A Hopf symmetry approach to the physics of ordinary and liquid crystals

Topological interactions and phase transitions in condensed matter

Charles Jean-Marc Mathy

Supervisor: Prof.dr.ir. F.A. Bais

University of Amsterdam
Institute for Theoretical Physics
Valckenierstraat 65
1018 XE Amsterdam
The Netherlands

June 2005

A word of thanks

I am very grateful to my supervisor, Sander Bais, for giving me an inspiring topic, and for guiding me all the way through. He has taught me a ”local” approach to research: his advice was to plunge straight into the World of Ideas, and look around. This approach has turned out to be more effective, productive and enjoyable than my initial ”global” approach. Our thorough sparring sessions have been pivotal to the evolution and completion of this work. I am also very grateful for his support for my next international career step.

I thank my roommates for making time at the Wibauthuis more enjoyable. Thanks to Aaron for quantum double jamming sessions, and to Liza, Mark, Michaelangelo for bringing merriment, photosynthesis, and etymology into the office. Thanks to Sheer, Hendrik, Eric and Balt for letting me come into their offices and rob their books once in a while, and shamelessly take up their time with my questions. I have also profited greatly from the sudden appearances of Vincent, and the ensuing jamming sessions, be they musical or mathematical.

Finally thanks to my family for supporting (and putting up with) me during the development of this work, and for being there for me.

Abstract

We analyze the modes (excitations) of phases that arise via spontaneous global symmetry breaking from the isotropic phase. We focus on ordinary and liquid crystals. The modes can be classified into three categories: *regular* or *electric* modes, which refer to regular excitations of the ground state; *topological* or *magnetic* modes, which have a core in which the symmetry is (partially) restored; and *dyons*, which carry both electric and magnetic quantum numbers. We study their interactions, in particular their fusion and braiding properties. These interactions can be captured by a mathematical structure called a Hopf algebra, which we analyze in detail. Once the Hopf symmetry of a phase has been established, we study symmetry breaking induced by the condensation of electric, magnetic or dyonic modes. The use of Hopf symmetry breaking to analyze the phase structure of media with topological excitations was pioneered by Bais, Schoers and Slingerland. In this thesis we use a novel more refined criterion to define the residual symmetry, which in certain cases leads to a richer analysis of the physics of the broken phase.

Some of the excitations are confined, meaning that they cost an infinite amount of energy to create. These confined excitations may form unconfined "hadronic" composites, in analogy with the hadrons in the theory of strong interactions of quarks. We determine the unconfined symmetry algebra of the broken phase, and develop a method to analyze the hadronic composites in condensed matter systems. We apply our symmetry breaking analysis to ordinary and liquid crystals, and predict a wealth of new phases, in particular phases resulting from nonabelian defect condensates.

Contents

1 Introduction 11

2 Ordered systems 15
- 2.1 Description of ordered systems 15
- 2.2 Excitations and modes 22
  - 2.2.1 Dispersion relation 22
  - 2.2.2 Continuity modes 23
  - 2.2.3 Goldstone modes and massive gauge fields 25
- 2.3 Topological defects 27
  - 2.3.1 Classification of topological defects 27
  - 2.3.2 Interactions between defects in d=3 31
  - 2.3.3 Some energetics of defects 34
- 2.4 Dyonic configurations 39
  - 2.4.1 Global implementation of $h \in H$ 39
  - 2.4.2 A dyon 41
- 2.5 Phase transitions and correlation functions 42
  - 2.5.1 Order of a phase transition 42
  - 2.5.2 Landau theory of phase transitions 42
  - 2.5.3 Correlation functions and order 43

3 Ordinary and liquid crystals 47
- 3.1 Introduction 47
  - 3.1.1 Definition in terms of symmetry 47
- 3.2 Liquid crystals 48
  - 3.2.1 Nematic liquid crystals 48
  - 3.2.2 Smectics 59
  - 3.2.3 Columnar phases 64
  - 3.2.4 Quantum Liquid Crystals 65
- 3.3 The hexatic phase 65
  - 3.3.1 Crystals 65
  - 3.3.2 The hexatic phase 71
  - 3.3.3 Kosterlitz Thouless phase transition 72

4 Hopf algebra symmetry 75
- 4.1 Representation theory of electric, magnetic and dyonic modes . . 75
  - 4.1.1 Electric modes 75
  - 4.1.2 Magnetic modes 78
  - 4.1.3 Dyonic modes 79
- 4.2 Hopf symmetry of liquid crystals and crystals 82
  - 4.2.1 Liquid crystals 82
  - 4.2.2 The hexagonal crystal 85
- 4.3 Braiding 87
  - 4.3.1 Braiding in a uniaxial nematic 93

5 Hopf symmetry breaking and confinement 99
- 5.1 Condensate and confinement 100
- 5.2 Hopf Symmetry breaking 105
  - 5.2.1 The criterion for symmetry breaking 105
  - 5.2.2 Relationship between $\mathcal{T}$, $\mathcal{T}_{\mathcal{R}}$ and $\mathcal{T}_{\mathcal{L}}$ 109
  - 5.2.3 $\mathcal{T}_{\mathcal{R}}$ and $\mathcal{T}_{\mathcal{L}}$: Hopf or not? 112
  - 5.2.4 Requirement on the condensate $|\phi_{\mathcal{R}}>$ 112
  - 5.2.5 Unconfined excitations and $\mathcal{U}$ 113
- 5.3 Electric condensates 117
- 5.4 Defect condensates 119
  - 5.4.1 Single defect condensate 119
  - 5.4.2 Class sum defect condensates 121
  - 5.4.3 Combined defect condensates 122
- 5.5 Defect-mediated melting 127
- 5.6 Defining a coproduct for $\mathcal{T}_{\mathcal{R}}$ 132
- 5.7 Dyonic condensates 137
- 5.8 Conclusions and outlook 141

A SO(3) and SU(2) 151
- A.1 The subgroups of SO(3) 151
- A.2 Double cover groups 152

B Hopf algebra's 155
- B.1 Quasitriangular Hopf algebras 155
  - B.1.1 Algebra's 155
  - B.1.2 Coalgebra's 156
  - B.1.3 Quasitriangular Hopf algebra's 156
- B.2 Generalized quantum doubles 162

C Defect condensates in achiral nematics 167
- C.1 Achiral octahedral nematic 167
- C.2 Achiral icosahedral nematic 169

D Irreducible representations 173
- D.1 Irreducible representations of semidirect product groups 173
  - D.1.1 Irreps of a finite group 173
  - D.1.2 Subduced and induced representations 174
  - D.1.3 Semi-direct product groups 174
  - D.1.4 Irreps of $N \rtimes H$, with N abelian 175
- D.2 Irreducible representations of algebras 176

Chapter 1

Introduction

> *The enormous usefulness of mathematics in the natural sciences is something bordering on the mysterious and there is no rational explanation of it.*
> - Eugene P. Wigner

One of the major drives of modern day theoretical physics is the search for unifying principles. An important tool in this search is the concept of symmetry. Over the past century physicists have had to generalize the concept of symmetry, from global to local, from the Lorentz group to supersymmetry, and from ordinary groups to Hopf algebras. In this thesis we will study new applications of Hopf symmetry to condensed matter systems, in particular to describe the rich variety of interactions and phase transitions in liquid crystals.

When the temperature of a condensed matter system is lowered, a common phenomenon is the development of order in the system. The system enters a phase with less symmetry than the high temperature phase. To characterize symmetry breaking phase transitions, physicists introduce *order parameters*, which determine the symmetries which are broken of manifest.

The spectrum of systems with the same symmetry breaking pattern share important properties. The spontaneous breaking of global symmetries implies the existence of low energy excitations called *Goldstone modes*, and of excitations called *topological defects*. The Goldstone modes are labelled by *quantum numbers* that characterize their transformation properties under the residual symmetry group in the broken phase. The topological excitations carry so called "topological quantum numbers", which are quite different from the quantum numbers of Goldstone modes. However, using the extended symmetry concept of Hopf algebras both types of quantum numbers can be treated *on equal footing*. The Hopf algebra captures the transformation properties of ordinary and topological modes under symmetry transformations, and their topological interactions. Recently it has been discovered that many physical systems can be described using Hopf algebras, from crystals to Quantum Hall states to 2+1 dimensional gravity. Our application to liquid crystals is novel.

Liquid crystals are systems with a symmetry in between that of a liquid and a crystal. In nature a wide variety of liquid crystals have been discovered, and they have subsequently found important applications in industry, in flat screens. In theoretical physics, they offer an ideal testing ground for certain models of the early Universe. Namely, as the early Universe expanded it cooled down and went through a sequence of phase transitions that are analogous to phase transitions in liquid crystals. Liquid crystals are particularly convenient materials to analyze phase transitions because they are relatively cheap compounds, their phase transitions occur at temperatures that can be reached with a common kitchen oven, and they can be observed with a simple microscope.

Our goal is to study symmetry breaking in crystals and liquid crystals using the Hopf symmetry. The most common phase transitions in crystals and liquid crystals are associated with the development of order, and thus the breaking of symmetries. These are described by order parameters. There are also phase transitions induced by topological defects, which break no conventional symmetries. It is possible to introduce *disorder parameters* to study these phase transitions, but until now it has not been possible to treat defect-mediated phase transitions in terms of symmetry breaking. In this thesis, we will see that the defect-mediated phase transitions break Hopf symmetries, and thus we can treat these phase transitions as *Hopf symmetry breaking phase transitions*. The Hopf symmetry description allows one to unify the description of conventional and defect-mediated phase transitions, and it allows for a systematic investigation of all the possible phase transitions from a given phase.

The Hopf symmetry approach to symmetry breaking was pioneered by Bais, Schoers and Slingerland [6]. We will carry out the analysis of the Hopf symmetry description, and of Hopf symmetry breaking in liquid crystals and crystals. We choose these phases because they exhibit a rich variety of symmetry breaking patterns and defect-mediated phase transitions, so that they provide the perfect testing ground for the Hopf algebra description of symmetry breaking.

Outline

Chapter 2

In chapter 2 we study the physics of condensed matter systems with spontaneously broken symmetries. We introduce order parameter fields, and explain how to set up the free energy of a system. We distinguish between global and local symmetries, and in both cases analyze the mode content. We discuss continuity modes, Goldstone modes, massive gauge bosons, topological defects and dyons, and we analyze their interactions. We analyze the transformation properties of these modes under residual global symmetry transformations. Finally we describe the tools used to study conventional phase transitions: Landau's theory of phase transitions. We introduce correlation functions, and use them to characterize the order of a phase. We illustrate this with the XY model.

Chapter 3

In chapter 3 we classify the different types of liquid crystals, and study their mode content, using the tools described in chapter 2. The most important categories of liquid crystals are *nematics* and *smectics*. We discuss the different types of nematics and smectics, and analyze the dynamics of the modes. In particular, we explain why the Goldstone modes are diffusive in these systems (i.e. they decay in time). We study the free energy of liquid crystals, and find an effective gauge theory in smectics, which leads to massive Goldstone modes. We also comment on the energetics of defects.

Finally, we discuss the hexatic phase. This liquid crystal phase is obtained from a two-dimensional crystal by the condensation of defects. The defect condensate restores translational symmetry, while rotational symmetry remains unbroken. In order to understand defect condensates, we also treat the Kosterlitz-Thouless phase transition, which is a simpler example of defect condensate, in the XY model.

Chapter 4

In chapter 4 we introduce the Hopf symmetry description of condensed matter systems. We give the generic name *electric modes* to low energy excitations of the order parameter field, such as Goldstone and continuity modes. The topological defects are called *magnetic modes*. This denomination is borrowed from Quantum Electrodynamics. We show that the representation theory and fusion rules of electric and magnetic modes is reproduced by a Hopf algebra called *quantum double*. Then we show which Hopf algebra is appropriate for the liquid crystals phases discussed in chapter 3. We find that we need to slightly modify the quantum doubles, thus we define *generalized quantum doubles*.

Then we discuss the *braiding* of modes: when one mode is adiabatically (i.e. slowly) transported around another mode, it may return in a slightly different state. The most famous example of this behaviour is the *Aharonov-Bohm effect*: an electron braiding with a infinitely long coil carrying a current picks up a phase factor, even though the electric and magnetic fields outside the coil are zero. The Aharonov-Bohm effect is an example *abelian* braiding, because the gauge group $U(1)$ of electromagnetism is abelian. In the case of non-abelian gauge theories, the outcome of braiding may be more complicated than a simple phase factor. Once we've described braiding in local theories, we will discuss what the analogous phenomenon is in a global theories. Adiabatic transport in global theories also leads to a change of the state of a mode, but the interpretation is different. The mode is being *frame dragged*, meaning that it is following a geodesic in a curved space. We analyze the braiding of modes in liquid crystals

The braiding can be captured by the Hopf symmetry description. Part of the Hopf structure is a *universal R-matrix*, which acts as a braid operator of the modes. A Hopf algebra with a universal R-matrix is called a *quasitriangular Hopf algebra*. We introduce the universal R-matrix of the quantum double, and show that it reproduces the braiding of modes in liquid crystals.

Finally we analyze the braiding in a uniaxial nematic. We find a rather interesting phenomenon: the defects actually have a continuous label, and we can define *coherent sums* of defects, thereby obtaining a set of magnetic states with $U(1)$ charge. We show how charge can be transferred from these defect states to electrically charged particles by braiding the electric particle around the magnetic state. This phenomenon is the global analogue of *Cheshire charge*.

Chapter 5

In the final chapter we study we study symmetry breaking in a phase with Hopf symmetry $\mathcal{A}$. A phase transition is accompanied by a *condensate* $|\phi>$, which is a particle that fills the ground state. We define a *residual symmetry algebra* $\mathcal{T}_{\mathcal{R}}$, which is the set of operators that are well defined with respect to the condensate particle $|\phi>$. Excitations of the ground state are irreps of $\mathcal{T}_{\mathcal{R}}$. We show that there are two types of irreps of $\mathcal{T}_{\mathcal{R}}$: *confined* irreps, whose excitation requires introducing a half-line singularity in the condensate. The state $|\phi>$ of the condensate is different to the left and right of this half-line. This implies that the half-line costs a finite amount of energy per unit length. Thus in an infinite system a confined excitation costs and infinite amount of energy to excite. However, the half-line may end on another confined excitation, thus giving it finite extent. We call such an combination of confined excitations a *hadronic composite*, in analogy with hadrons in Quantum Chromodynamics.

The other type of irreps are *unconfined* excitations, and they are not connected to a half-line singularity. We will show that there is a Hopf algebra called the *unconfined symmetry algebra*, whose irreps are precisely the unconfined excitations. We work out $\mathcal{U}$ for electric condensates, and defect condensates in liquid crystals. We find what we expect for electric condensates, namely that Landau's theory of phase transitions is reproduced. We also find natural answers for the defect condensates, and we show that these results reproduce the defect condensates discussed in chapter 3. As an example, we work out all electric and defect phase transitions from a tetrahedral nematic.

We then take a closer look at the confined excitations, and at $\mathcal{T}_{\mathcal{R}}$. $\mathcal{T}_{\mathcal{R}}$ is not always a Hopf algebra, which is due to the fact that the tensor product of irreps of $\mathcal{T}_{\mathcal{R}}$ is not associative. We discuss how to deal with this.

Finally, we study a simple example of dyonic condensate. The calculations are rather involved in the case of dyonic condensates. We work out $\mathcal{T}_{\mathcal{R}}$ and $\mathcal{U}$ for a relatively simple dyon in a dihedral nematic.

Chapter 2

Ordered systems

In this chapter, we describe the physics of ordered condensed matter systems. We first discuss how to capture the *state* of a system using *fields*. Then we discuss the symmetries of a phase, and the phenomenon of *spontaneous symmetry breaking* to an ordered phase with less symmetry. At high temperatures, the symmetry of the state of the system is G. As the temperature is lowered, the state may change to a state with less symmetries. The symmetry group of the ordered phase is called the *residual symmetry group* H. We introduce *order parameters* which determine which symmetries are broken, and which symmetries are unbroken in the ordered phase. To describe excitations in the ordered phase, the order parameters are allowed to vary in space, thus becoming *order parameter fields.*

Once the fields that characterize the state of the system have been established, we can study *low energy excitations* of the state. We consider plane wave excitations of the fields, and using the *equations of motion* of the fields we can determine the *dispersion relations* of the plane wave excitations. By *linearizing* the equations of motion, we obtain the long wavelength, low frequency behaviour of the excitations. Plane wave excitations that satisfy the linearized equations of motion are called *modes.* We will discuss continuity modes and Goldstone modes.

In an ordered phase, there are also excitations called *topological defects*, which are configurations of the fields that are stable for topological reasons. These defects are classified using *homotopy theory.*

Finally, it is also possible to have *dyonic excitations*, which are a combination of topological defect and low energy excitation.

2.1 Description of ordered systems

States and fields

The configuration of a condensed matter system at some time t is called the *state* of the system at time t. The state of the system is determined by the positions, velocities and configurations of all its constituents. Since we cannot keep track of every constituent, we consider local averages of certain quantities to describe the state, such as mass density. We introduce *fields* which describe these local averages. Examples of fields are the mass density field $\rho(\vec{x},t)$ in a fluid, the average local magnetization field $\vec{m}(\vec{x},t)$ in a magnet, and the displacement field $u(\vec{x},t)$ in a crystal.

In a quantum mechanical treatment, the state of the system at time t is a *vector* $|\phi(t)>$ in a Hilbert space, and the fields are *expectation values* of operators in this state. We denote operators with a hat $\hat{}$. For example, $\hat{\rho}(\vec{x})$ is the local density operator at position x, and in the state $|\phi(t)>$ the density field $\rho(\vec{x},t)$ is given by

$$\rho(\vec{x},t) =< \phi(t)|\hat{\rho}(\vec{x})|\phi(t)> .$$

Hamiltonian and symmetries

The time evolution of a d-dimensional condensed matter system is determined by its *Hamiltonian.* The Hamiltonian is an operator $\mathcal{H}$ acting on the Hilbert space of the system. The state of the system $|\phi(t)>$ at time t satisfies Schr odinger's equation:

$$\frac{\partial|\phi(t)>}{\partial t} = -\frac{i}{\hbar}\mathcal{H}|\phi(t)>,$$

whose solution is [1]

$$|\phi(t)>= e^{-\frac{i}{\hbar}\mathcal{H}t}|\phi>, \tag{2.1}$$

where $|\phi>=|\phi(0)>$ is the state of the system at time $t=0$. We say that $\mathcal{H}$ *generates* time evolution.

A *ground state* of the system is an eigenvector of the Hamiltonian with the smallest eigenvalue of $\mathcal{H}$. According to (2.1), a system that starts in a ground state stays in the ground state (the state picks up an irrelevant phase factor).

The Hamiltonian is invariant under the transformations of some group G, called the *symmetry group* of the system. For example, the Hamiltonian of an ideal gas is invariant under time translation, time reversal, and the Euclidean group $ISO(3) = \mathbb{R}^3 \rtimes O(3)$ in three dimensions consisting of translations, rotations and reflections. We assume that G is a Lie group, so that it is fully characterized by its *generators*[2] L^a, which are infinitesimal symmetry transformations that can be exponentiated to give elements of G.

In quantum theory, these symmetry transformations are operators, called *symmetry operators*, that commute with the Hamiltonian. The states $|\phi>$ in the Hilbert space of the system transform under a *representation* of the symmetry group G. We denote the action of $g \in G$ on $|\phi>$ by $g \cdot |\phi>$.

Since the symmetry operators commute with $\mathcal{H}$, and $\mathcal{H}$ generates time evolution, an eigenvector $|\phi>$ of $g \in G$ with eigenvalue λ_g evolves into a state

---

[1] We assume the Hamiltonian is not explicitly dependent on time.

[2] In mathematical terminology, the generators form a basis of the Lie algebra of the Lie group G.

$|\phi(t)>$ with the same eigenvalue under the action of g:

$$g \cdot |\phi >= \lambda_g |\phi > \text{ and } [H, g] = 0 \Rightarrow g e^{-i\mathcal{H}t} = e^{-i\mathcal{H}t} g \Rightarrow g \cdot |\phi(t) >= \lambda_g |\phi(t) > \tag{2.2}$$

where we used (2.1) in the last line.

Thus we can label eigenstates of symmetry operators with the eigenvalues under the action of the corresponding symmetry operators. These eigenvalues are called *quantum numbers* of the state, and are conserved in time. It is not possible to choose states that are eigenstates of all the symmetry operators, unless the symmetry operators commute.

We distinguish between different types of symmetries. Symmetries can be *spacetime* or *external* symmetries, which means that they act on the space in which the system lives, or they can be *internal*, meaning that they act on some internal space. External symmetries act on the coordinates $\vec{x}$. If $g \in G$ is some external symmetry, then there is a matrix $\Gamma_{ij}(g)$ associated to it that transforms $\vec{x}$ according to $x^i \mapsto \Gamma_{ij}(g) x^j$. Then the field $\phi(\vec{x}, t)$ transforms into a new field $\phi^{'}(\vec{x}, t)$ under the action of g as follows:

$$\phi(\vec{x}, t) \mapsto \phi^{'}(\vec{x}, t) = \phi(\Gamma(g)^{-1}\vec{x}, t) \tag{2.3}$$

The reason for the appearance of $\Gamma(g)^{-1}$ is that we are transforming the space "from under" the field. For example, if we translate the space in the x-direction by one unit, The value of the new field at $x = 1$ is equal to the value of the old field at $x = 0$. This is called an *active* transformation of space[3].

Internal symmetries act on some internal label i of the fields. An internal symmetry transformation $g \in G$ acts on ϕ^i according to a matrix $\Gamma(g)^i_j$:

$$\phi^i \mapsto \Gamma(g)^i_j \phi^j. \tag{2.4}$$

We denote the action of g on ϕ by $g \cdot \phi$.

We further subdivide internal symmetries into *global* symmetries and *local symmetries*. Global symmetries act uniformly on the system, while local symmetries can act differently on different parts of the system[4]. Therefore, while a global symmetry is set by an element $g \in G$, a local symmetry is defined by a spacetime dependent $g(\vec{x}, t) \in G$, i.e. a map:

$$g : \mathbb{R}^{d+1} \mapsto G, \tag{2.5}$$

where $\mathbb{R}^d$ is space. To write down a Lagrangian that is invariant under local symmetry transformations, we must introduce a *gauge field*, of which the electromagnetic field A_μ is the paramount example.

---

[3] Note that when some authors refer to external symmetries, they mean an external symmetry coupled to an internal one. For example, a rotation of a space with a vector in it not only rotates the starting point of the vector, but also the vector itself! We will not adopt this point of view. Instead, we will describe such a transformation as a simultaneous internal and external rotation.

[4] This is actually a subtle distinction. For example, it is known that general relativity can be treated as a theory with the Lorentz group $SO(3, 1)$ as local internal symmetry. But the Lorentz transformations act on the tangent space at every point! Therefore, in some cases external symmetries may be treated as internal. The same applies to crystals. We will discuss this later.

Spontaneous symmetry breaking and order parameter fields

A condensed matter system is said to be in *equilibrium* when we can identify a ground state, such that the configurations of the fields of the system as a function of time only deviate slightly from the configuration of the fields corresponding to the ground state.

We denote the ground state at high temperatures by $|0>$. $|0>$ is invariant under the full symmetry group G:

$$g \cdot |0>=|0> \quad \forall g \in G. \tag{2.6}$$

As the system is cooled down, the ground state may change to a new ground state that is only invariant under a subgroup H of G. The low temperature phase is then called an *ordered phase*, or a *broken phase*, and the symmetry is said to be *spontaneously broken*. Spontaneously symmetry breaking is different from *explicit symmetry breaking*, where a term is added in the Hamiltonian that is only invariant under H.

In the case of spontaneous symmetry breaking, the Hamiltonian is invariant under all of G, but the ground state isn't. Spontaneous symmetry breaking is ubiquitous in condensed matter physics. It appears in a wide variety of phases, such as ferromagnets, crystals, liquid crystals, superfluids, and superconductors. In this thesis we will significantly extend the notion of spontaneously broken symmetry to include so called *Hopf symmetries*.

The difference between the high and low temperature phases is characterized by a change in the expectation value of an operator $\hat{\phi}$. $\hat{\phi}$ may have a label i. For example, if $\hat{\phi}$ is a vector operator: $\hat{\phi} = (\hat{\phi}^1, \hat{\phi}^2, \hat{\phi}^3)$. We will usually omit this label, and write it down when necessary.

We label states with the expectation value of $\hat{\phi}$. For example, $|\phi>$ corresponds to a state that satisfies

$$<\phi|\hat{\phi}|\phi>=\phi. \tag{2.7}$$

The state may have other labels, such as electric charge, but we neglect these for the moment, since we are only interested in the phase transition to the ordered phase.

We always choose our operators $\hat{\phi}$ such that $<0|\hat{\phi}|0>=0$, i.e. their expectation value is zero in the high temperature phase. Thus our notation of $|0>$ for the ground state of the high temperature phase is consistent with (2.7).

In the ordered phase, the ground state is $|\phi_0>$ with $\phi_0 =<\phi_0|\hat{\phi}|\phi_0>= \phi_0 \neq 0$. Thus the expectation value ϕ_0 of $\hat{\phi}$ is nonzero in the ordered phase. The expectation value of $\hat{\phi}$ is called an *order parameter*. Its expectation value distinguishes the high temperature phase from the ordered phase.

As we saw in (2.4), the order parameter ϕ transforms under a representation Γ of the symmetry group. Generally the representation Γ can be decomposed into *irreducible representations*, which are disjoint subsets of ϕ^i's that are only transformed into combinations of each other under the action of G. It is advantageous to work with order parameters in an irreducible representation.

The symmetry group H in the ordered phase is the subgroup of G whose elements leave ϕ_0 invariant:

$$h \cdot \phi_0 = \phi_0 \quad \forall h \in H. \tag{2.8}$$

H is called the *stabilizer* of ϕ_0. Note that H is also the the subgroup of G whose elements leave the ground state invariant. Namely,

$$h \cdot |\phi_0 >= |\phi_0 > \Longleftrightarrow \; h \cdot \phi_0 = \phi_0. \tag{2.9}$$

To study fluctuations of the ground state, we allow for a spatial dependence of the operator, so that we can compute its local expectation value $\phi(\vec{x}) =< \hat{\phi}(\vec{x}) >$ at a point $\vec{x}$. $\phi(\vec{x})$ is then called an *order parameter field*.

Free energy

Once the order parameter fields have been established, we can set up the *free energy* of the system. The free energy captures the competition between energy and entropy in a system, and it is minimized when the system is in thermal equilibrium. The ground state is a configuration that minimizes the free energy. We can find such a configuration using standard techniques from the calculus of variations. This is analogous to field theory, where the free energy is replaced by an action, and the equations that the fields have to satisfy in order to minimize the action are called the *Euler-Lagrange equations*. There is one minimizing equation for every field. In general, the free energy will be of the form

$$F[\phi] = \int d^3x \; f[\phi](\vec{x}) = \int d^3x \; (\; f_{el}[\phi](\vec{x}) + V[\phi](\vec{x}) \;) \tag{2.10}$$

where f is called the *free energy density*. f_{el} is called the *elastic free energy density*, and all its terms contain derivatives of the order parameter fields. V is the *potential term*, and contains no derivatives[5]. f_{el} and V depend on the *coupling constants* of the theory. In thermodynamics a system has at least two coupling constants, which can be taken to be pressure P and temperature T.

All the terms in the free energy respect the symmetries of the Hamiltonian. We use this fact to write down the possible terms in f: we seek combinations of the fields ϕ that are invariant under the symmetry group G. For example, if ϕ is a vector and $G = SO(3)$, then we could include $\phi^i\phi_i$ in the free energy.

The elastic free energy and potential term are separately invariant under G. For any configuration ϕ of the fields, and any $g \in G$, we have

$$\int d^3x \; f_{el}[g \cdot \phi] = \int d^3x \; f_{el}[\phi] \qquad \int d^3x \; V[g \cdot \phi] = \int d^3x \; V[\phi]. \tag{2.12}$$

---

[5] The free energy may depend on other fields χ than order parameter fields, such as mass density. We will assume that the total free energy F_t can be split into two parts

$$F_t[\phi, \chi] = F_0[\chi] + F[\phi]. \tag{2.11}$$

We will only study $F = F_t - F_0$. There may also be several order parameter fields, in which case we would include a sublabel ϕ_a.

This implies that

$$f_{el}[g \cdot \phi] = f_{el}[\phi] + \nabla \cdot g_{el}[\phi] \qquad V[g \cdot \phi] = V[\phi] + \nabla \cdot g_V[\phi], \tag{2.13}$$

where $\nabla \cdot g_{el}[\phi]$ and $\nabla \cdot g_V[\phi]$ are total derivative terms, that disappear when we the take the integral over all space (neglecting surface effects). We will ignore eventual total derivative terms.

The potential term plays a crucial role in spontaneous symmetry breaking. Namely, the ground state is a configuration of the fields, which we denote by ϕ_0, that minimizes the free energy. We assume the ground state is uniform, i.e. the fields are all constant in the ground state[6]. Since the terms in f_{el} contain derivatives, $f_{el}[\phi_0] \equiv 0$, so that to minimize f, ϕ_0 must be a minimum of $V[\phi_0]$. At high temperatures, the minimum of $V[\phi]$ occurs at $\phi_0 = 0$. To obtain spontaneous symmetry breaking, $V[\phi]$ must change shape as the temperature is lowered, so that below some critical temperature T_c (at fixed pressure) the global minimum occurs at nonzero values of ϕ_0. This signals that we have entered the ordered phase.

Say the system is in the ordered phase with ground state ϕ_0 and symmetry group H. Then ϕ_0^i is a global minimum of $V[\phi]$, the elements of H are defined as those $h \in G$ that satisfy

$$h \cdot \phi_0 = \phi_0. \tag{2.14}$$

H is called the *stabilizer* of ϕ_0.

Since, according to (2.13), for any $g \in G$ we have $V[g \cdot \phi] = V[\phi]$, if ϕ_0 is a minimum of $V[\phi]$ then so is $g \cdot \phi_0$. By letting all of G act on ϕ_0, we obtain a continuous family of ground states, called the *ground state manifold*. We assume that all the minima of $V[\phi]$ are obtained by acting with elements of G on a chosen minimum ϕ_0. Then G/H is *equal* to the order parameter space, since the order parameter ϕ_0 takes value in the ground state manifold. Since H leaves the ground state invariant, the ground state manifold is equal to[7] G/H.

There is one type of term in $V[\phi]$ we would like to mention: if $\frac{1}{2}m^2C(\phi^i)^2$ is present in $V[\phi]$, with m^2 some constant and i fixed, then we say that ϕ^i is *massive*, and m is the *mass* of ϕ^i. There can be different masses for different i.

The story until now applies equally well to spontaneous symmetry breaking in a theory with global and local symmetries. The only difference in a local theory is that the derivatives in f_{el} are *covariant derivatives* D_i, which means that we redefine the derivatives so that f_{el} energy is invariant under local symmetry transformations. To show how this is done, pick one field ϕ that transforms under a representation Γ of G. For $g \in G$, the matrix $\Gamma(g)$ can be written as

$$\Gamma(g) = exp(i\epsilon_a T^a) \tag{2.15}$$

where the T^a form an $n \times n$ matrix representation of the generators of G, and the ϵ_a are real numbers. In a local symmetry transformation, the ϵ_a are dependent

---

[6] The ground state can be nonuniform, given certain boundary conditions. We impose the boundary condition "the ground state is uniform at infinity".

[7] Elements of G/H are left cosets gH, defined by $gH = \{gh | h \in H\}$. Given that G is a Lie group and acts smoothly on the order parameter space, the G/H is a manifold.

on position. Now we introduce a *gauge field*

$$A_i(\vec{x}) = A_i^a(\vec{x})T^a, \tag{2.16}$$

where the $A_i^a(\vec{x})$ are real numbers. $A_i(\vec{x})$ is an $n \times n$ matrix at every point in space. Under a local symmetry transformation $g(\vec{x}) = exp(i\epsilon_a(\vec{x})L^a)$, we postulate that A_μ transforms according to

$$A_i(\vec{x}) \mapsto \Gamma(g)(\vec{x})A_i(\vec{x})\Gamma(g(\vec{x}))^{-1} + \frac{i}{q}\Gamma(g)(\vec{x})\nabla_i\Gamma(g(\vec{x}))^{-1} \tag{2.17}$$

ϕ transforms as

$$\phi(\vec{x}) \mapsto \Gamma(g(\vec{x}))\phi(\vec{x}). \tag{2.18}$$

Now if we define the covariant derivative as

$$D_j = \nabla_j - iqA_\mu, \tag{2.19}$$

then we verify that under the local transformation $g(\vec{x})$, D_j has a simple transformation law:

$$D_j(\vec{x}) \mapsto \Gamma(g(\vec{x}))D_j. \tag{2.20}$$

We can now construct a free energy density that is invariant under local symmetry transformations:

$$f_{gauge} = D_j\phi \cdot D^j\phi - V[\phi] \tag{2.21}$$

where the dot "$\cdot$" is an inner product that satisfies

$$(\Gamma(g)D_j\phi) \cdot (\Gamma(g)D^j\phi) = D_j\phi \cdot D^j\phi. \tag{2.22}$$

for any $g \in G$. We will see examples of such an inner product later on.

Since the potential term contains no derivatives and it is invariant under global symmetries, it is also invariant under local ones. Therefore this free energy is invariant under local transformations. If G is a nonabelian group, then we are dealing with a *nonabelian gauge theory.*

In analogy with the global case, if the order parameter acquires a nonzero ground state expectation value ϕ_0, and the stabilizer of ϕ_0 is H, then the residual local symmetry transformations are those that take values in H everywhere:

$$\begin{aligned} g: \quad & \mathbb{R}^d \to H \\ & \vec{x} \mapsto g(\vec{x}) \in H, \end{aligned} \tag{2.23}$$

where R^d is space. A special case that will occur frequently in this thesis is the case of a discrete[8] H. In this case, for the map $g : \mathbb{R}^d \to H$ to be continuous, it must be constant, since there is no way to "move around" continuously in H, since H is discrete. Thus we are only left with global symmetry transformations $h \in H$. A gauge theory that is spontaneously broken to a discrete H is called a *discrete gauge theory.*

---

[8] H is discrete if it is countable. There are no "infinitesimal" transformations, arbitrarily close to the identity.

2.2 Excitations and modes

Once the symmetries of the ground state of a phase have been determined, we can study the low energy excitations of the ground states. Given the ground state ϕ_0, we write

$$\phi(\vec{x},t) = \phi_0 + \delta\phi(\vec{x},t), \tag{2.24}$$

where $\delta\phi(\vec{x},t)$ is the excitation from the ground state ϕ_0. Note that we've included time dependence in the fields here, while in the free energy we had no time dependence in the fields. In thermodynamics, the free energy is obtained by integrating out time, and time evolution is determined by the second law of thermodynamics. However, when we study excitations we are interested in their time evolution. If we have the Hamiltonian of our system, we can determine the Hamiltonian equations of motion for every field ϕ. Then we fill in $\phi(\vec{x},t) = \phi_0 + \delta\phi(\vec{x},t)$ into the equations of motion for ϕ, and linearize them (i.e. we only consider first order terms in $\delta\phi$). The ensuing equations are called the *linearized equations of motion*. Solutions to the linearized equations of motion for ϕ are called the *modes* of ϕ.

We don't always know the exact Hamiltonian of our system. It may be very complicated, involving interactions of atomic orbitals. We may, however, have an expression for the free energy in terms of coarse grained variables. The free energy has no explicit time dependence, so we don't have equations of motion if we only have a free energy. In that case we need a separate analysis to determine the time evolution equations.

2.2.1 Dispersion relation

The common prescription to study the modes of a field ϕ is to consider a plane wave configuration of the excitation $\delta\phi$ of the field from the ground state ϕ_0:

$$\delta\phi(\vec{x},t) = (\delta\phi)_0 \; e^{i\vec{k}\cdot\vec{x} - i\omega t} \tag{2.25}$$

where $(\delta\phi)_0$ is constant configuration of the field. We can justify studying only plane wave excitations by noting that any field configuration can be written as an integral of plane waves, using a Fourier transform.

To obtain the *dispersion relations* of the modes of ϕ, we need an equation that determines the time derivative of $\delta\phi(\vec{x},t)$. This is a nontrivial problem. In this equation, we have to write down all terms that contribute to changes in time of the field $\phi(\vec{x},t)$. We will see a simple example when we discuss continuity modes. Once we have obtained this equation, we plug in the plane-wave configuration, to obtain the *dispersion relation* of the mode, which gives ω as a function of $\vec{k}$:

$$\omega(\vec{k}) = a(\vec{k}) + ib(\vec{k}) \tag{2.26}$$

From the form of the dispersion relation we learn a lot about a mode. We can distinguish between different cases:

- $b(\vec{k}) \neq 0, a(\vec{k}) \neq 0$
  The mode oscillates, and the oscillations damp out exponentially as a function of time. This is called a *diffusive* mode. It signals that our mode is losing energy to the incoherent degrees of freedom of the system.

- $b(\vec{k}) \neq 0, a(\vec{k}) = 0$
  The mode is then *overdamped.* It will not oscillate, but simply die out exponentially.

- $a(\vec{k}) = c|\vec{k}|$, and $b(\vec{k})$ goes to zero faster than $a(\vec{k})$ as $k \mapsto 0$
  c is a constant. This is called a *sound mode*, or a *phonon.* c is called the *sound velocity* of the phonon.

- $b(\vec{k}) \sim |\vec{k}|^2$
  This is called a *heat mode.*

- $a(\vec{0}) \neq 0$
  This is a *massive mode.* There is an energy gap for such a mode: it can only be excited if a minimum finite amount of energy is put into it. The energy gap is of order $a(\vec{0})$.

The precise form of the dispersion relation of the modes has dramatic consequences on the physics of the system. For example, as Landau first pointed out [36], the existence of a sound mode in a superfluid accounts for the superfluidity! In superconductors, the massive photon is responsible for the Meissner effect.

2.2.2 Continuity modes

A condensed matter system satisfies certain conservation laws. For example, in a simple fluid, mass, energy and momentum are conserved. The variables associated with these conversation laws are called conserved quantities, and are measured by operators that commute with the Hamiltonian. Analogously to the order parameter fields, we introduce *continuity fields* that correspond to local expectation values of these operators. In a simple fluid, the continuity fields are the mass density $\rho(\vec{x}, t)$, the energy density $\epsilon(\vec{x}, t)$, and the momentum density $\rho\vec{v}(\vec{x}, t)$. In the ground state, all these fields are constant. These fields satisfy *continuity equations* that express their local conservation:

$$\frac{\partial \epsilon}{\partial t} + \nabla_i J_i^{\epsilon} = 0 \tag{2.27}$$

$$\frac{\partial \rho}{\partial t} + \nabla_i \rho \vec{v}_i = 0 \tag{2.28}$$

$$\frac{\partial \rho \vec{v}_j}{\partial t} + \nabla_i \Pi_{ij} = 0 \tag{2.29}$$

There are five conservation laws: one for energy, one for mass, and three for momentum. The modes associated to these equations are called *continuity modes*, and there are five of them in a simple fluid[9]. The latter two equations are coupled, but the first equation is independent, so we can treat it independently. We will heuristically derive the dispersion relation associated to this equation. To do this, we need an equation for J_i^ϵ. If the energy density is constant, the current should be zero. If it is nonuniform, then J_i^ϵ should try to reestablish equilibrium, so to lowest order the simplest expression we can write down is

$$J_i^\epsilon(\vec{x}) = -D_T \nabla_i \epsilon(\vec{x}) \tag{2.30}$$

where D_T is called the *heat diffusion constant.* Plugging this into (2.27) we get

$$\frac{\partial \epsilon}{\partial t} = D_T \nabla^2 \epsilon, \tag{2.31}$$

which is the famous *diffusion equation.* Note that this equation is not invariant under time reversal $t \mapsto -t$. This signals that the flow determined by this equation is irreversible. We use this criterion in more general flow equations to see if the flow is reversible or not.

The diffusion equation yields the following dispersion relation:

$$\omega = -iD_T k^2, \tag{2.32}$$

which is, as defined above, a heat mode.
We can Fourier transform this equation back to position space, to see what happens. Given an initial configuration $\phi(\vec{x}, t = 0) = \delta(x)$ (an excitation at the origin), at later times ϕ is given by

$$\phi(\vec{x}, t) = \frac{1}{(4\pi D_T |t|)^{\frac{d}{2}}} e^{-\frac{\vec{x}^2}{4D_T|t|}} \tag{2.33}$$

where d is the number of space dimensions (here $d = 3$). The time evolution shows that the initial excitation broadens and decays.

We have four modes left to discuss. The derivation can for example be found in [15]. We will only describe the dispersion relations.

Two of these four modes are diffusive transverse momentum modes, corresponding to excitations of $\rho\vec{v}(\vec{x})$. Transverse means that the momentum is perpendicular to the wave-number $\vec{k}$ of the plane wave. In other words, the plane wave excitation is of the form $\rho\vec{v}_\perp e^{i\vec{k}\cdot\vec{x} - i\omega t}$ with $\vec{v}_\perp \cdot \vec{k} = 0$. There are two transverse directions, thus two modes, with the same dispersion relation:

$$\omega = -i\frac{\eta}{\rho} k^2 \tag{2.34}$$

---

[9] These conservation laws follow from the external symmetries of the system! Namely, time translation invariance implies conservation of energy, spatial translation invariance implies momentum conservation, rotational invariance implies angular momentum conservation, etc. This is a consequence of Noether's theorem, which states that there is a conserved quantity associated to every global symmetry of the Hamiltonian. Mass is a special case: in a relativistic theory it is not conserved (since it can be converted into energy), but mass is approximately conserved in the nonrelativistic limit.

where η is a viscosity parameter.

Finally, we have two diffusive sound modes, which correspond to combined excitations of the mass density and longitudinal momentum. Their dispersion relations are:

$$\omega = \pm ck - i\frac{1}{2}\Gamma k^2 \tag{2.35}$$

where Γ and c are constants. Thus given a momentum $\vec{k}$, the sound mode can propagate in the negative or positive $\vec{k}$ direction.

Note that our analysis of the energy mode was rather simplified. Actually the energy density *is* coupled to the mass density. The energy mode and sound modes are coupled excitations of the mass density, energy density and longitudinal momentum. Still, our simplified analysis does give the right dispersion relation.

The mode structure is already quite intricate in a simple fluid, and when we add order parameter fields it will become even more complex. In liquid crystals, we will see that the Goldstone modes, corresponding to the order parameter fields, couple to the continuity modes, thereby forming composite modes.

