude that one element of $\pi_2(P^2)$ on the sphere can give two elements on the torus, characterized by elements of the fundamental group $\pi_1(P^2)$ around curves of type $\alpha$. Notice that if this element is trivial, then we can pinch the curves $\alpha$ and $\beta$ to a point and obtain nothing but a sphere. So in this case nothing has really happened. We have the same situation in this case as in the trivial example. But if we have a non-trivial element of $\pi_1(P^2)$ along $\alpha$, this cannot be done, and we have obtained a different situation: the ring carries charge and flux. It exemplifies the essential interrelation between magnetic flux and charge that is present in this model!

What about the overall quantum-number of a map $f : S^2 \to P^2$, which is associated to the magnetic charge ? How does this behave under the extension. A first, rather obvious observation, but therefore not less true, is the fact that the overall quantumnumber of a map $f : T^2 \to P^2$ is just the same as the quantumnumber we had on $S^2$, when we extended this map over $B$. Alternatively, one could consider figure 7, where $\tilde{T}^2$ is the single - or double covering of $T^2$, depending on which ele-

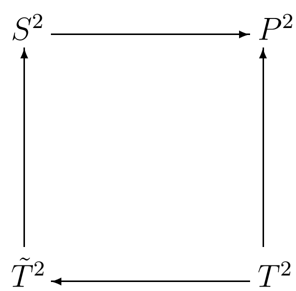


Figure 7: Commutative diagram, expressing the fact that the overall quantumnumber is unchanged after extension.

ments of the fundamental group we have on curves $\alpha$ and $\beta$, in figure 6. So we have four different classes $C_{m,n}$ with $m, n = 0, 1$. From the diagram we see that for each $m, n$ the homotopy classes are in one-to-one correspondence with maps $S^2 \to P^2$, due to the covering space property of homotopy groups. So we get the earlier stated result

$$[T^2; P^2] \simeq (\mathbf{Z}_2 \oplus \mathbf{Z}_2) \otimes \mathbf{Z} \tag{28}$$

Notice that (for trivial curves $\beta$) confirms our obvious observation.

We finally conclude that the deformed monopole has a magnetic charge associated with the normal $\pi_2$ classification, but there are two kinds of monopoles for every element of $\pi_2$. They can be distinguished by the fundamental group $\pi_1$ along the curves of type $\alpha$.

If the element of $\pi_1$ is non-trivial we have a closed fluxtube in the form of a ring, because the loop cannot be pinched to a point. So in this three dimensional object we have both flux and magnetic charge. The flux can be compared with the strings formed in type-II superconductors, whose existence follws from topological considerations as well. It is clear that the different core states can only be distinguished physically by performing experiments where charged particles are scattered of the core. In the case of an extended singularity one would be able to observe particle--antiparticle transitions due to the presence of the Alice flux. Indeed, it turns out that this monopole has exotic properties some of which are discussed in [9].

# 5 Conclusion

In this paper we investigated core deformations of topological defects in spontaneously broken gauge theories. In particular in theories where more than a single homotopy group is non-vanishing, i.e. where several types of topological excitations can coexist. In theories where this is the case, it turns out to be possible to have topologically stable core deformations.

The admissible core deformations can be found by establishing relations between elements of $\pi_k(G/H)$ and elements of $[A; G/H]$, where $A$ is the manifold enclosing the topological singularity. This relation was established with the help of obstruction theory. Obstruction theory tells us whether there exist extensions of maps defined on a space $A$ into $Y$, to maps from $X \supset A$ to $Y$. This is only the case if the obstruction is zero. This obstruction is an element of a certain cohomology group.

We investigated two simple examples in detail. We found a topologically stable core deformation in the case of $G = SO(3)$ and $H = \mathcal{N}(U(1))$. It is interesting to exted this discussion to theories where the discrete part of the gaugegroup is more complicated and nonabelian.

We note further that non-trivial examples can be found in the case of instantons. One particularly interesting model might be the model with $G = SO(3)$ and $H = U(1)$, such that $G/H \simeq S^2$. In this case, the third and second homotopy group are isomorphic to $\mathbf{Z}$. The idea is that we can deform the core, such that the manifold which encloses the extended topological singularity is $S^1 \times S^2$. We might then obtain a ring of monopoles, because $\pi_2$ is non-trivial.