2.2.3 Goldstone modes and massive gauge fields

We've seen that after spontaneous symmetry breaking, the ground state manifold is G/H. Now we will discuss *Goldstone's theorem*, which states that for every broken generator of G there is a massless mode of the order parameter field. After that, we will discuss the *Higgs phenomenon*, which is the analogon of Goldstone's theorem in a gauge theory.
The ground state is ϕ_0. Expand the potential $V[\phi]$ around ϕ_0 to second order:

$$V[\phi] = V[\phi_0] + (\phi - \phi_0)^i M_{ij} (\phi - \phi_0)^j \tag{2.36}$$

where M_{ij} is the *mass matrix* defined by

$$M_{ij} = \left.\frac{\partial^2 V}{\partial\phi_i \partial\phi_j}\right|_{\phi^i = \phi_0^i} \tag{2.37}$$

There are no first order terms in the expansion, since ϕ_0 is a minimum of $V[\phi]$, so that

$$\left.\frac{\partial V}{\partial \phi^i}\right|_{\phi^i = \phi_0^i} = 0 \tag{2.38}$$

Denote the generators of H by t^b. These are a subset of the generators T^a of G (see 2.15), that satisfy

$$(t^b)^i_j \phi_0^j = 0, \tag{2.39}$$

and are called the *unbroken generators.* The other generators are said to be *broken.*

We will now prove that for every T^a that is not a generator of H, $T^a\phi_0$ is an eigenvector of M_{ij} with eigenvalue zero. This corresponds to a mode with zero mass, called a Goldstone mode. Since it is massless, we learn from the free energy that it costs no energy to uniformly excite such a mode, i.e.

$$f[\phi_0(\vec{x})] = f[\phi_0(\vec{x}) + T^a\phi_0]. \tag{2.40}$$

Nonuniform excitations, however, will cost energy because of the derivatives in f_{el}.

The proof runs as follows. An element g of G can be written as[10] $g = exp(i\epsilon_a T^a)$, with ϵ_a real numbers. From (2.13) we know that for any configuration ϕ^i of the fields

$$V[g\cdot\phi] = V[\phi]$$

Take small ϵ_a, so that we can expand g: $g = 1 + i\epsilon_a T^a$. Plugging this into the previous equation, and Taylor expanding, we get

$$\begin{aligned} & V[\phi] + \epsilon_a (T^a)^i_j \phi^j \frac{\partial V}{\partial \phi^i} = V[\phi] \\ \Rightarrow \qquad & \epsilon_a (T^a)^i_j \phi^j \frac{\partial V}{\partial \phi^i} = 0 && (2.41) \\ & && (2.42) \end{aligned}$$

Now derive this equation with respect to ϕ^k, and evaluate the equation at $\phi^i = \phi^i_0$. Using (2.38) we obtain

$$\epsilon_a M_{ki} (T^a)^i_j \phi^j_0 = 0 \tag{2.43}$$

For the unbroken generators t^b this equation is trivial. For the broken generators, $(T^a)^i_j \phi^j_0 \neq 0$, so that $(T^a)^i_j \phi^j_0$ is a zero eigenvalue eigenvector of M_{ij}. Therefore the Goldstone modes form a vector space, the zero-eigenvalue eigenspace of M_{ij}. We diagonalize M_{ij} in the free energy, and we see that the Goldstone modes do not appear in $V[\phi]$. They are only present in derivative terms in f_{el}.

We will now briefly discuss what happens in a spontaneously broken local theory. The Goldstone modes are unphysical degrees of freedom that can be gauged away. The gauge bosons associated to the would-be Goldstone bosons (i.e. the $A^a_i T^a$ with T^a broken) become massive, while the other gauge bosons remain massless. This is the celebrated *Higgs mechanism*, and it is the only mechanism we know that gives mass to gauge bosons. It plays a crucial role in Glashow-Salam-Weinberg theory of weak interactions, where the local symmetry is sopntaneously broken from $SU(2) \times U(1)$ to $U(1)$. Three generators are broken, leading to three massive gauge bosons W^+, W^-, Z that mediate the weak interaction. The unbroken $U(1)$ corresponds to a massless gauge boson, which is the photon. In superconductors, the $U(1)$ of electromagnetism is broken to $\mathbb{Z}_2$, which leads to a massive photon. The *Meissner effect* ensues: magnetic

---

[10] This is only true for group elements that are connected to the identity. For the analysis of the Goldstone modes we only need to study such transformations.

fields aimed at the superconductor only penetrate a finite depth, and decay exponentially inside the superconductor. Later on we will see analogs of the Higgs mechanism in liquid crystals and crystals, without a gauge theory!

2.3 Topological defects

Topological defects are a special type of configuration of the order parameter fields. To obtain such a configuration in our d-dimensional space, remove a manifold C of dimension $n < d$ from the space. For example, remove a line or a point in three dimensions. Then there may exist continuous configurations of the order parameter field, that it take values in G/H everywhere outside of C, and *cannot* be given a value in G/H on C such that the overal configuration is continuous. Sometimes we can smoothly vary the order parameter field close to C such that the configuration is continuous everywhere, in which case the singularity is said to be *removable*. It this is not possible, then the configuration is called a *topological defect* of dimension n, and C is called the *core* of the defect. Close to the core, the order parameter *can* be defined continuously if we allow it to leave G/H. For example, we can have $\phi = 0$ at the core, and ϕ increases smoothly from zero to its value in G/H outside the core. Some authors refer to the area around C in which ϕ is still outside of G/H as the core. There is a length scale in the system, associated to the size of this core.

Defects are given names according to the dimensionality of their core. For example, a defect with a one-dimensional core in a three-dimensional space is called a *line defect*, and if the core is a point (a zero-dimensional manifold) it is called a *monopole*. More generally, in a d-dimensional space, the names of defects are given in table 2.1.

Dimension of the core	Type	
$d-1$	Domain wall	
1	Line defect	Flux, vortex
0	Point defect	Monopole

Table 2.1: The different defects in a d-dimensional space. We give the dimension of the core in space.

2.3.1 Classification of topological defects

For an excellent review on the theory behind topological defects in condensed matter, see [42].

Homotopy groups

We study a defect by surrounding the core by S^k, with an appropriate k. A line defect in three dimensions is surrounded by a circle S^1, for example. More generally, an n-dimensional defect in d-dimensional space is surrounded by S^{d-n-1}.

On every point of this S^k we have a value of the order parameter field in G/H. Thus this gives a map from S^k to G/H. We are interested in the *homotopy classes* of these maps. A homotopy between two maps h_0 and h_1 from S^k to G/H is a continuous family F_t of maps from S^k to G/H, with $t \in [0,1]$, such that $F_0 = h_0$ and $F_1 = h_1$.

One can prove that if the map around the core of a defect is homotopic to the map around the core of another defect, then we can smoothly change the configuration of the field around one core so that it becomes the other core. We say that the cores of homotopic defects can be turned into each other by *local surgery*. This local surgery only costs a finite amount of energy. If the maps are not homotopic, then to turn one defect into the other we would have to change the configuration of the fields out to infinity, which would cost an infinite amount of energy. Thus nonhomotopic defects are separated by infinitely high energy barriers, and we say they belong to different *topological sectors*.

Homotopy theory studies these embeddings of maps. $\Pi_n(G/H)$ is called the *n-th homotopy group*, and classifies the homotopy classes of embeddings of S^n into G/H. Remarkably, $\Pi_n(G/H)$ can always be given a group structure. The multiplication of homotopy classes corresponds to the composition of homotopies. For examples, two loops in $\Pi_1(G/H)$ are composed by traversing both paths in succession. One can similarly define the composition of maps from S^k for any k. We call the group element corresponding to the homotopy class of the defect the *topological charge* of the defect.

Note that we can only compose loops if one loop ends where the other one starts. Thus we are actually studying *based homotopies*, which means that we pick a distinguished point x of S^k, and a distinguished point y of G/H, and require that for all our mappings x gets mapped to y. If the space G/H is connected, then the homotopy groups for different basepoints are isomorphic, so it doesn't matter which basepoint we pick. For convenience, we pick H as the basepoint in G/H (remember that "points" in y are left H cosets).
Homotopy theory gives us isomorphism between homotopy groups of different dimension. We will need two of them. We state them first, and then explain them.

Theorem 1. *If G is a connected and simply connected Lie group, then*

$$\Pi_1(G/H) \simeq \Pi_0(H) \tag{2.44}$$

$$\Pi_2(G/H) \simeq \Pi_1(H_0), \tag{2.45}$$

where H_0 is the component of H that is connected to the identity[11].

$\Pi_0(H)$ is the set of connected components of H, which also has a group structure.

G is called *simply connected* if all loops in G are contractible, i.e. $\Pi_1(G) = 0$. If this is not the case for a group we are interested in, then there is a theorem which tells us that we can always enlarge G to $\overline{G}$, called the *universal*

---

[11] It can be shown that H_0 is a normal subgroup of H, so that H/H_0 is a group.

covering group of G. This group covers G, which means that there is a surjective homomorphism $\rho : \overline{G} \mapsto G$, whose kernel consists of a finite set. Therefore $\overline{G}$ should be thought of as a number of copies of G. A famous example is $\rho : SU(2) \mapsto SO(3)$ with kernel $\{1, -1\}$, so that $SU(2)$ is a *double cover* of $SO(3)$. Once we've enlarged G to $\overline{G}$, we must also enlarge H to $\overline{H}$. This has no effect on the ground state manifold, because $= \overline{G}/\overline{H} = G/H$. Using this covering group, we can rewrite the two theorems without assuming that G is simply connected:

$$\Pi_1(G/H) \simeq \Pi_0(\overline{H}) \tag{2.46}$$

$$\Pi_2(G/H) \simeq \Pi_1(\overline{H_0}). \tag{2.47}$$

There is a convenient to picture these isomorphisms. Assume G is simply connected. Take a representative f of a homotopy class in $\Pi_1(G/H)$, which is a map from S^1 to G/H that starts at H. To this f we can associate a path $\tilde{f}$ in G called a *covering path*, that starts at the identity e and ends in a point in H, such that $\tilde{f}(modH) = f$. Studying homotopies of loops f in G/H is equivalent to studying homotopies of paths in G that end in H. From fig.2.1 (a), it is clear that two paths in G that end in H are homotopic if and only if[12] *their endpoints are in the same connected component of* H. Therefore $\Pi_1(G/H) = H/H_0$.

An analogous analysis applies to monopoles. In this case, we first note that instead of embedding S^2 into G/H with a distinguished point mapped to H, we can equivalently consider embeddings of $I^2 = [0,1]^2$ to G/H with the surface ∂I^1 of I^2 mapped to H. This is equivalent because S^2 is topologically equivalent to I^2 with the boundary identified to a point. Take a representative α of a homotopy class in $\Pi_2(G/H)$. This is a map from I^2 to G/H with the boundary taken to H. To this α we can associate a map $\tilde{\alpha}$ from I^2 to G, called the *covering map of* α, such that a distinguished point x on the boundary is mapped to e. The boundary ∂I^1 is then a loop in G that is connected to e, so that it sits in H_0. Call this loop f_α. Given two maps α and β from I^2, respective covering maps $\tilde{\alpha}$ and $\tilde{\beta}$, and corresponding loops on the boundary f_α and f_β, one can prove[13] that α *and* β *are homotopic if and only if* f_α *and* f_β *are homotopic loops in* H_0.

An important example is when the residual symmetry group H is discrete, meaning that $H_0 = \{e\}$. In this case, we get

$$\Pi_1(G/H) \simeq \Pi_0(H) \simeq H \tag{2.48}$$

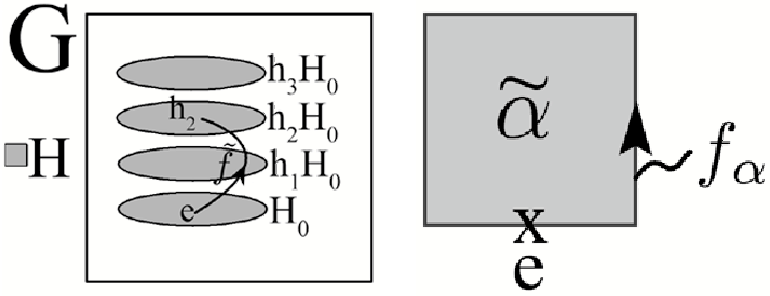
$$\Pi_2(G/H) \simeq \Pi_1(H_0) \simeq 0. \tag{2.49}$$

So the line defects are characterized by elements of H, and there are no monopoles.

---

[12] Here we use the fact that G is simply connected. All paths in G are homotopic, but under the restriction of the endpoint ending in H, two loops with endpoints ending in disconnected components are not homotopic, because the homotopy would contain loops that don't end in H.

[13] The theorem follows from the lifting property of homotopies[23]. The same theorem is used to prove the theorem for $\Pi_1(G/H)$ discussed above.



(a) Representation of a homotopy class in $\Pi_1(G/H)$, as a path $\tilde{f}$ in G that starts at e and ends in some disconnected component h_iH_0 of H.

(b) Representation of a homotopy class in $\Pi_2(G/H)$, as a map $\tilde{\alpha}$ from I^2 to G, with a distinguished point x on the boundary mapped to e.

Figure 2.1: The isomorphisms of homotopy theory. The maps from I^n to G/H can be replaced by maps from I^n to G, with appropriate boundary conditions.

If we drop the assumption that G is simply connected, then $\Pi_1(G/H) \simeq \overline{H}$, and $\Pi_2(G/H) \simeq 0$.

In three dimensions, line defects are divided into two categories: *Dislocations* are defects whose characteristic loop is a noncontractible loop in the *translational* part of the symmetry group. They only occur if the translational group is broken *Disclinations* are defects whose loop is noncontractible in the *rotational* part of the defect. This may be the external of the internal rotational group.

Defect multiplets

If the group $\Pi_1(G/H)$ is non-abelian, then the defects are organized in multiplets, and the different defects in different multiplets are separated by infinitely high energy barriers. In other words, the cores of the defects in the same multiplet can be interchanged, which means that we can turn one defect into the other by local surgery, with only costs a finite amount of energy.

To see which defects are part of the same multiplet, we must act on the defect with global symmetry transformations. We've seen that a line defect can be characterized by a path $\tilde{f}$ in G that starts at e and ends in an element h_i of H. The homotopy class is set by the left coset in H/H_0 to which h_i belongs. A global symmetry transformation $h \in H$ conjugates the path [14], turning it into $\tilde{f}'$: $\tilde{f}' = h\tilde{f}h^{-1}$. This is again a loop that starts at e and ends in a new element of H, namely hh_ih^{-1}. Therefore the charge of the defect has been conjugated.

---

[14] A global symmetry transformation h actually multiplies the loop by h from the left, since h at acts on every point of the space. This gives a loop that starts in h and ends in $h\tilde{f}(1)$. We can multiply this loop by h^{-1} from the right, because we are actually studying loops in G/H, which are defined modulo multiplication by elements of H from the right. This gives the loop $\tilde{f}'$, which starts at e and ends at $h\tilde{f}(1)h^{-1}$. Therefore its charge is given by the left coset in H/H_0 to which $h\tilde{f}(1)h^{-1}$ belongs.

So *line defects whose charges are in the same conjugacy class of* H/H_0 *are in the same multiplet.* For these multiplets to consist of more than one defect, H/H_0 has to be non-abelian: If H/H_0 is abelian, then the conjugacy classes consist of single elements.

We've already discussed that defects with the same charge are homotopic. We now further specify that *defects whose charges are in the same conjugacy class are freely homotopic* [15]. The defects in different classes are separated by infinitely high energy barriers in configuration space, thus they correspond to different topological sectors.

The cores of defects in the same conjugacy class can be interchanged because the global symmetry transformation h is connected to the identity (we assumed G was simply connected). Thus we have a path $\overline{h}(t)$ in G going from e to h:

$$\overline{h}(0) = e \quad \overline{h}(1) = h.$$

By conjugating $\tilde{f}$ with $\overline{h}(t)$, we obtain a family of paths:

$$\tilde{f}_t = \overline{h}(t)\tilde{f}\overline{h}(t)^{-1}.$$

This is a continuous family of paths, that starts at $\tilde{f}_0 = \tilde{f}$ and ends at $\tilde{f}_1 = h\tilde{f}h^{-1}$. Thus we have just defined a homotopy between $\tilde{f}$ and $\tilde{f}'$.

A similar analysis applies to monopoles. To describe the action of global symmetry transformations, we use the isomorphism (2.45). If the loop in H_0 characterizing the monopole is f_α, then a global symmetry transformation h conjugates the loop, giving a monopole charcterized by the loop $f'_\alpha = hf_\alpha h^{-1}$. Therefore *the monopoles are arranged in classes*, consisting of based homotopy classes of loops in H_0 that are conjugates under the action of H/H_0. The monopoles in the same class are homotopic. For the monopole classes to consist of more than one element, H must be non-abelian.

2.3.2 Interactions between defects in d=3

Now that we've classified the different charges of defects, we must study what happens when several defects are present in the system. The defects can influence each other's charge, by moving around each other, which we call *braiding*. Furthermore, two defects can also *fuse* to form a new defect. We discuss both types of interactions.

The braiding of defects

Consider a configuration with two line defects. If we remove the cores of these line defects from the space, then the resulting space is not simply connected. Namely, a loop around one of the cores cannot be to shrunk to the trivial loop. Not that this is only true if the line defects have infinite extent. If the line defects end on monopoles, then the space is simply connected.

---

[15] A free homotopy is a homotopy that is not based, i.e. we don't require that a distinguished point of S^n be mapped to e.

The charges of the defects may interact with each other, and the defects in the same multiplet can be transformed into one another. Fig.2.2 tells us that if a defect h is transported halfway around g, then it is conjugated by g, and its charge becomes ghg^{-1}. This transport halfway around is called *half braiding*. Thus line defects in the same multiplet transform into each other when they braid with other line defects. If H/H_0 is abelian, then the braiding is trivial, since then $ghg^{-1} = h$. Thus these interactions between charges only occur when the first homotopy group is non-abelian.

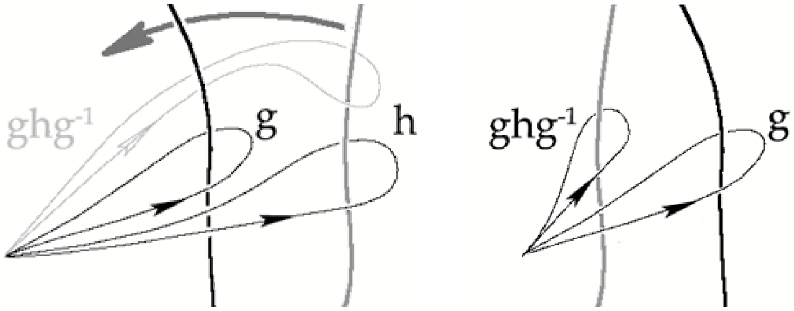


Figure 2.2: The conjugation of the defect h by the presence of g. This conjugation is the effect of braiding one defect around another.

Monopoles in the same class can also transform into each other by braiding around a line defect. If a monopole is described by the loop f_α in H_0 under the isomorphism (2.45), then after transporting the monopole around a line defect of charge $h_i H_0 \in H/H_0$ the monopole is characterized by the loop[16] $f'_\alpha = h_i f_\alpha h_i^{-1}$. Therefore the monopoles in the same class can turn into each other by braiding around a line defect.

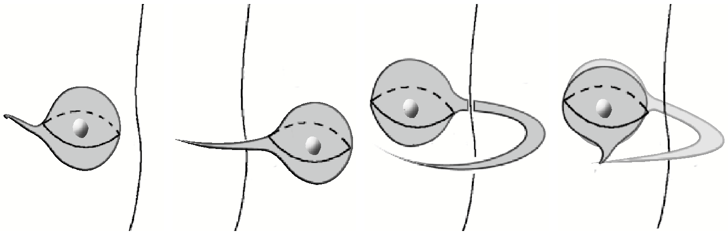


Figure 2.3: The conjugation of a monopole by a line defect. The charge of the monopole is an element of $\Pi_2(G/H)$. It is acted on by the charge of the line defect, an element of $\Pi_1(G/H)$. Homotopy theory is used to calculate the action.

---

[16] This action is well defined because H_0 is a normal subgroup of H, and the action of H_0 on H/H_0 is therefore trivial: Take $h_0 \in H_0$, and $hH_0 \in H/H_0$, then $h_0 h H_0 h_0^{-1} = hH_0$.

Note that the conjugation of the monopole is achieved by a *full braiding* of the monopole around the line defect, in contrast to the half braiding of the line defects.

The braiding of two monopoles does not affect their charge, since such a braiding is homotopic to doing nothing (the path can be contracted ”over” the other monopole).

The fusion of defects

Let us take a cross section of the line defects discussed until now. If we cut fig. 2.2 horizontally, the line defects become points in two dimensions.
When two line defects g and h are brought so close together that their cores fuse, the charge of the resulting defect is gh, as shown in fig.2.3.2. In the presence of another defect f, the fusion is ambiguous. Namely, if g is transported around f in a clockwise fashion, the fusion becomes $f^{-1}gfh$. Similarly, if h were transported around f in a clockwise path, the fusion would give $gf^{-1}hf$. From

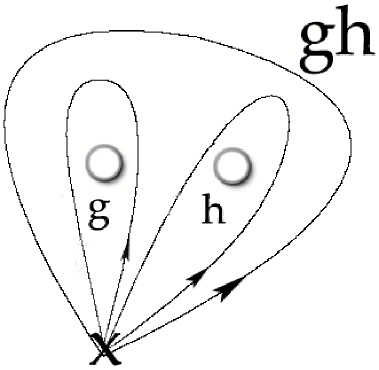


Figure 2.4: The fusion of two defects.

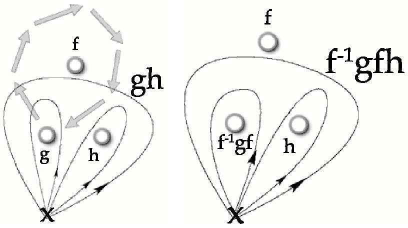


Figure 2.5: The fusion of defects g and h in the presence of another defect f. The fusion originally gives gh. If g travels around f in a clockwise path, then g is conjugated by f, and the fusion with h becomes $f^{-1}gfh$.

these observations we conclude that the possible outcomes of the fusion of g and h are given by the multiplication of all elements in the conjugacy of g in H/H_0 with all elements in the conjugacy class of h. Denote the conjugacy class of an element g by C_g. We call the *fusion rules* of two particles the possible outcomes of their fusion. We've learned that *the fusion rules of g and h are given by*[17] $C_g \times C_h$. One can prove that the multiplication of conjugacy classes is commutative, and that it is composed of entire conjugacy classes, although some conjugacy classes may be repeated in the decomposition. We denote the fusion rules as

$$C^A \times C^B = \sum_C {N^{AB}}_C C^C \tag{2.50}$$

where C^A is the conjugacy class of an element g_A, C^B the conjugacy of class g_B, etc. We are adopting the notation that will be used in the next chapter, where we will pick a preferred element out of each conjugacy class.${N^{AB}}_C$ is an integer that tells us how often C^C appears in the multiplication of elements of C^A with those of C^B.

2.3.3 Some energetics of defects

In a global theory with spontaneous symmetry breaking, the line defects have a free energy per unit length outside the core which diverges logarithmically with the size of the system. In a gauge theory, the energy per unit length outside the core is finite. We will look at examples that show how this comes about.

The energy of a defect has a contribution from the core, where the symmetry is higher and therefore the system is "hotter". To calculate it one needs an expression for the free energy that is valid at the core. The free energies one writes down are usually not valid at the core, since they are explicitly written for a phase with the broken symmetry. However one can often make good estimates of the core energy [15]. For our purposes, we need only remember that is makes a finite contribution to the total energy.

Outside the core, we need an expression for the free energy to calculate the energy. We will now consider an explicit example to illustrate the energetic behaviour mentioned above.

We consider the free energy of a two-dimensional space with a complex order parameter field $\phi = |\phi| e^{i\theta}$:

$$F = \frac{1}{2} \int d^2x (\mu (\nabla_i \phi)^* (\nabla_i \phi) - \lambda (T - T_c)(|\phi|^2 - v^2)^2), \tag{2.51}$$

where v and μ are real numbers, and λ is a positive real number. It has the following global symmetry:

$$\phi \mapsto \phi e^{i\theta} \tag{2.52}$$

with $\theta \in \mathbb{R}$. μ and λ depend on coupling constants. The second term is sometimes referred to as the 'Mexican hat' potential. If $T < T_c$, it is energetically

---

[17] $C_g \times C_h$ is obtained by multiplying all elements of C_g with all elements of C_h.

favorable for ϕ to acquire a vacuum expectation value $<\phi>= ve^{i\theta}$, and the system enters the ordered phase. By using a global symmetry transformation we can set θ to zero, such that $<\phi>= v$. Excitations of $|\phi|$ away from v are massive excitations, and at low temperatures we can set $|\phi| = v$ in the Lagrangian, obtaining an effective free energy F_{eff}:

$$F_{eff} = \frac{1}{2}\int d^2x\mu v^2(\nabla\theta)^2 = \frac{1}{2}\int d^2x\rho_s(\nabla\theta)^2 \tag{2.53}$$

We defined the spin-wave stiffness $\rho_s = \mu v^2$. It dictates how rigid the phase is. (2.53) is the celebrated *XY model* in two dimensions. It plays a pivotal role in the theory of superfluidity, where ϕ is the wavefunction of the system. It also models ferromagnets in two dimensions. Finally, we will see that hexatic liquid crystals are also described by this effective free energy.

In this phase, $G = SO(2)$, $H = \{e\}$, and we have line defects characterized by an integer:

$$\Pi_1(G/H) = \mathbb{Z}.$$

In this theory, the charge of the defect is called its *winding number*, because for a defect with charge n, loop around the defect winds around the order parameter space S^1 n times (see fig.2.6). The defects in this theory are called *vortices*.

We would like to find configurations of the fields that minimize the free energy, and correspond to the defects of different winding number. Minimizing (2.53) yields the equation

$$-\rho_s\nabla^2\theta = 0, \tag{2.54}$$

which is Laplace's equation in two dimensions. Aside from the constant solution $\theta(x, y) = \theta_0$, it also has singular solutions:

$$\theta_{n,\vec{x}_0}(x, y) = n\ arctan(\frac{y - y_0}{x - x_0}) = n\varphi, \tag{2.55}$$

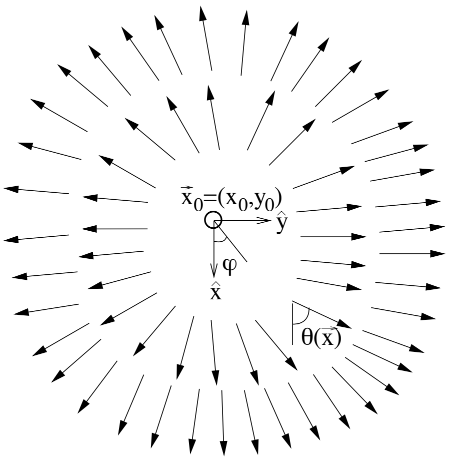
where n is an integer, $vecx_0 = (x_0, y_0)$ is a two-dimensional vector, and φ is the polar angle of the vector $(x - x_0, y - y_0)$.

$\theta_{n,\vec{x}_0}$ is the field of a defect of charge n whose core is at $\vec{x_0}$. We can calculate the free energy of this defect using (2.53). We impose a short and a long distance cutoff of the integral, a and R respectively, and get

$$F_{el}(n) = \frac{1}{2}\rho_s\int d^2x(\nabla\theta)^2 = \frac{1}{2}\rho_s 2\pi n^2\int_a^R\frac{dr}{r} = \rho_s\pi n^2 ln(R/a) \tag{2.56}$$

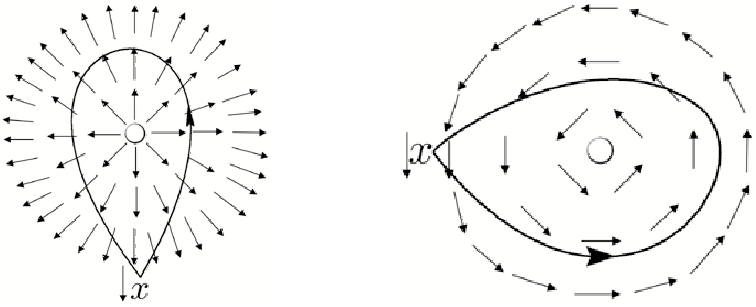
The free energy is infrared (long distances) and ultraviolet (short distances) divergent. We needn't worry about the ultraviolet divergence, since this is the core region, where the effective free energy is invalid. We can obtain a regular solution at the core if we demand that the symmetry is restored, meaning that $|\phi| \mapsto 0$ as we go to the center of the core.

The infrared divergence signals that the defect has an infinite energy if the system has infinite extent. This divergence is traced back to the fact that the



(a)

A charge 1 defect.

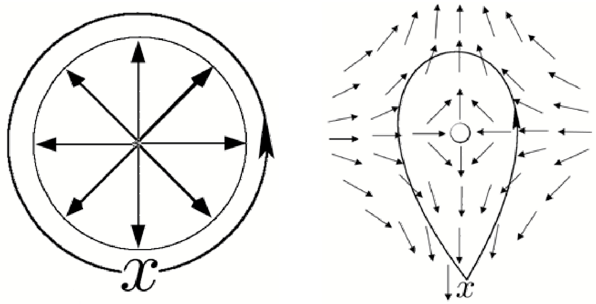


(b)

The loop around a charge 1 defect

(c)

Another charge 1 defect.



(d)

The order parameter space $SO(2)$, and the loop around the charge 1 defects. x is the basepoint, where all loops must start and end.

(e)

A charge -1 defect.

Figure 2.6: Examples of defects in the XY model. We denote the angle θ by a unit vector that makes an angle θ with the x-axis. A loop drawn around the defect gives a closed loop in order parameter space, with winding number equal to the charge of the defect.

spins have a non zero gradient out to infinity. However, a defect-antidefect pair has finite energy, because far away from the pair the gradient goes to zero. More generally, if there are many vortices present of different winding numbers n_i, we will show that as long as $\sum_i n_i = 0$ the energy is finite. In jargon, a vortex gas (a finite density of defects) in this system must be neutral to have finite energy. Therefore the only thermally excited configuration will have zero net vorticity. To prove this, first note that if $\theta_{n_i,\vec{x}_i}$ are solutions to Laplace's equation, then so is $\sum_i \theta_{n_i,\vec{x}_i}$, since Laplace's equation is linear. Define $\vec{r}_{ij} = (x_j - x_i, y_j - y_i)$, the vector from vortex i to vortex j. The elastic free energy of the configuration $\sum_i \theta_{n_i,\vec{x}_i}$ with multiple vortices is

$$F_{vortices} = \pi\rho_s ln(\frac{R}{a})(\sum_i n_i)^2 + 2\pi\rho_s \sum_{i<j} n_i n_j ln(\frac{a}{\vec{r}_{ij}}) \tag{2.57}$$

Therefore the energy is logarithmically divergent with the sample size, unless $\sum_i n_i = 0$. The second term is the interaction energy. From it we learn that vortices with vorticity of like sign repel, while they attract if their vorticities have opposite sign. The defects interact via a Coulomb potential[18] in two dimensions, thus they behave like electrically charged particles in two dimensions.

In three dimensions, the point defects become line defects. The free energy formulas we've just derived correspond to the free energy per unit length of a line defect.

Now take the original free energy before symmetry breaking, and promote it to a gauge theory (for example, by coupling the field to electrodynamics):

$$F_{gauge} = \frac{1}{2}\int d^2x(\mu(D_j\phi)^*(D^j\phi) + \lambda(|\phi|^2 - v^2)^2) \tag{2.59}$$

The covariant derivative D is defined by $D_j\phi = (\partial_j + iA_j)\phi$ $(j = x, y)$, and[19] $\vec{A} \in \mathbb{R}^2$. The free energy has the following local symmetry:

$$\phi(\vec{x}) \mapsto \phi(\vec{x})e^{i\alpha(\vec{x})}, \quad A_j(\vec{x}) \mapsto A_j(\vec{x}) + \partial_j\alpha(\vec{x}) \tag{2.60}$$

where $\alpha : \mathbb{R}^2 \mapsto U(1)$.
In this theory, there are finite energy configurations that correspond to defects of any winding number. Go to polar coordinates (r, φ), and choose the following values for the fields outside the core region[20]:

$$\phi(r,\varphi) = ve^{in\varphi} \tag{2.61}$$

$$A_r = 0 \tag{2.62}$$

$$A_\varphi = \frac{n}{r} \tag{2.63}$$

---

[18] A Coulomb potential V in d dimensions is a potential that satisfies Poisson's equation:

$$\nabla^2 V(\vec{x}) = -\lambda\rho(\vec{x}) \tag{2.58}$$

where λ is a constant, and $\rho(\vec{x})$ is the local charge density.

[19] F_{gauge} is not Lorentz invariant. This is due to the fact that we are considering the system at finite temperatures. The relativistic version of this Lagrangian is called the planar Abelian Higgs model[5] (see also [58]).

[20] We've neglected radial dependence. If we wanted a solution that is also valid at the core,

The free energy vanishes everywhere with this configuration! The energy of this defect is concentrated in the core region. Note that the field ϕ is locally flat! By this we mean that it satisfies the parallel transport equation of the gauge field:

$$D_j\phi \equiv 0. \tag{2.64}$$

This local flatness is the reason that the free energy vanishes everywhere. To explicitly see that ϕ is flat, we can choose a gauge where ϕ is constant. Apply the gauge transformation

$$\phi \mapsto \phi e^{-in\varphi},\ \vec{A} \mapsto \vec{A} + n\nabla\varphi. \tag{2.65}$$

This gives

$$\phi = v,\ A_r = 0,\ A_\varphi + n\nabla\varphi. \tag{2.66}$$

ϕ is now constant, but note that $\nabla\varphi$ is not well defined everywhere. If we let φ run from 0 to 2π, there is a singularity in $\nabla\varphi$ at 2π. To resolve this, we need to realize that we can only define our gauge transformation *locally.* To be mathematically precise, we needs to work with charts, and realize that we are dealing with a nontrivial line bundle. But it isn't necessary to get too technical. There is a physically intuitive way of dealing with this singularity. Just introduce a cut in our space along the $\varphi = 0$ line. This is called a *Dirac string.* It starts at the core and goes out to infinity. The Dirac string should be thought of as a reminder that we have a nontrivial line bundle. A more physical way to picture the Dirac string is to say that we've used a gauge transformation to concentrate the windings of the Higgs field into a thin tube.

If we consider the analogous theory in 3 spatial dimensions, then there will be line defects, with a finite energy per unit length. The energy equations for the vortex in two dimensions correspond to the energy per unit length of the line defects in the three-dimensional theory.

In this discussion of defects in gauge theory, we have glossed over one important point: is the topological charge of the defects a gauge invariant quantity? We will prove it for our abelian case. Go back to the original gauge choice (2.61). If we draw any closed loop C around a defect of winding number n, the integral of the gauge field along C gives $\oint_C \vec{A}\cdot d\vec{l} = 2\pi n$. Thus we can also define the winding to be

$$n = \frac{1}{2\pi}\oint_C \vec{A}\cdot d\vec{l} \tag{2.67}$$

---

we would write

$$\begin{aligned} \phi(r,\varphi) &= f(r)e^{in\varphi} \\ A_r &= 0 \\ A_\varphi &= \frac{n}{r}g(r) \end{aligned}$$

with $f(0) = 0, f(\infty) = v$, and $g(\infty) = 1$. Then we would use the field equations to set $f(r)$ and $g(r)$. We won't go into this. This is done in [52]. We also haven't included an energy term for the gauge field, which in relativistic notation is of the form $F_{\mu\nu}F^{\mu\nu}$. One can check it is also finite[58].

Now apply the gauge transformation $\phi(\vec{x}) \mapsto \phi(\vec{x})e^{i\lambda(\vec{x})}$, $A_j \mapsto A_j - \nabla\lambda$. Then this integral is invariant, because $\oint_C \nabla\lambda \cdot \vec{dl} = 0$, since λ is a regular function. So the integral is gauge invariant, and so is the winding number. Thus we can study the defects with homotopy theory, just like in the global case.
For general G and H, and defects of general dimensions, the topological charge is also related to an integral of the gauge field[58], although for the non-abelian case we must define the integral differently. For example, the charge g of a line defect is equal to the exponential of the *path ordered integral* of the gauge field along a closed path C encircling the defect:

$$g = Pe^{i\oint \vec{A}\cdot dx}. \tag{2.68}$$

When the gauge group is abelian path ordering is trivial.

If H is discrete, then, as we discussed earlier, the only residual symmetry transformations are global symmetry transformations $h \in H$. Under this transformation, the gauge field transforms according to $\vec{A} \mapsto h\vec{A}h^{-1}$. Using (2.68) and the properties of the path-ordered integral, we can determine the transformation of the charge g under the symmetry transformation h:

$$g = Pe^{i\oint \vec{A}\cdot dx} \mapsto Pe^{i\oint h\vec{A}h^{-1}\cdot dx} = hPe^{i\oint \vec{A}\cdot dx}h^{-1} = hgh^{-1}. \tag{2.69}$$

This is precisely the transformation property in the global case. Thus when H is discrete the topological classification of defects is the same in the case of global and local symmetries. When H is continuous the comparison of both cases is more subtle.

2.4 Dyonic configurations

In this section we assume G is simply connected.

If there is a defect g in our space, corresponding to a configuration of ϕ, it is interesting to study another field Ψ with this defect in the background. Ψ transforms under a representation $\Gamma_\Psi$of G, i.e.

$$\Psi^i(\vec{x}) \mapsto \Gamma_\Psi(g)^i_j\Psi^j(\vec{x}) \tag{2.70}$$

for any $g \in G$. We want to answer the following question: which symmetry transformations in H can we *globally* implement on Ψ with the defect g in the background? A globally implemented transformation must leave the defect configuration invariant, and act only on Ψ. If H is abelian, then all of H is always globally implementable. If H is non-abelian, then we will see that in some cases there are *topological obstructions* to globally implementing certain transformations.

2.4.1 Global implementation of $h \in H$

Outside the core of the defect g the order parameter field $\phi(\vec{x})$ takes values in G/H. The stabilizer $H_{\vec{x}}$ of $\phi(\vec{x})$ depends on position. Therefore to globally implement a transformation $h \in H$, we must make it dependent on position. To $h \in H$ we want to associate a smooth function $h_{\vec{x}}$ from the space to G, such $h_{\vec{x}} \in H_{\vec{x}}$ for all x (since then, and only then, does the transformation leave the defect configuration invariant). This choice should respect the multiplication in H, i.e. if $h, h' \in H$ then

$$h_{\vec{x}} h'_{\vec{x}} = (hh')_{\vec{x}}. \tag{2.71}$$

We can then let h act on $\Psi(\vec{x})$ according to

$$\psi(\vec{x}) \mapsto \Gamma(h_{\vec{x}})\psi(\vec{x}). \tag{2.72}$$

However not all of H can be made to satisfy (2.71).
To see which elements of H *can* be globally implemented, we draw a loop around the defect, and parametrize it with an angle φ (see fig. 2.4.1).

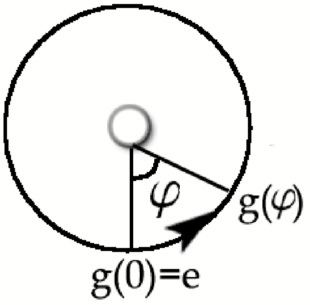


Figure 2.7: The path in G characterizing the line defect. φ parametrizes the loop. We take $g(0) = e$, so that $g(2\pi)$ is in the left coset in H/H_0 that corresponds to the topological charge of the defect.

Along the loop we have a path in G that characterizes the defect. The path starts at $g(0) = e$ and ends at $g(2\pi)$, with $g(2\pi) \in H$. Then *the only globally implementable transformations are those $h \in H$ that commute with $g(2\pi)$, i.e.*

$$hg(2\pi)h^{-1} = g(2\pi) \tag{2.73}$$

To see why, note that the order parameter field around the defect is defined by acting with $g(\varphi)$ on $\phi(0) \equiv \phi$:

$$\phi(\varphi) = g(\varphi)\phi. \tag{2.74}$$

This configuration is smooth because the path $g(\varphi)$ is smooth, and $g(2\pi) \in H$ so that $g(2\pi)\phi = \phi$. Therefore there is no singularity at $\varphi = 0$.
Now the stabilizer of $\phi(\varphi)$ is

$$H_\varphi = g(\varphi)Hg(\varphi)^{-1} \tag{2.75}$$

with $H_0 = H$ the stabilizer of ϕ. This tells us how to globally implement an $h \in H$. Define

$$h(\varphi) = g(\varphi) h g(\varphi)^{-1} \tag{2.76}$$

With this definition, $h(\varphi) \in H_\varphi$ for all φ, so that when $h(\varphi)$ acts on the space it leaves the defect invariant. However, this may be ill defined at $\varphi = 0 \sim 2\pi$. For the action to be well defined at $\varphi = 0$ we must demand that

$$g(2\pi) h g(2\pi)^{-1} = h, \tag{2.77}$$

thus h is only globally implementable if it commutes with $g(2\pi)$.

Thus we can only globally implement the *centralizer* of $g(2\pi)$. The centralizer of a defect of charge g is denoted by N_g.

We note that a similar analysis applies to defects of other dimensions, such as monopoles. The analysis is more subtle in that case, since we must globally implement h on all of S^2 [8].

2.4.2 A dyon

If Ψ transforms nontrivially under the action of N_g, we call the configuration a *dyon*.

We mention two subtleties. First, there may exist $h \in H$ that do not commute with $g(2\pi)$, but that *can* be globally implemented on Ψ if Ψ *doesn't distinguish between the action of* h *and the action of* $g(2\pi)hg(2\pi)^{-1}$, i.e.

$$h \cdot \phi = g(2\pi) h g(2\pi)^{-1} \cdot \phi. \tag{2.78}$$

Second, $g(2\pi)$ is not well defined if $H_0 \neq 0$. Namely, we saw that for any $h_0 \in H_0$, $g(2\pi)h_0$ represents the same defect. But the centralizers of $g(2\pi)$ and $g(2\pi)h_0$ needn't be equal, and they needn't even be isomorphic [3]! Thus, to study dyons, $\Pi_1(G/H)$ is too coarse a description, and we need to specify the actual endpoint of the loop. However, for all cases we will be studying the centralizers are isomorphic for different choices of endpoint.

For example, if H is discrete there is no ambiguity. $\Pi_1(G/H) = H$ in this case, and for $h \in H$, N_g is simply the centralizer of h in H.

Finally, it may be unnecessary to include another field Ψ to get a dyonic configuration. In the case of the uniaxial nematic, we will see that the defect configuration already dyonic, because the field configuration transforms nontrivially under the centralizer of the defect.

There are dyons in gauge theory as well, and the analysis is the same. The dyon is then an electrically and magnetically charged configuration of the gauge fields. The theory behind electric modes with a defect in the background is well established, for an introduction see [55].

2.5 Phase transitions and correlation functions

We can use the free energy to study phase transitions to an ordered state, and the fluctuations around the ordered state.

2.5.1 Order of a phase transition

If the order parameter jumps discontinuously at the phase transition, we say the transition is *first order*. If it changes continuously, the phase transition is *second order*.

2.5.2 Landau theory of phase transitions

Landau's theory is a *mean field approach* to the study of phase transitions. We neglect fluctuations of the fields ϕ^i, so that the free energy (2.10) becomes

$$F[\phi] = \int d^d x V[\phi]. \tag{2.79}$$

Landau developed general arguments to write down the possible terms in this free energy. He then used it to predict the *order* of the phase transitions. We will only mention the salient features of Landau's theory. For a detailed treatment, see [37].

First we assume that the ϕ^i transform under an irreducible representation (irrep) of G. If the representation is reducible, we decompose it into irreps, and consider the irreps separately. All the terms in $V[\phi]$ must be invariant under G. Thus we must form scalar invariants using combinations of the ϕ^i. There cannot be a first order invariant[21]. The second order term $M_{ij}\phi^i\phi^j$, where M_{ij} is the mass matrix. It is positive definite in the isotropic phase, i.e. its eigenvalues are positive, to assure that $\phi_0 = 0$ is a minimum of the potential. Thus we can diagonalize the second order term, by defining $\phi^{'i}$'s as linear combinations of ϕ^j's such that

$$M_{ij}\phi^i\phi^j = \phi' i\phi'_i \equiv \lambda^2. \tag{2.80}$$

There is only one second order invariant, which we've called λ^2.

If we work to fourth order, we must also determine all third order and fourth order invariants. We write them in terms of rescaled fields

$$\tilde{\phi} = \lambda\phi^{'}. \tag{2.81}$$

We index the third order invariants with an index α, and the fourth order invariants with an index β. The Landau free energy then takes the general form

$$F[\phi] = \lambda^2 A(P,T) + \lambda^3 \sum_\alpha C_\alpha(P,T) f_\alpha^3(\tilde{\phi}) + \lambda^4 \sum_\beta B_\beta(P,T) f_\beta^4(\tilde{\phi}). \tag{2.82}$$

---

[21] A first order invariant would imply the representation is reducible (it contains the trivial representation in its decomposition). We're assuming the representation is not trivial.

We now change the pressure and/or temperature, and we have a phase transition when there is a new *global* minimum of the free energy away from $\phi = 0$. A global minimum is a minimum of the overal function, not simply a local minimum. By studying this expression for the free energy as a function of λ, we can derive the following behaviour at the phase transitions:

- If there are no third order invariants, the phase transition is of second order. The ground state leaves 0 continuously.
- If there are third order invariants the transition is of first order. The ground state jumps from 0 to a nonzero value.

2.5.3 Correlation functions and order

We want to study fluctuations from the ground states. We assume the departures are isothermal. In that case, when the field configuration goes from 0 to ϕ, the work done on the system is equal to the change in free energy. This work done is proportional to the change in entropy (see [37], p.62):

$$\Delta F[\phi] = W[\phi] = -T\Delta S[\phi] \tag{2.83}$$

where T is the temperature of the system. From this we derive the behaviour of the *weight function* $w(\phi) \sim e^{S[\phi]/k}$, which determines the relative probability of the system to be in state ϕ, as a function of the free energy:

$$w(\phi) \sim e^{-\Delta F[\phi]/kT} \tag{2.84}$$

To check these statements, see any standard book on statistical mechanics such as [37].

Thus we can study the fluctuations of our order parameter fields around the ground state, using the free energy. The *correlation function* between ϕ^i at point $\vec{x_1}$ and ϕ^j at $\vec{x_2}$ is defined as

$$< \phi^i(\vec{x_1})\phi^j(\vec{x_2}) >= \frac{\int D\phi\ \phi^i(\vec{x_1})\phi^j(\vec{x_2})\ w(\phi)}{\int D\phi\ w(\phi)} = \frac{\int D\phi\ \phi^i(\vec{x_1})\phi^{(}\vec{x_2})\ e^{-\Delta F[\phi]/kT}}{\int D\phi\ e^{-F[\phi]/kT}} \tag{2.85}$$

The integrals are *path integrals*, which are integrals over all configurations of the fields ϕ. To make this well defined, we can give our system finite extent and periodic boundary conditions, and Fourier transform to momentum space. Then the field $\phi(\vec{x})$ becomes characterized by a countable set $\phi_{\vec{k}_i}$, and $\int D\phi$ becomes $\int\ \Pi_i d\vec{k_i}$, which is a countable number of integrals.

If $F[\phi]$ only has terms quadratic in the ϕ^i, i.e. it is of the form

$$F[\phi] = \frac{1}{2}\int d\vec{x_1}d\vec{x_2}\phi^i(\vec{x_1})G^{-1}_{ij}(\vec{x_1},\vec{x_2})\phi^j(\vec{x_2}) \tag{2.86}$$

then we can calculate the correlation functions exactly:

$$< \phi^i(\vec{x_1})\phi^j(\vec{x_2}) >= kTG^{ij}(\vec{x_1},\vec{x_2}), \tag{2.87}$$

where $G^{ij}(\vec{x_1}, \vec{x_2})$ is defined by

$$\int d\vec{x_3} G^{-1}_{ij}(\vec{x_1}, \vec{x_3}) G_{ij}(\vec{x_3}, \vec{x_2}) = \delta(\vec{x_1} - \vec{x_2}) \tag{2.88}$$

If $F[\phi]$ has terms of higher order in ϕ, then we can only calculate correlations perturbatively, using Feynman diagrams. We will restrict ourselves to quadratic order.

The behaviour of the correlation function at large separations of $\vec{x_1}$ and $\vec{x_2}$ is used to characterize the system:

- *Long Range Order* (LRO)

  $$< \phi^i(\vec{x_1}) \phi^j(\vec{x_2}) > \mapsto \text{Constant} \quad \text{as } |\vec{x_2} - \vec{x_1}| \mapsto \infty. \tag{2.89}$$

- *Short Range Order* (SRO)

  $$< \phi^i(\vec{x_1}) \phi^j(\vec{x_2}) > \mapsto \frac{e^{-\frac{|\vec{x_2} - \vec{x_1}|}{\xi}}}{|\vec{x_2} - \vec{x_1}|^\eta} \quad \text{as } |\vec{x_2} - \vec{x_1}| \mapsto \infty, \tag{2.90}$$

  where ξ and η are positive constants.

- *Quasi Long Range Order* (QLRO)

  $$< \phi^i(\vec{x_1}) \phi^j(\vec{x_2}) > \mapsto \frac{1}{|\vec{x_2} - \vec{x_1}|^\eta} \quad \text{as } |\vec{x_2} - \vec{x_1}| \mapsto \infty, \tag{2.91}$$

  where η is a positive constant.

We can use the correlation functions to study the fluctuations in an ordered phase. Let us carry this out for the XY model in d dimensions. The free energy given by (2.51), with d^2x replaced by d^dx. We want to calculate $< |\phi|^2 >$ in the ordered phase, were $< |\phi| >= v$. This cannot be done exactly, however if we set $|\phi| = v$ and neglect fluctuations in $|\phi|$, we can use the effective free energy, (2.53) with d^2x replaced by d^dx:

$$F_{eff} = \frac{1}{2} \int d^dx \rho_s (\nabla\theta)^2.$$

By going to Fourier space and transforming back, the correlation function of θ is calculated to be [22]

$$< \theta(\vec{x}) \theta(\vec{0}) >= \frac{kT}{\rho_s} \int_{|k|<\Lambda} d^dk \frac{1}{k^2} e^{i\vec{k}\cdot\vec{x}} \tag{2.93}$$

---

[22] This equation actually needs to be regularized to be well defined. We won't go into details, but note that the behaviour $< \theta(\vec{x})\theta(0) > \sim |vecx|^{2-d}$ as $|\vec{x}| \mapsto \infty$ can be inferred by looking at the behaviour of the correlation function when $\vec{x} \mapsto \alpha\vec{x}$:

$$< \theta(\alpha\vec{x})\theta(0) > \sim \alpha^{2-d} < \theta(\vec{x}\theta(0) > \tag{2.92}$$

.

We introduced a cutoff Λ of the momenta: this integral is not valid at very high momenta. We need to know the behaviour of this integral as $k \mapsto 0$. By noting that the volume element in momentum space $d^dk \sim k^{d-1}dk$, we infer the following behaviour as $|\vec{x}| \mapsto \infty$:

- $d < 2$: the integral behaves like $\frac{kT}{\rho_s}|\vec{x}|^{2-d}$.
- $d = 2$: the integral behaves like $\frac{kT}{\rho_s}ln(|\vec{x}|)$.
- $d > 2$: the integral converges, we call it $2W$.

From this integral we can derive $< \phi(\vec{x})^* \phi(\vec{0}) >$:

$$< \phi(\vec{x})^* \phi(\vec{0}) >=< e^{i\theta(\vec{x})} e^{-i\theta(\vec{0})} >= e^{-\frac{1}{2}<\theta(\vec{x})\theta(\vec{0})>}. \tag{2.94}$$

In proving the second equal sign, we use a standard Gaussian integral[23]. Thus

- $d < 2$: $< \phi(\vec{x})^* \phi(\vec{0}) >\sim e^{-\frac{kT}{2\rho_s}|\vec{x}|^{2-d}}$.
- $d = 2$: $< \phi(\vec{x})^* \phi(\vec{0}) >\sim |\vec{x}|^{-\frac{kT}{2\rho_s}}$.
- $d > 2$: $< \phi(\vec{x})^* \phi(\vec{0}) >\sim e^{-W}$

For $d > 2$ the fluctuations do not destroy the Long-Range Order. For $d = 2$, the ordered phase has Quasi-Long Range Order: the fluctuations destroy the Long-Range Order! This is a special case of the *Mermin-Wagner theorem*[24][43], which states that if a continuous abelian symmetry is broken in two dimensions, there can never be long-range order. However, if we consider a small enough region of our system, then the system is effectively ordered. Namely, the order only decays algebraically. For $d < 2$, the fluctuations destroy the order completely, and only Short-Range Order is left over.

The lowest dimension d_c, such that LRO is maintained for $d > d_c$, is called the *lower critical dimension*. Thus $d_c = 2$ for the XY model.

For the XY model in $d = 2$ we saw that $< \phi(\vec{x})^* \phi(\vec{0}) >$ does not exhibit LRO. There is another field that *does* exhibit LRO in $d = 2$, namely $\nabla\theta(\vec{x})$:

$$< \nabla\theta(\vec{x})\nabla\theta(\vec{0}) >= \frac{kT}{\rho_s}\int_{|k|<\Lambda} d^dk e^{i\vec{k}\cdot\vec{x}} \sim 2W. \tag{2.97}$$

---

[23] If our weight function is

$$w(x_1, \ldots, x_n) = A\int dx_1 \ldots dx_n exp(-\frac{1}{2}G^{-1}_{ij}x_i x_j) \tag{2.95}$$

where A is a normalization factor, then

$$< exp(a^i x_i) >= exp(\frac{1}{2}a^i a^j < x_i x_j >). \tag{2.96}$$

[24] Sometimes the theorem is also attributed to Coleman, Berezinskii and/or Hohenberg.

One of the applications of the XY model is to superfluidity, where it is the simplest effective energy. When applied to superfluidity, $\phi(\vec{x})$ is the wavefunction of the system, and $\theta(\vec{x})$ is the phase angle. In the application to superfluidity, $\nabla\theta(\vec{x})$ is directly related to the *superfluid velocity* v_s:

$$v_s = \frac{\hbar}{m}\nabla\theta, \tag{2.98}$$

where $m = \frac{\rho_s}{v^2}$ is the 'mass'. This follows from the momentum operator in quantum mechanics: $\hat{p} = \frac{\hbar}{i}\nabla$. Applied on the wavefunction $\phi = ve^{i\theta}$ (where we set the amplitude to the constant the value v), we get $\hat{p}\phi = \hbar(\nabla\theta)\phi$. Since the velocity is obtained by dividing the momentum by the mass, the superfluid velocity is then the eigenvalue of the operator $\frac{\hat{p}}{m}$ when applied to ϕ, which is (2.98).

According to (2.97), the superfluid velocity exhibits LRO.

Chapter 3

Ordinary and liquid crystals

3.1 Introduction

Liquid crystals are phases with a symmetry between that of a liquid and a crystal. A veritable zoology of liquid crystals have been discovered, the most famous one being the nematic crystal used in flat screens. In biology liquid crystals are quite common, the cell membrane is an ubiquitous example.

In high school we all learn that there are three phases of matter: solid, liquid and gas. Now we know that there are many phases in between solids and liquids, called liquid crystals. They share properties of both phases.

In the previous chapter we developed the formalism used to study the fluctuations in such systems. We introduced order parameter fields, and found that Goldstone modes and topological defects are excitations of the order parameter fields. We also discussed the possible existence of other fields and dyons.

Liquid crystals exhibit a wide variety of orders, and we will study their symmetry properties, order parameter fields, and modes. First we give a clear definition of liquid crystals in terms of symmetry.

3.1.1 Definition in terms of symmetry

A *liquid* is a phase with unbroken external symmetry and internal symmetry. The external symmetry group is $O(d) \rtimes \mathbb{R}^d$, the Euclidean group in d dimensions, including reflections. At this point we needn't consider an internal symmetry: Either the molecules are spherically symmetric, or their orientations are random. Note that this doesn't distinguish a liquid from a gas.

A *crystal* is a phase in which the external symmetry is a discrete group. There are no infinitesimal rotations of translations in the symmetry group. The atoms that make up the crystal may have internal structure, such as spin. We will neglect the internal symmetries of the crystal for the moment.

Discrete external symmetry groups are called *space groups.* These have been completely classified: there 230 different space groups.

A *liquid crystal* is a phase with less symmetries than a liquid, and more symmetries than a crystal. More specifically, its external symmetry group is not discrete, but it is smaller than $O(3) \rtimes \mathbb{R}^3$. The internal symmetry group may or may not be broken.

This definition is subtle, because some liquid crystals have *coupled* external and internal symmetries, i.e. the phase is only invariant under simultaneous internal and external symmetry transformations. This will become clear in examples to come.

The symmetries of a phase can be determined experimentally using scattering experiments (with neutrons, or x-rays for example).

3.2 Liquid crystals

We now turn our attention to the most famous liquid crystals: nematics, smectics and columnar phases. A good reference book on liquid crystals is the book by de Gennes[18].

3.2.1 Nematic liquid crystals

Symmetry

A nematic liquid crystal is a phase with complete translational symmetry, and incomplete rotational symmetry. The phase then inherits the name of the residual rotational group: If the residual rotational group is the tetrahedral group, for example, the phase is called a tetrahedral nematic.

The residual internal rotational symmetry group H can be any proper subgroup of $G = SO(3)$. The possibilities are (see Appendix A)

$$G = SO(3) \mapsto H \in \{C_n, D_n, T, O, I, SO(2) \rtimes \mathbb{Z}_2\} \tag{3.1}$$

where we use the notation employed in the crystallography literature[14]. $C_n = \mathbb{Z}_n$, the abelian cyclic group of order n. D_n is the dihedral group of order n, T is the tetrahedral group, O the octahedral group, and I the icosahedral group. The most famous example is the uniaxial nematic, see fig.3.1. One possible realization of a uniaxial nematic is a fluid in which the molecules have a rod-like structure. The symmetry of one rod pointing along an axis $\vec{n}$ is denoted by $SO(2) \rtimes \mathbb{Z}_2$. This group contains rotations around $\vec{n}$, and rotations over π around any axis perpendicular to $\vec{n}$.

However, the phase is not invariant under purely *internal* rotations. Rotating the rods without rotating the system need not give a configuration of same energy. In the isotropic phase, the rotational symmetry group is $(SO(3)_{ext} \rtimes \mathbb{R}^3) \times SO(3)_{int}$, consisting of translations $\mathbb{R}^3$, an external rotational group $SO(3)_{ext}$ which rotates the space, and an internal rotational group which rotates the rods:

$$\begin{aligned} R_{ext} \in SO(3)_{ext} &: \vec{x} \mapsto R_{ext}\vec{x} \\ R_{int} \in SO(3)_{int} &: \vec{n} \mapsto R_{int}\vec{n} \end{aligned} \tag{3.2}$$

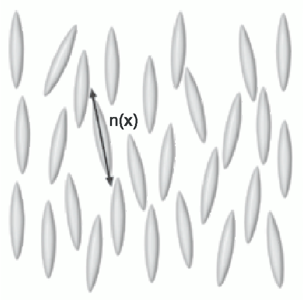


Figure 3.1: Schematic representation of a uniaxial nematic liquid crystal.

At low temperatures, the translational part is unbroken, while the external and internal rotational symmetries are coupled. The phase is only invariant under transformations such that the internal and external rotations are the same. In other words, the residual rotational group is the diagonal subgroup of $SO(3)_{ext} \times SO(3)_{int}$, i.e. all elements of the form $\{R(\hat{l},\theta), R(\hat{l},\theta)\}$, where $R(\hat{l},\theta)$ is a clockwise rotation of angle θ around the axis of the unit vector $\hat{l}$. We denote the diagonal subgroup as

$$SO(3)^{(\hat{l},\theta)}_{ext} \times SO(3)^{(\hat{l},\theta)}_{int}. \tag{3.3}$$

We can also describe the diagonal subgroup in terms of generators. Denote as L_x, L_y and L_z the generators of external rotations in $SO(3)_{ext}$ around the x-, y- and z-axes respectively. We say that $SO(3)_{ext}$ is generated by the vector $\vec{L} = (L_x, L_y, L_z)$, and write $SO(3)_L = SO(3)_{ext}$. Denote as T_x, T_y and T_z the generators of internal rotations in $SO(3)_{int}$ around the x-, y- and z-axes respectively. $SO(3)_{int}$ is generated by $\vec{T} = (T_x, T_y, t_z)$, and we write $SO(3)_T = SO(3)_{int}$. Then the diagonal subgroup is generated by $\vec{J} = \vec{L} + \vec{T} = (L_x + T_x, L_y + T_y, L_z + T_z)$, thus $SO(3)_J$ is the diagonal subgroup.

Then the residual symmetry of the uniaxial nematic is

$$H = \mathbb{R}^3 \rtimes SO(3)_J = (\mathbb{R}^3 \rtimes SO(3)^{(\hat{l},\theta)}_{ext}) \times SO(3)^{(\hat{l},\theta)}_{int}. \tag{3.4}$$

A similar statement holds for any nematic.

The locally gauged version of the uniaxial nematic is called *Alice electrodynamics*. For analysis of Alice electrodynamics see [64][63].

Order parameter space and field

To describe the order parameter, we must treat each nematic separately.
For the uniaxial nematic, we can describe the orientation of the rod as a bidirectional arrow $\overleftrightarrow{n}$, or a vector $\vec{n}$ with the equivalence $\vec{n} \sim -\vec{n}$. The order parameter space $\mathcal{O}$ is therefore S^2, the surface of a sphere, with antipodal points identified. This manifold is called $\mathbb{RP}^2$, the projective sphere. We cannot use a vector as an

order parameter field, since it does not invariant under $\vec{v} \mapsto -\vec{v}$. We must use a rank two tensor : take a symmetric traceless rank-two tensor, which transforms in the five dimensional irreducible representation of SO(3). A symmetric matrix is always diagonalizable and it has three orthonormal eigenvectors $(\hat{n}, \hat{l}, \hat{m})$, so we can write

$$Q_{ij} = \frac{2}{3} S n_i n_j + (\eta - \frac{1}{3} S) l_i l_j + (-\eta - \frac{1}{3} S) m_i m_j \tag{3.5}$$

where S and η are numbers. This parametrization is redundant: we need only specify $\hat{n}, \hat{l}, S$ and η to set Q_{ij}.

In the case of the uniaxial nematic, $\eta = 0$. Using $l_i l_j + m_i m_j = \delta_{ij} - n_i n_j$ (which is true for any triple of orthonormal vectors), we get

$$Q_{ij} = S(n_i n_j - \frac{1}{3}\delta_{ij}) \tag{3.6}$$

If, on the other hand, $\eta \neq 0$, we say the nematic is *biaxial.* The rotational symmetry will then be D_2, the point group of a rectangle.

If we want to describe another nematic liquid crystal, such as a tetrahedral nematic, then we need to use higher order rank tensors. We need an order parameter that is only invariant under the rotational group H of the nematic we are considering. In terms of representation theory, that means we seek a vector in a vector space on which a representation of SO(3) acts, whose stabilizer is H. This has to be worked out case by case (for a smart method, see [49]). For a tetrahedral nematic, we need a third order tensor. Take the third tensor power of the $l = 1$ (vector) representation of $SO(3)$, and consider the vector

$$T_{ijk} = \hat{x}_i \hat{y}_j \hat{z}_k + \hat{x}_i \hat{z}_j \hat{y}_k + \hat{y}_i \hat{x}_j \hat{z}_k + \hat{y}_i \hat{z}_j \hat{x}_k + \hat{z}_i \hat{x}_j \hat{y}_k + \hat{z}_i \hat{y}_j \hat{x}_k, \tag{3.7}$$

where $\hat{x}, \hat{y}, \hat{z}$ are orthonormal vectors in $\mathbb{R}^3$. T_{ijk} is only invariant under the tetrahedral group, to be precise the symmetry transformations of the tetrahedron given in fig.3.2.

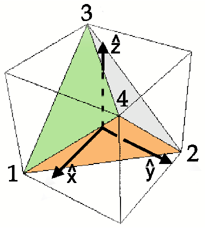


Figure 3.2: The tetrahedron with the axes discussed in the text.

Using fig.3.2, we can use a convenient notation of the symmetry transformations of the tetrahedron. Namely, $T \simeq A_4$, the even permutations of the four vertices. So we write the symmetries as permutations. For example, (12)(34) corresponds to a π rotation around the z-axis, and (12) corresponds to a mirror symmetry, with respect to the plane that contains the z-axis and passes through the vertices 3 and 4. Note that a similar notation, in terms of permutations, can be used for all the discrete subgroups[1] of $SO(3)$.

Until now, we had neglected mirror symmetries. The reason is that we haven't specified if the constituent molecules are *chiral*, i.e. different from their mirror image. If our tetrahedron isn't chiral, then its symmetry group is $T_d \simeq S_4$, the group of permutations of the four vertices. T_d is a subgroup of $O(3)$, so we must take $G = O(3)$ as the symmetry in the isotropic phase to describe chirality. If all the phases we are considering are achiral, then we can work with $SO(3)$. Note that (3.7) is invariant under all of T_d.

T_d includes symmetry transformations that are not connected to the identity. Later on we will study the effect of such *large* symmetry transformations, as they are called.

Experimentally, a wide variety of different nematics have been discovered. Not necessarily as a phase with constituents with such a symmetry, but also as models that capture the essential physics. For example, icosahedral nematics have been invoked to describe supercooled liquid prior to the glass transition [66]. The idea is that as the fluid is cooled down, locally the molecules want to form an icosahedral structure (because they pack closer together this way). Since they cannot maintain this icosahedral structure macroscopically, frustration occurs, and the phase remains liquid below the freezing point. It is therefore translationally invariant, but locally the rotational symmetry is icosahedral. This is a prime example of a nematic. There are also quantum systems with nematic symmetries, which we discuss later.

Free energy and phase transitions

Let us form the free energy for a uniaxial nematic[2]. The order parameter field is $Q_{ij}(\vec{x})$. To fourth order, the potential term is

$$V[Q] = B_1 Tr(Q^2) - B_2 Tr(Q^3) + B_3 Tr(Q^4) \tag{3.8}$$

These three terms are the only invariants one can build out of Q_{ij} to this order. Other invariants are not independent of these ones: for example[3], $TrQ^4 = \frac{1}{2}(TrQ^2)^2$, and $Tr(Q^3) = 3detQ$.

To form the elastic free energy density, one must form invariants under simultaneous rotations of the space and the orientation of the molecules. The possible terms are

$$F_{kin} = A_1 \nabla^i Q_{jk} \nabla^i Q_{jk} + A_2 \nabla^i Q_{ij} \nabla^k Q_{jk} + A_3 \nabla^i Q_{jk} \nabla^k Q_{ij} + A_4 \nabla^i \nabla^j Q_{ij} \tag{3.9}$$

---

[1] It can in fact be used for any finite group, since all finite groups are a subgroup of S_n for some n. We prove this in AppendixA.

[2] The locally gauged version of the uniaxial nematic is called "Alice electrodynamics". For an analysis see [64] and [63].

[3] These equations are easily checked in a basis where Q is diagonal.

If we plug this expression into (3.6), we get

$$F_{el} = K_1(\nabla.\vec{n})^2 + K_2(\vec{n}.\nabla\times\vec{n})^2 + K_3(\vec{n}\times(\nabla\times\vec{n})^2 + K_4\nabla\cdot((\vec{n}\cdot\nabla)\vec{n} - \vec{n}(\nabla\cdot\vec{n})). \tag{3.10}$$

This is called the *Frank free energy density*, and K_i are called the *Frank constants*. The four terms are called the *splay*, *twist*, *bend* and *saddle-splay curvature* terms, respectively. The saddle-splay term is a total derivative term, and is only important if one considers surface effects. We will set K_4 to zero.
A common simplification in the literature is to take $K_1 = K_2 = K_3 \equiv K$, known as the *one constant approximation*. In this approximation the elastic free energy density is called the *field-theory free energy density* f_{ft}, and is equal to

$$F_{ft} = K\nabla_i n_j \nabla^i n^j \tag{3.11}$$

In this approximation, the external and internal rotational groups decouple, and in the ordered phase the external rotational group is unbroken, while the internal rotational group is $SO(2) \rtimes \mathbb{Z}_2$. This is a common situation in field theory, where only the internal symmetry group is broken.
According to Landau's theory of phase transitions, due to the third-order term in (3.8) the phase transition from the isotropic to the uniaxial nematic phase is first order. Experiments reveal that it is only *weakly* first order, meaning that B_2 is small.
For other nematics, we have to form a free energy using whichever order parameter fields are necessary to describe the order. for example, for a tetrahedral nematic, using the order parameter field T_{ijk} we get

$$\begin{aligned} F \quad = \quad & A_1\nabla_i T_{jkl}\nabla^i T^{jkl} + A_2\nabla_i T^i{}_{jk}\nabla_l T^{ljk} + A_3 T_{ijk}T^{ijk} \\ & + A_4(T_{ijk}T^{ijk})^2 + A_5 T^{ijk}T_{ilm}T_j{}^{ln}T_k{}^{mn} \end{aligned} \tag{3.12}$$

There are no other terms because T_{ijk} is completely symmetric with respect to the interchange of its indices. The analog to the one-constant approximation in this case would be setting $A_2 = 0$, because the A_2 is the only term that couples internal and external rotations. With $A_2 = 0$ only the internal rotational group is broken.
To see the details on the theory behind Landau phase transitions between the different possible nematics, see [39].
Since there are no third order terms, Landau's theory predicts a second order phase transition. However, it has been proven in [54] that there is a runaway of the coupling constant A_4 in front of the fourth order term $(T^{ijk}T^{ijk})^2$ towards negative values, signaling a first order phase transition. Note that the Landau theory is still precious, as it is the basis for the Renormalization Group calculation.

Modes

There are 7 conventional modes in the uniaxial nematic: 5 continuity modes just as in a simple fluid, and 2 Goldstone modes.

In the low temperature phase, the rods are oriented on average along an axis, which we choose to be the z-axis. This leads to the existence of two Goldstone modes, δn_x and δn_y (see fig.3.3). They transform under a two-dimensional irreducible representation of the residual symmetry group $SO(2) \rtimes \mathbb{Z}_2$. However, the 2 transverse momentum modes couple to the Goldstone modes, because an excitation of a Goldstone mode leads to a transverse momentum current, and therefore the Goldstone modes imply the presence of transverse momentum diffusion. The dispersion relations are given in [15] (they are approximate). There are two "slow" (low frequency) and two "fast" modes (high frequency). We reproduce the dispersion relations for one slow, and one fast mode:

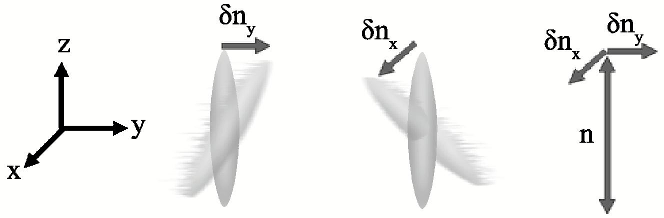


Figure 3.3: The Goldstone modes in a uniaxial nematic liquid crystal. These low-energy excitations are extrapolations into neighboring vacua. The Goldstone modes δn_x and δn_y span a two-dimensional vector space. Note that although $\vec{n}$ is a headless vector (with a symmetry $\vec{n} \mapsto -\vec{n}$), the Goldstone modes δn_x and δn_y aren't.

$$i\omega_s (K_2 k_x^2 + K_3 k_z^2)^{-1} = C_1 + C_2 \frac{k_z^2}{C_3 k_z^2 + C_4 k_x^2} \tag{3.13}$$

$$i\omega_f = \frac{C_3 k_z^2 + C_4 k_x^2}{\rho} \tag{3.14}$$

where C_i are constants related to the viscosity tensor of the phase. The frequency is purely imaginary, so these modes are overdamped, i.e. they do not propagate. The slow mode dies out on a longer timescale than the fast mode.

For other nematics, there are 5 continuity modes, and 3 Goldstone modes, since all 3 generators of $SO(3)$ are broken. The analysis of the couplings between continuity and Goldstone modes would be complicated, but analogous to the uniaxial case.

Defects

The defect structure is much more intricate for uniaxial nematics than for other nematics. For nematics other than the uniaxial nematic, H is discrete. Thus there are no monopoles, and line defects are characterized by $\overline{H}$, the double cover of H in $SU(2)$.

Now we consider the uniaxial nematic, for which H is continuous.

1 - The homotopy groups of the uniaxial nematic

To study the defects, we need the double cover $\overline{H}$ of $SO(2) \rtimes \mathbb{Z}_2$ in $SU(2)$. If the uniaxial nematics are on average ordered along the z-axis, then

$$\overline{H} = \{e^{i\frac{\theta}{2}\sigma_z}, i\sigma_x e^{i\frac{\theta}{2}\sigma_z}, \theta \in [0, 4\pi)\} \tag{3.15}$$

where $\sigma_x, \sigma_y, \sigma_z$ are the Pauli matrices (see appendix A). $\overline{H}$ consists of a component $\overline{H}_0$ that is connected to the identity, and a component $\overline{H}_d$ that is disconnected from the identity. They subsets of $\overline{H}$ consist of the following elements:

$$\begin{aligned} \overline{H}_0 &= \{e^{i\frac{\theta}{2}\sigma_z}, \theta \in [0, 4\pi)\} \\ \overline{H}_d &= \{i\sigma_x e^{i\frac{\theta}{2}\sigma_z}, \theta \in [0, 4\pi)\}. \end{aligned}$$

The homotopy groups of the uniaxial nematic are

$$\Pi_1(G/H) = \Pi_0(\overline{H}) = \mathbb{Z}_2 \tag{3.16}$$

$$\Pi_1(G/H) = \Pi_1(\overline{H}_0) = \mathbb{Z} \tag{3.17}$$

Thus there is only one nontrivial line defect, a disclination. The monopoles are labelled by an integer.

3 - The disclination of the uniaxial nematic

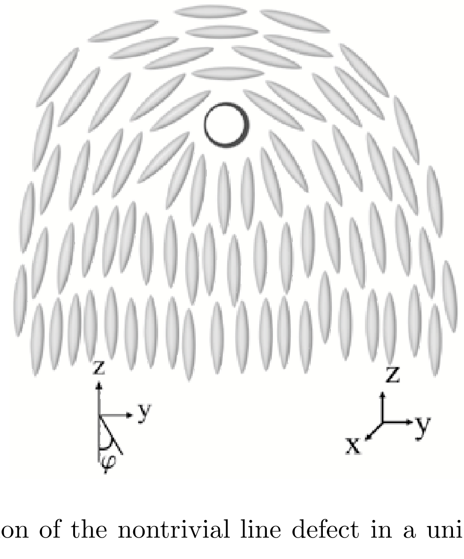


Figure 3.4: A section of the nontrivial line defect in a uniaxial nematic liquid crystal.

An explicit realisation of the line defect, with the core oriented along the x-axis, is

$$\vec{n}(\vec{x}) = (0, -sin(\frac{1}{2}\varphi), cos(\frac{1}{2}\varphi))$$

with φ the angle defined in fig.3.4. The kinetic free energy of this defect is easily calculated in the one constant approximation. Namely, in polar coordinates we get

$$\begin{aligned} \nabla n_x &= 0 \\ \nabla n_y &= -\frac{1}{2r}cos(\frac{1}{2}\varphi)\hat{\varphi} \\ \nabla n_y &= -\frac{1}{2r}sin(\frac{1}{2}\varphi)\hat{\varphi} \end{aligned}$$

The free energy density becomes $\frac{K}{4r^2}$. We can calculate the energy per unit length in the x-direction, which is the energy per unit length of the defect:

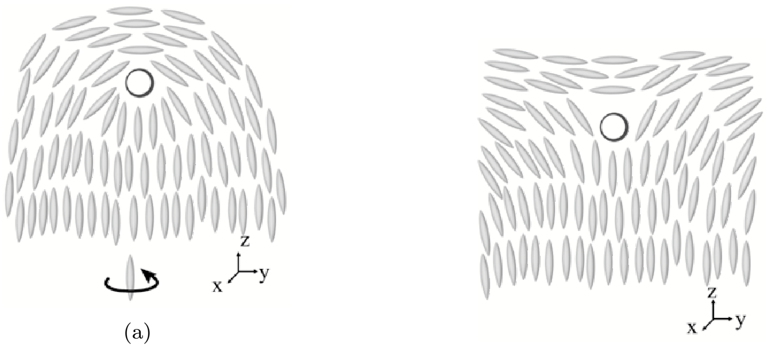
$$E_{unitlength} = \int_a^R rdrd\varphi\frac{K}{4r^2} = \frac{\pi}{2}Kln(\frac{R}{a}) \tag{3.18}$$

This result is analogous to the XY model which we discussed earlier: The integral of this energy is ultraviolet (short distances) and infrared (large distances) divergent. We needn't worry about the infrared divergence, since we can always place and antidefect at a certain distance, or give the system a finite extent. The ultraviolet divergence is a signal that our the Frank free energy description breaks down at the core. The integral is only valid outside the core of the defect. Inside the core, another form of the free energy is needed.

What is the symmetry at the core? The system could be isotropic at the core of the defect, but numerical investigations[62] have led to the conclusion that the core is biaxial. This biaxiality can be captured by the tensor order parameter description. The tools of homotopy theory discussed previously allow us to understand why the singularity can be removed.

The argument runs as follows: Now, the B_i are temperature dependent quantities, and they dictate the order of the phase transition. As B_1 goes from being positive to negative, a phase transition occurs. If $B_2 = 0$, the phase transition is of second order, otherwise it's of first order. The second term is present because there is no $Q \to -Q$ symmetry in a uniaxial nematic ($Q \to -Q$ does not correspond to $\vec{n} \to -\vec{n}$). Experiments reveal that the transition from the isotropic to the nematic phase is only weakly first order, meaning that B_2 is small relative to B_1 and B_3. So an approximation commonly used (which is only qualitatively valid close to the phase transition) is that TrQ^2 takes on the value that minimizes the first and third terms in the potential free energy. Using $TrQ^4 = \frac{1}{2}(TrQ^2)^2$, one gets $TrQ^2 = -\frac{B_1}{B_3}$. Now, in the five-dimensional space of traceless symmetric matrices, this is the equation of an S^4. And since $\Pi_1(S^4) = 0$, any loop can be shrunk. So if we start with the nontrivial defect in the uniaxial nematic, the singularity at the core can be removed by assuming biaxial symmetry at the core.

There is an important observation to make about the uniaxial nematic: if we consider the configuration in fig. 3.4, and rotate all the rods around the z-axis by an angle θ, then we obtain a new configuration, which is topologically a disclination, with the same line defect charge. Thus for every $\theta \in 2\pi$ we have a configuration that corresponds to a disclination. In the one-constant approximation, all these configurations have the same energy. We can now imagine all the rods rotating around the z-axis, see 3.5. This costs no energy, so it is an excitation of zero energy around the defect configuration, just like a Goldstone mode! We have discovered a *zero mode* around the defect. A zero mode is the generic name given to an excitation that costs zero energy in the long wavelength, low frequency limit. The zero mode implies that the disclination is dyonic in the one-constant approximation. Namely, rotations around the z-axis correspond to transformations of the residual symmetry group. Thus, the defect configuration transforms nontrivially under the residual symmetry group, which is how we defined dyons. We will come back to this discussion at the end of chapter 4.



(a)

The non trivial defect in a uniaxial nematic. Imagine rotating the configuration around the z-axis. This gives a continuous family of configurations which correspond to the same defect. In the one-constant approximation these configurations all have the same energy.

(b)

The same defect after a π rotation around the z-axis. Rotations around the z-axis expose the zero mode: the rods can rotate around the z-axis, at low energy cost. This is a massless excitation.

Figure 3.5: Schematic explanation of the dyonic nature of the disclination in a uniaxial nematic.

2 - Monopoles in the uniaxial nematic

The monopoles are labelled by a charge $n \in \mathbb{Z}$, which corresponds to the winding number of the loop in $\overline{H}_0$ that characterizes the monopole, under the isomorphism (2.45). Note that $\overline{H}_0$ is topologically a circle, which is why homotopy classes of loops in $\overline{H}_0$ are characterized by a winding number.

In chapter 2 we discussed the possibility of having theories where the charge of a monopole changes when it is transported around a line defect. In the uniaxial nematic, there is only one nontrivial line defect, a disclination. When a monopole of charge n is transported around the disclination, it returns with the opposite charge $-n$. This is not too difficult to prove. Namely, a charge n monopole is characterized by a loop $f_n : S^1 \to \overline{H}_0$ of winding number n in $\overline{H}_0$:

$$f_n(\theta) = e^{in\theta\sigma_z} \tag{3.19}$$

where θ is the angle that parametrizes the points on the circle S^1. In chapter 2 we saw that a line defect is characterized by a path in G that ends in a disconnected component of $\overline{H}$. Thus in this case the path ends in a point $i\sigma_x e^{i\frac{\theta}{2}\sigma_z}$ in $\overline{H}_d$. We also showed that the outcome of transporting a monopole around a line defect is the conjugation of the loop in $\overline{H}_0$ characterizing the monopole, by the endpoint of the path characterizing the line defect. This, in this case, if we transport a monopole of charge n, characterized by $f_n(\theta)$ given (3.19), around a disclination whose endpoint is $i\sigma_x e^{i\frac{\theta}{2}\sigma_z}$, the outcome is a monopole with loop $g(\theta)$ given by

$$g(\theta) = i\sigma_x e^{i\frac{\theta}{2}\sigma_z} f_n(\theta)(i\sigma_x e^{i\frac{\theta}{2}\sigma_z})^{-1} = f_{-n}(\theta). \tag{3.20}$$

Note that the outcome is independent of the endpoint of the disclination. Thus a monopole with charge n becomes a monopole with charge $-n$ when transported around a disclination. This has remarkable consequences. For example, if a monopole of even charge $2n$ decays into two monopoles of charge n (which is topologically possible), and one of the monopole is transported around a disclination, then it returns with charge $-n$. Thus we obtain a monopole of charge n and an antimonopole of charge $-n$, and there is no topological obstruction to their decay. Determining whether these processes actually takes place in a system requires a dynamical analysis. For a dynamical analysis in the gauge theoretical equivalent of the uniaxial nematic, called "Alice Electrodynamics", see [64].

If we leave the one-constant approximation, the topological analysis is the same, but the energetic analysis becomes very subtle. Namely, one has to explicitly solve the equations that minimize the free energy to find out if a certain configuration is stable. This has been done extensively for uniaxial nematics. It has been shown[16], for example, that the flux of the monopoles is not isotropically distributed around the monopole. Instead, it concentrates itself into two flux tubes.

We make one last comment on monopoles. The disclination has a core which is a line. This line can form a closed loop, and it turns out that this loop can carry monopole charge! To measure this monopole charge, we draw a surface that encompasses the loop. We will briefly explain why the loop can carry monopole charge. For a more detailed discussion see [13].

Under the isomorphism $\Pi_1(G/H) \simeq \overline{H}/\overline{H}_0$, every point of the core of a line defect can be associated to an element $h(t)$ in $\overline{H}$, where t parametrizes the core. As we move along the core, the element $h(t)$ may change, as long as it corresponds to the same left coset in $\overline{H}/\overline{H}_0$, since that doesn't affect the charge in $\Pi_1(G/H)$. The elements associated to the points on the core are given by $h(0)h'(t)$, with $h'(t) \in \overline{H}_0$. Thus if we consider a core that is a closed loop, $h'(t)$ will traced out a closed loop in $\overline{H}_0$ as we move around the loop. In the case of the uniaxial nematic, $\overline{H}_0$ is topologicall a circle, so that the homotopy classes of loops in $\overline{H}_0$ are characterized by a winding number. It turns out that this winding number is precisely the monopole charge of the line defect [13].

One of the implication of the possibility of closed line defect loops to carry monopole charge is that there is no topological obstruction to the decay of a monopole into a closed line defect loop [63]. Another implication is that a monopole can travel through a closed loop of line defect, and its charge will change sign. This follows from the analysis above of the transport of monopoles around a line defect.

Dyons

When we studied dyons in the previous chapter, we stated that in the presence of a defect characterized by a path $\tilde{f}$ in G the only elements of $\overline{H}$ that can be globally implemented are those that commute with $\tilde{f}(2\pi)$. For nematics other than the uniaxial one, the endpoint of the path is unique, so the centralizer is well defined.

For the uniaxial nematic, the endpoint of the path that characterizes the disclination is not unique. It can be any element of the form $i\sigma_x e^{i\frac{\theta}{2}\sigma_z}, \theta \in [0, 4\pi)$. The centralizer of such an element is $\{\pm 1, \pm i\sigma_x e^{i\frac{\theta}{2}\sigma_z}\}$ (see (3.15)). Therefore the centralizers for different θ are isomorphic to D_2. We could therefore have a field Ψ with this defect in the background, that transforms nontrivially under D_2, which gives a dyon.

Fluctuations and order

A nematic liquid has translational short-range order, and orientational long-range order. The orientational long-range order survives fluctuations. This is obvious for a uniaxial nematic in the one-constant approximation, where the free energy is that of an XY model in three dimensions. In the previous chapter we saw that in such a model long-range order survives fluctuations.

Cholesteric elastomers

Cholesteric elastomers are a special type of nematic liquid crystals, and we briefly describe them here.

The cholerestic phase is very similar to the uniaxial nematic. The molecules are rodlike so that they are characterized by a bidirectional arrow $\vec{n} \sim -\vec{n}$. In every plane perpendicular to the x-axis, the rods have the same average orientation. However, as we go along the x-axis the average orientation of the rods twists around the x-axis:

$$\vec{n} = \hat{y} cosqx + \hat{z} sinqx. \tag{3.21}$$

q is called the *pitch* of the cholesteric, which determines the period along the x-axis. If we set q to 0, we get a uniaxial nematic, so that *a uniaxial nematic is a cholesteric with infinite pitch.* This twist is achieved by introducing q into the Frank free energy (3.10) , in the following way:

$$F_{kin} = K_1(\nabla.\vec{n})^2+K_2(\vec{n}.\nabla\times\vec{n}+q)^2+K_3(\vec{n}\times(\nabla\times\vec{n})^2+K_4\nabla.((\vec{n}.\nabla)\vec{n}-\vec{n}(\nabla.\vec{n}))$$

Minimizing the K_2 term, i.e. setting

$$\vec{n}.\nabla \times \vec{n} + q = 0, \tag{3.22}$$

we find that (3.21) minimizes the K_2 term.

The symmetries of the phase consist of translations a distance a along the x-axis coupled to rotations of angle qa around the x-axis, and of π rotations are the x-, y- and z- axes. The coupled rotations and translations are connected to the identity, i.e. they sit in H_0. Thus $\Pi_1(G/H) = \overline{HH}_0$ is equal to the double cover of the group spanned by the π rotations, which are not connected to the identity. This group is D_2, so that $\Pi_1(G/H) = \overline{D_2} \simeq Q$, the quaternion group. There are no monopoles, since there are no nontrivial loops in $\overline{H}_0$.

Since locally the phase looks like a uniaxial nematic, the structure of the continuity and modes is similar.

Experimentally this phase can be obtained by adding chiral molecules to a uniaxial nematic.