In this paper we have restricted ourselves to a topological analysis, which determines what kind of deformations are possible in principle. It is now interesting to proceed and see whether in explicit models these deformations can be realized by an appropriate choice of parameters in the Higgs potential.

# A The Calculation of Cohomology Groups

Originally the cohomology groups used to be defined in terms of homology groups in the following way (see [10]): For a chain complex $C$ and module $G$ we define the cohomology module $H^*(C; G)$=$\{H^q(C; G)\}$ of $C$ with coefficients $G$ by

$$H^q(C; G) = H^q(\mathrm{hom}(C, G)). \tag{29}$$

The properties of the cohomology groups were derived from this definition. It is however easier to give an axiomatic description of cohomology, since we want to do calculations more than that we want to have a perfect mathematical perception of these matters. This description reads as follows (see [3]):

Let $G$ be an $R$ module. A cohomogy theory with coefficients $G$ consists of a contravariant functor $H^* = \{H^q\}$ from the category of topological pairs to graded $R$ modules and a natural transformation $\delta^* : H^*(A) \to H^*(X, A)$ of degree +1 such that the following axioms hold

1. **Homotopy Axiom** If $f_0, f_1 : (X, A) \to (Y, B)$ are homotopic,then
$$H^*(f_0) = H^*(f_1) : H^*(Y, B) \longrightarrow H^*(X, A). \tag{30}$$
2. **Exactness Axiom** For any pair $(X, A)$ with inclusion maps $i : A \subset X$ and $j : X \subset (X, A)$ there is an exact sequence
$$\cdots \xrightarrow{\delta^*} H^q(X, A) \xrightarrow{H^q(j)} H^q(X) \xrightarrow{H^q(i)} H^q(A) \xrightarrow{\delta^*} H^{q+1}(X, A) \cdots . \tag{31}$$
3. **Excision Axiom** For any pair $(X, A)$ if $U$ is an open subset of $X$ such that $\bar{U} \subset intA$, the excision map $j : (X - U, A - U) \subset (X, A)$ induces an isomorphism
$$H^*(j) : H^*(X, A) \simeq H^*(X - U, A - U). \tag{32}$$
4. **Dimension Axiom** On the category of one-point spaces, there is a natural equivalence of the constant functor $G$ with the functor $H^*$.

We will now calculate the cohomology groups which play a role in this paper. To calculate $H^2(B, S^2; \mathbf{Z}_2)$ we use axiom 1 with the deformation (see figure 5) of $(B, S^2)$ into $(S^2 \cup L)$ and axiom 3 to get

$$H^2(B, S^2; \mathbf{Z}_2) \overset{(1)}{\simeq} H^2(S^2 \cup L, S^2; \mathbf{Z}_2) \overset{(3)}{\simeq} H^2(L, \partial L; \mathbf{Z}_2) = 0 \tag{33}$$

The same goes for $H^3(B, S^2; \mathbf{Z})$

$$H^3(B, S^2; \mathbf{Z}) \simeq H^3(L, \partial L; \mathbf{Z}) = 0. \tag{34}$$

The last step in the calculation follows from dimensional considerations. The space is 1-dimensional so the second- and third cohomology group vanishes.

To calculate $H^2(B, T^2; \mathbf{Z}_2)$ we again use axiom 1, but now with the deformation $(B, T^2)$ into $(T^2 \cup D^2, T^2)$, see figure 6, followed by the excision axiom

$$H^2(B, T^2; \mathbf{Z}_2) \simeq H^2(T^2 \cup D^2, T^2; \mathbf{Z}_2) \simeq H^2(D^2, \partial D^2; \mathbf{Z}_2) \simeq \mathbf{Z}_2, \tag{35}$$

where we have used an exact sequence to calculate the last step. The same axioms give

$$H^3(B, T^2; \mathbf{Z}) \simeq H^3(D^2, \partial D^2; \mathbf{Z}) = 0. \tag{36}$$

The third cohomology group is zero because the dimension of $(D^2, \partial D^2)$ is 2.

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