3.2.2 Smectics

Symmetry

A smectic liquid crystal is obtained by breaking translational symmetry in a nematic liquid crystal, in one direction. If a nematic is cooled down, the molecules may form layers. Within a layer the system is a fluid. Note that the layer need not have the thickness of one molecule as we in suggest fig. 3.6. In some cases its thickness is that of two molecules.

There are different kinds of smectics, corresponding to different orientations of the rods within the layers, which changes their symmeties properties. We will discuss the smectic A, A* and C phases, because these are the most common smectics. At the end of this chapter we will also briefly mention the smectic B phase.

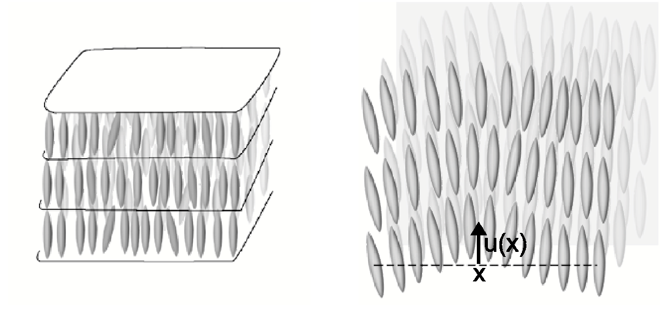
We call the axis normal to the layers the z-axis.

In a smectic A, the long axis of the molecules points along the z-axis.

The forming of the layers breaks the symmetry to a group that couples internal and external rotations. The residual symmetry group is

$$H = ((\mathbb{R}^2 \times \mathbb{Z}) \rtimes SO(3)^{(\hat{l},\theta)}_{ext}) \times SO(3)^{(\hat{l},\theta)}_{int}. \tag{3.23}$$

The only difference with the symmetry group of the uniaxial nematic is that translational symmetry in the z-direction is broken to $\mathbb{Z}$.



(a) The ground state of the smectic A phase.

(b) The Goldstone mode of the smectic A phase.

Figure 3.6: Schematic representation of a Smectic A liquid crystal. The rods are confined to planes, and in microscopic times they orient themselves perpendicularly to the planes. The only mode left is transverse oscillation of the planes, which is captured by the field $u(\vec{x})$.

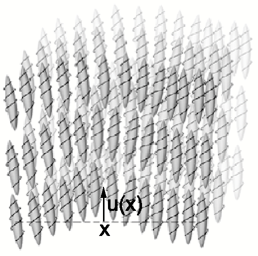


Figure 3.7: The smectic A* phase. The spirals on the molecules indicate the phase is chiral.

A smectic A* is a smectic A in which the molecules are chiral, i.e. they cannot be superimposed on their mirror image. The chirality affects the optical activity of the system. It is known that if a molecule is going to be optically active (i.e. rotate the plane of polarization of an incoming monochromatic light beam), it must be chiral because of symmetry arguments. Namely, if the molecule rotates the polarization plane in a specific direction, and the molecule is achiral, this violates the mirror symmetry.
The star is a general way of denoting a chiral phase, and applies to any phase. For example, a nematic* phase is a nematic made of chiral molecules.
To pictorially indicate a molecule is chiral, we will draw a spiral on the molecules. The only difference between the symmetry groups of the smectic A and A* phases is that reflections are broken in the A* phase. We had neglected reflections in the smectic A, but they were actually present.

In the smectic C phase, the rods do not point towards the normal to the layers.

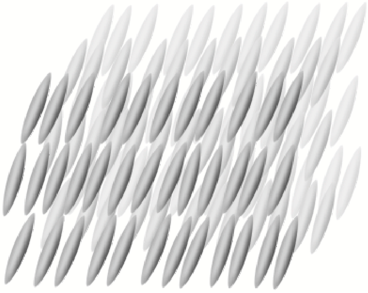


Figure 3.8: The smectic C phase.

Instead, their average direction is in a direction that is tilted with respect to the normal. The residual symmetry is $(\mathbb{R}^2 \times \mathbb{Z}) \rtimes \mathbb{Z}_2$, where $\mathbb{Z}_2$ corresponds to a π rotation around a vector in the x-y plane perpendicular to the rods, the $\mathbb{R}^2$ symmetries are translations within a plane, and $\mathbb{Z}$ translates perpendicularly to the layers.

Order parameter

To find the order parameter of a smectic, consider the Fourier transform of the density operator:

$$\rho(\vec{x}) = \int \frac{d^3q}{(2\pi)^3} \rho_{\vec{k}} e^{i\vec{k}\cdot\vec{x}}$$

In a nematic, we have $< \rho_{\vec{k}} >= \delta_{\vec{k},\vec{0}} \rho_{\vec{0}}$, where " $<>$ " denotes a thermal average. This simply means that the ground state has a constant density, which is an infinite wavelength density wave. In the smectic phase, a density wave with finite wavelength sets in, such that $< \rho_{\vec{k_0}} > \neq 0$, for some vector $\vec{k_0}$.

$\Psi \equiv \rho_{\vec{k_0}} = |\rho_{\vec{k_0}}|e^{-ik_0u}$ is the complex order parameter of the smectic. Increasing u uniformly corresponds to a translation along the z-axis. Thus u is actually a vector that points along the z-axis.

Modes

The particle content is interesting: one might expect three Goldstone modes, since a nematic crystal has two, and the breaking of translational symmetry should give a third mode. However, there is only one mode! This is the mode associated with the breaking of translational symmetry, see fig.3.6. What happened to the other two?

The figures we have provided are very suggestive. Namely, the rods are always perpendicular to the layer, and it costs a lot of energy to rotate away from the normal. If they slightly deflect from the normal, they decay back to it in microscopic time. In the language of field theory, the two modes corresponding to such an excitation (excitation in x- and in y-direction) are massive. Two Goldstone modes have acquired mass, via the Higgs mechanism!

This requires clarification, because the Higgs mechanism is a very specific mechanism, which requires a gauge theory, and the associated covariant derivative. In smectics, there is also a covariant derivative. A dynamically generated gauge field will indeed appear when we construct the smectic free energy F_s. The order parameter Ψ is now allowed to vary in space:

$$\Psi(\vec{x}) = \rho_{\vec{k_0}}(\vec{x})e^{-ik_0u(\vec{x})}.$$

Define $\nabla_\perp \equiv (\nabla_x, \nabla_y, 0)$, the derivative parallel to the undeformed planes; and $\nabla_{//} \equiv (0, 0, \nabla_z)$, the derivative perpendicular to the planes. If we assume that $\vec{n}$ does not fluctuate, then Landau theory tells us that the only terms in the free energy are now

$$\frac{1}{2}\int d^3x(r|\Psi|^2 + c_{//}|\nabla_{//}\Psi|^2 + c_\perp|\nabla_\perp\Psi|^2 + \frac{1}{2}g|\Psi|^4)$$

$c_{//} \neq c_\perp$ because the presence of the rods sets up an anisotropic background in the smectic.

If we allow for fluctuations of $\vec{n}$, then to lowest order only the $c_\perp$ term is modified. Define $\delta\vec{n}_\perp \equiv \vec{n} - \hat{z}$. Since $n^2 = 1$, the variations of $\vec{n}$ in the z-direction is second order, and we neglect it.

The $c_\perp$ term becomes

$$c_\perp|(\nabla_\perp - ik_0\delta\vec{n}_\perp)\Psi|^2,$$

which is very similar to the covariant derivative term in the free energy of a superconductor:

$$\frac{1}{m}|(\hbar\nabla - ie\vec{A})\Psi|^2.$$

To arrive at this term, note that $\delta\vec{n}_\perp$ should enter the $c_\perp$ in such a way that it is invariant under rigid rotations of the whole system, which involves rotating the layers as well as the molecules. If we rotate the system by an angle θ around a unit vector $\hat{l}$ in the x-y plane, then $\delta_{n_\perp} = -\theta \hat{l} \times \hat{z}$, and $u(\vec{x}) = -\delta_{\vec{n}_\perp} \cdot \vec{x}$. Therefore Ψ changes according to:

$$\Psi \mapsto \Psi' = \Psi e^{ik_0 \delta_{n_\perp} \cdot \vec{x}},$$

and $(\nabla_\perp - ik_0 \delta\vec{n})\Psi$ is invariant under global rotations. The smectic free energy becomes

$$F_s = \frac{1}{2} \int d^3x (r|\Psi|^2 + c_{//}|\nabla_{//}\Psi|^2 + c_\perp |(\nabla_\perp - ik_0\delta\vec{n})\Psi|^2 + \frac{1}{2} g|\Psi|^4) \tag{3.24}$$

The action is almost identical to the action for superconductivity [65]:

$$F_{SC} = \frac{1}{2} \int d^3x (\alpha|\Psi|^2 + \frac{1}{2m}|(\frac{\hbar}{i}\nabla - \frac{e}{c}\vec{A})\Psi|^2 + \frac{1}{8\pi\mu}(\nabla \times \vec{A})^2 + \frac{\beta}{2}|\Psi|^4)$$

It is invariant under

$$\begin{cases} \delta\vec{n}_\perp \mapsto \delta\vec{n}_\perp - \nabla\lambda \\ \Psi \mapsto \Psi e^{ik_s \lambda} \end{cases}$$

with $\lambda : \mathbb{R}^3 \mapsto \mathbb{R}$. An important difference is that only one gauge is physical: $(\delta_{\vec{n}_\perp})_z = 0$.

Summarizing, the Goldstone modes of the nematic are massive in the ordered phase due to the Higgs mechanism. The variation in the director is no longer an independent variable, as it relaxes in microscopic time to an orientation normal to the layers. Just like in the nematic liquid crystals, the Goldstone mode is diffusive, because it couples to continuity modes. See [15] for the precise coupling. The dispersion relations are anisotropic: they depend on the orientation of the wave number of the modes with respect to the layers.

Topological defects

The topological line defects are dislocations characterized by an integer:

$$\Pi_1(ISO(3)/ISO(2) \times \mathbb{Z}) = \Pi_0(ISO(2) \times \mathbb{Z}) = \mathbb{Z}.$$

The topological charge should be interpreted as a vector pointing perpendicularly to the layers, whose length is an integer times the distance d between consecutive layers. This vector is called the *Burgers* vector $\vec{b}$ of the defect. For a defect of charge n the Burgers vector is

$$\vec{b} = nd\hat{z} \tag{3.25}$$

We've treated the smectic in the continuum theory, so that $u(\vec{x})$ is a field that measures the local displacement of the layers along the z-axis. We consider $u(\vec{x})$ to be a vector $\vec{u}(\vec{x})$. Then a configuration of the field that corresponds to a dislocation of with Burgers vector b must satisfy the following property; for any loop L around the core of the dislocation

$$\oint_L \vec{u} \cdot d\vec{x} = \vec{b}. \tag{3.26}$$

Thus the Burgers vector measures the displacement of an observer inside the smectic turning around the dislocation (see fig. **??**). While the observer "thinks" he's walked a close loop around the dislocation, he is actually displaced by $\vec{b}$. The Burgers vector can point in any direction with respect to the orientation of the singular line. If it is parallel to the singular line, it is called a *screw dislocation*, and if it is perpendicular we are dealing with an *edge dislocation.* In general, the Burgers vector can be decomposed into a parallel and a perpendicular part, and the defect is simultaneously screw and edge.

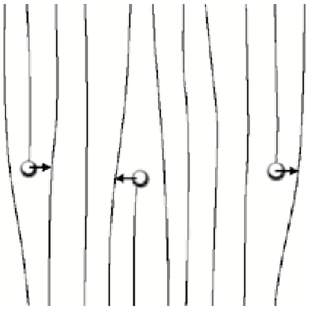


Figure 3.9: The cross section of a three dimensional smectic. There are three dislocations, and their Burgers vectors are drawn. This cross section is also a phase in itself, a two-dimensional smectic, also known as a stripe phase.

There are no monopoles, since $\Pi_2(ISO(3)/ISO(2)\times\mathbb{Z}) = \Pi_1(\mathbb{Z}_0) = 0$.

Fluctuations and order

There is only quasi-long range order due to the fluctuations of u. This is a general phenomenon for three-dimensional phases with periodic order in only one direction [37], and once again follows from convergence considerations of the fluctuation integrals.

3.2.3 Columnar phases

We can break translational symmetry in two directions, which corresponds to columnar phases. This phase is composed of lines along which the phase is liquid. These lines are placed on the vertices of a lattice. If we take a cross section of the liquid lines, we obtain a two dimensional crystal.

There is true long-range order, and there are 2 Goldstone modes, associated with the breaking of translational symmetry in two directions (The rotons are massive).

The topological defects are line defects, and no monopoles. The cross section gives a two-dimensional crystal with a certain homotopy group, which is equal to the homotopy group of the columnar phase. The defects are precisely the

Liquid crystal	Order parameter field	Symmetry	Goldstone modes	Defects
Uniaxial	Q_{ij}	$SO(2) \rtimes \mathbb{Z}_2$	2	$\mathbb{Z}_2$
Dihedral D_2	Q_{ij}	D_2	3	$\overline{D_2}$
Tetrahedral	T_{ijk}	T_d	3	$\overline{T}$
Octahedral	O_{ijkl}	O_h	3	$\overline{O}$
Icosahedral		I_h	3	$\overline{I}$
Cholesteric	$\overleftrightarrow{n}$	Complicated	2	$\overline{D_2}$
Smectic A	$\rho_{\vec{k_0}}$	$\mathbb{Z} \times ISO(2)$	1	$\mathbb{Z}$
Smectic A*	$\rho_{\vec{k_0}}$	$\mathbb{Z} \times ISO(2)$	1	$\mathbb{Z}$
Smectic C	$\rho_{\vec{k_0}}$	$\mathbb{Z} \times ISO(2)$	1	$\mathbb{Z}$
Smectic F	$\rho_{\vec{k_0}}$	$\mathbb{Z} \times ISO(2)$	1	$\mathbb{Z}$
Hexatic Smectic	$\{e^{i6\theta}, \rho_{\vec{k_0}}\}$	$\mathbb{Z} \times (Z_6 \rtimes \mathbb{R}^2)$	2	$\mathbb{Z} \times \mathbb{Z}_6$
Hexagonal	$e^{i6\theta}$	$(\mathbb{Z}_6 \rtimes \mathbb{Z}^2) \times \mathbb{R}$	3	$\mathbb{Z} \times \mathbb{Z}_6$

Table 3.1: The different liquid crystals, and their mode content. We omitted the continuity modes, since there are five of them in all these phases (though they may be coupled to Goldstone modes, see text). We also omitted the massive Goldstone modes in smectics.

dislocations and disclinations of this two dimensional crystal. The only difference with the defects of the two dimensional crystal is that the dislocations can have both screw and edge character.

3.2.4 Quantum Liquid Crystals

It has been theoretically predicted that Quantum Hall Systems at certain fractional filling levels behave like a one-dimensional array of Luttinger liquids [53][21]. This shares the properties of a two-dimensional smectic. The same happens in High-Tc superconductors[9]. These two-dimensional smectics are called stripe phases. We illustrate a 2D smectic in fig. 3.2.2.

3.3 The hexatic phase

The hexatic phase is a rather remarkable phase. It has unbroken translational symmetry but broken rotational symmetry. Its most important characteristic for us is that there are crystals that enter this phase *when defects condense.* Before we can explain this, we must review the theory behind crystals.

3.3.1 Crystals

Symmetry, order parameter field and free energy

A crystal is a phase in which the symmetry group $G = O(d) \rtimes \mathbb{R}^d$ has been spontaneously broken to a discrete group H, called the space group of the crystal.

The positions of the atoms in the crystal form a lattice. Severe restrictions on the possible symmetries arise from the observation that the crystal should look the same from every lattice point. For example, a crystal cannot have a fivefold symmetry axis. This forbids icosahedral symmetry, since an icosahedron has a fivefold symmetry axis. A tetrahedral crystal is also impossible in three dimensions, because we can't tile $\mathbb{R}^3$ with tetrahedra.
In the ground state, the i-th atom is on lattice point $\vec{x_i}$. At finite temperatures, its position fluctuates around $\vec{x_i}$. We denote its position by $\vec{x_i'}$, and its displacement from $\vec{x_i}$ by $u(\vec{x_i})$:

$$\vec{x_i'} = \vec{x_i} + \vec{u}(\vec{x_i}) \tag{3.27}$$

The order parameter is the *structure function*, defined by

$$S(\vec{q}) =< |\hat{\rho}(\vec{q})|^2 > \tag{3.28}$$

where $\hat{\rho}(\vec{q})$ is the Fourier transform of the density. It is equal to [15]

$$S(\vec{q}) = \sum_i e^{i\vec{q}\cdot\vec{x_i}} < e^{i\vec{q}\cdot(\vec{u}(x_i)-\vec{u}(0)}) > \tag{3.29}$$

To write down a free energy, we must take the lattice spacing to zero. $\vec{u}$ then becomes a field $\vec{u}(\vec{x})$, called the *displacement field*.
The free energy can only depend on derivatives of this field, since a uniform $\vec{u}$ is equivalent to a uniform translation, which costs no energy. For small deviations, the deformation energy density depends on the *strain tensor*

$$u_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}. \tag{3.30}$$

We build up the free energy density out of u_{ij}. The potential term will be of the form

$$V = u^{ij}\lambda_{ijkl}u^{kl} \tag{3.31}$$

The rank four tensor λ_{ijkl} is called the *elastic modulus tensor*. It is *invariant* under the action of the space group. For every space group we can determine the number of independent components of λ_{ijkl} by using its symmetry properties.

Continuity and Goldstone modes

Of the continuity modes in a simple fluid, only the heat mode is present in a crystal. Namely, momentum and mass cannot flow since the the atoms are restricted to oscillate around their equilibrium position. We will have to assume that heat flow happens *slowly*, i.e. the time scale of heat flow is long compared to the time scale of oscillations of the atoms. Thus we consider *adiabatic* excitations from the ground state. Under this assumption, we can actually write down a

kinetic term for the crystal: the kinetic energy of the i-th atom is $\frac{1}{2}m\dot{u_i}^2$, where the dot denotes a time derivative.
In the continuum limit mass becomes a mass density ρ. Therefore the *internal energy density* U of the crystal is

$$U = \frac{1}{2}\rho\dot{u_i}^2 + u^{ij}\lambda_{ijkl}u^{kl}. \tag{3.32}$$

Minimizing this equation, we obtain

$$\rho\ddot{u}_i = \lambda_{ijkl}\frac{\partial^2 u^l}{\partial x_j \partial x_k}. \tag{3.33}$$

To find the dispersion relation of the Goldstone modes, we substitute a plane wave

$$\vec{u} = \vec{u_0}e^{i(\vec{k}\cdot\vec{x}-\omega t)}, \tag{3.34}$$

to obtain

$$(\rho\omega^2\delta_{il} - \lambda_{ijkl}k^j k^k)u_l = 0. \tag{3.35}$$

For this equation to have solutions, the matrix acting on u_l must have zero determinant:

$$|\rho\omega^2\delta_{il} - \lambda_{ijkl}k^j k^k| = 0 \tag{3.36}$$

This equation in ω^2 of order d, where d is the dimension of the space. It has d solutions, which may be complex, and may be degenerate. These solutions are the dispersion relations of the d Goldstone modes.

We have obtained d Goldstone modes, but note that this does not equal the number of broken generators! Namely, the group $O(d) \rtimes \mathbb{R}^d$ as d generators of translations, and $\frac{1}{2}d(d-1)$ generators of rotations. We have obtained the translation Goldstone modes, so what happened to the rotational Goldstone modes? They don't appear because we didn't include rotations in the field $u_i(\vec{x})$. This field only has d components, so obviously it cannot have more than d independent excitations. If we wanted to include rotational Goldstone modes, called *rotons*, we would have to include a field that describes the local rotation of the crystal.

However, for crystals encountered in nature this is unnecessary, because the "rotons" turn out to be high-energy excitations. It costs relatively little energy to stretch the bonds between neighbouring atoms, while bending different parts of the crystal relative to each other is energetically very costly. Thus the rotons are massive.

The situation is analogous to the case of smectics. There is a term in the free energy that resembles a covariant derivative so that we effectively have a Higgs mechanism that gives mass to the rotons. Although it is analogous to the term for the smectic, it is more complicated, and involves 6 momenta, corresponding to 6 reciprocal lattice vectors. We will not discuss it here. It can be found in [46].

We make an important observation about our massive Goldstone modes: *the rotons have acquired mass because of translational symmetry breaking.* It is the rigidity of the crystal, associated with the breaking of translational symmetry, that makes it energetically costly to rotate different parts of the crystal relative to each other. After finishing the discussion of crystals, we will return to this point, when we discuss the *hexatic phase*.

Let us determine the Goldstone modes for a two-dimensional crystal with hexagonal symmetry. The symmetry is broken to $\mathbb{Z}^2 \rtimes \mathbb{Z}_6$. Symmetry under $\mathbb{Z}_6$ implies that λ_{ijkl} only has two independent components, and the free energy density becomes

$$f = \frac{1}{2}\lambda({u^i}_i)^2 + \mu u_{ij}u^{ij} \tag{3.37}$$

where λ and μ are constants. This is a special case, because we would have obtained the same free energy if we had demanded that λ be invariant under *any* rotation! Thus the elastic properties of a two-dimensional hexagonal crystal are equivalent to those of an *isotropic* crystal. An isotropic crystal may sound like a contradiction in terms, but it is perfectly possible that in the continuum limit the crystal has a free energy density of the form above.

Using (3.33), we obtain the equations that minimize the internal energy density:

$$\rho\ddot{u}_i = -\lambda\nabla_i\nabla_j u^j - \mu(\nabla^2 u_i + \nabla_i\nabla_j u^j) \tag{3.38}$$

To obtain the dispersion relations, we look for a plane wave solution $\vec{u}(\vec{x}) = \vec{u}e^{i(\vec{k}\cdot\vec{x}-wt)}$. The resulting equations simplify if we write $\vec{u} = \vec{u}_{//} + \vec{u}_\perp$, with $\vec{u}_{//}$ parallel to $\vec{k}$ ($\nabla \times \vec{u}_{//} = 0$), and $u_\perp$ perpendicular to $\vec{k}$ ($\nabla \cdot \vec{u}_\perp = 0$). The dispersion relations become

$$\vec{u}_{//} : \omega = \sqrt{\frac{\mu}{\rho}}k \tag{3.39}$$

$$\vec{u}_\perp : \omega = \sqrt{\frac{\lambda+2\mu}{\rho}}k \tag{3.40}$$

Thus we arrive at one longitudinal mode called the *compression mode*, with velocity $c_{//} = \sqrt{(\lambda+2\mu)/\rho}$; and one transversal Goldstone mode, with velocity $c_\perp = \sqrt{\mu/\rho}$. Note that the longitudinal mode is faster.

Topological defects

The homotopy groups are:

$$\Pi_1(G/H) = \overline{H} \tag{3.41}$$

$$\Pi_2(G/H) = 0 \tag{3.42}$$

In three dimensions, there are no monopoles because the space group H is discrete. The line defects are characterized by elements of the covering group of H.

Once again, we consider a two-dimensional hexagonal crystal. To determine $\overline{H}$ we need the covering group of $\mathbb{R}^2 \rtimes SO(2)$. The translational part $\mathbb{R}^2$ is already simply connected. The covering group of $SO(2)$ is $\mathbb{R}$. An element θ of $\mathbb{R}$ corresponds to a rotation over an angle θ. In the covering group, however, a rotation over 2π is not equivalent to the identity, like it is in $SO(2)$. The covering of Z_6 is

$$\overline{\mathbb{Z}_6} = \mathbb{Z} \times \mathbb{Z}_6 \simeq \mathbb{Z} \tag{3.43}$$

Therefore $\overline{\mathbb{Z}^2 \rtimes \mathbb{Z}_6} = \mathbb{Z}^2 \rtimes \mathbb{Z}$, and

$$\Pi_1(G/H) = \mathbb{Z}^2 \rtimes \mathbb{Z}. \tag{3.44}$$

Therefore a defect is an element of $\mathbb{Z}^2 \rtimes \mathbb{Z}$. We denote it by $(n,m)r^p$, with $n,m,p \in \mathbb{Z}$. r is a $\frac{2\pi}{6}$ rotation, (1,0) a translation by one lattice vector in the x-direction, and (0,1) a translation by one lattice vector in the y-direction. A defect of the form (n,m) (i.e. with $p=0$) is a dislocation. An r^p defect $(n=m=0)$ is a disclination. Now, we said that the rotational Goldstone modes were massive, because it cost a lot of energy to rotate different parts of the crystal relative to each other. For the same reason, *disclinations are massive*, and at low temperatures we only have dislocations.

For a three-dimensional crystal, the analysis is very similar. For example, in a crystal made up of layers of hexagonal crystal, $G = \mathbb{R}^3 \rtimes SO(3)$ is broken to $H = \mathbb{Z}^3 \rtimes Z_6$, so the defects have an extra label, which represents translational defects in the direction perpendicular to the layers:

$$\Pi_1(G/H) = \mathbb{Z}^3 \rtimes \mathbb{Z} \tag{3.45}$$

For a dislocation characterized by (n,m,k), the vector (n,m,k) is the Burgers vector of the defect. Just like in the smectic case, a general dislocation has both screw and edge character, and the Burgers vector is given by the integral of the field $\vec{u}(\vec{x})$ along a loop L encircling the core.

The fusion rules of line defects have some interesting features, which are crucial for the story to come. We illustrate these features for the hexagonal crystal. To determine the fusion rules, we need the conjugacy classes of the defects. This requires knowledge of the multiplication in $\mathbb{Z}^2 \rtimes \mathbb{Z}_6$, which is set by

$$r(a,b)r^{-1} = (-b, a+b). \tag{3.46}$$

Thus we can classify all the conjugacy classes, see Table 3.2. Now we can prove that two disclinations can fuse to form any dislocation in this crystal. Namely, the fusion rules are given by multiplying entire conjugacy classes. If we fuse r and r^{-1}, we can get any dislocation (m,n) because $(m,n)r$ is an element of C_r. Thus, from a topological point of view a dislocation can decay into two disclinations.

Fluctuations and orientational order

The structure function $S(\vec{q}) = \sum_i e^{i\vec{q}\cdot\vec{x_i}} < e^{i\vec{q}\cdot(\vec{u}(x_i)-\vec{u}(0))} >$ is our order parameter. Exactly analogous to the discussion of the XY model of the previous chapter, fluctuations in u_i affects the order of the phase:

Representative element	Conjugacy class
$r^{6k},\ k \in \mathbb{Z}$	$C_{r^{6k}} = \{r^{6k}\}$
$(a,b)r^{6k}$ $k \in \mathbb{Z},\ a,b \geq 0$	$C_{(a,b)r^{6k}} = \{(a,b)r^{6k}, (-b,a+b)r^{6k}, (-a-b,a)r^{6k},$ $(-a,-b)r^{6k}, (b,-a-b)r^{6k}, (a+b,-a)r^{6k}\}$
$r^{1+6k},\ k \in \mathbb{Z}$	$C_{r^{1+6k}} = \{(m,n)r^{1+6k} : (m,n) \in \mathbb{Z}^2\}$
$r^{-1+6k},\ k \in \mathbb{Z}$	$C_{r^{-1+6k}} = \{(m,n)r^{-1+6k} : (m,n) \in \mathbb{Z}^2\}$
$r^{2+6k},\ k \in \mathbb{Z}$	$C_{r^{2+6k}} = \{(m,n)r^{2+6k} : m-n \in 3\mathbb{Z}\}$
$(1,0)r^{2+6k},\ k \in \mathbb{Z}$	$C_{(1,0)r^{2+6k}} = \{(m,n)r^{2+6k} : m-n \notin 3\mathbb{Z}\}$
$r^{3+6k},\ k \in \mathbb{Z}$	$C_{r^{3+6k}} = \{(m,n)r^{3+6k} : (m,n) \in 2\mathbb{Z} \times 2\mathbb{Z}\}$
$r^{3+6k},\ k \in \mathbb{Z}$	$C_{(1,0)r^{3+6k}} = \{(m,n)r^{3+6k} : (m,n) \notin 2\mathbb{Z} \times 2\mathbb{Z}\}$
$r^{4+6k},\ k \in \mathbb{Z}$	$C_{r^{4+6k}} = \{(m,n)r^{4+6k} : m-n \in 3\mathbb{Z}\}$
$(1,0)r^{4+6k},\ k \in \mathbb{Z}$	$C_{(1,0)r^{4+6k}} = \{(m,n)r^{4+6k} : m-n \notin 3\mathbb{Z}\}$

Table 3.2: The defect classes in $\mathbb{Z}^2 \rtimes \mathbb{Z}_6$.

- If $d > 2$ the correlation of the fluctuations is finite, thus LRO survives.
- If $d = 2$ we have QLRO.
- If $d < 2$ we have SRO

The reason we get the same behaviour is that when considering fluctuations, we only need consider the derivative terms in the free energy. These terms, when Fourier transformed, are of the form $u_i^*(\vec{k})u_j(\vec{k})\tilde{f}^{ij}(k_x,k_y,k_x)$, where $\tilde{f}$ is a quadratic function of the k_i. Just like in the XY model calculation,

$$< u_i(\vec{x})u_j(\vec{0}) >\sim \int d^dk\tilde{f}_{ij}^{-1}(k_x,k_y,k_x) \tag{3.47}$$

which exhibits the behaviour discussed above.

Thus a two-dimensional crystal cannot have infinite extent. However, since the divergence is only logarithmic, for a finite system we can speak of order.

A more advanced treatment [47] has led to the following: one can define two correlation functions. The first one is called the *translation correlation function* Γ_T, which is the one we defined. The second one is the *rotational correlation function.* to define it, we first define an angle θ according to

$$\theta(\vec{x}) = \frac{1}{2}(\frac{\partial u_y(\vec{x})}{\partial x} - \frac{\partial u_x(\vec{x})}{\partial y}). \tag{3.48}$$

In this equation we encounter the xy component of the *rotational field*

$$\omega_{ij} = \frac{\partial u_j(\vec{x})}{\partial x_i} - \frac{\partial u_i(\vec{x})}{\partial y_j}. \tag{3.49}$$

By considering infinitesimal rotations, one can prove that it does indeed describe the local rotation of the crystal[4]. Using θ we define a new order parameter field

$$\Psi(\vec{x}) = e^{6i\theta(\vec{x})} \tag{3.50}$$

---

[4] In the Einstein-Cartan formulation of crystals, ω is the *spin connection.*

In the ordered phase $< \Psi >= 1$, meaning that the phase has hexagonal symmetry. It can be proven that Ψ *does* exhibit long range order in the crystal phase. In the hexatic phase, which we discuss in the next section, it exhibits QLRO. The heuristic derivation of this is that θ contains only first-order derivative, which gives only one power of k in the denominator of the correlation function of θ. Thus the correlation function converges, because $\int d^2k\frac{1}{k}$ converges for small k.

3.3.2 The hexatic phase

The phase is two dimensional. It is translationally invariant, but the rotational group is broken to C_6. Such a phase looks strange at first sight, because it seems that to break rotational symmetry we need to put the atoms on a lattice, thereby also breaking translational symmetry. But such a hexatic phase occurs in nature!

Not that it *is* impossible to break translational symmetry without breaking rotational symmetry. This can be proved by studying the representation theory[5] of $\mathbb{R}^2 \rtimes SO(2)$.

An order parameter field is provided by $\phi = e^{i6\theta}$, where θ measures the angle the bonds between neighbouring molecules makes with the x-axis. In the high-temperature isotropic phase, $< e^{i6\theta} >= 0$. In the ordered phase, $< e^{i6\theta} >\neq 0$ because the angle of the bonds is on average a multiple of $\frac{2\pi}{6}$.

The hexatic phase can arise via spontaneous symmetry breaking from the isotropic phase, but it can also be the outcome of the *melting* of a two-dimensional hexagonal crystal. This is what happens in nature: as we increase the temperature of the crystal, at some point the translational symmetry is restored, but locally the molecules are still arranged hexagonally. Of course, locally there is some translational symmetry, but the correlations of translations decay exponentially.

Remember that in the crystal there were two massless translational Goldstone modes, and there were dislocations. The disclinations and roton were massive. As the translational symmetry is *restored*, the hexatic phase is reached. The translational Goldstone modes and dislocations disappear. At the same time, roton becomes massless, and disclinations are the defects of this phase. We say that the roton and disclinations are *liberated*.

The traditional phase diagram of a two-dimensional crystal contains a line separating the crystal phase and the liquid phase. The phase transition is first order. However, for the hexagonal crystal the transition to the liquid phase can occur via two second-order phase transitions: first a phase transition to the translationally invariant phase with broken rotational symmetry, and then a transition to the liquid.

The mechanism behind the phase transition from crystal to hexatic is quite re-

---

[5] If an irrep is trivial when restricted to $\mathbb{R}^2$, when restricted to $SO(2)$ it can be any irrep of $SO(2)$. If, on the other hand, it is trivial on $SO(2)$, then the irrep must act the same on all vectors of the same length, since we can turn any two unit vectors into each other with a transformation of $SO(2)$. This leads to the conclusion that the irrep is trivial on $\mathbb{R}^2$.

markable: it is a *dislocation condensation.* As we've seen, defects in a phase with global symmetry breaking cost an infinite amount of energy. However, defect-antidefect pairs cost a finite amount energy, and at finite temperatures they have a finite density, and an average separation between the defects in a pair, which we call a. At the phase transition, a diverges, so that the single defects fill the ground state. This *restores translational symmetry.* In the last chapter of this work we will discuss why it restores symmetries. For now we note what happens next: the ground state is now filled with single dislocations. We've seen that these dislocations can decay into a disclination-antidisclination pair. They also have an average separation, b. At the second phase transition, the disclinations blow out, thus *restoring rotational symmetry.*

The hexatic phase has been experimentally verified in numerous three dimensional phases, such as smectics, dense solutions of DNA, flux arrays of High-Tc superconductors, superfluids[44], etc. All these phases have some kind of layered order. The hexatic smectic is also called the smectic B phase. The bond-orientational order is within the layers of the smectic. For references on hexatic phases, see [46].

3.3.3 Kosterlitz Thouless phase transition

We will now discuss another example of defect-mediated phase transition, which occurs in the XY model discussed in the previous chapter. It is called the *Kosterlitz Thouless* phase transition.

In chapter 2, we saw that the XY model is the effective free energy at low temperatures of (2.51), where ϕ acquires a nonzero ground state expectation value: $<\phi>=v$. We also proved that fluctuations of θ diverge logarithmically, which leads to QLRO, i.e. algebraic decay of $<\phi(\vec{x})\phi(\vec{0})>$. Our analysis of the XY model was actually incomplete, because we didn't take fluctuations of the vortices into account. According to (2.56), the energy of a single vortex in the XY model is logarithmically divergent with the system size, thus it is infinite in a system of infinite extent. If we have multiple vortices in the system with charges n_i, then according to (2.57) the configuration only has finite energy if the net vorticity is zero, i.e. if $\sum_i n_i = 0$. Thus, if we consider the system at finite temperatures, the only configurations of vortices that will be thermally activated are configurations of zero net vorticity. Since the vortices of high charge are energetically disfavored, we will only take vortices of charge ± 1 into account.

When we wrote down the free energy expression (2.56) for a single vortex, we neglected the entropy of the configuration. The complete expression for the free energy is given by $F = E - TS$, where E is the energy of the single vortex configuration, T is the temperature, and S the entropy. We include the energy E_C of the core of the defect in E, which we neglected in (2.56). Thus E is given by

$$E = \pi \rho_s ln(\frac{R}{a}) + E_C. \tag{3.51}$$

For a two dimensional system of linear dimension R, the number of possible positions for a vortex whose core has linear dimension a is $(R/a)^2$ (we notice this by dividing the space up into squares of linear dimension a). Thus the entropy is $2ln(R/a)$, and the free energy for a single vortex of charge ± 1 is

$$F = (\pi \rho_s - 2T) ln(\frac{R}{a}) + E_C. \tag{3.52}$$

Thus this simple analysis, originally due to Kosterlitz and Thouless [35], predicts a phase transition at the temperature $T_c = \pi\rho/2$. If $T < T_c$ the free energy of a single vortex is positive, thus the free energy is minimized when there are no vortices. If $T > T_c$ the presence of vortices reduces the free energy.

A more detailed analysis involves a *Renormalization group* calculation [34]. The idea of the calculation is to calculate the contribution to the partition function of closely bound vortices. The vortices cannot be closer to each other than the the size a of their cores, so one calculates the contribution to the partition function from vortices separated by a distance between a and $a + da$, with da small. We neglect the possibility of having three vortices close to each other: we only consider pairs of vortices. If the vortices in a pair have vorticities of the same sign, then they repel and the energy of the configuration is high, thus the contribution of these pairs to the partition function is suppressed. Thus we only consider vortex-antivortex pairs.

By integrating out the contribution of vortex-antivortex pairs separated by a distance between a and $a + da$, we obtain a new theory where the vortices have a minimum separation of $a + da$. We then rescale the theory with the scaling parameter $b = 1 + \frac{da}{a}$. This rescaled theory is described by an effective free energy of the same form as the free energy of the original, except that the parameters in the free energy have changed, due to the contribution of closebound vortex-antivortex pairs. The free energy of the original is determined by two parameters. We can take these to be K and y, defined by

$$K = \frac{\rho_s}{kT} \quad y = e^{-\frac{E_C}{kT}}. \tag{3.53}$$

As we renormalize the theory, K and y *flow*, i.e. their values change. The Renormalization group flow is schematically shown in fig.3.10.

The dashed line corresponds to points at the critical temperature $T_c = \pi\rho/2$. Indeed, the simple derivation of T_C given above gives the correct expression for T_c.

If $T < T_c$, $y = e^{-\frac{E_C}{kT}}$ gets renormalized to zero. This means that $E_C \mapsto \infty$, so that the vortices can no longer be excited. Thus the vortices get renormalized right out of the problem, and the only relevant degrees of freedom are the spin waves, whose correlation functions decay algebraically.

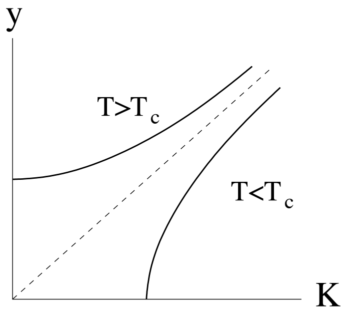


Figure 3.10: The renormalization group flow of the Kosterlitz Thouless phase transition. The dashed line corresponds to points at the critical temperature $T_c = \pi\rho/2$. If $T < T_c$, then $y = e^{-E_C/kT}$ gets renormalized to zero, which means that the vortices get renormalized right out of the problem. The only degrees of freedom are the spin waves. If $T > T_c$, y flows to some fixed value, while K flows to zero. Thus the vortices are the only relevant degrees of freedom. They are free mobile charges interacting via a Coulomb potential.

If $T > T_c$, K flows to 0, and y flows to a constant. Thus the spin wave stiffness goes to zero, so that there are no more spin waves. We are left with free vortices interacting via a Coulomb potential.

In chapter 5, we will analyze a different approach to symmetry breaking, using *Hopf symmetries*. We will show that the Hopf symmetry approach reproduces the results of the Kosterlitz-Thouless condensate given here.

Chapter 4

Hopf algebra symmetry

We have discussed the different modes in ordinary and liquid crystals, as well as their topological defects. We will now call the smooth excitations of the ground state, such as continuity and Goldstone modes, *electric* or *ordinary* modes. As we have seen, electric modes are the excitations of fields ϕ that transform under irreps of the symmetry group that characterizes the phase. We will consider defects to be excitations as well, and we call them *magnetic* or *topological* modes. In this chapter, we will show that the electric and magnetic modes can be treated on equal footing, by introducing a *Hopf symmetry.* This Hopf symmetry can also describe *dyons*, which as we saw are combinations of magnetic and electric modes. The treatment in this chapter is inherently quantum mechanical, namely we want to find the *quantum numbers* of our modes. These are labels we can assign to our modes that are invariant in time. Examples are the spin of an electron, and the topological charge of a defect.

First we summarize the properties of electric, magnetic, and dyonic modes in terms of representation theory. Then we discuss the braiding of the different types of modes. Once we've studied the similarities and differences of the properties of the different modes, we show how these properties can be algebraically encoded in a Hopf algebra, called a *quantum double.* We uncover the appropriate quantum double for liquid crystals and crystals discussed in the previous chapter. Finally, we discuss effect of *braiding*, i.e. the transport of different types of modes around each other. We illustrate this with the uniaxial nematic.

4.1 Representation theory of electric, magnetic and dyonic modes

4.1.1 Electric modes

If the symmetry group of a phase is H, the fields ϕ in the system transform under representations of H. The electric modes of ϕ are the smooth excitations of ϕ.

We've seen that it is advantageous to decompose ϕ into irreducible representations (irreps). The irreps ϕ_a transform into each other under the action of H, and don't mix with the other irreps. Thus we write

$$\phi = \bigoplus_a \phi_a \tag{4.1}$$

where ϕ_a belongs to the irrep Π_a. This implies that we can consider the modes of the different irreps separately: excitations of one irrep transform into each other under the action of H. The different vectors the vector space on which an irrep acts are called the *states* of the irrep.

Now the symmetry operators commute with the Hamiltonian. Thus if a state $|v>$ has an eigenvalue λ_g under a symmetry operator $g \in H$, then this eigenvalue is preserved in time, since the Hamiltonian generates time evolution (see chapter 2). We then call λ_g a *quantum number* of the state $|v>$.

The different states of one irrep need not have the same eigenvalue under the action of g. Still, in general one can define operators, called *Casimir operators*, that commute with all the symmetry operators and the Hamiltonian. By Schur's lemma, they act as a constant times the identity matrix on an irrep. Thus Casimir operators have one eigenvalue when they act on all the states of one irrep, and we can label the irrep with this eigenvalue, which is called a *quantum number* of the irrep. In general, one can define a set of independent Casimir operators such that the set eigenvalues of these operators when acting on an irrep completely specifies the irrep. Once we have determined a complete set $J_k, k = 1, \ldots, n$ of independent Casimir operators , then the irreps are labelled by the eigenvalues $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ under the action of J_k. Two irreps are equivalent if and only if they have the same eigenvalues under the Casimir operators. A prime example of a Casimir operator is the operator J^2 that measures the total spin (internal plus angular) and squares it.

An electric mode is a state of an irrep of the symmetry group. We write it as $|\xi^j>$, where the ξ^j are all the numbers we need to characterize the state: This includes spatial coordinates and quantum numbers. The spatial coordinates are actually also quantum numbers.

Now imagine two modes are present in our system. Their states are $|\xi_1^j>$ and $|\xi_2^j>$. The state of the combined system is[1] $|\xi_1^j, \xi_2^j>=|\xi_1^j> \otimes |\xi_2^j>$, the *tensor product* of both states. In general this tensor product is *reducible*:

$$|\xi_1^i> \otimes |\xi_2^j>= \bigoplus_a |\xi_a^k>, \tag{4.2}$$

where $|\xi_2^k>$ belongs to the irrep Π_a. (4.2) called the *fusion rule* of $|\xi_1^i>$ and $|\xi_2^j>$. It implies that as particles 1 and 2 in states $|\xi_1^i>$ and $|\xi_2^j>$ are brought

---

[1] Note that if the particles are indistinguishable, i.e. their quantum numbers are equal, then we have to be careful, because the state of the combined system must satisfy the spin-statistics theorem: if the particles are anyons, then we must require that under the adiabatic transport of the coordinates of Ψ_1 around the coordinates of Ψ_2 the state of the combined system is acted on by the braid operator R. If the action of this adiabatic transport is nontrivial, then the wavefunction corresponding to the state is multiply valued.

close together, if we measure the quantum numbers of the combined system we can get different outcomes, given by the $|\xi_a^k>$ in the decomposition (4.2).

The states live in a vector space. The global symmetry transformations are linear operators on this vector space. Consider one of the irreps α. Denote a basis of this vector space by $|e_j^\alpha>$, where j labels the different basis vectors of the vector space. Then $g \in H$ acts as follows:

$$\alpha(g) \cdot |e_j^\alpha> = \alpha(g)_j^k \cdot |e_k^\alpha>, \tag{4.3}$$

thus $\alpha(g)$ is matrix acting on the vector space.

We have to specify certain properties of our irreps. Firstly, we demand that our irreps are *unitary*, because the state should stay normalized under the action of G. Secondly, if H is not simply connected, then the states can transform under *projective* representations of H. This means that the action of H on the states is not a group homomorphism: the action of g_2 first and then g_1 is not equal to the action of $g_1 g_2$. However, the actions only differ a phase factor:

$$\alpha(g_1)\alpha(g_2)\Psi = e^{\varphi(g_1,g_2)}\alpha(g_1 g_2)\Psi \tag{4.4}$$

This is allowed because the phase factor disappears when we calculate probabilities $|\Psi|^2$. For example, electrons transform under a projective representation of $SO(3)$. To be explicit, a rotation of 2π around any axis $\hat{n}$ is trivial in $SO(3)$, but it acts nontrivially on an electron: $\alpha(R(\hat{n}, 2\pi))\Psi = -\Psi$. This is the definition of a fermion: a particle that picks up a minus sign under 2π rotation. A boson, on the other hand, picks up no minus sign.

Mathematicians have exhaustively studied projective representations, and they have proven that *projective representations of H correspond to representations of the universal covering group $\overline{H}$.* For example, a spin $\frac{1}{2}$ particle transforms under the doublet, which is the two dimensional faithful representation of $SU(2)$, but corresponds to a projective representation of $SO(3)$.

In three dimensions, the spin is either integer or half integer. This follows from the fact that $SU(2)$ covers $SO(3)$ twice. Now if we are in two spatial dimensions, the spin analysis changes drastically, because the rotation group is the abelian group $SO(2)$. The universal covering group of $SO(2)$ is $\mathbb{R}$, which covers $SO(2)$ an infinite number of times. Representations of $SO(2)$ are labelled by an integer, while representations of $\mathbb{R}$ are labelled by any real number r. The action of the irrep ρ_r, with $r \in \mathbb{R}$ on $x \in \mathbb{R}$ is:

$$\rho_r(x) = e^{irx} \tag{4.5}$$

If r is integer, then $\rho_r(2\pi) = 1$, thus the irrep is bosonic. If r is half-integer, $\rho_r(2\pi) = -1$, and the irrep is fermionic. For general r, $\rho_r(2\pi) = e^{i2\pi r}$. It is called an anyon. Note that this wasn't possible in the $SO(3)$ case precisely because $SU(2)$ is only a double cover of $SO(3)$. $\mathbb{R}$ covers $SO(2)$ an infinite number of times, which is why the fields can have any charge $\theta \in \mathbb{R}$.
Summarizing, when we discuss electric modes in a phase with H symmetry, we will look at representations α of $\overline{H}$.

4.1.2 Magnetic modes

Our discussion in the previous chapter on line defects was very similar to the discussion of electric modes just given. Namely, we saw that the line defects are arranged in classes, and that global symmetry transformations transform defects in the same class into each other. Furthermore, the fusion rules are given by 'multiplying' the classes with each other. Therefore we can consider the defects in the same conjugacy class as being part of a *multiplet*.

Let us assume that H is discrete. We denote the defect g by a ket $|g>$. In analogy with the theory of electric modes, we put the defects in a vector space spanned by the $|g>$ with $g \in \overline{H}$: $V = \{\sum_j \lambda_j |g_j>: \lambda_j \in \mathbb{C}, g_j \in H\}^2$.

In our quantum treatment we can add defects. There is no obvious classical interpretation of $|g_1> + |g_2>$, for example. It corresponds to a superposition of the defects g_1 and g_2.

We've seen that global symmetry transformations conjugate the defect. The action of a global symmetry transformation $g \in \overline{H}$ on the defect $|g_1>$ is

$$g \cdot |g_1> = |g g_1 g^{-1}> \tag{4.6}$$

Thus the global symmetry transformations are linear operators on this vector, just as in the electric case.

The representations are labelled by conjugacy classes A: Namely, global symmetry transformations transform defects in the same conjugacy class into each other. We denote the representation corresponding to the conjugacy class A by Π^A.

The representation Π^A can be reducible[3]! However, there are other operators in our theory that make the representation irreducible. Namely, *we can measure the precise flux of a defect, using Aharonov-Bohm scattering experiments*[24]. Thus we have *projection operators* P_h in our theory, that act on the defects according to

$$P_h|g> = \delta_{h,g}|g> . \tag{4.7}$$

The projection operators span a vector space $F(\overline{H})$, which is isomorphic to the vector space of functions from $\overline{H}$ to $\mathbb{C}$. Namely P_h can be associated with the function on $\overline{H}$ defined by $P_h(g) = \delta_{h,g}$.

The conjugacy classes form irreps under the combined action of $\overline{H}$ and $F(\overline{H})$. Note that we've described the action of $F(\overline{H})$ and $\overline{H}$ separately, and we need to know what happens when a projection operator and a global symmetry transformation are applied in succession. Thus we want to turn the combination of $\overline{H}$ and $F(\overline{H})$ into an *algebra*, i.e. we want to be able to multiply elements of $\overline{H}$ and $F(\overline{H})$. Physics dictates what the answer is: the multiplication is this algebra set by

$$gP_h = P_{ghg^{-1}}g. \tag{4.8}$$

---

[2] Note that we could have taken $\lambda_i \in \mathbb{R}$, but in the next chapter when we treat our theory quantum mechanically, the vector space will be $\mathbb{C}$, so we may as well use this now.

[3] In fact, if H is non-Abelian, then the representation for a conjugacy with more than one element is always reducible: take the sum over the defects in A: $\sum_{g \in A} |g>$. $\overline{H}$ acts trivially on the one dimensional subspace spanned by this vector, thus the representation is reducible.

The reason for this equation is: if we measure a flux h with P_h, and then conjugate the defect with g, we have a flux ghg^{-1}. This action is equivalent to first conjugating the defect with g, and then measuring ghg^{-1} with P_{ghg-1}.
The algebra defined by this multiplication is called the *quantum double of* $\overline{H}$, and is denoted by $D(\overline{H})$. As a vector space, $D(\overline{H}) = F(\overline{H}) \otimes \mathbb{C}\overline{H}$, i.e. all elements are a sum of elements of the form $P_h g$. The multiplication is set by the action defined above. thus we can say that *the defects transform under irreps of* $D(\overline{H})$.
The tensor product $|g_1 > \otimes|g_2 >$ of two defects is to be interpreted as "a configuration with defect g_1 to the left of defect g_2". The order is important: if we measure the total flux of $|g_1 > |g_2 >$ we get $g_1 g_2$, while the total flux of $|g_2 > |g_1 >$ is $g_2 g_1$. Thus we define the action of the projection operators on the tensor product as follows:

$$P_h(|g_1 > \otimes|g_2 >) = \delta_{h,g_1 g_2}(|g_1 > \otimes|g_2 >). \tag{4.9}$$

4.1.3 Dyonic modes

For every conjugacy class A we pick a *preferred element* g_A. then all defects can be written as gg_Ag^{-1} for some A and some $g \in H$. Call N_A the centralizer of g_A. The centralizers of elements in the same conjugacy class are isomorphic: $N_{gg_Ag^{-1}} = gN_Ag^{-1}$.
A dyonic mode is an electric mode with a defect g_A in the background. We saw that we can only globally implement N_A, thus the electric mode transforms under an irrep α of N_A.
We also use the ket notation for dyons. The configuration with electric mode $|e^\alpha_j >$ and defect $h \in A$ in the background is denoted by $|h, e^\alpha_j >$. The $|e^\alpha_j >$ form a basis of the vector space on which α acts, so that the $|h, e^\alpha_j >$ are a basis of the vector space associated to the dyon. We denote the irrep of our dyon by Π^A_α.
The action of global symmetry transformations on this vector space is subtle. If we take a transformation $n \in N_A$, then

$$n \cdot |h, e^\alpha_j > \equiv \Pi^A_\alpha(n)|h, e^\alpha_j >= |h, \alpha(n)e^\alpha_j >, \tag{4.10}$$

i.e. it acts on the electric mode.
Now if the transformation $g \notin N_A$, it conjugates the defect, and it can also act on the electric mode! To describe this action, it is convenient to define another notation for the vectors in Π^A_α.
First note that the elements of the conjugacy class A are in one-to-one correspondence with left N_A-cosets in $\overline{H}$. Choose representatives x^A_i of left N_A cosets, such that $x^A_1 = e$. Then x^A_i corresponds to $h^A_i = (x^A_i)g_A(x^A_i)^{-1}$, where h^A_i is an element of A. THis association is well defined because it is independent of the particular choice of representative x^A_i of the left N_A coset, since by definition the elements of N_A commute with g_A. Furthermore, different x^A_i correspond to different elements h^A_i of A, and we have $A = \{h^A_1 = g_A, h^A_2, h^A_3, \ldots, h^A_n\}$. Now a basis of the vector space on which Π^A_α acts is given by $\{|h^A_i, e^\alpha_j >\}$.

Alternately, we can denote $|h_i^A, e_j^\alpha >$ by $x_i^A \otimes |e_j^\alpha >$. This last notation allows us to calculate the action of $g \in \overline{H}$ on this dyon. In this new notation, conjugating the defect h_i^A by a transformation g corresponds to *multiplying* x_i^A by g from the left. Thus

$$g \cdot |h_i^A, e_j^\alpha > \equiv g \cdot x_i^A \otimes |e_j^\alpha > = g x_i^A \otimes |e_j^\alpha > = x_k^A n \otimes |e_j^\alpha > \equiv |h_k^A, \alpha(n) e_j^\alpha > \quad (4.11)$$

where $g x_i^A = x_k^A n$, with $n \in N_A$. In other words, $g x_i^A$ sits in some left N_A coset. Since the x_k^A form representatives of left N_A cosets, $g x_i^A$ is equal to $x_k^A n$ for some k and some $n \in N_A$. This n then acts on the electric part of the dyon. This notation is the most transparent notation we can adopt for the action of g on the dyon. In sloppy language, we say that g conjugates the defect, and the part of g that "shoots through" the defect, i.e. n, acts on the electric mode.
The action of the projection operators P_h on the dyon is

$$P_h \cdot |h_i^A, e_j^\alpha > = \delta_{h, h_i^A} |h_i^A, e_j^\alpha >, \quad (4.12)$$

thus it projects the defect part.
Summarizing, the Π_α^A are irreps of $D(\overline{H})$. It turns out that these are *all* the irreps of $D(\overline{H})$ (see Appendix B). We denote the vectors on which Π_α^A acts by V_α^A.

Note that the electric and magnetic modes discussed are also irreps of $D(\overline{H})$! Namely, electric modes are irreps $\Pi_\alpha^{C_e}$, with C_e the conjugacy class of the identity e. $C_e = \{e\}$, and corresponds to having no defect. Thus electric modes are dyons with the conjugacy class of the identity element. Magnetic modes are irreps Π_{id}^A, dyons with a trivial representation of N_A. Thus *the quantum double offers a unified description of electric, magnetic and dyonic modes.*
We haven't exactly proved that all the properties of the modes are captured by $D(\overline{H})$. For example, we would like $D(\overline{H})$ to reproduce the fusion rules of the modes. This can be done, by introducing a *coproduct*. The coproduct Δ defines the product of the irreps. It does so in the following way: Δ is a map from $D(\overline{H})$ to $D(\overline{H}) \otimes D(\overline{H})$, that respects the multiplication (i.e. it's an algebra morphism):

$$\Delta(P_h g P_{h'} g') = \Delta(P_h g) \Delta(P_{h'} g') \quad (4.13)$$

Given an element $P_h g$, the coproduct can be written out in a basis of $D(\overline{H}) \otimes D(\overline{H})$:

$$\Delta(P_h g) = \sum_{h_1, g_1, h_2, g_2 \in \overline{H}} \lambda_{h_1, g_1, h_2, g_2} P_{h_1} g_1 \otimes P_{h_2} g_2.$$

This is rather cumbersome notation. Instead, we adopt *Sweedler's notation.* For any $a \in D(\overline{H})$ we write

$$\Delta(a) = \sum_{(a)} a^{(1)} \otimes a^{(2)}. \quad (4.14)$$

This means we can write $\Delta(a)$ as a sum of elements of the form $a^{(1)} \otimes a^{(2)}$, with $a^{(1)}$ and $a^{(2)} \in D(\overline{H})$.

Now if we have two irreps Π^A_α and Π^B_β, their tensor product is a representation of $D(\overline{H})$ whose action on a is given by

$$(\Pi^A_\alpha \otimes \Pi^B_\beta)a \equiv (\Pi^A_\alpha \otimes \Pi^B_\beta)\Delta(a) = \sum_{(a)} \Pi^A_\alpha(a^{(1)}) \otimes \Pi^B_\beta(a^{(2)}). \tag{4.15}$$

Now the idea is to choose the coproduct so that it produces the right fusion rules. The answer turns out to be (see Appendix B)

$$\Delta(P_h g) = \sum_{h' \in \overline{H}} P_{hh'^{-1}} g \otimes P_{h'} g. \tag{4.16}$$

$D(\overline{H})$ is also a *Hopf algebra*. This means there is more structure on $D(\overline{H})$ than we have defined until now. We discuss this in Appendix B. Here we will only introduce the structures that are relevant for this chapter and the next.
A Hopf algebra has a *counit* ϵ, which corresponds to the *vacuum representation*. For $D(\overline{H})$, ϵ is defined by

$$\epsilon(P_h g) = \delta_{h,e} \tag{4.17}$$

ϵ is a one-dimensional representation of the $D(\overline{H})$, whose tensor product with any irrep Π^A_α gives Π^A_α:

$$\Pi^A_\alpha \otimes \epsilon = \epsilon \otimes \Pi^A_\alpha = \Pi^A_\alpha. \tag{4.18}$$

We prove this in Appendix B.
We introduce one more structure: the *antipode* S, defined for $D(\overline{H})$ by

$$S(P_h g) = P_{g^{-1}h^{-1}g} g^{-1}. \tag{4.19}$$

It is used to define the *antiparticle representation* $\overline{\Pi^A_\alpha}$ of Π^A_α:

$$\overline{\Pi^A_\alpha}(P_h g) = {\Pi^A_\alpha}^t S(a), \tag{4.20}$$

where t denote the transpose. The properties of the antipode imply that this $\overline{\Pi^A_\alpha}$ is a representation, and that the vacuum representation ϵ appears in the decomposition of $\overline{\Pi^A_\alpha} \otimes \Pi^A_\alpha$:

$$\overline{\Pi^A_\alpha} \otimes \Pi^A_\alpha = \epsilon \oplus \bigoplus_{B,\beta} \Pi^B_\beta. \tag{4.21}$$

This explains our denomination of antiparticle irrep: an irrep and its anti-irreps can decay into the vacuum representation ϵ. This discussion applies to general Hopf algebras.
Finally we introduce a different notation for the elements of $D(\overline{H})$, which we used frequently in the next chapter. We want to write elements of $F(\overline{H}) \otimes \mathbb{C}\overline{H}$ as functions from $F(\overline{H} \times \overline{H})$ to $\mathbb{C}$. To do this, we associate to $h \in \overline{H}$ the function $\delta_h \in F(\overline{H})$, defined by $\delta_h(g_1) = \delta_{h,g_1} \forall g_1 \in \overline{H}$. Then the element $P_h g$ of $D(\overline{H})$ corresponds to the function $\delta_h \otimes \delta_g$ defined as follows: for all $g_1, g_2 \in \overline{H}$

$$\delta_h \otimes \delta_g(g_1, g_2) = \delta_{h,g_1} \delta g, g_2. \tag{4.22}$$

We can rewrite all the structures on $D(\overline{H})$, such as multiplication and coproduct, in this function notation. This is done in Appendix B.

4.2 Hopf symmetry of liquid crystals and crystals

The Hopf symmetry of the crystals and liquid crystals with residual symmetry group H is a quantum double $D(\overline{H})$, or a slight variation of it.

4.2.1 Liquid crystals

We take a nematic liquid crystal with discrete symmetry group H ($H \in SO(3)$). The Goldstone modes, defects and dyons are irreps of $D(\overline{H})$.
Classifying all the irreps of $D(\overline{H})$ gives us the possible particle spectrum of any phase with $D(\overline{H})$ symmetry[4].
The steps towards classifying all the irreps of $D(\overline{H})$ are

- Determine the conjugacy classes A of $\overline{H}$
- Pick a preferred element g_A for every A, and determine the centralizer N_A of g_A
- Determine the irreps α of N_A
- The irreps of $D(\overline{H})$ are Π^A_α.

As an example, let us carry out this analysis for a tetrahedral nematic with $H = T$. We saw that elements of T are permutations of the four vertices of the tetrahedron (see fig. 4.1). We also need a notation for the elements of $\overline{T}$. It is similar to the notation for T: The defects can be noted as a permutation and a plus of minus sign in front of it. For example, $\pm[(12)(34)], \pm[123]$ are defects (we use square brackets for the defects). We describe the meaning of this notation in Appendix B. The minus sign is itself a defect: $-e$ is the defect corresponding to a 2π rotation. The conjugacy classes are given in table 4.1, and the centralizers and irreps in table 4.3.

We now return to a point alluded to in the previous chapter: what is the relevance of "large" global symmetry transformations, i.e. transformations that are not connected to the identity? In other words, how does the representation theory change if $H = T_d$? Note that the Hopf symmetry of such a phase isn't simply the quantum double $D(\overline{T_d})$. Namely, there are no defects associated to the reflections, because it is not connected to e, and defects are characterized by drawing loops from e.

---

[4] Note we haven't mentioned the continuity modes. The reason is that they transform under irreps of the external symmetry group, and we are neglecting that part of the symmetry group. All our modes transform under a representation of the external symmetry group. Take for example a momentum mode $\vec{v_0} e^{i\vec{k}\cdot\vec{x} - i\omega t}$. Its transformation properties under external symmetries follow from the fact it is a vector under the action of the external rotation group. Under an external rotation R_{ext}, $\vec{v_0}$ and $\vec{k}$ are rotated by R_{ext}. Under translations T, $\vec{k}$ is translated by $T\vec{k}$. When we expand modes into plane waves, all plane waves have the same transformation properties under the external symmetry group. Thus, in the rest of our story we neglect the external symmetry group, and the continuity modes.

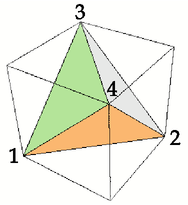


Figure 4.1: The tetrahedron, with labelled vertices.

Preferred element g_A	Conjugacy class A
e	$C_e = \{e\}$
$-e$	$C_{-e} = \{-e\}$
$[123]$	$C_{[123]} = \{[123],[134],[142],[243]\}$
$-[123]$	$C_{-[123]} = \{-[123],-[134],-[142],-[243]\}$
$[124]$	$C_{[124]} = \{[124],[132],[234],[143]\}$
$-[124]$	$C_{-[124]} = \{-[124],-[132],-[234],-[143]\}$
$[(12)(34)]$	$C_{[(12)(34)]} = \{[(12)(34)],[(13)(24)],[(14)(23)],$ -[(12)(34)],-[(13)(24)],-[(14)(23)]$\}$

Table 4.1: The conjugacy classes of $\overline{T}$, and the preferred elements g_A.

To resolve this issue, let us first discuss how these large global symmetry transformations act on the defects. They act just like any other group elements, namely they conjugate the loop that characterizes the defect. Under the action of these reflections on a defect g we may get a defect outside the conjugacy class of g! Thus the defect classes under the action of T_d are in general larger than the conjugacy classes. We show the defect classes for the tetrahedral nematic case in Table 4.2. This result requires a careful interpretation. The defects in the same class have the same energy, since the symmetry transformations commute with the Hamiltonian. They also have isomorphic centralizers. However, *the cores of defects only related by a large symmetry cannot be interchanged*. This is due to the fact that the symmetry isn't connected to the identity. Thus these defects are *not* topologically equivalent: they cannot be deformed into each other with

Classes of $\overline{T}$ under the action of T_d
C_e
C_{-e}
$C_{[123]} \cup C_{[124]}$
$C_{-[123]} \cup C_{-[124]}$
$C_{[(12)(34)]}$

Table 4.2: The defect classes in a theory with T_d symmetry. The classes are unions of conjugacy of $\overline{T}$. The conjugacy classes of $\overline{T}$ are defined in table 4.1.

a finite amount of energy. So we might conclude that we shouldn't act with large symmetries on the defects, and we should work with conjugacy classes. However, we cannot simply neglect the large global symmetries, since they act on the electric modes.

This phase with T_d symmetry has defects labelled by $\overline{T}$, and the electric modes transform under representations of $\overline{T_d}$, the double cover of T_d in $O(3)$. We call the defect group the *magnetic group*, and the group under which the electric modes transform the *electric group*. So this T_d phase has different electric and magnetic groups. We ask the more general question: How do we deal with a phase with electric group H_{el}, and magnetic group H_m, with H_{el} and H_m unequal?

We found that it *is* possible to define a Hopf symmetry that captures the physics of such phases, given some restrictions on H_m and H_{el}. In all the physical systems we've discussed these restrictions are met.

We call the Hopf symmetry of such a phase a *generalized quantum double*. As a vector space, it is $F(H_m) \otimes \mathbb{C}H_{el}$. This generalized quantum double is a *transformation group algebra* (see Appendix B). To define the multiplication in this algebra, we need an *action* of the electric group H_{el} on the magnetic group H_m. For the normal quantum double, this action is simply conjugation. In the T_d case, for example, we've had to calculate the action of reflections on the magnetic group separately. The transformations connected to the identity act as a conjugation. The generalized quantum double of the achiral tetrahedral nematic, for example, is $F(\overline{T}) \otimes \mathbb{C}T_d$. Note that we are not considering projective representation of the electric group T_d, i.e. we are only considering bosonic irreps of the electric group. If we want to include fermionic irreps, the appropriate generalized quantum double is $F(\overline{T}) \otimes \mathbb{C}\overline{T}_d$.

$g_{\mathcal{O}}$	Centralizer $N_{\mathcal{O}}$	Irreps
$\pm e$	T	Π^e_α
$\pm[123]$	$\{e,(123),(132)\} \simeq \mathbb{Z}_3$	$\Pi_k^{\pm[123]}$, $k \in \mathbb{Z}_3$
$\pm[124]$	$\{e,(124),(142)\} \simeq \mathbb{Z}_3$	$\Pi_k^{\pm[124]}$, $k \in \mathbb{Z}_3$
$[(12)(34)]$	$\{e,(12)(34),(13)(24),(14)(23)\} \simeq \mathbb{Z}_4$	$\Pi_k^{\pm[123]}, k \in \mathbb{Z}_4$

Table 4.3: The irreps of $F(\overline{T}) \otimes \mathbb{C}T_d$. The α are irreps of T_d.

Denote the action of $h \in H_{el}$ on $g \in H_m$ as $h \cdot g$. Then the multiplication in the generalized quantum double is set by

$$hP_g = P_{h \cdot g}h. \tag{4.23}$$

The Hopf symmetry $F(H_m) \otimes \mathbb{C}H_{el}$ of our phase is well defined, and since it is a transformation group algebra we can use the theorem in Appendix B to determine its irreps. The calculation is similar to the quantum double case. Given an element $g_{\mathcal{O}} \in H_m$, define the *normalizer* $N_{\mathcal{O}}$ of $g_{\mathcal{O}}$ to be the subgroup of

H_{el} whose elements $h \in N_{\mathcal{O}}$ satisfy $h \cdot g_{\mathcal{O}} = g_{\mathcal{O}}$. Then the steps for determining the irreps of $F(H_m) \otimes \mathbb{C}H_{el}$ are:

- Determine the *orbits* $\mathcal{O}$ of H_m under the action of H_{el}
- Pick a preferred element $g_{\mathcal{O}}$ for every $\mathcal{O}$, and determine the normalizer $N_{\mathcal{O}}$ of $g_{\mathcal{O}}$
- Determine the irreps α of $N_{\mathcal{O}}$
- The irreps of $F(H_m) \otimes \mathbb{C}H_{el}$ are $\Pi^{\mathcal{O}}_{\alpha}$.

4.2.2 The hexagonal crystal

We reconsider the planar hexagonal crystal discussed in the previous chapter. We argued there that the symmetry was $\mathbb{Z}^2 \rtimes \mathbb{Z}_6$. If the crystal is achiral, the symmetry group is actually $\mathbb{Z}^2 \rtimes D_6$. We've just discussed how to deal with such a phase, so we consider an achiral hexagonal crystal. The defect classes are larger than the conjugacy classes, but as we have argued above we still consider conjugacy classes.

The first Brillouin zone (rescaled) is shown in Fig.4.3, which is a hexagon.

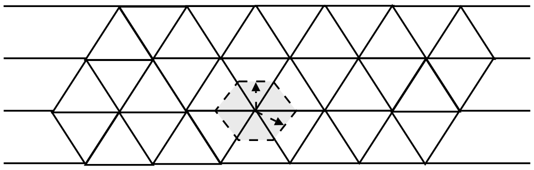


Figure 4.2: A triangular lattice, representing a crystal with hexagonal symmetry. The dashed vectors are vectors in the reciprocal lattice, and the shaded hexagon represents the first Brillouin zone.

Points in the Brillouin zone correspond to irreps of the translation group. Namely, a point in the Brillouin is a momentum vector $\vec{k} = (k_x, k_y)$, and the associated irrep of $\mathbb{Z}^2$ is:

$$(n, m) \mapsto e^{ik_x n + k_y m}. \tag{4.24}$$

The whole of k-space is the whole plane, and is tiled by hexagons (note that position space is tiled by triangles). Equivalent points of k-space (corresponding to equivalent irreps of the translation group) are obtained by translations that leave the tiling invariant.

In the previous chapter, we saw that the magnetic group of a hexagonal crystal with large symmetry transformations is $\mathbb{Z}^2 \rtimes \mathbb{Z}$. Now we are considering a hexagonal crystal with large symmetry transformations (reflections), and the magnetic group is the same: $H_m = \mathbb{Z}^2 \rtimes \mathbb{Z}$. This follows from the observation, made earlier, that large symmetry transformations do not affect the magnetic group.

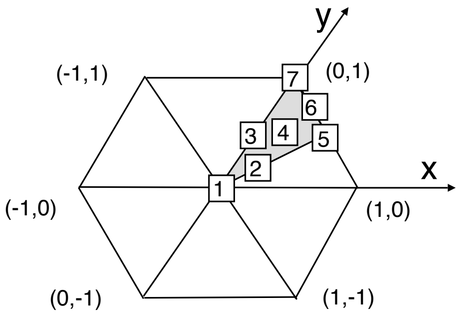


Figure 4.3: The first Brillouin zone of the planar hexagonal crystal. The gray area is a set of representatives of orbits under rotations and reflections. The numbered squares are representatives of the different types of orbits. The x- and y-axes have been rescaled. The unit distance on these rescaled axes corresponds to a distance of $\frac{\pi}{a}$, where a is the lattice spacing in the crystal.

The electric group is $H_{el} = \overline{\mathbb{Z}^2 \rtimes D_6} = (\mathbb{Z}^2 \rtimes D_6) \times \mathbb{Z}$. The last $\mathbb{Z}$ part corresponds to rotations of $2\pi n$, $n \in \mathbb{Z}$, which we denote by r^{6n}. These rotations are in the center of the group[5]. Thus the Hopf symmetry is[6] $F(\mathbb{Z}^2 \rtimes \mathbb{Z}) \otimes \mathbb{C}(\mathbb{Z}^2 \rtimes D_6) \times \mathbb{Z}$.

In the previous chapter we denoted an element of $(\mathbb{Z}^2 \rtimes \mathbb{Z}_6) \times \mathbb{Z}$ as $(k,l)r^p$, with $k, l, p \in \mathbb{Z}$. We adopt a similar notation here. For $(\mathbb{Z}^2 \rtimes D_6) \times \mathbb{Z}$ we write an element as $(k,l)s^j r^{k+6n}$, with $k, l, n \in \mathbb{Z}$, $j \in \mathbb{Z}_2$, $n \in \mathbb{Z}$. The reflection in D_6 is denoted by s. We take s to be the reflection with respect to the x-axis.
To analyze the particles of our theory, we proceed just as in the case of the tetrahedral nematic. We need the orbits $\mathcal{O}$ of $((\mathbb{Z}^2 \rtimes \mathbb{Z}_6) \times \mathbb{Z})$ under the action of $(\mathbb{Z}^2 \rtimes D_6) \times \mathbb{Z}$, the centralizer N of a chosen preferred element of each orbit, and the irreps of the centralizer. See table 4.5. The action of $(\mathbb{Z}^2 \rtimes D_6) \times \mathbb{Z}$ on $\mathbb{Z}^2 \rtimes \mathbb{Z}_6$ is set by:

$$\begin{aligned} r(a,b)r^{-1} &= (-b, a+b), \\ s(a,b)s^{-1} &= (a+b, -b). \\ srs^{-1} &= r^{-1} \end{aligned}$$

There is one class of orbits, $\mathcal{O}_{r^{6k}}$, for which determining the irreps of the centralizer $(\mathbb{Z}^2 \rtimes D_6) \times \mathbb{Z}$ is a little complicated. We can determine the irreps

---

[5] The center of a group is the subgroup of elements that commutes with all other elements.

[6] There is a subtlety due to the infinity of the groups. Drawing on the analogy with generalized quantum doubles with finite groups, as a vector space this Hopf algebra is $F(((\mathbb{Z}^2 \rtimes \mathbb{Z}_6) \times \mathbb{Z}) \times (\mathbb{Z}^2 \rtimes D_6))$. However, to make the representation theory tractable we only take functions with compact support in the second variable. One can check that this is an algebra (i.e. the finite sum and product of such elements also has compact support in the second variable). We don't take compact support in the first variable because of our analysis of defect condensates, to come. If we condense the dislocation $|(1,0)>$, for example, then we need functions that are constant on left $(1,0)$ cosets, and these left cosets have an infinite number of elements. The reader is referred to the next chapter for details.

Orbits
$\mathcal{O}_{r^{6k}} = C_{r^{6k}}$
$\mathcal{O}_{(a,b)r^{6k}} = \{C_{(a,b)r^{6k}}, C_{(a+b,-b)r^{6k}}\}$
$\mathcal{O}_{r^{1+6k}} = \{C_{r^{1+6k}}, C_{r^{-1-6k}}\}$
$\mathcal{O}_{r^{2+6k}} = \{C_{r^{2+6k}}, C_{r^{-2-6k}}\}$
$\mathcal{O}_{(1,0)r^{2+6k}} = \{C_{(1,0)r^{2+6k}}, C_{(1,0)r^{-2-6k}}\}$
$\mathcal{O}_{r^{3+6k}} = \{C_{r^{3+6k}}, C_{r^{-3-6k}}\}$
$\mathcal{O}_{(1,0)r^{3+6k}} = \{C_{(1,0)r^{3+6k}}, C_{(1,0)r^{-3-6k}}\}$

Table 4.4: The orbits of $\mathbb{Z}^2 \times \mathbb{Z}$ under the action of $(\mathbb{Z}^2 \rtimes D_6) \times \mathbb{Z}$. The conjugacy classes are defined in table 3.2.

of $\mathbb{Z}^2 \rtimes D_6$ and those of $\mathbb{Z}$ separately, because the irreps of a direct product of groups is the tensor product if the irreps of the separate groups. Determining the irreps of $\mathbb{Z}^2 \rtimes D_6$ requires the little group method described in D.1.4. We must pick a momentum vector $\vec{k}$ which corresponds to an irrep of the translation group $\mathbb{Z}^2$, and act on it with D_6. The elements that transform $\vec{k}$ into a vector that corresponds to the same irrep form a subgroup of D_6, which we denote by $(D_6)_{\vec{k}}$. Thus we are considering orbits in the first Brillouin zone (which has nothing to do with the orbits described above!). We must find all possible cases of vectors in the first Brillouin, with different $(D_6)_{\vec{k}}$. For the hexagonal crystal, there are seven different types of orbits, and a representative element of each type is given in fig. 4.3. We then call $\mathbb{Z}^2 \rtimes (D_6)_{\vec{k}}$ the *little group* of $\vec{k}$. We must determine the irreps of the little group, and then *induce* irreps of the whole group (see D.1.4). This procedure gives all the irreps.

As an example, We explicitly give the irrep corresponding to orbit 5 in fig. 4.3. In D.1.4 we illustrate how to derive this. The matrices of the irrep are

$$(a,b) \mapsto \begin{pmatrix} e^{i\frac{\pi}{2}a}e^{i\frac{\pi}{2}b} & 0 & 0 \\ 0 & e^{i\frac{\pi}{2}(a+b)}e^{i\frac{\pi}{2}(-a)} & 0 \\ 0 & 0 & e^{i\frac{\pi}{2}b}e^{i\frac{\pi}{2}(-a-b)} \end{pmatrix},$$

$$r \mapsto \begin{pmatrix} 0 & 0 & e^{i\pi m} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, s \mapsto \begin{pmatrix} 0 & & e^{i\pi n} \\ 0 & e^{i\pi n} & 0 \\ e^{i\pi n} & 0 & 0 \end{pmatrix}$$

4.3 Braiding

In this section we discusse the braiding of electric, magnetic, and dyonic modes. In other words, we address the following question: What happens when one type of mode is adiabatically (i.e. slowly) transported around another mode? We introduce a braid operator and discuss its properties.

Let us first assume that H is discrete. We've actually already discussed the braiding of defects. Namely, we saw that when a line defect h is transported halfway around a line defect g in an anticlockwise direction (viewed from above),

Conjugacy classes	N	Irreps
$C_{r^{6k}}$ $k \in \mathbb{Z}$	$(\mathbb{Z}^2 \rtimes D_6) \times \mathbb{Z}$	$\Pi^{r^{6k}}_{\rho,\lambda}$ ρ irrep of $\mathbb{Z}^2 \rtimes D_6$ (see table 4.2.2)
$C_{(a,b)r^{6k}}, C_{(a+b,-b)r^{6k}}$ $a, b \geq 0; b \geq a; a, b \geq 0; k \in \mathbb{Z}$	$(\mathbb{Z}^2) \times \mathbb{Z}$	$\Pi^{(a,b)r^{6k}}_{\vec{k},\lambda}$ $\vec{k} \in \mathbb{R}^2$
$C_{r^{1+6k}}, C_{r^{-1-6k}}$ $k \in \mathbb{Z}$	$(\mathbb{Z}_6) \times \mathbb{Z}$	$\Pi^{r^{1+6k}}_{m,\lambda}$ $m \in \mathbb{Z}_6$
$C_{r^{2+6k}}, C_{r^{-2-6k}}$ $k \in \mathbb{Z}$	$(\mathbb{Z}_6) \times \mathbb{Z}$	$\Pi^{r^{2+6k}}_{m,\lambda}$ $m \in \mathbb{Z}_6$
$C_{(1,0)r^{2+6k}}, C_{(1,0)r^{-2-6k}}$ $k \in \mathbb{Z}$	$(\mathbb{Z}_3) \times \mathbb{Z}$	$\Pi^{(1,0)r^{2+6k}}_{n,\lambda}$ $n \in \mathbb{Z}_3$
$C_{r^{3+6k}}, C_{r^{-3-6k}}$ $k \in \mathbb{N}$	$(\mathbb{Z}_6) \times \mathbb{Z}$	$\Pi^{r^{3+6k}}_{m}$ $m \in \mathbb{Z}_6$
$C_{(1,0)r^{3+6k}}, C_{(1,0)r^{-3-6k}}$ $k \in \mathbb{N}$	$(\mathbb{Z}_2) \times \mathbb{Z}$	$\Pi^{(1,0)r^{3+6k}}_{l,\lambda}$ $l \in \mathbb{Z}_2$

Table 4.5: The irreps of $F((\mathbb{Z}^2 \rtimes \mathbb{Z}) \otimes \mathbb{C}(\mathbb{Z}^2 \rtimes D_6) \times \mathbb{Z}$. $\lambda \in U(1)$ corresponds to the irreps of the $\mathbb{Z}$ part of the centralizers, which represents the rotations by an angle $2\pi n$.

then h gets conjugated by g, and becomes ghg^{-1}. We can encode this behaviour by defining a *braid operator* $\mathcal{R}$. If $|g>$ lives in V^A and $|h>$ in V^B, then $\mathcal{R}$ is a map from $V^A \otimes V^B$ to $V^B \otimes V^A$ whose action is defined by

$$\mathcal{R} \cdot |g>|h>= |ghg^{-1}>|g> \tag{4.25}$$

The braid operator encodes the braiding properties of the defects. Note that it braids the defect to the right halfway around the other defect, and we call this *half-braiding*. To achieve a full braiding, we have to apply $\mathcal{R}^2$.
From our previous discussions we know that the equation for the braiding of defects $|g>$ and $|h>$ we have just discussed applies equally well to the cases of

Orbit number	$(D_6)_{\vec{k}}$	Number of irreps	Dimension of irreps
1	D_6	6	1, 1, 1, 1, 2, 2
2	$\mathbb{Z}_2$	2	6, 6
3	$\mathbb{Z}_2$	2	6, 6
4	e	1	12
5	$\mathbb{Z}_2 \rtimes \mathbb{Z}_2$	4	3, 3, 3, 3
6	$\mathbb{Z}_2$	2	6, 6
7	D_3	3	2, 2, 4

Table 4.6: The irreps of $\mathbb{Z}^2 \rtimes D_6$. The number of an orbit corresponds to the number in Fig. 4.3. $(D_6)_{\vec{k}}$ is the Little Group of the orbit.

global and gauge symmetry breaking.

Electric modes braid trivially with each other[7]:

$$\mathcal{R}|v_1 > |v_2 >= |v_2 > |v_1 > . \tag{4.26}$$

Now for the braiding of an electric mode with a defect. We will discuss the gauge theory case first, because the global case is more subtle. We will be brief, because it is well covered in the literature [19].
Consider a gauge theory with a defect g. This topological charge is equal to the path-ordered exponential of the gauge field around the defect (see (2.68)):

$$g = Pe^{i\oint \vec{A}\cdot dx}.$$

Now imagine we have a particle coupled to the gauge field. Then the calculation of the change of its wavefunction Ψ as it is adiabatically transported around the defect is called a *Berry's phase* calculation [11]. The outcome is

$$\Psi \mapsto \Psi' = Pe^{i\oint \vec{A}\cdot dx}\Psi = g\cdot\Psi. \tag{4.27}$$

Ψ' solves the equation $D_i\Psi = 0$[68].

Thus *as the electric mode is transported around the defect, it is acted on by the topological charge g of the defect.* We can choose a gauge such that the field Ψ is constant everywhere except along a line going "up" from the origin. Then half-braiding already gives $\Psi \mapsto g\cdot\Psi$, because underneath the defect Ψ is constant, i.e. Ψ braids trivially with the defect if it passes below the defect.

Now for the case of global symmetry breaking. What is the braiding of an electric mode with a line defect? For example, what happens when a Goldstone mode is transported around the nontrivial defect in the uniaxial nematic?

The modern approach to defects is to consider them to be sources of curvature and torsion in a metric space. In other words, the defects curve the space around them, so that geodesics are deformed. This idea has been applied extensively to crystals, and the geometry is called Riemann Cartan geometry[8]. For an introduction, see [27] and [29]. To be able to apply a geometry, one must take the continuum limit of a crystal, which turns the theory into an $ISO(3)$ gauge theory (just like gravity is an $SO(3,1)$ gauge theory). Note that we briefly encountered the continuum theory of defects when we discussed smectics and crystals. We introduced continuum fields, and saw that the Burgers vector of a dislocation is given by integrating the field around the defect.
In this theory, the outcome of braiding is the same as in the local case, but the

---

[7] Actually, if we are braiding two indistinguishable electriccparticles, then the braiding may give a phase factor. Under half-braiding the wavefunction of the system picks up a phase factor $e^{i2\pi s}$, where s is the spin of the particles.

[8] The difference with standard Riemann geometry is that one doesn't assume that the metric is torsion free, i.e. the Christoffel symbols $\Gamma^{\kappa}_{\mu\nu}$ are not required to satisfy $\Gamma^{\kappa}_{\mu\nu} = \Gamma^{\kappa}_{\nu\mu}$. Dropping this requirement makes the theory more complicated. For example, in normal general relativity geodesics (paths along with the covariant derivative is zero) are the shortest paths between the endpoints. This is no longer the case if there is torsion! There is a difference between extremals and geodesics of the metric.

interpretation is different here. In the gauge theory case, the wavefunctions are multiply valued, and the phase factor follows from the path dependent formulation of symmetry operators. In the global case, during the parallel transport the particle is following a curved path in its internal space. It is being "frame dragged", as it is called[40]. To be specific, one defines a local coordinate frame somewhere at the start of the path in G characterising the defect. Then one lets the elements of the path act on this initial frame, to obtain a new frame everywhere on the path. An electric mode is then parallel transported around the defect if its coordinates are constant with respect to the local frames.

This idea has been applied to many phases, such as superfluid helium, where the symmetry is also global [28]. It has also been applied to uniaxial nematic liquid crystals in the one constant approximation[41] (in the absence of diffusion).

The braid operator can be used to study scattering. In the gauge theory case, the equation for the scattering cross section is [69]:

$$\frac{d\sigma}{d\theta} = \frac{1}{4\pi p sin^2\theta/2}[1 - Re < \psi_{in}|\mathcal{R}^2|\psi_{in} >] \tag{4.28}$$

where $|\psi_{in} >$ is the initial internal wavefunction of the whole system, and p the momentum. For example, for the case of an electric mode $|v >$ to the left of a vortex $|g >$, we have $|\psi_{in} >= |v > \otimes|g >$. If the braiding is trivial, i.e. $\mathcal{R}^2|\psi_{in} >= |\psi_{in} >$, then there is no scattering.

Let us look at a specific example of braiding of an electric mode with a magnetic mode, in a hypothetical uniaxial nematic without diffusion. This example will show that frame dragging leads to the same formula for braiding as in the gauge theory case: $\mathcal{R} \cdot |h > |v >= |\alpha(h)v > |h >$.

Place the disclination somewhere in space, and consider the braiding of the two Goldstone modes with the defect. The defect is characterized by a loop around it. At every point of the loop there is an element of G, so that the loop corresponds to a path in G that starts at e and ends in the disconnected component of $SO(2) \rtimes \mathbb{Z}_2$. Now to parallel transport a Goldstone mode along the loop around the defect, let the group elements along the loop act on the Goldstone mode! The endpoint of the loop *is* an element H, so we know where the Goldstone mode will end up: it ends up being $g \cdot |v >$, where g is the endpoint of the loop, and $|v >$ is the Goldstone mode. See fig. 4.3. One of the Goldstone modes doesn't transform, the other one picks up a minus sign. Note that in the figure we have done half the braiding, i.e. we applied R. Once the Goldstone modes have arrived at the left side of the defect, if we braid them underneath the defect nothing changes, because the space under the defect is flat.

Summarizing, the action of braid operator on a defect next to an electric mode is

$$\mathcal{R} \cdot (|g > \otimes|v >) = (|\alpha(g)|v >) \otimes |g > \tag{4.29}$$

$$\mathcal{R} \cdot (|v > \otimes|g >) = |g > \otimes|v > \tag{4.30}$$

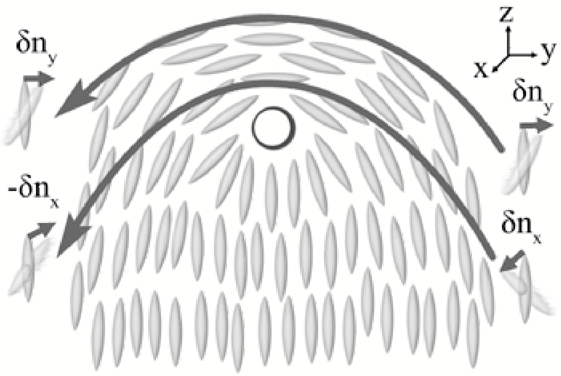


Figure 4.4: The braiding of the Goldstone modes with the disclination of the uniaxial nematic liquid crystal. One Goldstone mode is left unchanged, while the other one picks up a minus sign. In both cases, the result is the action of the group element at the end of the path in G characterizing the defect, which in this case is a rotation of π around the x-axis.

To incorporate the braiding into a Hopf algebra $\mathcal{A}$, we introduce a *universal R matrix* R. R is an element of $\mathcal{A} \otimes \mathcal{A}$. It encodes the braiding of states the irreps of $\mathcal{A}$: to braid two states, $|\phi_1>$ in Π_1 and $|\phi_2>$ in Π_2, act with R on $|\phi_1> \otimes |\phi_2>$, and then apply the flip operator τ:

$$\mathcal{R}(|\phi_1> \otimes |\phi_2>) = \tau \circ (\Pi_1 \otimes \Pi_2) \circ R \circ |\phi_1> \otimes |\phi_2> . \tag{4.31}$$

τ flips any two vectors $|\phi_1>$ and $|\phi_2>$ around:

$$\tau(|\phi_1> \otimes |\phi_2>) = |\phi_2> \otimes |\phi_1> . \tag{4.32}$$

If $|\phi_1>$ is in the vector space V_1, and $|\phi_2>$ in V_2, then $|\phi_1> \otimes |\phi_2>$ is a vector in $V_1 \otimes V_2$. Then $\mathcal{R}(|\phi_1> \otimes |\phi_2>) = \tau \circ (\Pi_1 \otimes \Pi_2) \circ R|\phi_1> \otimes |\phi_2>$ is a vector in $V_2 \otimes V_1$.

The universal R matrix is an invertible element of $\mathcal{A} \otimes \mathcal{A}$, i.e. there is an $R^{-1} \in \mathcal{A} \otimes \mathcal{A}$ which satisfies

$$RR^{-1} = R^{-1}R = 1 \otimes 1. \tag{4.33}$$

$\mathcal{R}$ corresponds to braiding the particle on the right in a counterclockwise fashion halfway around the particle on the left. Using R^{-1}, we can define the inverse braiding, which is the clockwise braiding of the particle on the right halfway around the particle on the left:

$$\mathcal{R}^{-1} = R^{-1} \circ \tau. \tag{4.34}$$

We can write R in Sweedler's notation:

$$R = \sum_{(R)} R^{(1)} \otimes R^{(2)}. \tag{4.35}$$

We can let R act on n-particle states. To do this, We define

$$R_{ij} = \sum_{(R)} 1 \otimes \cdots \otimes R^{(1)} \otimes \cdots \otimes R^{(2)} \otimes \cdots \otimes 1 \tag{4.36}$$

where $R^{(1)}$ is in the i-th, and $R^{(2)}$ in the j-th position. R_{ij} implements the half-braiding of particles i and j. i needn't be smaller than j. For example, on a two particle state $R_{21} = \sum_{(R)} R^{(2)} \otimes R^{(1)}$.

In this notation, we have $R^{-1} \circ \tau = \tau \circ R_{21}$, so that we can rewrite $\mathcal{R}^{-1}$:

$$\mathcal{R}^{-1} = \tau \circ R_{21}^{-1}. \tag{4.37}$$

We will this equation in the next chapter.

For the $D(H)$ case, the universal R matrix is given by

$$R = \sum_{g \in G} P_g e \otimes g. \tag{4.38}$$

The braid operator $\mathcal{R}$ that is derived from this R reproduces the braiding of the different modes discussed in this section.

The universal R matrix satisfies certain axioms discussed in Appendix B. From the axioms it follows that R satisfies the *Yang-Baxter equation*:

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \tag{4.39}$$

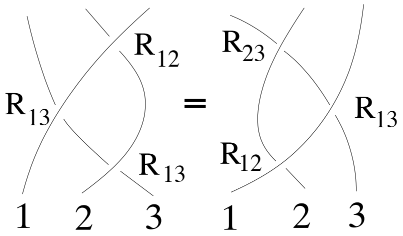


Figure 4.5: The Yang-Baxter equation.

A Hopf algebra with a universal R matrix is called a *quasi-triangular Hopf algebra*. Thus the quantum double $D(\overline{H})$ is quasitriangular.

We do not believe that scattering in real liquid crystals can be treated with this equation, because diffusion dominates. For example, we saw that the Goldstone modes are diffusive, so the cross sections will be of no use here. However, since the Landau theory of phase transitions still applies to liquid crystals, and this theory is independent of diffusion, we do expect that symmetry breaking in liquid crystals can be studied using Hopf algebra's. Namely, phase transitions involve universality classes, i.e. phases with the same symmetry breaking scheme often have the same properties at the phase transition. Diffusion does not alter the properties of the phase transition itself, such as the scaling properties of quantities such as specific heat, and the order of the phase transition. It only alters the properties of the phase obtained after the phase transition. We study symmetry breaking in the next chapter.

Even though diffusion may dominate, it is still interesting to consider braiding in a uniaxial nematic when diffusion is neglected. We called this the one-constant approximation in chapter 3. In that chapter, we analyzed the nontrivial line defect in a uniaxial nematic, and we saw that given a defect configuration, there is a continuous family of defect configurations, labelled by $\theta \in [0, 2\pi)$, that are obtained from the original configuration by applying rotations in the residual symmetry group. We will now show that we can define *coherent superpositions* of line defects in this family that carry an integer charge with respect to the rotations in the residual symmetry group. Furthermore, when a charged particle is braided around a coherent superposition of defects with charge n, the sign of the charge of the particle changes, and the coherent superposition of defects picks up a compensating charge, so that the overal charge is conserved. Thus charge is transferred to the defect. This phenomenon is the *Global analogue of Cheshire Charge.* For a reference on this phenomenon see [41], and for the analogue in Alice Electrodynamics see [13].

4.3.1 Braiding in a uniaxial nematic

The defect classes

We consider the defect configuration shown in fig. 3.4. By rotating by angle θ around the z-axis, we obtain a new configuration with the same line defect charge. Using the ket notation for defects introduced in this chapter, we will denote the defect obtained by a rotation of θ by $|\theta>$. Thus $|0>$ is the original configuration, and $|2\pi>=|0>$. We have a continuous labelling of defects, by an angle $\theta \in [0, 2\pi)$. This case is therefore different from all the cases we have encountered until now in this chapter, because all defect labels encountered until now were discrete.

The residual symmetry group is (see 3.15):

$$\overline{H} = \overline{U(1) \rtimes \mathbb{Z}_2} = \{e^{i\frac{\theta}{2}\sigma_z}, i\sigma_x e^{i\frac{\theta}{2}\sigma_z}, \theta \in [0, 4\pi)\}.$$

to determine the class of the defect $|0>$, we must act on $|0>$ with elements of $\overline{H}$, and see which defects we get. $|0>$ is characterized by a path in G that ends in a point in $\overline{H}_d$, the part of $\overline{H}$ that is not connected to the identity:

$$\overline{H}_d = \{i\sigma_x e^{i\frac{\theta}{2}\sigma_z}, \theta \in [0, 4\pi)\}.$$

We can choose which endpoint corresponds to $|0>$. This choice sets the endpoints for the defects $|\theta>$, as we will now see.

We take the path in G that ends in $i\sigma_x$ as the path that characterizes $|0>$. Then the elements of $\overline{H}$ act on the defect by conjugating the endpoint of the loop (see chapter 2). An element $e^{i\frac{\theta}{2}\sigma_z}$ acts as follows on $|0>$:

$$e^{i\frac{\theta}{2}\sigma_z} i\sigma_x (e^{i\frac{\theta}{2}\sigma_z})^{-1} = e^{i\theta\sigma_z} i\sigma_x. \tag{4.40}$$

Since we defined $|\theta>$ as the defect obtained from $|0>$ by the action of $e^{i\frac{\theta}{2}\sigma_z}$, we see that the endpoint of the path that characterizes $|\theta>$ is $e^{i\theta\sigma_z} i\sigma_x$. We can use a more convenient terminology: the defect $|\theta>$ has a *charge in* $\overline{H}$ given by $e^{i\theta\sigma_z} i\sigma_x$. Thus the charge of the defects is no longer a left coset in $\overline{H}/\overline{H}_0$, but is an element of $\overline{H}$. The classification of line defects in terms of the first homotopy group is too coarse in this case.

Now that we have the charge of the defects $|\theta>$, we can answer any question: for example, how does the residual symmetry transformation $i\sigma_x$ act on the defects? The answer is obtained by conjugating the charge in $\overline{H}$ of the defect. The outcome is

$$i\sigma_x \cdot |\theta>= |-\theta> . \tag{4.41}$$

The residual symmetry transformation $e^{i\frac{\alpha}{2}\sigma_z}$ act as follows:

$$e^{i\frac{\alpha}{2}\sigma_z} \cdot |\theta>= |\theta + \alpha> . \tag{4.42}$$

Since we can obtain any defect $|\theta>$ from any other $|\theta'>$ by applying a global symmetry transformation, all the defects $|\theta>$ are in the same class. The defects $|\theta>$ transform under a representation of $\overline{H}$ with a continuous family of basis vectors. In the next section we will see that this representation is reducible.

Coherent superposition of defects

We can adopt a difference basis for the representation of $U(1) \rtimes \mathbb{Z}_2$ spanned by the $|\theta>$: define a *charge n defect* $|n>_m$ as

$$|n>_m= \int_0^{2\pi} d\theta e^{-in\theta} |\theta> \tag{4.43}$$

We have taken a *coherent superposition* of defects. This transformation is similar to a Fourier transform. We can now calculate the action of $\overline{H}$ on $|n>_m$:

$$i\sigma_x \cdot |n>_m= |-n>_m \tag{4.44}$$

$$e^{i\frac{\alpha}{2}\sigma_z} \cdot |n>_m= e^{i\alpha n} |n>_m . \tag{4.45}$$

Thus $|n>_m$ transforms under an irreducible representation of the subgroup $\overline{U(1)} = \{e^{i\frac{\alpha}{2}\sigma_z} : \alpha \in [0, 2\pi)\}$ of $\overline{H}$, since it picks up a phase factor under the action of elements of $\overline{U(1)}$. Furthermore, the 2π rotation in $\overline{U(1)}$: $u(\hat{z}, 2\pi) = e^{i\frac{2\pi}{2}\sigma_z}$ acts trivially on $|n>_m$. This implies that $|n>$ transforms under an irrep on $U(1) \in SO(3)$ corresponding to rotations around the z-axis. Namely, $\overline{U(1)}$ is the double cover of $U(1) = \{R(\hat{z}, \theta) : \theta \in [0, 2\pi)\}$, and irreps of $\overline{U(1)}$ that are trivial on the 2π rotation correspond to irreps of $U(1)$ (see appendix A for an explanation of the notation). Irreps of $U(1)$ are labelled by an integer, which is called the *charge* of the irrep. Thus from (4.45) we conclude that $|n>_m$ has $U(1)$ *charge* n.

The coherent superposition of defects is a purely quantum mechanical entity, with no classical analogue. Classically, the $|\theta>$ form a degenerate set of states with the same energy. The quantum theory may paint a different picture, depending on the Hamiltonian of the system. It may be the case (as it is in the theory discussed in [41]) that the coherent superpositions of defects, with $U(1)$ charge n, are eigenvectors of the Hamiltonian, in which case they each carry a certain energy. The energy levels are then quantized.

Braiding a defect with a charged particle

The question arises how we should interpret the charge n on the defect $|n>_m$. To study this, we will analyze the braiding of regular excitations with defects, using (4.30).

A regular particle in this system is an irrep of $\overline{H} = \overline{U(1) \rtimes \mathbb{Z}_2}$. We will only consider irreps of $\overline{H}$ that are also irreps of $H = U(1) \rtimes \mathbb{Z}_2$, i.e. we consider only bosonic irreps (just as we did with the tetrahedral nematic, discussed above). H is precisely the semidirect product of an abelian group $U(1)$ with a finite group $\mathbb{Z}_2$. Thus, we can use the Little Group Method described in appendix D to determine its irreps. The outcome is as follows: denote an element of H as $R_\theta X^j$ $(j = 0, 1)$. R_θ corresponds to a rotation of angle θ around the z-axis, and X corresponds to a π rotation around the x-axis. Then the irreps are:

1. One dimensional irreps $\Pi_{0,0}$ and $\Pi_{0,1}$:

$$\Pi_{0,0}(R_\theta X^j) = 1 \tag{4.46}$$

$$\Pi_{0,1}(R_\theta X^j) = (-1)^j. \tag{4.47}$$

2. Two dimensional irreps $\Pi_k,\ k \in \mathbb{Z}$

$$\Pi_k(R_\theta) = \begin{pmatrix} e^{ik\theta} & 0 \\ 0 & e^{-ik\theta} \end{pmatrix} \tag{4.48}$$

$$\Pi_k(X) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{4.49}$$

We denote the basis of the irrep Π_k that corresponds to the matrix representation of Π_k given in (4.49) by $\{|k>, |-k>\}$. The vector $\begin{pmatrix} a \\ b \end{pmatrix}$ on which the matrices act corresponds to $a|k> + b|-k>$. This is very suggestive notation: we see from (4.49) that $|k>$ transforms under an irrep of $U(1)$: $R_\theta|k> = e^{in\theta}|k>$. We say that $|k>$ has $U(1)$ *charge* k. $|-k>$ then has $U(1)$ charge $-k$, since $R_\theta|-k> = e^{in\theta}|-k>$. The operator X maps $|k>$ into $|-k>$ and vice versa. Thus X acts as a *flip operator* on these irreps: it flips the sign of the charge.

Now consider a defect in the state $|n>_m$, and the particle Π_k in the state $|k>$ to the right of it. Using (4.30), we can determine the outcome of counterclockwise half-braiding of these two particles. To do so, we first determine the half-braiding of defect $|\theta>$ with $|k>$. $|\theta>$ has $\overline{H}$ charge ${}^{i\theta\sigma_z} i\sigma_x$. The outcome of half-braiding is the action of $e^{i\theta\sigma_z} i\sigma_x$ on $|k>$. This action is determined by first using the homomorphism from $SU(2)$ to $SO(3)$, that acts as

$$e^{i\theta\sigma_z} i\sigma_x \mapsto R_\theta X.$$

Thus the outcome of half-braiding is

$$\mathcal{R} \cdot (|\theta> \otimes |k>) = \Pi_k(R_\theta X) \cdot |k> \otimes |\theta> = e^{-ik\theta}| - k> \otimes |\theta> . \qquad (4.50)$$

Using the definition of $|n>_m$ $|n>_m = \int_0^{2\pi} d\theta e^{-in\theta} |\theta>$, we can now determine the outcome of half- braiding $|k>$ around $|n>_m$:

$$\begin{aligned} \mathcal{R} \cdot (|n>_m \otimes |k>) &= \int_0^{2\pi} d\theta e^{-in\theta} \mathcal{R} \cdot (|\theta> \otimes |k>) \\ &= \int_0^{2\pi} d\theta e^{-in\theta} e^{-ik\theta} | - k> \otimes |\theta> \\ &= | - k> \otimes |n + k>_m . \end{aligned} \qquad (4.51)$$

Thus the regular particle is now in a state with charge $-k$, and the defect has charge $|n + k>_m$. The overal charge is conserved.

This charge is called *Cheshire charge.*In the gauge theory equivalent of the uniaxial nematic, Alice Electrodynamics, this charge is rather mysterious, because it cannot be localized anywhere. There is a sheet attached to the line defect, and the sheet acts like a charged perfect conductor. However, the position of the sheet is not gauge-invariant: gauge transformation can move the sheet around. Thus the Cheshire is non localizable. See [63] for further discussion.

In the case of the uniaxial nematic, the charge *is* localizable. To prove this requires a knowledge of the formula for the Noether current associated to rotations around the z-axis. It is a function of derivatives of the field $\vec{n}(\vec{x})$ of the uniaxial nematic. We will not go into discuss, see [41] for a derivation of the current.

Braiding of two defects

We can also calculate the outcome of braiding two defects, of charge $|n>_m$ and $|k>_m$, using (4.25). First we braid $|\theta_1>$ with $|\theta_2>$. We introduce a new notation for the defects: $|\theta_i> \equiv |e^{i\theta\sigma_z} i\sigma_x>$, i.e. we can denote a defect by its $\overline{H}$ charge. The action of braiding is the conjugation of the $\overline{H}$ charge. Using

(4.25), we get

$$\begin{aligned}
\mathcal{R} \cdot (|\theta_1 > |\theta_2 >) &= \mathcal{R} \cdot (|e^{i\theta_1 \sigma_z} i\sigma_x > |e^{i\theta_2 \sigma_z} i\sigma_x >) \\
&= |(e^{i\theta_1 \sigma_z} i\sigma_x) e^{i\theta_2 \sigma_z} i\sigma_x (e^{i\theta_1 \sigma_z} i\sigma_x)^{-1} > |e^{i\theta_1 \sigma_z} i\sigma_x > \\
&= |e^{i(2\theta_1 - \theta_2)\sigma_z} i\sigma_x > ||e^{i\theta_1 \sigma_z} i\sigma_x > \\
&= |2\theta_1 - \theta_2 > |\theta_1 > . \qquad (4.52)
\end{aligned}$$

Using this equation, we calculate the outcome of half-braiding $|n >_m$ and $|k >_m$:

$$\begin{aligned}
\mathcal{R} \cdot (|n >_m |k >_m) &= \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 e^{-in\theta_1} e^{-ik\theta_2} \mathcal{R} \cdot (|\theta_1 > |\theta_2 >) \\
&= \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 e^{-in\theta_1} e^{-ik\theta_2} |2\theta_1 - \theta_2 > |\theta_1 > \\
&= | - k >_m |n + 2k >_m . \qquad (4.53)
\end{aligned}$$

Thus the braiding of defects is analogous to the braiding of a defect with a charged particle.

Hopf symmetry description

The Hopf symmetry description requires a continuouos generalized quantum double, $F(\overline{U(1) \rtimes \mathbb{Z}_2}) \otimes \mathbb{C}(U(1) \rtimes \mathbb{Z}_2)$. Quantum doubles of compact groups are described in [32]. The representation theory is analogous to the representation theory of generalized quantum doubles with discrete groups: the irreps are labelled by an orbit in $\overline{U(1) \rtimes \mathbb{Z}_2}$ under the action of $U(1) \rtimes \mathbb{Z}_2$, and by an irrep of the centralizer of a chosen preferred element in the orbit. The orbits, centralizers and irreps are given in table 4.7.

Preferred element	Orbit	Centralizer	Irreps
e	$\{e\}$	$U(1) \rtimes \mathbb{Z}_2$	$\Pi^e_{0,0}$ $\Pi^e_{0,1}$ Π^e_k, $k \in \mathbb{Z}$
$-e$	$\{e^{i\pi\sigma_z}\}$	$U(1) \rtimes \mathbb{Z}_2$	$\Pi^{-e}_{0,0}$ $\Pi^{-e}_{0,1}$ Π^{-e}_k, $k \in \mathbb{Z}$
$e^{i\frac{\theta}{2}\sigma_z}$, $\theta \in [0, 2\pi)$	$\{e^{i\frac{\theta}{2}\sigma_z}, e^{-i\frac{\theta}{2}\sigma_z}\}$	$U(1)$	Π^θ_k, $k \in \mathbb{Z}$
$i\sigma_x$	$\{e^{i\theta\sigma_z} i\sigma_x : \theta \in [0, 2\pi)\}$	$\{e, X\} \simeq \mathbb{Z}_2$	$\Pi^{i\sigma_x}_j$, $j \in \mathbb{Z}_2$

Table 4.7: The irreps of $F(\overline{U(1) \rtimes \mathbb{Z}_2}) \otimes \mathbb{C}(U(1) \rtimes \mathbb{Z}_2)$. For the irreps of $U(1) \rtimes \mathbb{Z}_2$, we used the notation given in (4.47) and (4.49).

The irreps Π^e_α, with α an irrep of $U(1) \rtimes \mathbb{Z}_2$ are the regular irreps. The Π^{-e}_{alpha} irreps are dyons with the 2π rotation in the background, or simply the $-e$ defect if α is trivial. Since we assumed that α was a bosonic irrep, with $-e$ in the background we can still implement the full group $U(1) \rtimes \mathbb{Z}_2$. The Π^θ_k

are intriguing: they correspond to irreps with 'defect' $e^{i\frac{\theta}{2}\sigma_z}$. But $e^{i\frac{\theta}{2}\sigma_z}$ is not a defect, so these irreps can safely be neglected. In the gauge theory case, they are to be interpreted as a configuration with a flux $e^{i\frac{\theta}{2}\sigma_z}$ of the gauge field. This flux is not topological stable, thus $e^{i\frac{\theta}{2}\sigma_z}$ is not a topological quantum number, and we should neglect these irreps.

The $\Pi_j^{i\sigma_x}$ irrep corresponds to the irrep spanned by the $|\theta >$, discussed above. The j label labels an irrep of $\mathbb{Z}_2$. If the irrep of $\mathbb{Z}_2$ is nontrivial, we have a dyonic configuration. However, note that we had already interpreted $\Pi_j^{i\sigma_x}$ as a dyonic configuration, because we defined states coherent superpositions $|n >_m$ of defects, with $U(1)$ charge n. These states can carry a further nontrivial irrep of $\mathbb{Z}_2$.

Chapter 5

Hopf symmetry breaking and confinement

In the previous chapters we thoroughly analyzed the modes in liquid crystals, and found that their fusion rules and braiding properties are captured by a generalized quantum double $F(H_m) \otimes \mathbb{C}H_{el}$. In this chapter we will show that this Hopf symmetry description allows for a systematic investigation of phase transitions in these phases. Since electric, magnetic and dyonic modes are treated on equal footing, we will unify the study of phase transitions induced by all three types of modes. Such an analysis was carried out for the first time in [61]. In that work, certain physical and mathematical constraints were imposed on the residual symmetry after a phase transition. We will encounter situations where some of these constraints must be relaxed, which will require the development of a novel criterion for symmetry breaking.

The electric phase transitions we've come across are characterized by an order parameter ϕ, that acquires a nonzero ground state expectation value ϕ_0 in the ordered phase. The residual symmetry in the ordered phase is the *stabilizer* of ϕ_0, i.e. the subgroup H_{el} of the original symmetry group G whose elements $h \in H_{el}$ satisfy

$$h \cdot \phi_0 = \phi_0. \tag{5.1}$$

The excitations of the ordered phase transform under irreducible representations of the residual symmetry group H_{el}.

As we saw in chapter 2, ϕ_0 corresponds to the expectation of some order parameter $\hat{\phi}$ with respect to a state called the ground state and denoted by $|\phi_0>$:

$$<\phi_0|\hat{\phi}|\phi_0>=\phi_0.$$

In a quantum treatment, the ordered phase corresponds to a phase in which $|\phi_0>$ fills the space, with a certain density. The stabilizer of $|\phi_0>$ is equal to ϕ_0, because $|\phi_0>$ transforms has the same transformation properties under H_{el} as ϕ_0 does:

$$H_{el} = \{h \in G : h \cdot |\phi_0>\}. \tag{5.2}$$

In this chapter we will generalize the concept of a stabilizer to the case of a phase described by a Hopf symmetry $\mathcal{A}$. Given a state $|\phi_0>$ in an irrep of $\mathcal{A}$, we will define a Hopf stabilizer $\mathcal{T}_\mathcal{R}$ of ϕ. $\mathcal{T}_\mathcal{R}$ is a subalgebra of $\mathcal{A}$, and excitations from the ordered phase transform under irreps of $\mathcal{T}_\mathcal{R}$. We give $|\phi_0>$ the suggestive name of *condensate*, and say that $|\phi_0>$ has *condensed.* By studying the residual symmetry for all the possible condensates in a system, we obtain a systematic analysis of all conceivable phases. Determining whether these phases are actually realized requires a dynamical analysis.

A careful analysis of the irreps of the residual symmetry algebra $\mathcal{T}_\mathcal{R}$ reveals that some excitations are not *local* excitations of the condensate. These excitations braid nontrivially with the condensate $|\phi_0>$. Thus the state of the condensate is not singly valued around them. We will see that to create these excitations we must introduce a half line singularity in the condensate. To the left and to the right of this half-line the condensate takes on a different value. This half line goes from the excitation out to infinity, and it costs a finite amount of energy per unit length. We call such an excitation a *confined excitation.* In an infinite system the confined excitations cost an infinite amount of energy. They cannot be excited, unless the half-line ends on another confined excitation, so that the singular line only has finite extent. We call a configuration consisting of confined excitations connected by half-line singularities, such that the overal configuration is unconfined, a *hadronic composite*, in analogy with hadrons in Quantum Chromodynamics, where hadrons are unconfined composites of confined quarks.

The excitations that braid trivially with the condensate are called *unconfined excitations*, and they can be created locally.

In the next section we will introduce confinement by taking a new look at the XY-model. We will reinterpret the phase transition to the ordered state as a *confinement mechanism.* After that we will study symmetry breaking for a phase described by a general Hopf algebra $\mathcal{A}$. Then we will specialize to the case of $\mathcal{A}$ being a generalized quantum double $F(H_m) \otimes \mathbb{C}H_{el}$. We will show that our symmetry breaking formalism reproduces the standard theory of electric condensates. After that, we will address the case of defect condensates. We will find general formulae that describe the residual symmetry, and show that these formulas reproduce the results of the defect phase transitions that we discussed in chapter 3: the Kosterlitz-Thouless phase transition, the transition from a hexagonal crystal to the hexatic phase, and the transition from the hexatic phase to the isotropic phase. Finally, we briefly discuss dyonic condensates.

5.1 Condensate and confinement

The XY model is described by (2.53):

$$F_{eff} = \frac{1}{2}\int d^2x \mu v^2 (\nabla\theta)^2 = \frac{1}{2}\int d^2x \rho_s (\nabla\theta)^2,$$

where θ is a real number with the identification $2\pi n \equiv 0$. This free energy is the low temperature effective free energy of (2.51):

$$F = \frac{1}{2}\int d^2x(\mu(\nabla_i\phi)^*(\nabla_i\phi) - \lambda(T - T_c)(|\phi|^2 - v^2)^2),$$

where $\phi = |\phi|e^{i\theta}$ is a complex order parameter field.

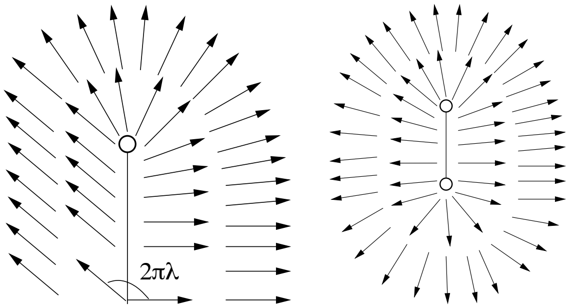
We briefly review the excitations in the ordered phase discussed in chapter 2.

At low temperatures $|\phi| = v$. The symmetry is spontaneously broken from $G = U(1)$ to $H = \{e\}$. This implies the existence of defects labelled by $\Pi_1(\mathbb{R}/\mathbb{Z}) \simeq \mathbb{Z}$. The electric particles transform under irreps of $\mathbb{Z}$. These irreps are labelled by $\lambda \in U(1)$.

The equation that $\theta(\vec{x})$ has to satisfy in order to minimize F_{eff} is Laplace's equation in two dimensions (2.54). We found singular solutions $\theta_{n,(\vec{x_0})}$ that corresponded to a defect of charge n centered at $\vec{x_0}$. There are in fact other singular solutions to (2.54), in which the singularity is a half-line. They are labelled by a $\lambda \in \mathbb{R}$ and a vector $\vec{x_0} = (x_0, y_0)$. Their field configuration is given by

$$\theta_{\lambda,\vec{x}_0}(x, y) = \lambda\ arctan(\frac{x - x_0}{y - y_0}) = \lambda\varphi, \tag{5.3}$$

with φ the polar angle.



(a) A confined charge λ vortex in the XY model

(b) An unconfined composite of two confined charge $\frac{1}{2}$ vortices

Figure 5.1: Confinement of vortices of non-integer charge λ in the ordered phase of the XY-model. In (a) we see a half-line singularity which starts at $\vec{x_0}$ and goes out to infinity. It carries a finite amount of energy per unit length. (b) shows an unconfined composite of two confined charge $\frac{1}{2}$ vortices. The charge $\frac{1}{2}$ vortices are attached by a singular line. The overall configuration has charge 1.

For $\lambda = n \in \mathbb{Z}$, there is no line singularity. For $\lambda \notin \mathbb{Z}$, there is a line starting at $\vec{x_0}$ and going out to infinity, along which $\phi = e^{i\theta}$ is discontinuous. To see that there is a discontinuity, we follow a loop around $\vec{x_0}$, and notice that as we go full circle θ turns by an angle $2\pi\lambda$. If $\lambda \notin \mathbb{Z}$, θ does not return to its original value as we finish travelling along our loop (remember that θ is defined modulo 2π). Thus there is a half line singularity in θ, which implies a half line singularity in ϕ. This line singularity carries a finite amount of energy per unit length. To prove this, fill $|\phi| = v$ into (2.51), and obtain

$$F = \frac{1}{2}\int d^2x\mu(\nabla_i\phi)^*(\nabla_i\phi). \tag{5.4}$$

Call $\phi_{\mathcal{L}}$ the value of the order parameter field just to the left of the singular half-line and $\phi_{\mathcal{R}}$ its value just to the right of it (see fig. 5.1). At the line singularity, we have

$$\nabla\phi = (\phi_{\mathcal{R}} - \phi_{\mathcal{L}})\delta(\varphi), \tag{5.5}$$

since we orient the singular line along $\varphi = 0$, and integrating $\nabla\phi$ across the singularity gives $\phi_{\mathcal{R}} - \phi_{\mathcal{L}}$. If the line L has a length l, its free energy content is[1]

$$F_{line} = \frac{1}{2}\int_L d^2x\mu(\nabla_i\phi)^*(\nabla_i\phi) = \frac{1}{2}\mu|\phi_{\mathcal{R}} - \phi_{\mathcal{L}}|^2 l. \tag{5.6}$$

Thus, if $\lambda \notin \mathbb{Z}$, the free energy of the configuration $\theta_{\lambda,\vec{x}_0}$ increases linearly with the system size[2]. This is not a topological defect, since it does not carry a $\Pi_1(G/H)$ charge. We will call it a vortex with non-integer charge λ, and say that this vortex is *confined*. It is attached to a half-line singularity which corresponds to a *domain wall*, because the line is of one dimension lower than the dimension of the space (see table 2.1). More generally, confinement is defined as follows: an excitation is confined if its energy increases linearly with the system size.

Thus the confined vortex configurations carry an energy which increases linearly with the system sizes. Now remember that the integer charge vortices carry an energy that increases logarithmically with the system size (see (2.56)). We say that the integer charge vortices are *logarithmically confined*. The noninteger charge defects also carry energy outside the half-line singularity, and this energy increases logarithmically with the system size (the calculation is analogous to (2.56)). However, for a large system we can neglect this logarithmically increasing energy compared to the linearly increasing energy of the half-line singularity. We say that the noninteger charge vortices are *linearly confined*, in contrast to logarithmic confinement. In this thesis when we say confinement, we mean linear confinement, unless stated otherwise. Also, when we say a particle is *confined*, we mean that is not linearly confined. It may still be logarithmically confined.

The different confinements for the integer and noninteger charge vortices may have important physical consequences. If the system is at finite temperatures, then the vortex configurations that are thermally excited are those with neutral

---

[1] To be precise, we are integrating the free energy within a thin stroke containing the half-line singularity.

[2] When we talk of system size, we mean the length of one axis of the system.

overal vorticity, because of the logarithmic confinement of the vortices (see chapter 3). In chapter 3 we stated that we can consider only vortex-antivortex pairs at low temperatures, because configurations with more than two vortices cost more energy, and vortices of the same sign also cost more energy. There may be vortex-antivortex pairs where the constituent vortices have integer charge, and others where the constituents have noninteger charge. Because the integer charge vortices are logarithmically confined, and the noninteger charge are linearly confined, it is reasonable to assume that there are different length scales associated to the integer and noninteger vortex-antivortex pairs. The average separation at a certain temperature between the integer charge vortices may be much larger than the average separation between the noninteger charge vortices. Thus there may be temperatures where the only degrees of freedom are the integer charge vortices. In the ordered phase of the XY model, for example, we can safely neglect the noninteger charge vortices, even though they can be thermally activated.

In the equivalent of the XY model with local symmetries discussed in chapter 2, the integer charge vortices are not logarithmically confined, and there is no need to consider different length scales, since there only is one type of confinement: linear confinement. Later on, we will discuss the implications of logarithmic confinement, but for now we can neglect, and consider the effect on linear confinement only.

Using the concept of confinement, the phase transition from the high temperature phase to the ordered phase of the XY model can be interpreted as follows: before symmetry breaking, vortices of any charge are unconfined. Namely, we can have $|\phi| \mapsto 0$ as we approach the line at which θ is singular, so that there is no singularity in the order parameter field $\phi = |\phi| e^{i\theta}$. After symmetry breaking, $|\phi| = v$, and the vortices of non-integer charge are confined. It is conceivable that there are noninteger charges in this phase, connected together by half-line singularities, such that the overal charge is integer. We call a composite of confined particles that is unconfined as a whole a *hadronic composite*. See fig. 5.1 for an example of a hadronic composite of two charge $\frac{1}{2}$ vortices.

The crucial characteristic of confined excitations is that the condensate takes on a different value to the left and right of the excitation. We had started by condensing the order parameter field, such that it took on the value ϕ_0 everywhere. Now we have an excitation such that the order parameter field takes the value $\phi_{\mathcal{L}}$ to the left, and $\phi_{\mathcal{R}}$ to the right of the half-line singularity connected to the excitation. So we ask ourselves: where does the condensate take on the value ϕ_0? To answer this, we must make a choice with respect to the excitations. We must specify whether the excitation comes in from the left, or from the right. We choose to have all particles coming in from the left, both confined and unconfined particles. Thus, we set $\phi_0 = \phi_{\mathcal{R}}$, and as a confined excitation comes in from the left the condensate to the left of the excitation takes on the value $\phi_{\mathcal{L}} \neq \phi$. If an unconfined excitation comes in from the left, then $\phi_{\mathcal{L}} = \phi_{\mathcal{R}} = \phi_0$.

We now restate these statements on the value of the condensate in terms of the *state* $|\phi_0>$ of the condensate. We specify that the state of the condensate is $|\phi_{\mathcal{R}}>=|\phi_0>$ to the right of the system. Particles come in from the left, and they are confined if the state of the condensate to the left of the particle is not equal to $|\phi_{\mathcal{R}}>$, thus $|\phi_{\mathcal{L}}>\neq|\phi_{\mathcal{R}}>$.

Imagine a particle $|v>$ coming in from the left. We want to know how to tell if $|v>$ is confined of not. Thus we need to know if $|\phi_{\mathcal{L}}>=|\phi_{\mathcal{R}}>$, in which case $|v>$ is unconfined. This is where braiding comes in: $|\phi_{\mathcal{L}}>$ is the outcome of half-braiding $|\phi_{\mathcal{R}}>$ counterclockwise around $|v>$. Thus, to find out whether $|v>$ is confined, we braid the condensate around the excitation, using the braid operator $\mathcal{R}$. For $|v>$ to be unconfined, the condensate $|\phi_{\mathcal{R}}>$ has to braid trivially both clockwise and anticlockwise around $|v>$. We must check that both braidings are trivial, because the half-line singularity could run along the positive or the negative x-axis in fig. 5.1. This leads to the following test of whether an excitation is confined or unconfined:

$$|v> \text{ is unconfined}$$
$$\Longleftrightarrow$$
$$\left\{\begin{array}{r}\mathcal{R}\cdot(|v>\otimes|\phi_{\mathcal{R}}>)=|\phi_{\mathcal{R}}>\otimes|v>\\ \mathcal{R}^{-1}\cdot(|v>\otimes|\phi_{\mathcal{R}}>)=|\phi_{\mathcal{R}}>\otimes|v>.\end{array}\right.$$

Let us apply this criterion to the vortices in the XY model, to see which ones are unconfined. Denote a vortex of charge λ by $|\lambda>$. We set $|\phi_{\mathcal{R}}>=v$. Then $|\phi_{\mathcal{R}}>$ gets frame dragged as we braid it counterclockwise around $|\lambda>$, and picks up a phase factor $e^{i2\pi\lambda}$. Clockwise braiding is trivial, since we've chosen the convention that the clockwise braiding of an electric mode to the right of a defect is trivial[3]. Thus

$$\begin{aligned}\mathcal{R}\cdot(|\lambda>\otimes|\phi_{\mathcal{R}}>)&=e^{i2\pi\lambda}|\phi_{\mathcal{R}}>\otimes|\lambda>\\ \mathcal{R}^{-1}\cdot(|\lambda>\otimes|\phi_{\mathcal{R}}>)&=|\phi_{\mathcal{R}}>\otimes|\lambda>,\end{aligned}$$

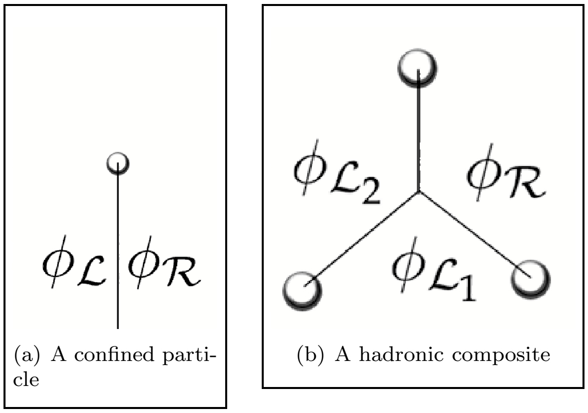
and a vortex is unconfined $\Longleftrightarrow$ $\lambda\in\mathbb{Z}$. Thus the defects are precisely the unconfined vortices.

In the high temperature phase, the "electric" excitations are irreps of the electric group $U(1)$. If we also consider projective irreps, then the electric excitations are irreps of the universal covering group $\overline{U(1)}=\mathbb{R}$ of $U(1)$. The irreps are denoted by ρ_r, they are labelled by an $r\in R$. All these irreps braid trivially with the condensate, because the condensate is in a state $|\phi_{\mathcal{R}}>$ in an electric irrep, and electric irreps braid trivially with each other.

In summary, the phase transition from the high temperature to the ordered phase in the XY model can be described as a confinement mechanism. In this chapter, we will stress the importance of confinement in the discussion of conceivable phase transitions.

---

[3] In a gauge theory, this amounts to picking a gauge such that the Dirac string points 'upwards' in the drawings. In the global case, we orient the defect such that the frame dragging happens in the upper half plane.



(a) A confined particle

(b) A hadronic composite

Figure 5.2: Confinement of particles in the ordered phase, and hadronic composites. The dark lines are singularities in the condensate and cost a finite amount of energy per unit length.

5.2 Hopf Symmetry breaking

In this section we will first study symmetry breaking for a phase described by a general Hopf algebra $\mathcal{A}$. Then we will specialize to the cases where $\mathcal{A}$ is a generalized quantum double: $\mathcal{A} = F(H_m \times H_{el})$. The quantum double $D(H)$ is a special case of the generalized quantum double, with $H_m = H_{el} = H$.

5.2.1 The criterion for symmetry breaking

We consider any physical system whose particles are irreps of a quasitriangular Hopf algebra $\mathcal{A}$. A state $|\phi_{\mathcal{R}}>$ of an irrep Π of $\mathcal{A}$ condenses, so that our ground state is a state filled with the particles in the state $|\phi_{\mathcal{R}}>$. We will now define the *residual symmetry algebra* $\mathcal{T}_{\mathcal{R}}$, which is an algebra such that the excitations from the condensate are irreps of $\mathcal{T}_{\mathcal{R}}$.

The residual symmetry algebra $\mathcal{T}_{\mathcal{R}}$ consists of the operator that are 'well defined with respect to the condensate'. Before we explain what that means, let us first consider the case of a system whose symmetry is a group G. G is broken spontaneously to H by the condensation of a state $|\phi_{\mathcal{R}}>$ of an irrep of G. H is the stabilizer of $|\phi_{\mathcal{R}}>$, i.e. the set of symmetry transformations that leave the condensate invariant. This means that we can implement these transformations on excitations of the condensate, because they leave the condensate invariant. Thus they only act on the excitations. Since the symmetry transformations commute with the Hamiltonian, they transform low energy excitations into low energy excitations.

Now consider a particle $|v>$ in an irrep of the original symmetry G. If we fuse $|v>$ with $|\phi_{\mathcal{R}}>$, and act on the outcome $|v>|\phi_{\mathcal{R}}>$ of this fusion with an $h \in H$, the result is the same as when we act on $|v>$ with h first, and then fuse the outcome with $|\phi_{\mathcal{R}}>$. This follows from

$$h \cdot (|v> \otimes |\phi_{\mathcal{R}}>) = (h \cdot |v>) \otimes (h \cdot |\phi_{\mathcal{R}}>) = (h \cdot |v>) \otimes |\phi_{\mathcal{R}}>, \qquad (5.7)$$

since $h \cdot |\phi_{\mathcal{R}}> = |\phi_{\mathcal{R}}>$. Thus we can also define the residual symmetry group H to be the set of transformations that *don't notice fusion with the condensate*. Whether a particle $|v>$ in an irrep of G fuses with the condensate or not, the action of H is the same. Thus we define *residual symmetry operators* to be operators that don't notice fusion with the condensate. When the symmetry is spontaneously broken from G to H, the residual symmetry operators are the elements of H.

This definition of residual symmetry operator can be generalized to the case of a condensate $|\phi_{\mathcal{R}}>$ in an irrep of a Hopf symmetry $\mathcal{A}$. The residual symmetry operators are the operators that don't notice fusion with the condensate. The residual symmetry operators form a subalgebra of $\mathcal{A}$, which we call the *residual symmetry algebra* $\mathcal{T}_{\mathcal{R}}$. There is a subtlety in the definition of $\mathcal{T}_{\mathcal{R}}$: an operator $a \in \mathcal{A}$ is an element of $\mathcal{T}_{\mathcal{R}}$ if its action on any particle $|v>$ in any irrep of $\mathcal{A}$ is the same whether $|v>$ has fused with the condensate $|\phi_{\mathcal{R}}>$ or not. But we must specify whether $|v>$ fuses with $|\phi_{\mathcal{R}}>$ from the left or the right! Namely, in the systems we are considering $|v> \otimes |\phi_{\mathcal{R}}>$ and $|\phi_{\mathcal{R}}> \otimes |v>$ are not (necessarily) the same state, because of the possibility of nontrivial braiding. Thus we have to fix a convention. This convention is set by our earlier choice of having all particles come in from the left. Remember that we had to make this choice because some particles are confined. Thus we define $\mathcal{T}_{\mathcal{R}}$ as follows: $a \in \mathcal{T}_{\mathcal{R}} \iff$ for any particle $|v>$ in any irrep of $\mathcal{A}$, we have

$$a \cdot (|v> \otimes |\phi_{\mathcal{R}}>) = (a \cdot |v>) \otimes |\phi_{\mathcal{R}}> . \qquad (5.8)$$

Since $|v>$ and $|\phi_{\mathcal{R}}>$ are states of particles of $\mathcal{A}$, their fusion is set by the coproduct Δ of $\mathcal{A}$:

$$a \cdot (|v> \otimes |\phi_{\mathcal{R}}>) = (\Pi \otimes \Pi_{\phi}) \circ \Delta(a)(|v> \otimes |\phi_{\mathcal{R}}>), \qquad (5.9)$$

where $|v>$ is in the irrep Π of $\mathcal{A}$, and $|\phi_{\mathcal{R}}>$ in the irrep Π_{ϕ} of $\mathcal{A}$. Since this equation has to hold for all vectors $|v>$ in all irreps Π of $\mathcal{A}$, it is equivalent to[4]

$$(1 \otimes \Pi_{\phi})\Delta(a)(1 \otimes |\phi_{\mathcal{R}}>) = a \otimes |\phi_{\mathcal{R}}> . \qquad (5.10)$$

$\mathcal{T}_{\mathcal{R}}$ consists of all operators that satisfy this criterion. We will now prove that $\mathcal{T}_{\mathcal{R}}$ is a subalgebra of $\mathcal{A}$. We also prove two more properties of $\mathcal{T}_{\mathcal{R}}$ which will play a role later on.

Lemma 1. *The elements of a finite dimensional Hopf algebra $\mathcal{A}$ that satisfy*

$$(1 \otimes \Pi_{\phi})\Delta(a)(1 \otimes |\phi_{\mathcal{R}}>) = a \otimes |\phi_{\mathcal{R}}>$$

form a subalgebra $\mathcal{T}_{\mathcal{R}}$ of $\mathcal{A}$ that satisfies:

---

[4] We note that Joost Slingerland has independently discovered this mathematical structure, and some of its properties, and we are grateful for private discussions.

1. $\Delta(\mathcal{T_R}) \subset \mathcal{A} \otimes \mathcal{T_R}$
2. *The elements of* $\mathcal{T_R}$ *leave tensor products of the vacuum invariant.*

Proof. That $\mathcal{T_R}$ is an algebra follows from the fact that Δ is an algebra morphism.

1. Using the definition of $\mathcal{T_R}$ and the coassociativity of the coproduct,

$$\begin{aligned}
&(1 \otimes 1 \otimes \Pi_\phi)(1 \otimes \Delta)\Delta(a)(1 \otimes |\phi_{\mathcal{R}} >) = (1 \otimes 1 \otimes \Pi_\phi)(\Delta \otimes 1)\Delta(a)(1 \otimes |\phi_{\mathcal{R}} >) = \\
&= (\Delta \otimes 1)(1 \otimes \Pi_\phi)\Delta(a)(1 \otimes |\phi_{\mathcal{R}} >) = (\Delta \otimes 1)(a \otimes |\phi_{\mathcal{R}} >= \sum_{(a)} a^{(1)} \otimes a^{(2)} \otimes |\phi_{\mathcal{R}} > \\
&\Rightarrow \sum_{(a)} a^{(1)} \otimes (\Delta(a^{(2)}) \cdot (1 \otimes |\phi_{\mathcal{R}} >)) = \sum_{(a)} a^{(1)} \otimes a^{(2)} \otimes |\phi_{\mathcal{R}} > \\
&\Rightarrow \Delta(a^{(2)}) \cdot (1 \otimes |\phi_{\mathcal{R}} >) = a^{(2)} \otimes \phi \\
&\Rightarrow a^{(2)} \in \mathcal{T_R} \\
&\Rightarrow \Delta(a) \in \mathcal{A} \otimes \mathcal{T_R}
\end{aligned}$$

2. We just proved that $(1 \otimes \Pi_\phi)\Delta(a^{(2)})(1 \otimes |\phi_{\mathcal{R}} >) = a^{(2)} \otimes |\phi_{\mathcal{R}} >$. Using this, we get

$$\begin{aligned}
&(1 \otimes \Pi_\phi \otimes \Pi_\phi)(1 \otimes \Delta)\Delta(a)(1 \otimes |\phi_{\mathcal{R}} > \otimes |\phi_{\mathcal{R}} >) \\
&= \sum_{(a)} a^{(1)} \otimes ((\Pi_\phi \otimes 1)(1 \otimes \Pi_\phi)\Delta(a^{(2)})(1 \otimes |\phi_{\mathcal{R}} >)(|\phi_{\mathcal{R}} > \otimes 1)) \\
&= \sum_{(a)} a^{(1)} \otimes ((\Pi_\phi \otimes 1)(a^{(2)} \otimes |\phi_{\mathcal{R}} >)(|\phi_{\mathcal{R}} > \otimes 1)) \\
&= ((1 \otimes \Pi_\phi)\Delta(a)(1 \otimes |\phi_{\mathcal{R}} >)) \otimes |\phi_{\mathcal{R}} > \\
&= a \otimes |\phi_{\mathcal{R}} > \otimes |\phi_{\mathcal{R}} >
\end{aligned}$$

□

We call $\mathcal{T_R}$ the *right residual symmetry algebra*.
What if we condense the sum of two vectors in different irreps, $|\phi_1 > + |\phi_2 >$? According to (5.10), $a \in \mathcal{A}$ is part of the right residual symmetry algebra $\mathcal{T_R}$ of $|\phi_1 > + |\phi_2 >$ if

$$(1 \otimes \Pi_{\phi_1})(\Delta(a))(1 \otimes |\phi_1 >) + (1 \otimes \Pi_{\phi_2})(\Delta(a))(1 \otimes |\phi_2 >) = a \otimes |\phi_1 > + a \otimes |\phi_2 > . \tag{5.11}$$

Since we are dealing with irreps, the only way to get an equality is by equating the first terms on the left-hand side and right-hand side of this equation, and the last terms, separately. So $\mathcal{T_R} = (\mathcal{T_R})_1 \cap (\mathcal{T_R})_2$, the intersection of the right residual symmetry algebras of $|\phi_1 >$ and $|\phi_2 >$. Therefore, we need only treat the condensation of vectors in one irrep, since we can then take intersection of the right residual symmetry algebras of condensates in different irreps to get the right residual symmetry algebra for any condensate.

We will now specialize to the case where $\mathcal{A}$ is a generalized quantum double: $\mathcal{A} = F(H_m) \otimes \mathbb{C}H_{el}$, where H_m and H_{el} are groups (see Appendix B).

The irreps of $\mathcal{A} = F(H_m) \otimes \mathbb{C}H_{el}$ are labelled by an orbit A in H_m under the action of H_{el}, and an irrep of the normalizer N_A of a preferred element $g_A \in A$.

In the previous chapter we showed how to write elements of $F(H_m) \otimes \mathbb{C}H_{el}$ as functions $f \in F(H_m \times H_{el})$. In this notation the derivations to come are more elegant. Consider symmetry breaking by a condensate $|\phi_{\mathcal{R}}>$ in an irrep Π^A_α of $\mathcal{A}$.

Lemma 2. *Take* $f \in F(H_m \times H_{el})$*. Then*

$$f \in \mathcal{T}_{\mathcal{R}} \iff f(x_1(x{\cdot}g_A), y_1)|\phi_{\mathcal{R}}(y_1^{-1}x) >= f(x_1, y_1)|\phi_{\mathcal{R}}(x)> \quad \forall x_1 \in H_m, x, y_1 \in H_{el} \tag{5.12}$$

Proof. We use the formulase in Appendix B.

$$\begin{aligned}
& (id \otimes \Pi^A_\alpha)\Delta(f)(1 \otimes |\phi_{\mathcal{R}}(x)>) = f \otimes |\phi_{\mathcal{R}}(x)> \\
\iff \quad & 1 \otimes \sum_{z \in H_{el}} \Delta(f)(x_1, y_1; x \cdot g_A, z)|\phi_{\mathcal{R}}(z^{-1}x) >= f(x_1, y_1) \otimes |\phi_{\mathcal{R}}(x)> \\
\iff \quad & 1 \otimes \sum_{z \in H_{el}} f(x_1(x \cdot g_A), y_1)\delta_{y_1}(z)|\phi_{\mathcal{R}}(z^{-1}x) >= f(x_1, y_1) \otimes |\phi_{\mathcal{R}}(x)> \\
\iff \quad & 1 \otimes f(x_1(x \cdot g_A), y_1)|\phi_{\mathcal{R}}(y_1^{-1}x) >= f(x_1, y_1) \otimes |\phi_{\mathcal{R}}(x)>
\end{aligned}$$

□

$\mathcal{T}_{\mathcal{R}}$ was obtained by condensing $|\phi_{\mathcal{R}}>$ to the right of our system. If we choose to condense $|\phi_{\mathcal{R}}>$ to the left, we get another residual symmetry algebra:

$$\mathcal{T}_{\mathcal{L}} = \{a \in \mathcal{A} : (\Pi_\phi \otimes id)\Delta(a)(|\phi_{\mathcal{R}} > \otimes 1) = |\phi_{\mathcal{R}} > \otimes a\}. \tag{5.13}$$

We call $\mathcal{T}_{\mathcal{L}}$ the *left residual symmetry algebra.*

It is interesting to compare this criterion to the one in [61]. There, the *residual symmetry algebra* is denoted by $\mathcal{T}$, and is defined as the largest Hopf subalgebra of $\mathcal{A}$ whose elements satisfy

$$a \cdot |\phi_{\mathcal{R}} >= \epsilon(a)|\phi_{\mathcal{R}}> . \tag{5.14}$$

The motivation for this criterion is: The residual symmetry operators act on the condensate like the vacuum irrep ϵ does.

The important difference between $\mathcal{T}$ and $\mathcal{T}_{\mathcal{R}}$ is that we don't require $\mathcal{T}_{\mathcal{R}}$ to be a Hopf algebra. In particular, we don't expect the residual symmetry to have a coproduct. We can give a heuristic reason why $\mathcal{T}_{\mathcal{R}}$ needn't have a coproduct: we have chosen thr condensate to be $|\phi_{\mathcal{R}}>$ on the right. Bring in a confined excitation from the left. The condensate takes on the value $|\phi_{\mathcal{L}}>$ to the left of this excitation. Now consider a second particle coming in from the left. It sees the condensate $|\phi_{\mathcal{L}}>$, and it is therefore an excitation of the residual symmetry of $|\phi_{\mathcal{L}}>$, which needn't be equal to the residual symmetry of $|\phi_{\mathcal{R}}>$! Thus, there is an *ordering* in the particles, i.e. it is crucial to know in which order we brought in the particles from the left. We will see that this translates itself in the absence of a coproduct in $\mathcal{T}_{\mathcal{R}}$, namely we can't simply fuse irreps of $\mathcal{T}_{\mathcal{R}}$. Before we can discuss this, we need to know more about our residual symmetry algebras.

5.2.2 Relationship between $\mathcal{T}$, $\mathcal{T}_{\mathcal{R}}$ and $\mathcal{T}_{\mathcal{L}}$

There are very interesting connections between $\mathcal{T}$, $\mathcal{T}_{\mathcal{R}}$ and $\mathcal{T}_{\mathcal{L}}$. All the operators in $\mathcal{T}_{\mathcal{R}}$ and $\mathcal{T}_{\mathcal{L}}$ satisfy (5.14):

$$a \cdot |\phi_{\mathcal{R}}>= \epsilon(a)|\phi_{\mathcal{R}}> .$$

Thus the operators of $\mathcal{T}_{\mathcal{R}}$ and $\mathcal{T}_{\mathcal{L}}$ act on the condensate like the vacuum irrep ϵ does, just as the operators of $\mathcal{T}$ do. At the same time, we have the inclusions

$$\mathcal{T} \subset \mathcal{T}_{\mathcal{R}} \quad \mathcal{T} \subset \mathcal{T}_{\mathcal{L}}. \tag{5.15}$$

The left and right residual symmetry algebras encompass $\mathcal{T}$. We prove these statements in the following lemma.

Lemma 3. *$\mathcal{T}_{\mathcal{R}}, \mathcal{T}_{\mathcal{L}}$ and $\mathcal{T}$ satisfy the following:*

1. *All elements of $\mathcal{T}_{\mathcal{R}}$ and of $\mathcal{T}_{\mathcal{L}}$ satisfy (5.14):*

$$a \cdot |\phi_{\mathcal{R}}>= \epsilon(a)|\phi_{\mathcal{R}}>$$

2. *$\mathcal{T} \subset \mathcal{T}_{\mathcal{R}} \cap \mathcal{T}_{\mathcal{L}}$*

Proof. 1. $a \in \mathcal{T}_{\mathcal{R}}$ implies:

$$\begin{aligned} & (id \otimes \Pi_\phi)\Delta(a) 1 \otimes |\phi_{\mathcal{R}}>= a \otimes |\phi_{\mathcal{R}}> \\ \Rightarrow\quad & (\epsilon \otimes 1)(1 \otimes \Pi_\phi)\Delta(a) 1 \otimes |\phi_{\mathcal{R}}>= \epsilon(a) \otimes |\phi_{\mathcal{R}}> \\ \Rightarrow\quad & (1 \otimes \Pi_\phi)(\epsilon \otimes 1)\Delta(a) 1 \otimes |\phi_{\mathcal{R}}>= \epsilon(a) \otimes |\phi_{\mathcal{R}}> \\ \Rightarrow\quad & 1 \otimes \Pi_\phi(a)|\phi_{\mathcal{R}}>= \epsilon(a) \otimes |\phi_{\mathcal{R}}> \\ \Rightarrow\quad & \Pi_\phi(a)|\phi_{\mathcal{R}}>= \epsilon(a)|\phi_{\mathcal{R}}>, \end{aligned}$$

where we used one the axioms of a Hopf algebra: $(\epsilon \otimes id)\Delta(a) = 1 \otimes a$. This proves the claim for $\mathcal{T}_{\mathcal{R}}$. The proof for $\mathcal{T}_{\mathcal{L}}$ is analogous, and left to the reader.

2. $\mathcal{T}$ is a Hopf algebra, so

$$\begin{aligned} a \in \mathcal{T} \quad \Rightarrow\quad & \Delta(a) = \sum_{(a)} a^{(1)} \otimes a^{(2)} \in \mathcal{T} \otimes \mathcal{T} \Rightarrow a^{(2)} \in \mathcal{T} \Rightarrow a^{(2)} \cdot |\phi_{\mathcal{R}}>= \epsilon(a^{(2)})|\phi_{\mathcal{R}}> \\ \Rightarrow\quad & (id \otimes \Pi_\phi)\Delta(a)(1 \otimes |\phi_{\mathcal{R}}>) = (id \otimes \epsilon)\Delta(a)(1 \otimes |\phi_{\mathcal{R}}>) = a \otimes |\phi_{\mathcal{R}}> \\ \Rightarrow\quad & a \in \mathcal{T}_{\mathcal{R}}. \end{aligned}$$

This proves $\mathcal{T} \subset \mathcal{T}_{\mathcal{R}}$. We leave the proof that $\mathcal{T} \subset \mathcal{T}_{\mathcal{L}}$ to the reader.

□

$\mathcal{T}_{\mathcal{R}}$ and $\mathcal{T}_{\mathcal{L}}$ are not necessarily Hopf algebras, while $\mathcal{T}$ is a Hopf algebra by its definition. It turns out that $\mathcal{T}_{\mathcal{R}}$ is a Hopf algebra $\iff \mathcal{T}_{\mathcal{R}} = \mathcal{T}$! Similarly, $\mathcal{T}_{\mathcal{L}}$ is a Hopf algebra $\iff \mathcal{T}_{\mathcal{L}} = \mathcal{T}$. Also, $\mathcal{T}_{\mathcal{R}} = \mathcal{T} \iff \mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{L}}$. Thus $\mathcal{T}_{\mathcal{R}}$ and $\mathcal{T}_{\mathcal{L}}$ are very interesting extensions of $\mathcal{T}$: if they are equal to each other, then they are equal to $\mathcal{T}$. Thus, the difference between $\mathcal{T}_{\mathcal{R}}$ and $\mathcal{T}_{\mathcal{L}}$ is a measure of the departure of $\mathcal{T}_{\mathcal{R}}$ (and $\mathcal{T}_{\mathcal{L}}$) from being a Hopf algebra.

We need one assumption about $\mathcal{A}$ to prove these claims: the antipode S of $\mathcal{A}$ must satisfy $S^2 = id$. Generalized quantum doubles satisfy this assumption, for example.

First we prove a little lemma.

Lemma 4. *If the antipode S of $\mathcal{A}$ satisfies $S^2 = id$, then $S(\mathcal{T}_{\mathcal{L}}) = \mathcal{T}_{\mathcal{R}}$ and $S(\mathcal{T}_{\mathcal{R}}) = \mathcal{T}_{\mathcal{L}}$.*

Proof. According to $(B.4)$, and using $S^2 = id$:

$$\begin{aligned} &\Delta^{op} \circ S = (S \otimes S) \circ \Delta \\ &\Rightarrow \Delta^{op} = (S \otimes S) \circ \Delta \circ S \end{aligned}$$

Say $a \in \mathcal{T}_{\mathcal{L}}$, then using the last equation

$$\begin{aligned} &(\Pi_\phi \otimes 1)\Delta(a)|\phi_{\mathcal{R}} > \otimes 1 = |\phi_{\mathcal{R}} > \otimes a \\ &\Rightarrow (1 \otimes \Pi_\phi)\Delta^{op}(a) 1 \otimes |\phi_{\mathcal{R}} >= a \otimes |\phi_{\mathcal{R}} > \\ &\Rightarrow (1 \otimes \Pi_\phi)(S \otimes S)\Delta(S(a)) 1 \otimes |\phi_{\mathcal{R}} >= (a \otimes 1)(1 \otimes |\phi_{\mathcal{R}} >) \\ &\text{Apply } S \otimes S \text{ to left the and right, and use } S^2 = id \text{ and } S(1) = 1 \text{ to get} \\ &\Rightarrow (1 \otimes \Pi_\phi)\Delta(S(a)) 1 \otimes |\phi_{\mathcal{R}} >= S(a) \otimes |\phi_{\mathcal{R}} > \\ &\Rightarrow (1 \otimes \Pi_\phi)\Delta(S(a)) 1 \otimes |\phi_{\mathcal{R}} >= S(a) \otimes |\phi_{\mathcal{R}} > \\ &\Rightarrow S(a) \in \mathcal{T}_{\mathcal{R}} \end{aligned}$$

So $S(\mathcal{T}_{\mathcal{L}}) \subseteq \mathcal{T}_{\mathcal{R}}$. Similarly we can prove that $S(\mathcal{T}_{\mathcal{R}}) \subseteq \mathcal{T}_{\mathcal{L}}$. Since S is invertible, we have $dim(S(\mathcal{T}_{\mathcal{R}})) = dim(\mathcal{T}_{\mathcal{R}})$ and $dim(S(\mathcal{T}_{\mathcal{L}})) = dim(\mathcal{T}_{\mathcal{L}})$, where dim is the dimension as a vector space. Therefore, using $S(\mathcal{T}_{\mathcal{L}}) \subseteq \mathcal{T}_{\mathcal{R}}$ and $S(\mathcal{T}_{\mathcal{R}}) \subseteq \mathcal{T}_{\mathcal{L}}$, we get

$$dim(S(\mathcal{T}_{\mathcal{L}})) \leq dim(\mathcal{T}_{\mathcal{R}}) = dim(S(\mathcal{T}_{\mathcal{R}})) \leq dim(\mathcal{T}_{\mathcal{L}}) = dim(S(\mathcal{T}_{\mathcal{L}})). \tag{5.16}$$

Thus $dim(\mathcal{T}_{\mathcal{R}}) = dim(S(\mathcal{T}_{\mathcal{L}}))$. Since $S(\mathcal{T}_{\mathcal{L}}) \subseteq \mathcal{T}_{\mathcal{R}}$, we must have $S(\mathcal{T}_{\mathcal{L}}) = \mathcal{T}_{\mathcal{R}}$. Applying S to both side of this equations, we get $\mathcal{T}_{\mathcal{L}} = S(\mathcal{T}_{\mathcal{R}})$. □

This lemma states that the antipode S brings us from $\mathcal{T}_{\mathcal{R}}$ to $\mathcal{T}_{\mathcal{L}}$, and back. Remember that S is used to construct the antiparticle or conjugate irrep of a given irrep. Thus, going from $\mathcal{T}_{\mathcal{R}}$ to $\mathcal{T}_{\mathcal{L}}$ is tantamount to replacing all particles by their antiparticles!

Using lemma 4, we can prove all our claims about the relationships between $\mathcal{T}$, $\mathcal{T}_{\mathcal{R}}$ and $\mathcal{T}_{\mathcal{L}}$.

Proposition 1. *For $\mathcal{A}$ an n-dimensional Hopf algebra whose antipode S satisfies $S^2 = id$, we have*

$(1)\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{L}} \iff (2)$ *$\mathcal{T}_{\mathcal{R}}$ is a Hopf algebra* $\iff (3)\mathcal{T}_{\mathcal{R}} = \mathcal{T} \iff (4)\mathcal{T}_{\mathcal{L}} \subset \mathcal{T}_{\mathcal{R}}$

Proof.

- $(1) \Rightarrow (2)$

  We assume $\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{L}}$. Take $a \in \mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{L}}$. To prove that $\mathcal{T}_{\mathcal{R}}$ is a Hopf subalgebra of $\mathcal{A}$, we need to prove three things:

  $$1 \in \mathcal{T}_{\mathcal{R}}, \quad \Delta(a) \in \mathcal{T}_{\mathcal{R}} \otimes \mathcal{T}_{\mathcal{R}}, \quad S(\mathcal{T}_{\mathcal{R}}) \subset \mathcal{T}_{\mathcal{R}}. \tag{5.17}$$

  The first demand is trivial, because $\Delta(1) = 1 \otimes 1$, so that

  $$(1\otimes\Pi_\phi)\Delta(1)(1\otimes|\phi_{\mathcal{R}}>) = 1\otimes\Pi_\phi(1)|\phi_{\mathcal{R}}> = 1\otimes|\phi_{\mathcal{R}}> \Rightarrow 1 \in \mathcal{T}_{\mathcal{R}}. \tag{5.18}$$

  For the second demand: since $a \in \mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{L}}$, we have $\Delta(a) \in \mathcal{T}_{\mathcal{L}} \otimes \mathcal{A} = \mathcal{T}_{\mathcal{R}} \otimes \mathcal{A}$, and $\Delta(a) \in \mathcal{A} \otimes \mathcal{T}_{\mathcal{R}}$. Choose a basis $\{r_i\}_{1\le i\le k}$ of $\mathcal{T}_{\mathcal{R}}$, and a basis $\{a_j\}_{k+1\le j\le n}$ of $\mathcal{T}_{\mathcal{R}}^{\perp}$. Then

  $\Delta(a) \in \mathcal{T}_{\mathcal{R}} \otimes \mathcal{A} \Rightarrow \Delta(a) = \sum r_i \otimes a'_i$

  Write a'_i out in terms of the bases $\{r_i\}$ of $\mathcal{T}_{\mathcal{R}}$ and $\{a_j\}$ of $\mathcal{T}_{\mathcal{R}}^{\perp}$:

  $a'_i = B_{ij} r_j + C_{ik} a_k \quad B_{ij}, C_{ik} \in \mathbb{C}$

  $\Rightarrow \Delta(a) = \sum r_i \otimes B_{ij} r_j + \sum r_i \otimes C_{ik} a_k$

  $\Delta(a) \in \mathcal{A} \otimes \mathcal{T}_{\mathcal{R}} \Rightarrow C_{ik} a_k = 0$

  $\Rightarrow \Delta(a) = \sum r_i \otimes B_{ij} r_j \in \mathcal{T}_{\mathcal{R}} \otimes \mathcal{T}_{\mathcal{R}}$

  Now for $S(\mathcal{T}_{\mathcal{R}}) \subset \mathcal{T}_{\mathcal{R}}$. Using lemma 4, $S(\mathcal{T}_{\mathcal{R}}) = \mathcal{T}_{\mathcal{L}} = \mathcal{T}_{\mathcal{R}}$.

  Thus $\mathcal{T}_{\mathcal{R}}$ is a Hopf subalgebra of $\mathcal{A}$.

- $(2) \Rightarrow (3)$

  According to lemma 3, we have $\mathcal{T} \in \mathcal{T}_{\mathcal{R}}$, and all $a \in \mathcal{T}_{\mathcal{R}}$ satisfy $\Pi_\phi(a)|\phi_{\mathcal{R}}> = \epsilon(a)|\phi_{\mathcal{R}}>$. $\mathcal{T}$ was defined as the largest Hopf subalgebra of $\mathcal{A}$ whose elements satisfy $\Pi_\phi(a)|\phi_{\mathcal{R}}> = \epsilon(a)|\phi_{\mathcal{R}}>$, so if $\mathcal{T}_{\mathcal{R}}$ is a Hopf algebra we must have $\mathcal{T}_{\mathcal{R}} = \mathcal{T}$.

- $(3) \Rightarrow (1)$

  We already proved that $\mathcal{T} \subset \mathcal{T}_{\mathcal{L}}$. Thus if we assume $\mathcal{T} = \mathcal{T}_{\mathcal{R}}$ we have $\mathcal{T}_{\mathcal{R}} \subset \mathcal{T}_{\mathcal{L}}$.

  Since $\mathcal{T}_{\mathcal{R}} = \mathcal{T}$, $\mathcal{T}_{\mathcal{R}}$ is a Hopf algebra. Thus $S(\mathcal{T}_{\mathcal{R}}) \subset \mathcal{T}_{\mathcal{R}}$. From lemma 4, we know that $S(\mathcal{T}_{\mathcal{R}}) = \mathcal{T}_{\mathcal{L}}$. Thus $\mathcal{T}_{\mathcal{L}} \subset \mathcal{T}_{\mathcal{R}}$.

  Done: $\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{L}}$.

- $(1) \Rightarrow (4)$

  Obvious.

- $(4) \Rightarrow (1)$

  Lemma 4 taught us that $\mathcal{T}_{\mathcal{R}} = S(\mathcal{T}_{\mathcal{L}})$ and $S(\mathcal{T}_{\mathcal{R}}) = \mathcal{T}_{\mathcal{L}}$. Apply S to the left and right of $\mathcal{T}_{\mathcal{L}} \subset \mathcal{T}_{\mathcal{R}}$ to obtain $\mathcal{T}_{\mathcal{R}} \subset \mathcal{T}_{\mathcal{L}}$. Done: $\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{L}}$.

□

5.2.3 $\mathcal{T}_\mathcal{R}$ and $\mathcal{T}_\mathcal{L}$: Hopf or not?

We are interested in finding out which condensates yield a right residual symmetry algebra $\mathcal{T}_\mathcal{R}$ that is a Hopf algebra. From proposition 1 we know that $\mathcal{T}_\mathcal{R}$ is a Hopf algebra $\iff$ $\mathcal{T}_\mathcal{L}$ is the same Hopf algebra.

As an example, consider a phase with a generalized quantum double as its Hopf symmetry: $\mathcal{A} = F(H_m) \otimes \mathbb{C}H_{el}$. We write elements of $\mathcal{A}$ as functions $f \in F(H_m \times H_{el})$ (see appendix B). Now condense $|\phi_\mathcal{R}>$ in an irrep Π^A_α of $\mathcal{A}$. We saw in lemma 2 that a function $f \in F(H_m \times H_{el})$ is an element of $\mathcal{T}_\mathcal{R}$ if it satisfies

$$f(x_1(x \cdot g_A), y_1)|\phi_\mathcal{R}(y_1^{-1}x)>= f(x_1, y_1)|\phi_\mathcal{R}(x)> \quad \forall x_1 \in G, x, y_1 \in H_{el}$$

where g_A is the preferred element of A.

Analogously to the derivation of lemma 2, we can prove that the functions f in $\mathcal{T}_\mathcal{L}$ are precisely those f that satisfy

$$f((x \cdot g_A)x_1, y_1)|\phi_\mathcal{R}(y_1^{-1}x)>= f(x_1, y_1)|\phi_\mathcal{R}(x)> \qquad \forall x_1 \in G, \forall x, y_1 \in H_{el}. \tag{5.19}$$

Proposition 1(4) tells us that proving that $\mathcal{T}_\mathcal{R}$ is a Hopf algebra is equivalent to proving that $\mathcal{T}_\mathcal{L} \subseteq \mathcal{T}_\mathcal{R}$. Thus, to prove that $\mathcal{T}_\mathcal{R}$ is a Hopf algebra, we must prove that if a function f satisfies (5.19), it automatically satisfies (5.12):

$$\begin{aligned} & \forall x_1 \in H_m, \forall x, y_1 \in H_{el} : \\ & f(x_1 x \cdot g_A, y_1)|\phi_\mathcal{R}(y_1^{-1}x)>= f(x_1, y_1)|\phi_\mathcal{R}(x)> \\ \Rightarrow \quad & f((x \cdot g_A)x_1, y_1)|\phi_\mathcal{R}(y_1^{-1}x)>= f(x_1, y_1)|\phi_\mathcal{R}(x)> \end{aligned} \tag{5.20}$$

This implication is automatically satisfied if H_m is an abelian group, because then $x \cdot g_A$ and x_1 commute. Thus, if the magnetic group H_m is abelian, $\mathcal{T}_\mathcal{R}$ is necessarily a Hopf algebra.

Also, if g_A is in the center of H_m, and is acted on trivially by all of H_{el}, then (5.20) is satisfied. Electric condensates are an example, since $g_A = e$ for electric condensates. e is in the center of H_m, and H_{el} acts trivially on e.

If the original phase is a $D(H)$ phase, then we have similar results: $\mathcal{T}_\mathcal{R}$ is a Hopf algebra if H is abelian, or if g_A is in the center of H. We needn't demand that all of H acts trivially on g_A, since this immediately follows from g_A being in the center of H.

5.2.4 Requirement on the condensate $|\phi_\mathcal{R}>$

If the condensate is $|\phi_\mathcal{R}>$, then our ground state is filled with the particles in the state $|\phi_\mathcal{R}>$. We know that if an excitation of this ground state braids nontrivially with the condensate, then it is connected to a half-line singularity which costs a finite amount of energy per unit length. Thus, if $|\phi_\mathcal{R}>$ were to braid nontrivially with itself, then the ground would be filled with half-line singularities, and it wouldn't make sense to speak of a condensate. Thus we require of our condensate $|\phi_\mathcal{R}>$ that it braid trivially with itself:

$$\mathcal{R} \circ (|\phi_\mathcal{R}> \otimes |\phi_\mathcal{R}>) = |\phi_\mathcal{R}> \otimes |\phi_\mathcal{R}> . \tag{5.21}$$

Note that we are braiding indistinguishable particles, and as we discussed in chapter 4, if $|\phi_{\mathcal{R}}>$ has spin s, then the braiding picks up an extra phase factor $e^{i2\pi s}$. The spin factor should be taken into account when verifying the trivial self braiding condition.

Recently fermionic condensates have received considerable theoretical and experimental attention[22], and we could definitely treat those as well with our methods. We may then relax the trivial self braiding condition, since identical fermions don't braid trivially with each other: they pick up a minus sign under half-braiding.

5.2.5 Unconfined excitations and $\mathcal{U}$

The trivial braiding conditions

Now that we've learned how to derive $\mathcal{T}_{\mathcal{R}}$, we want to study the unconfined excitations of $\mathcal{T}_{\mathcal{R}}$. We found a criterion for determining whether an excitation was confined: If the excitation doesn't braid trivially with the condensate $|\phi_{\mathcal{R}}>$, then it is confined.

The condensate is in a state $|\phi_{\mathcal{R}}>$ of an irrep Π_ϕ of $\mathcal{A}$. Now consider an excitation of the ground state, sitting in an irrep Ω of $\mathcal{T}_{\mathcal{R}}$. Since the universal R matrix $R = \sum_{(R)} R^{(1)} \otimes R^{(2)} \in \mathcal{A} \otimes \mathcal{A}$, we cannot simply act with R on states in the tensor product representation $\Pi_\phi \otimes \Omega$, because we can only act with elements of $\mathcal{T}_{\mathcal{R}}$ on states of Ω. We need a projection P of $\mathcal{A}$ onto $\mathcal{T}_{\mathcal{R}}$, so that we can act with $(id \otimes P)R$ on states of $\Pi_\phi \otimes \Omega$. If $\mathcal{A}$ has an inner product, then we can use this inner product to define the projection P. Take an orthonormal basis $\{a_i\}$ of $\mathcal{T}_{\mathcal{R}}$, and an orthonormal basis[5] $\{b_j\}$ of $\mathcal{T}_{\mathcal{R}}^{\perp}$. Together, the a_i and b_j form an orthonormal basis of $\mathcal{A}$, such that the a_i span $\mathcal{T}_{\mathcal{R}}$, and for all i and j we have[6]

$$(a_i, b_j) = 0, \tag{5.22}$$

where (a, b) denotes the inner product between a and b. Now take any $a \in \mathcal{A}$, and write

$$a = \sum_{(a_i)} (a_i, a)a + \sum_{(b_j)} (b_j, a)b_j \equiv Pa + (1 - P)a. \tag{5.23}$$

Thus we have our projection: $Pa = \sum_{(a_i)} (a_i, a)a \in \mathcal{T}_{\mathcal{R}}$. It is known that this projection is in fact independent of our choice of basis: given a vector space with an inner product, the perpendicular projection onto a vector subspace is uniquely defined.

Generalized quantum doubles $F(H_m) \otimes \mathbb{C}H_{el}$ come equipped with an inner product (B.22):

$$(P_g h, P_{g'} h') = \delta_{g,g'} \delta_{h,h'}.$$

We use this inner product to define the projection operator $P : \mathcal{A} \to \mathcal{T}_{\mathcal{R}}$, just as we discussed above. Now we can define the braiding of a state $|v>$ of Ω with

---

[5] $\mathcal{T}_{\mathcal{R}}^{\perp}$ is the vector space consisting of all vectors in $\mathcal{A}$ perpendicular to all vectors in $\mathcal{T}_{\mathcal{R}}$.

[6] We can find such a basis of $\mathcal{A}$ using the Gram-Schmidt orthogonalization procedure.

$|\phi_{\mathcal{R}}>$:

$$\text{Counterclockwise} \quad : \tau \circ (\Omega \otimes \Pi^A_\alpha)(P \otimes id)R(|v> \otimes |\phi_{\mathcal{R}}>) \qquad (5.24)$$

$$\text{Clockwise} \quad : \tau \circ (\Omega \otimes \Pi^A_\alpha)(P \otimes id)R^{-1}_{21}(|v> \otimes |\phi_{\mathcal{R}}>) \qquad (5.25)$$

where in the second line we used (4.37): $\mathcal{R}^{-1} = \tau \circ R^{-1}_{21}$.

We want to find out which irreps Ω of $\mathcal{T}_{\mathcal{R}}$ braid trivially with the condensate, thus the braiding has to be trivial for all $|v>$ in the vector space on which Ω acts. We still need a definition of "trivial braiding". A natural definition is: Ω braids trivially with the condensate if it braids just like the vacuum irrep ϵ does. This definition immediately implies that the vacuum irrep ϵ is unconfined, since obviously ϵ braids like ϵ does. Thus the conditions for trivial braiding of an irrep Ω of $\mathcal{T}_{\mathcal{R}}$ with the condensate $|\phi_{\mathcal{R}}>$ living in the irrep Π^A_α of $F(H_m) \otimes \mathbb{C}H_{el}$ become

$$(\Omega \otimes \Pi^A_\alpha)(P \otimes 1)R(1 \otimes |\phi_{\mathcal{R}}>) = (\Omega(1)\epsilon \otimes \Pi^A_\alpha)(P \otimes 1)R(1 \otimes |\phi_{\mathcal{R}}>) \qquad (5.26)$$

$$(\Omega \otimes \Pi^A_\alpha)(P \otimes id)R^{-1}_{21}(1 \otimes |\phi_{\mathcal{R}}>) = (\Omega(1)\epsilon \otimes \Pi^A_\alpha)(P \otimes id)R^{-1}_{21}(1 \otimes |\phi_{\mathcal{R}}>). \qquad (5.27)$$

$\Omega(1)$ is an $n \times n$ unit matrix, where n is the dimensionality of the irrep Ω.

If these equations are satisfied, then we can replace $1 \otimes |\phi_{\mathcal{R}}>$ in these equations by $|v> \otimes |\phi_{\mathcal{R}}>$ for any state $|v>$ of Ω. Thus if Ω satisfies the trivial braiding conditions (5.26) and (5.27), then all the states $|v>$ of Ω braid trivially with $|\phi_{\mathcal{R}}>$.

An irrep Ω that satisfies these two equations is said to braid trivially with the condensate. If an irrep doesn't braid trivially with the condensate, it is a confined excitation, attached to a string that goes out to infinity and costs a finite amount of energy per unit length.

The unconfined symmetry algebra $\mathcal{U}$

The trivial braiding equations (5.26) and (5.27) divide the irreps of $\mathcal{T}_{\mathcal{R}}$ into confined and unconfined irreps. We cannot simply take the tensor product of irreps of $\mathcal{T}_{\mathcal{R}}$, since $\mathcal{T}_{\mathcal{R}}$ isn't a Hopf algebra. The reason for the absence of a coproduct is the presence of confined excitations. The condensate to the right of a confined excitation takes on the value $|\phi_{\mathcal{R}}>$, while it takes on a different value $|\phi_{\mathcal{L}}>$ to the left of the excitation. Thus particles coming in from the left see a different condensate: they are excitations of the residual symmetry algebra of $|\phi_{\mathcal{L}}>$.

The situation is actually a little more complicated, because the value $|\phi_{\mathcal{L}}>$ of the condensate to the left of a state $|v>$ of an irrep Ω of $\mathcal{T}_{\mathcal{R}}$ depends on the explicit state $|v>$ of Ω. Thus $|\phi_{\mathcal{L}}>$ is not unique for an irrep Ω.

We will discuss how to deal with these issues later. For now, we note that unconfined excitations do not suffer from such complications, since the condensate takes on a constant value around unconfined excitations. Thus we expect the fusion rules of unconfined excitations to be associative, if we only consider their composition with other unconfined excitations. There should be a Hopf algebra $\mathcal{U}$, called the *unconfined symmetry algebra*, whose irreps are precisely the unconfined irreps, and whose fusion rules give the fusion channels of the unconfined excitations into other unconfined excitations.

To obtain $\mathcal{U}$, we first determine all unconfined irreps of $\mathcal{T}_{\mathcal{R}}$. Then we take the intersection of the kernels of all unconfined irreps[7], and denote it by $\mathcal{I}$. Finally, we define the algebra

$$\mathcal{U} = \mathcal{T}_{\mathcal{R}}/\mathcal{I} \tag{5.28}$$

This is an algebra because $\mathcal{I}$ is an ideal of $\mathcal{T}_{\mathcal{R}}$. Its irreps are precisely the unconfined irreps.

Our claim is that $\mathcal{U}$ *is a Hopf algebra*. In all the cases we've worked out, this is indeed the case. We have therefore stumbled upon a physically motivated conjecture, and hopefully mathematicians will take up the challenge of proving that $\mathcal{U}$ is always a Hopf algebra. We note that a proof would have to make use of the trivial self braiding condition (5.21) that we formulated for the condensate ϕ, because if this condition is dropped then we have found cases where $\mathcal{U}$ isn't a Hopf algebra.

Trivial braiding for $F(H_m/B) \otimes \mathbb{C}N$

In the next section, we will see that for the electric and defect condensates in a phase with $F(H_m) \otimes \mathbb{C}H_{el}$ symmetry, the residual symmetry algebra $\mathcal{T}_{\mathcal{R}}$ takes on a special form:

$$\mathcal{T}_{\mathcal{R}} = F(H_m/B) \otimes \mathbb{C}N, \tag{5.29}$$

where B is a subgroup of H_m, and N is a subgroup of H_{el} whose elements $n \in N$ satisfy

$$n \cdot B = B. \tag{5.30}$$

This last equation tells us that the action of $n \in N$ on H_m/B is well defined: $F(H_m) \otimes \mathbb{C}H_{el}$ is a generalized quantum double, so that the action of $h \in H_{el}$ on H_m satisfies (B.24):

$$\forall g_1, g_2 \in H_m : h \cdot (g_1 g_2) = (h \cdot g_1)(h \cdot g_2).$$

Since $N \subset H_{el}$, all $n \in N$ also satisfy this equation. The action of n on H_m/B is given by:

$$n \cdot (gB) = (n \cdot g)(n \cdot B) = (n \cdot g)B. \tag{5.31}$$

$\mathcal{T}_{\mathcal{R}} = F(H_m/B) \otimes \mathbb{C}N$ is a transformation group algebra, so we can use the canonical theorem on the irreps of transformation group algebras given in Appendix B. The irreps are labelled by an orbit T in H_m/B under the action of N, and an irrep τ of the normalizer N_T of a preferred element g_T of T. We denote irreps of $F(H_m/B) \otimes \mathbb{C}N$ by Ω^T_τ. The conditions (5.26) and (5.27) for Ω^T_τ to braid trivially with $|\phi_{\mathcal{R}} >$ in Π^A_α reduce to

$$\sum_{h \in B} |\phi_{\mathcal{R}}(\Gamma(h(x_\eta \cdot g_T^{-1}))x) >= \sum_{h \in B} |\phi_{\mathcal{R}}(\Gamma(h)x) > \tag{5.32}$$

---

[7] The kernel of a map is the set of elements which the irrep maps to zero. These elements form an ideal I, meaning that if $i \in I$ and $a \in \mathcal{T}_{\mathcal{R}}$, then $ia \in I$ and $ai \in I$.

$$\begin{aligned}|\phi_{\mathcal{R}}(x)>\neq 0, \Gamma(x\cdot g_A)\in N \Rightarrow \quad & \tau_{ij}(x_\eta^{-1}\Gamma(x\cdot g_A^{-1})x_\zeta)1_{N_T}(x_\eta^{-1}\Gamma(x\cdot g_A^{-1})x_\zeta) \\ & = \tau_{ij}(x_\eta^{-1}x_\zeta)1_{N_T}(x_\eta^{-1}x_\zeta) \qquad (5.33)\end{aligned}$$

where the x_η and x_ζ are chosen representatives of left N_T cosets of H.

The proof of these equations is rather lengthy. Some intermediate steps are

- Proof of (5.32)

$$\begin{aligned}
& R = \sum_{h,g\in G} P_g e \otimes P_h\Gamma(g) \\
& P(P_h\Gamma(g)) = 1_N(\Gamma(g))\frac{1}{|B|}P_{hB}\Gamma(g) \\
& \sum_{g,h}\epsilon(P(P_g e))\Pi^A_\alpha(P_h\Gamma(g))|\phi_{\mathcal{R}}(x)> = \sum_{g\in B}\frac{1}{|B|}|\phi_{\mathcal{R}}(\Gamma(g)x)> \\
& \Omega^T_\tau(P(P_h g))^{i,j}_{\eta,\zeta} = \frac{1}{|B|}1_N(\Gamma(g))\sum_{N_T}P_{hB}(x_\eta\cdot g_T)\delta_g(x_\eta n x_\zeta^{-1})\beta_{i,j}(n) \\
& \sum_{h,g}\Omega^T_\tau(P(P_g e))^{i,j}_{\zeta,\eta}\Pi^A_\alpha(P_h\Gamma(g))|\phi_{\mathcal{R}}(x)> \\
& = \Omega^T_\tau(1)^{i,j}_{\zeta,\eta}\sum\epsilon(P(P_g e))\Pi^A_\alpha(P_h\Gamma(g))|\phi_{\mathcal{R}}(x)> \\
& \Longleftrightarrow\ \frac{1}{|B|}\tau_{i,j}(x_\eta^{-1}x_\zeta)1_{N_T}(x_\eta^{-1}x_\zeta)\sum_{h\in B}|\phi_{\mathcal{R}}(\Gamma(hx_\eta\cdot g_T^{-1})x)> \\
& = \frac{1}{|B|}\tau_{i,j}(x_\eta^{-1}x_\zeta)1_{N_T}(x_\eta^{-1}x_\zeta)\sum_{h\in B}|\phi_{\mathcal{R}}(hx)> \\
& \Longleftrightarrow\ \sum_{h\in B}|\phi_{\mathcal{R}}(\Gamma(h(x_\eta\cdot g_T^{-1}))x)> = \sum_{h\in B}|\phi_{\mathcal{R}}(hx)>
\end{aligned}$$

- Proof of (5.33)

$$\begin{aligned}
& \sum_{h,g}\epsilon(P(P_h\Gamma(g^{-1})))\Pi^A_\alpha(P_g e)|\phi_{\mathcal{R}}(x)> = 1_N(\Gamma(x\cdot g_A))|\phi_{\mathcal{R}}(x)> \\
& \sum_{h,g}\Omega^T_\tau(P(P_h\Gamma(g^{-1})))\Pi^A_\alpha(P_g e)\phi = \Omega^T_\tau(1)\sum_{h,g}\epsilon(P(P_h\Gamma(g^{-1})))\Pi^A_\alpha(P_g e)|\phi_{\mathcal{R}}(x)> \\
& \Longleftrightarrow\ 1_N(\Gamma(x\cdot g_A))\tau_{i,j}(x_\eta^{-1}\Gamma(x\cdot g_A^{-1})x_\zeta)1_{N_T}(x_\eta^{-1}\Gamma(x\cdot g_A^{-1})x_\zeta)|\phi_{\mathcal{R}}(x)> \\
& = 1_N(\Gamma(x\cdot g_A))1_{N_T}(x_\eta^{-1}x_\zeta)\tau_{i,j}(x_\eta^{-1}x_\zeta)\phi(x) \\
& \Longleftrightarrow\ |\phi_{\mathcal{R}}(x)> = 0 \text{ or } \Gamma(x\cdot g_A^{-1})\notin N \\
& \text{or } \tau_{i,j}(x_\eta^{-1}\Gamma(x\cdot g_A^{-1})x_\zeta)1_{N_T}(x_\eta^{-1}\Gamma(x\cdot g_A^{-1})x_\zeta) = 1_{N_T}(x_\eta^{-1}x_\zeta)\tau_{i,j}(x_\eta^{-1}x_\zeta)
\end{aligned}$$

When the generalized quantum double is a quantum double $D(H)$, the conditions for an irrep Ω^T_τ of $\mathcal{T}_{\mathcal{R}}$ to braid trivially with the condensate $|\phi_{\mathcal{R}}>$ become

$$\sum_{h\in B}|\phi_{\mathcal{R}}(hx_\eta g_T^{-1}x_\eta^{-1}x)> = \sum_{h\in B}|\phi_{\mathcal{R}}(hx)> \qquad (5.34)$$

$$\begin{aligned} |\phi_{\mathcal{R}}(x) >\neq 0, xg_A x^{-1} \in N \Rightarrow \quad & \tau_{ij}(x_\eta^{-1} x g_A^{-1} x^{-1} x_\zeta) 1_{N_T}(x_\eta^{-1} x g_A^{-1} x^{-1} x_\zeta) \\ & = \tau_{ij}(x_\eta^{-1} x_\zeta) 1_{N_T}(x_\eta^{-1} x_\zeta). \end{aligned} \tag{5.35}$$

We will now use these equations to work out $\mathcal{T}_{\mathcal{R}}$ and $\mathcal{U}$ for various condensates in generalized quantum doubles. We start with electric condensates, and show that the conventional theory of electric condensates (Landau's theory) is reproduced. After that we study a variety of defect condensates.

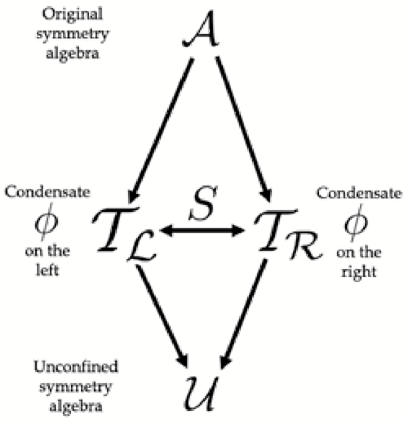


Figure 5.3: The symmetry breaking scheme.

5.3 Electric condensates

Consider a phase described by a quantum double $D(H)$, and condense a state $|\phi_{\mathcal{R}}>$ of an electric irrep Π^e_α of $D(H)$. Then equation (5.12) tells us that the functions$f$ in $\mathcal{T}_{\mathcal{R}}$ satisfy

$$\begin{aligned} & f(x_1, y_1)\alpha(x^{-1}y_1) = f(x_1, y_1)\alpha(x^{-1}) \\ & f(x_1, y_1)\alpha(y_1) = f(x_1, y_1)\mathbb{1} \\ & f(x_1, y_1) = 0 \vee y_1 \in N_{\phi_{\mathcal{R}}} \\ \Rightarrow \quad & \mathcal{T}_{\mathcal{R}} = F(H) \otimes \mathbb{C}N_{\phi_{\mathcal{R}}} \end{aligned} \tag{5.36}$$

where $N_{\phi_{\mathcal{R}}}$ is the stabilizer of $|\phi_{\mathcal{R}}>$, i.e. the set of elements $h \in H$ that satisfy $\Pi^e_\alpha(h)|\phi_{\mathcal{R}} >= \alpha(h)|\phi_{\mathcal{R}} >= |\phi_{\mathcal{R}}>$. $\mathbb{1}$ is the unit matrix. Since the condensate is purely electric, the magnetic group is unbroken, thus all of $F(H)$ is present in $\mathcal{T}_{\mathcal{R}}$. $\mathcal{T}_{\mathcal{R}}$ is a Hopf algebra in this case, which implies that the tensor product of irreps is well defined, and associative.

Some defects are confined. The trivial braiding conditions (5.34) and (5.35) tell us that only the defects $g \in N_{\phi_{\mathcal{R}}}$ are unconfined. Thus the unconfined symmetry algebra is

$$\mathcal{U} = D(N_{\phi_{\mathcal{R}}}). \tag{5.37}$$

One of the consequences of electric symmetry breaking is the lifting of degeneracies. Namely, excitations which used to be in the same irrep are now split into different irreps, which may have different energies. This splitting of energy levels is experimentally measurable, in principle.

To treat a phase with inversion symmetry, such as an achiral tetrahedral nematic with $F(\overline{T}) \otimes \mathbb{C}T_d$ symmetry, we need the formulas for electric symmetry breaking of generalized quantum doubles $F(H_m) \otimes \mathbb{C}H_{el}$. The derivation of $\mathcal{T}_{\mathcal{R}}$ and $\mathcal{U}$ after the condensation of a state $|\phi_{\mathcal{R}} >$ of an electric irrep of a generalized quantum double $F(H_m) \otimes \mathbb{C}H_{el}$ is analogous to the derivation given above for a quantum double $D(H)$. The result is

$$\mathcal{T}_{\mathcal{R}} = F(G) \otimes \mathbb{C}N_{\phi_{\mathcal{R}}} \tag{5.38}$$

$$\mathcal{U} = F(\Gamma^{-1}(N_{\phi_{\mathcal{R}}})) \otimes \mathbb{C}N_{\phi_{\mathcal{R}}} \tag{5.39}$$

For example, for $F(\overline{T}) \otimes \mathbb{C}T_d$, $\Gamma^{-1}(N_{\phi_{\mathcal{R}}}) = \overline{N_{\phi_{\mathcal{R}}}}$, so that $\mathcal{U} = F(\overline{N_{\phi_{\mathcal{R}}}}) \otimes \mathbb{C}N_{\phi_{\mathcal{R}}}$.

As an example, we've worked out all possible electric phase transitions from $F(\overline{T}) \otimes \mathbb{C}T_d$, see table 5.1. Although we haven't found references that systematically work out all electric condensates for all irreps as we have done, the theory behind electric condensates is well known, so there results are not innovative. Our innovation lies in the derivation of the theory of electric condensates from the Hopf symmetry description of liquid crystals. A reference that offers a detailed analysis of the theory of electric condensates in nematic liquid crystals is [39].

Original symmetry	Condensate in this irrrep of T_d	$\mathcal{T}_{\mathcal{R}}$	$\mathcal{U}$
$F(\overline{T}) \otimes \mathbb{C}T_d$	A_1	$F(\overline{T}) \otimes \mathbb{C}T_d$	$F(\overline{T}) \otimes \mathbb{C}T_d$
	A_2	$F(\overline{T}) \otimes \mathbb{C}T$	$F(\overline{T}) \otimes \mathbb{C}T$
	E	$F(\overline{T}) \otimes \mathbb{C}D_2$	$F(\overline{D_2}) \otimes \mathbb{C}D_2$
	E	$F(\overline{T}) \otimes \mathbb{C}D_{2d}$	$F(\overline{D_2}) \otimes \mathbb{C}D_{2d}$
	F_1	$F(\overline{T}) \otimes \mathbb{C}S_4$	$F(\overline{C_2}) \otimes \mathbb{C}S_4$
	F_1	$F(\overline{T}) \otimes \mathbb{C}C_{1v}$	$F(\overline{C_1}) \otimes \mathbb{C}C_{1v}$
	F_1	$F(\overline{T}) \otimes \mathbb{C}C_3$	$F(\overline{C_3}) \otimes \mathbb{C}C_3$
	F_2	$F(\overline{T}) \otimes \mathbb{C}C_{1v}$	$F(\overline{C_1}) \otimes \mathbb{C}C_{1v}$
	F_2	$F(\overline{T}) \otimes \mathbb{C}C_{2v}$	$F(\overline{C_2}) \otimes \mathbb{C}C_{2v}$
	F_2	$F(\overline{T}) \otimes \mathbb{C}C_{3v}$	$F(\overline{C_3}) \otimes \mathbb{C}C_{3v}$

Table 5.1: Electric condensates in a tetrahedral nematic. We use the notation for the irreps of T_d given in [57], and the standard crystallographic notation for groups (S_4 is not the permutation group of 4 elements, it is a rotary-reflection group).

The transition from $F(\overline{T}) \otimes \mathbb{C}T_d$ to $F(\overline{D_2}) \otimes \mathbb{C}D_2$ in table 5.1, induced by the condensation of a vector in the irrep E of T_d, is an example of spontaneous symmetry breaking from an achiral to a chiral phase, since D_2 does not contain any inversions or reflections, while T_d does. This may be the explanation of the experimental discovery of a phase built up of achiral molecules, whose symmetry is spontaneously broken to a chiral phase[48]. For a relevant discussion, see [39].

5.4 Defect condensates

There are different types of defect condensates which we wish to analyze. Consider a phase described by a quantum double $D(H)$, or a generalized quantum double $F(H_m) \otimes \mathbb{C}H_{el}$, and pick a magnetic representation Π_1^A (1 is the trivial representation of the centralizer N_A). A basis of the vector space on which this irrep acts is given by $\{|g_i^A>\}$, where the g_i^A are the different defects in A. We consider the following types of condensates:

- Single defect condensate
$$\phi = |g_i^A> \tag{5.40}$$
- Class sum defect condensate
$$\phi = \sum_{g_i \in A} |g_i> =: |C_{g_A}> \tag{5.41}$$
where C_{g_A} is a conjugacy class in the $D(H)$ case, and an orbit in G under the action of H in the $F(H_m) \otimes \mathbb{C}H_{el}$ case. We denote the condensate by $|C_{g_A}>$, where g_A is the preferred element of A.
- Combined defect condensate
$$\phi = \sum_{g_i \in E} |g_i> \tag{5.42}$$
where E is a subset of the defects in one class. We need only take the elements to be within one class because, as we mentioned earlier, we need only study the cases where the condensate is the sum of vectors in the same irrep.

The single defect and class sum defect condensates are a special case of combined defect condensate. The derivation of $\mathcal{T}_{\mathcal{R}}$ and $\mathcal{U}$ for a combined defect condensate is rather technical, so we will discuss the results for the single defect and class sum defect condensates first, and then derive the general formulae.

5.4.1 Single defect condensate

Consider a phase with $D(H)$ symmetry, and condense $|g_A>$ in the magnetic irrep Π_1^A. We're condensing the chosen preferred element in the conjugacy class A. This is not a restriction on our choice of defect, since g_A was chosen arbitrarily. The condensate $|g_A>$ satisfies the trivial self braiding condition (5.21).

The function $|\phi >: H \rightarrow \mathbb{C}$ that corresponds to the vector $|g_A>$ is (see appendix B)

$$|\phi(x) >= 1_{N_A}(x) \ \forall x \in H. \tag{5.43}$$

The criterion (5.12) that defines $\mathcal{T_R}$ becomes

$$\begin{aligned}
& & & f \in \mathcal{T_R} \\
&\iff \forall x, x_1, y_1 \in H : & & f(x_1 x g_A x^{-1}, y_1) 1_{N_A}(y_1^{-1} x) = f(x_1, y_1) 1_{N_A}(x) \\
& \forall x \notin N_A : & & f(x_1, y_1) = 0 \quad \forall y_1 \notin N_A \\
& \forall x \in N_A : & & f(x_1 g_A, y_1) = f(x_1, y_1) \quad \forall y_1 \in N_A \\
& \Rightarrow & & \mathcal{T_R} = F(H/(g_A)) \otimes \mathbb{C}N_A
\end{aligned} \tag{5.44}$$

where we define (g_A) to be the smallest subgroup of H that contains g_A.

This result for $\mathcal{T_R}$ has a very natural interpretation: the residual electric group is N_A, the subgroup of H that doesn't conjugate the defect. The magnetic part $H/(g_A)$ is not necessarily a group. It consists of left cosets of $(g_A) = \{\ldots g_A^{-1}, e, g_A, g_A^2, \ldots\}$. The defects are now defined modulo the condensate defect $|g_A>$. In other words, if a particle in a magnetic irrep of the residual symmetry $\mathcal{T_R}$ fuses with the condensate $|g_A>$, it is left unchanged. Thus its defect is defined modulo g_A.

Using our previous propositions, we can prove that $\mathcal{T_R}$ is a Hopf algebra $\iff$ (g_A) is a normal subgroup of H $\iff$ $H/(g_A)$ is a subgroup of H.

The unconfined symmetry algebra is

$$\mathcal{U} = D(N_A/(g_A)). \tag{5.45}$$

If we condense another defect $|k g_A k^{-1}>$ in the conjugacy class A, the symmetry algebras are [8]:

$$\mathcal{T_R} = F(H/k(g_A)k^{-1}) \otimes \mathbb{C}kN_A k^{-1} \tag{5.46}$$

$$\mathcal{U} = D(N_A/k(g_A)k^{-1}). \tag{5.47}$$

The results for a single defect condensate $|g_A>$ in a phase with $F(H_m) \otimes \mathbb{C}H_{el}$ symmetry are analogous:

$$\mathcal{T_R} = F(G/(g_A)) \otimes \mathbb{C}N_A \tag{5.48}$$

$$\mathcal{U} = F(N_A/(g_A)) \otimes \mathbb{C}N_A/\Gamma((g_A)). \tag{5.49}$$

As an example, we work out all single defect condensates in an achiral tetrahedral nematic in table 5.2. The single defect condensates in an achiral octahedral, and an achiral icosahedral nematic are given in appendix C.

---

[8] To prove this, use $< k g_A k^{-1} >= k < g_A > k^{-1}$ and $N_{k g_A k^{-1}} = k N_A k^{-1}$.

Single defect condensate in $\mathcal{A} = F(\overline{T}) \otimes \mathbb{C}T_d$	K	$\mathcal{T}_{\mathcal{R}}$	$\mathcal{U}$
$\vert -e>$	$\overline{C_1}$	$F(T) \otimes \mathbb{C}T_d$	$F(T) \otimes \mathbb{C}T_d$
$\vert [123]>$	$\overline{C_3}$	$F(T/C_3) \otimes \mathbb{C}C_3$	$D(e)$
$\vert [(12)(34)]>$	$\overline{C_2}$	$F(T/C_2) \otimes \mathbb{C}D_2$	$D(C_2)$
$\vert -[(12)(34)]>$	$\overline{C_2}$	$F(T/C_2) \otimes \mathbb{C}D_2$	$D(C_2)$

Table 5.2: Single defect condensates in a tetrahedral nematic.

5.4.2 Class sum defect condensates

Consider a phase with $D(H)$ symmetry, and condense the sum of the defects in the conjugacy class A:

$$\phi = \sum_{g_i^A \in A} |g_i^A >=: |C_{g_A} > .$$

A class sum defect condensate satisfies the trivial self braiding condition (5.21):

$$\begin{aligned}
\mathcal{R}(|C_{g_A} > \otimes |C_{g_A} >) &= \mathcal{R}(\sum_{g_i^A \in A} |g_i^A > \otimes \sum_{g_k^A \in A} |g_k^A >) \\
&= \sum_{g_i^A \in A} (\sum_{g_k^A \in A} |g_i^A g_k^A (g_i^A)^{-1} >) \otimes |g_i^A > \\
&= \sum_{g_i^A \in A} (\sum_{g_k^A \in A} |g_k^A >) \otimes |g_i^A > . \\
&= |C_{g_A} > \otimes |C_{g_A} >
\end{aligned}$$

In going from the second to the third line, we use the fact that $gAg^{-1} = A$ for any $g \in H$.

A class sum condensate doesn't break the electric group at all! Namely, conjugation acts trivially on a conjugacy class, since for any $g \in H$ we have

$$g \cdot \phi = g \cdot (\sum_{g_i^A \in A} |g_i^A >) = \sum_{g_i^A \in A} |g g_i^A g^{-1} >= \sum_{g_i^A \in A} |g_i^A >= \phi. \tag{5.50}$$

Thus this condensate is invariant under all residual symmetry transformations in H. For this reason, in the case of a gauge theory this condensate is called a *gauge invariant magnetic condensate.* Namely, in the $D(H)$ phase of a gauge theory, with H a discrete group, the only residual gauge transformations are global, because H is discrete and the gauge transformation must be continuously defined on the space (see chapter 2). These gauge transformations act trivially on the class sum defect condensates, thus these condensates are indeed gauge invariant.

The residual and unconfined symmetry algebras are

$$\mathcal{T}_{\mathcal{R}} = F(H/K) \otimes \mathbb{C}H \tag{5.51}$$
$$\mathcal{U} = D(H/K), \tag{5.52}$$

where K is the smallest subgroup of H that contains the class A. From this definition, it follows that K is a normal subgroup of H. Thus H/K is a group.

If we condense a class sum defect condensate $|C_{g_A} >$ in a generalized quantum double $F(H_m) \otimes \mathbb{C}H_{el}$, the outcome is

$$\mathcal{T}_{\mathcal{R}} = F(H_m/K) \otimes \mathbb{C}H_{el} \tag{5.53}$$
$$\mathcal{U} = F(H_m/K) \otimes \mathbb{C}H_{el}/\Gamma(K). \tag{5.54}$$

As an example, we work out all class sum defect condensates in an achiral tetrahedral nematic in table 5.3. The class sum defect condensates in an achiral octahedral, and an achiral icosahedral nematic are given in appendix C.

Defect conjugacy classes of $\mathcal{A} = F(\overline{T}) \otimes \mathbb{C}T_d$	K	$\mathcal{T}_{\mathcal{R}}$	$\mathcal{U}$
$\|C_{-e} >$	$\overline{C_1}$	$F(T) \otimes \mathbb{C}T_d$	$D(e)$
$\|C_{[123]} >, \|C_{-[123]} >, \|C_{[124]} >, \|C_{-[124]} >$	$\overline{T}$	$F(\mathbb{Z}_3) \otimes \mathbb{C}T$	$D(e)$
$\|C_{[(12)(34)]} >$	$\overline{D_2}$	$F(\mathbb{Z}_3) \otimes \mathbb{C}T_d$	$D(\mathbb{Z}_3)$

Table 5.3: Class sum defect condensates in a tetrahedral nematic.

5.4.3 Combined defect condensates

The formal derivation

We will now derive all the formulas for defect condensates we have come across.

Start with a phase with $F(H_m) \otimes \mathbb{C}H_{el}$ symmetry. Choose an irrep Π^A_α, and consider a condensate of the form $\sum_{g_i \in E} |g_i >$, with E a subset of the defects in one conjugacy class.

The demand of trivial self braiding (5.21) gives

$$\begin{aligned}
&\mathcal{R}(\sum_{g_i \in E} |g_i > \otimes \sum_{g_k \in E} |g_k >) = \sum_{g_i \in E} |g_i > \otimes \sum_{g_k \in E} |g_k > \\
\iff \quad & \sum_{g_k \in E} \sum_{g_i \in E} |g_i g_k g_i^{-1} > \otimes |g_i > = \sum_{g_i \in E} \sum_{g_k \in E} |g_i > \otimes |g_k > \\
\iff \quad & \forall g_i \in E : \{g_i g_k g_i^{-1}\}_{g_k \in E} = \{g_k\}_{g_k \in E}.
\end{aligned}$$

It interesting in itself to study how many different defect condensates satisfy this criterion. Defect-antidefect condensates $|g > + |g^{-1} >$ always satisfy this criterion[9], as do any set of commuting elements in a certain conjugacy class, and class sum defect condensates. This trivial self braiding condition will play a crucial role in determining $\mathcal{T}_{\mathcal{R}}$.

The derivation of $\mathcal{T}_{\mathcal{R}}$ and $\mathcal{U}$ is rather formal. We give the results first, and then we derive them:

$$\mathcal{T}_{\mathcal{R}} = F(H_m/K) \otimes \mathbb{C}M_E$$
$$\mathcal{U} = F(N_E/K) \otimes \mathbb{C}M_E/\Gamma(K).$$

For the derivation, we must introduce various definitions. Define the following subset of H (which needn't be a subgroup):

$$V_E \subset H_{el}: \; V_E = \{x_i N_A\}_{g_i \in E} \tag{5.55}$$

where $N_A \subset H_{el}$ is the normalizer of the chosen preferred element g_A in A, and x_i satisfies $x_i g_A x_i^{-1} = g_i$. In function notation, the condensate wavefunction is

$$|\phi_{\mathcal{R}}(x) >= 1_{V_E}(x) \quad \forall x \in H. \tag{5.56}$$

Define the following subgroup of H_{el}:

$$\begin{aligned} M_E \subset H_{el}: \; M_E &= \{m \in H : \{m \cdot g_i\}_{g_i \in E} = \{g_i\}_{g_i \in E}\} && (5.57)\\ &= \{m \in H_{el} : mV_E = V_E\}. && (5.58)\end{aligned}$$

M_E is composed of the global symmetry transformations that leave the condensate invariant.

Also define

$$N_E \subset H_m: \; N_E = \{n \in H_m : \{n g_i n^{-1}\}_{g_i \in E} = \{g_i\}_{g_i \in E}\}. \tag{5.59}$$

Using (B.37): $\Gamma(g_1) \cdot g_2 = g_1 g_2 g_1^{-1} \; \forall g_1, g_2 \in H_m$, we can prove that

$$\Gamma^{-1}(M_E) = N_E \quad \text{and} \quad \Gamma(N_E) = M_E. \tag{5.60}$$

From this equation we can derive that the elements of M_E satisfy

$$|\phi_{\mathcal{R}}(mx) >= |\phi_{\mathcal{R}}(x) > \quad \forall m \in M_E, x \in H_{el}. \tag{5.61}$$

Finally, we need one more definition:

$$K \subset H_m: \; K = (\{g_i\}_{g_i \in E}), \tag{5.62}$$

where $(\{g_i\}_{g_i \in E})$ is the smallest subgroup of H_m that all the $g_i \in E$, i.e. the defects in the condensate.

The trivial self braiding equation (5.55) implies that $K \subset N_E$. Thus, according to (5.60) and (5.61)

$$\forall k \in K: \; \Gamma(k) \in M_E \quad \text{and} \quad \phi(\Gamma(k)x) = \phi(x). \tag{5.63}$$

---

[9] Note that g and g^{-1} needn't be in the same conjugacy class.

The residual symmetry algebra $\mathcal{T}_{\mathcal{R}}$ is given by the set of functions $f \in F(H_m \times H_{el})$ that satisfy (5.12):

$$f(x_1(x \cdot g_A), y_1) 1_{V_E}(y_1^{-1}x) = f(x_1, y_1) 1_{V_E}(x). \tag{5.64}$$

We will now prove that

$$\mathcal{T}_{\mathcal{R}} = F(H_m/K) \otimes \mathbb{C}M_E. \tag{5.65}$$

Note that G/K need not be a group.

To prove (5.65), take $y_1 \notin M_E$. Then $\exists\, x \in V_E$ such that $y_1^{-1}x \notin V$. Namely, if such an x doesn't exist, then $y_1^{-1}V = V$, thus $y_1^{-1} \in M_E$ according to (5.58), and $y_1 \in M_E$.

Substitute an $y_1 \notin M_E$, and x with $y_1^{-1}x \notin V$, into (5.64). This gives

$$0 = f(x_1, y_1) \forall x_1 \in H_m \quad \text{if} \quad y_1 \notin M_E,$$

so that $\mathcal{T}_{\mathcal{R}} \subset F(H_m) \otimes \mathbb{C}M_E$.

Now substitute $y_1 \in M_E$ into (5.64): $1_{V_E}(y_1^{-1}x) = 1_{V_E}(x)$ so the equation implies $f(x_1(x \cdot g_A), y_1) = f(x_1, y_1)$ for all $x_1 \in G, x \in V_E$. Acting with all the $x \in V_E$ on g_A gives us all the $g_i \in E$, thus $f(x_1 g_i, y_1) = f(x_1, y_1)$ for all $g_i \in E$. Thus, in the first component f must be constant on left K cosets, since K is generated by the g_i. Thus $\mathcal{T}_{\mathcal{R}} = F(H_m/K) \otimes \mathbb{C}M_E$.

$\mathcal{U}$ is a little harder to extract. It is given by

$$\mathcal{U} = F(N_E/K) \otimes \mathbb{C}M_E/\Gamma(K) \tag{5.66}$$

For the case of a quantum double $D(H)$

$$\mathcal{U} = D(N_E/K). \tag{5.67}$$

To prove (5.66) and (5.67), we must find out which irreps Ω_τ^T of $F(H_m/K) \otimes \mathbb{C}M_E$ braid trivially with the condensate ϕ in the irrep Π_α^A of $F(H_m) \otimes \mathbb{C}H_{el}$.

Our residual symmetry algebra $\mathcal{T}_{\mathcal{R}}$ is of the form (5.29), with $B = K$ and $N = M_E$. Thus we can use the conditions (5.32) and (5.33) to determine the irreps of $\mathcal{T}_{\mathcal{R}}$ that braid trivially with the condensate. The unconfined symmetry algebra $\mathcal{U}$ is then the Hopf algebra whose irreps are precisely the unconfined irreps.

Equation (5.32) states that for an unconfined irrep Ω_τ^T, with g_T the preferred element in the orbit T:

$$\sum_{k \in K} |\phi_{\mathcal{R}}(\Gamma(k(x_\eta \cdot g_T^{-1}))x) >= \sum_{k \in K} |\phi_{\mathcal{R}}(\Gamma(k)x) > \quad \forall x \in H_{el}, \tag{5.68}$$

where the x_η are chosen representatives of left N_T cosets in M_E.

Using (5.63), equation (5.68) becomes

$$\begin{aligned}
&\sum_{k\in K} |\phi_{\mathcal{R}}(\Gamma(x_\eta \cdot g_T^{-1})x) >= \sum_{k\in K} |\phi_{\mathcal{R}}(x) > \ \forall x \in H_{el} \\
&\Rightarrow |\phi_{\mathcal{R}}(\Gamma(x_\eta \cdot g_T^{-1})x) >= |\phi_{\mathcal{R}}(x) > \ \forall x \in H_{el} \\
&\Rightarrow \Gamma(x_\eta \cdot g_T^{-1}) \in M_E \\
&\Rightarrow \Gamma(x_\eta \cdot g_T) \in M_E \\
&\Rightarrow x_\eta \cdot g_T \in N_E \quad \text{using (5.60).}
\end{aligned}$$

Choosing $x_\eta = e$, we get $g_T \in N_E$. (5.32) is actually equivalent to $g_T \in N_E$ in this case, because $x_\eta \cdot g_T \in N_E$ follows from $g_T \in N_E$. To prove this, note that $x_\eta \in M_E$, so $\{x_\eta \cdot g_i\}_{g_i\in E} = \{g_i\}_{g_i\in E}$. Thus

$$\begin{aligned}
&\{(x_\eta \cdot g_T)g_i(x_\eta \cdot g_T)^{-1}\}_{g_i\in E} = \{(x_\eta \cdot g_T)(x_\eta \cdot g_i)(x_\eta \cdot g_T)^{-1}\}_{g_i\in E} \\
&= \{(x_\eta \cdot (g_T g_i g_T)^{-1}\}_{g_i\in E} = \{x_\eta \cdot g_i\}_{g_i\in E} = \{g_i\}_{g_i\in E} \\
&\Rightarrow x_\eta \cdot g_T \in N_E
\end{aligned}$$

In proving the third equal sign we used the fact that $g_T \in N_E$.

Thus (5.32) has taught us that for an irrep Ω_τ^T to be unconfined, we must have $g_T \in N_E$. The magnetic part of the unconfined symmetry algebra $\mathcal{U}$ is therefore $F(N_E/K)$. From the definition of N_E and K, we can prove that

$$\forall n \in N_E : \ nKn^{-1} = K. \tag{5.69}$$

Thus K is a normal subgroup of N_E. N_E/K is the unconfined magnetic group.

Equation (5.33) further restricts Ω_τ^T:

$$\begin{aligned}
&|\phi_{\mathcal{R}}(x) >\neq 0, \ \Gamma(x \cdot g_A) \in M_E \\
\Rightarrow \quad &\tau_{ij}(x_\eta^{-1}\Gamma(x \cdot g_A^{-1})x_\zeta)1_{N_T}(x_\eta^{-1}\Gamma(x \cdot g_A^{-1})x_\zeta) = \tau_{ij}(x_\eta^{-1}x_\zeta)1_{N_T}(x_\eta^{-1}x_\zeta)
\end{aligned}$$

Choose an x such that $|\phi_{\mathcal{R}}(x) >\neq 0$. This is equivalent to saying that $x \cdot g_A = g_k$ for some $g_k \in E$. Now choose $x_\eta = x_\zeta$ in (5.33):

$$\tau_{ij}(x_\eta^{-1}\Gamma(g_k^{-1})x_\eta)1_{N_T}(x_\eta^{-1}\Gamma(g_k^{-1})x_\eta) = \tau_{ij}(e) = \delta_{ij}.$$

Now $x_\eta^{-1}\Gamma(g_k^{-1})x_\eta = \Gamma(x_\eta^{-1} \cdot g_k^{-1}) = \Gamma((x_\eta^{-1} \cdot g_k)^{-1})$. Since $x_\eta \in M_E$ and $g_k \in K$, we have $(x_\eta^{-1} \cdot g_k) \in K$, so $(x_\eta^{-1} \cdot g_k)^{-1} \in K$. Thus $\Gamma((x_\eta^{-1}g_k)^{-1}) \in \Gamma(K)$. Now $\Gamma(K)$ acts trivially on the magnetic group N_E/K, due to (B.37). Thus necessarily $\Gamma(K) \subset N_T$, since the elements of $\Gamma(K)$ are normalizers of all elements of N_E, so they are also normalizers of g_T. This means that $1_{N_T}(x_\eta^{-1}\Gamma(g_k^{-1})x_\eta) = 1$. 5.70) becomes

$$\begin{aligned}
&\tau_{ij}(x_\eta^{-1}\Gamma(g_k^{-1})x_\eta) = \tau_{ij}(\Gamma(x_\eta^{-1} \cdot g_k^{-1})) = \tau_{ij}(\Gamma(x_\eta^{-1} \cdot g_k))^{-1} = \delta_{ij}. \\
&\Rightarrow \tau_{ij}(\Gamma(x_\eta^{-1} \cdot g_k)) = \delta_{ij}.
\end{aligned}$$

Observe that the set $\{x_\eta^{-1} \cdot g_k\}_{\eta,k} = E$. Since τ must send all $\Gamma(x_\eta^{-1} \cdot g_k)$ to the unit matrix $\mathbb{1}$, τ must send all of $\Gamma(K)$ to the unit matrix (since K is generated by E). We conlude that $\Gamma(K)$ is in the kernel of τ, and the electric group is $M_E/\Gamma(K)$.

At the start of this last derivation, we filled in $x_\eta = x_\zeta$ in (5.33). The case $x_\eta \neq x_\zeta$ gives nothing new, because

$$1_{N_T}(x_\eta^{-1}\Gamma(x \cdot g_A^{-1})x_\zeta) = 1_{N_T}(x_\eta^{-1}\Gamma(x \cdot g_A^{-1})x_\eta x_\eta^{-1}x_\zeta) = 1_{N_T}(x_\eta^{-1}x_\zeta), \quad (5.70)$$

where in the last line we used a fact that we proved earlier: $x_\eta^{-1}\Gamma(x \cdot g_A)x_\eta \in N_T$. Thus (5.33) becomes

$$\begin{aligned} & \tau_{ij}(x_\eta^{-1}\Gamma(x \cdot g_A^{-1})x_\zeta)1_{N_T}(x_\eta^{-1}x_\zeta) = \tau_{ij}(x_\eta^{-1}x_\zeta)1_{N_T}(x_\eta^{-1}x_\zeta) \\ \Longleftrightarrow \quad & \tau_{ij}(x_\eta^{-1}\Gamma(x \cdot g_A^{-1})x_\eta)\tau_{ij}(x_\eta^{-1}x_\zeta)1_{N_T}(x_\eta^{-1}x_\zeta) = \tau_{ij}(x_\eta^{-1}x_\zeta)1_{N_T}(x_\eta^{-1}x_\zeta) \end{aligned}$$

This is equation is satisfied if $\tau_{ij}(x_\eta^{-1}\Gamma(x \cdot g_A^{-1})x_\eta)$, which we already proved.

Summarizing, the unconfined magnetic group is N_E/K, and the unconfined electric irreps are those that have $\Gamma(K)$ in their kernel, which means that the electric group is $M_E/\Gamma(K)$. Thus we have derived (5.66). Had we started with a quantum double $D(H)$ ($H = H_{el} = H_m$), the unconfined symmetry algebra becomes $\mathcal{U} = D(N_E/K)$, because in that case $M_E = N_E$.

Examples

We will now show some examples of combined defect condensates. We give all the defect condensates in a tetrahedral nematic, see table 5.4. By looking at this table, and looking at all the other tables of condensates in this chapter, we note that two different condensates never give simultaneously the same $\mathcal{T}_\mathcal{R}$ and $\mathcal{U}$. Some condensates give the same unconfined symmetry algebra, but $\mathcal{T}_\mathcal{R}$ is then different. Thus there are differences in the spectrum, be it the unconfined spectrum of the hadronic spectrum, of different condensates, so that in principle they are distinguishable. However, it is not at all obvious that we can set up an experiment that distinguishes between the different condensates. The problem with measuring defect condensates, for example, is that the conventional measuring techniques can measure the electric symmetry group (by looking at Bragg reflections, for example), but as far as we know there are no techniques yet to measure the magnetic symmetry group. If such techniques were developed, then we could use our tables to determine which condensates we are dealing with.

In appendix C, we work out all defect condensates from an achiral octahedral, and an achiral icosahedral nematic.

For example, table 5.4 predicts a defect-mediated phase transition from an octahedral nematic $F(\overline{O}) \otimes \mathbb{C}O$ to a biaxial nematic $D(D_2)$. Note that this biaxial nematic is rather special, namely there is no 2π defect. This is not because this defect is confined, but because it's condensed (since it's an element of K, or more to the point because $|[12][34] > \times |[12][34] >= | - e >$, thus the 2π defect is the outcome of fusion of the condensed defect with itself).

We see a general phenomenon appear in defect condensates: the electric group may be broken, or partially broken, or unbroken (if we condense a class

Combined defect condensate in $\mathcal{A} = F(\overline{T}) \otimes \mathbb{C}T_d$	K	$\mathcal{T}_{\mathcal{R}}$	$\mathcal{U}$
$\vert[123] > + \vert[132] >$	$\overline{C_3}$	$F(T/C_3) \otimes \mathbb{C}C_{3v}$	$\mathbb{C}C_{1v}$
$\vert[(12)(34)] > + \vert[(13)(24)] >$	$\overline{D_2}$	$F(T/D_2) \otimes \mathbb{C}C_{2v}$	$\mathbb{C}_{1v}$
$\{\vert[(12)(34)] > + \vert[-(12)(34)] > +$ $+ \vert[(13)(24)] > + \vert -[(13)(24)] >\}$	$\overline{D_2}$	$F(T/D_2) \otimes \mathbb{C}D_{2d}$	$\mathbb{C}C_{1v}$
$\vert[(12)(34)] > + \vert[-(12)(34)] >$	$\overline{C_2}$	$F(T/C_2) \otimes \mathbb{C}D_{2d}$	$F(C_2) \otimes \mathbb{C}C_{2v}$
$\vert[(12)(34)] > + \vert[(13)(24)] > + \vert[(14)(23)] >$	$\overline{D_2}$	$F(\mathbb{Z}_3) \otimes \mathbb{C}T$	$D(\mathbb{Z}_3)$

Table 5.4: Combined defect condensates in an achiral tetrahedral nematic (that satisfy trivial self braiding). In this table, $\mathcal{A}$ is the original symmetry algebra, K is the smallest subgroup of H_m that contains all the defects g_i that are in the condensate, $\mathcal{T}_{\mathcal{R}}$ is the residual symmetry algebra, and $\mathcal{U}$ is the unconfined symmetry algebra. No two defect condensates have simultaneously the same $\mathcal{T}_{\mathcal{R}}$ and $\mathcal{U}$. This in principle the different defect condensates are distinguishable.

sum defect condensate for example). More often than not, most electric particles are confined, which we notice by looking at the unconfined electric groups in $\mathcal{U}$.

Now that we have set up the mathematical theory behind symmetry breaking, and applied it to defect condensates, we must physically interpret our results. There are several aspects of defect condensates we would like to address. First of all, when we discussed defect condensates in chapter 3 we noticed that they restore symmetries. In the next section we will show how symmetry restoration fits into our formalism. Secondly, we mentioned three defect condensates: the Kosterlitz-Thouless condensate in the XY-model, the dislocation condensate in hexagonal crystals giving the hexatic phase, and the disclination condensate in the hexatic phase giving the isotropic phase. We will show that these phase transitions follow naturally from our formalism.

There is one last phenomenon we will address: the Abrikosov lattice in type II superconductors, which is a phase where the defects sit on a lattice, thus only restoring the translation group partially. This phase has an analog in liquid crystals, called the *Twist-Grain Boundary Phase*[15]. It consists of slabs of smectic with different orientations, and separated by an array of parallel screw dislocations.

5.5 Defect-mediated melting

Now that we've set up a mathematically and physically sensible theory of Hopf symmetry breaking, the time has come to take a closer look at defect condensates. The defect condensates discussed in chapter 3 restore electric symmetries. A careful analysis of the unconfined particles reveals that our theory actually predicts symmetry restoration! We will illustrate this with two simple but crucial examples.

Global internal abelian symmetry: Kosterlitz-Thouless condensate

The simplest example of defect condensate we have come across until now is the Kosterlitz Thouless phase transition in the XY model. In the isotropic phase, the internal symmetry group is $SO(2)$. Representations of $SO(2)$ are labelled by a $n \in \mathbb{Z}$. For reasons that will become clear shortly, we also introduce particles that transform under projective representations of $SO(2)$, which is equivalent to saying they transform under representations of the universal covering group $\mathbb{R}$ of $SO(2)$. The representations of $\mathbb{R}$ are labelled by a real number $\lambda \in \mathbb{R}$, giving the irrep ρ_λ:

$$\rho_\lambda(x) = e^{i\lambda x} \; \forall x \in \mathbb{R}.$$

If $\lambda \in \mathbb{Z}$, the particle is bosonic, because $\rho_\lambda(2\pi) = 1$. If $\lambda \in \mathbb{Z} + \frac{1}{2}$, the irrep is fermionic, because $\rho_\lambda(2\pi) = -1$. For any other value of λ, the irrep is an abelian anyon.

In the low temperature phase, $SO(2)$ symmetry is spontaneously broken to $\{e\}$. In the covering group, $\mathbb{R}$ is broken to $\mathbb{Z}$. The residual symmetry transformation $n \in \mathbb{Z}$ are multiples of the 2π rotation. The Hopf symmetry of this phase is $D(\mathbb{Z})$, because the fluxes are labelled by $\Pi_1(\mathbb{R}/\mathbb{Z}) = \Pi_0(\mathbb{Z}) = \mathbb{Z}$. The electric irreps are now labelled by $\mu \in U(1)$, labelling irreps of $\mathbb{Z}$. However, *we can still consider all the electric irreps of the isotropic theory!* In other words, we can restrict the irreps of $\mathbb{R}$ to $\mathbb{Z}$, giving representations of $\mathbb{Z}$, some of which are equivalent. We need to take all the electric irreps along, because we want to see which irreps of the isotropic phase are unconfined in the various phases. Note that the Goldstone boson transforms trivially under $\mathbb{Z}$, because 2π rotations don't affect it.

In this $D(\mathbb{Z})$ phase, the Kosterlitz-Thouless condensate is a condensate of the defect $|1>$. After $|1>$ has condensed, the residual phase has no defects, since according to (5.45):

$$\mathcal{U} = D(\mathbb{Z}/\mathbb{Z}) = D(e).$$

What about the electric irreps? The only irreps of $\mathbb{R}$ that braid trivially with the condensate $|1>$ are those labelled by an integer $n \in \mathbb{Z}$:

$$\rho_n(x) = e^{i2\pi nx} \; \forall x \in \mathbb{R}.$$

So the residual theory has no defects, and electric irreps are labelled by $n \in \mathbb{Z}$. A theory with no defects and charges labelled by $n \in \mathbb{Z}$ is an $SO(2)$ theory! This suggests that *the symmetry has been restored to* $SO(2)$, i.e. the original $\mathbb{R}$ theory has been compactified, and the electric charge is now quantized. More generally, condensing the $|n>$ defect gives a theory with $\mathbb{Z}_n$ defects (defects defined modulo n), and the electric irreps are labelled by $\frac{k}{n}$, $k \in \mathbb{Z}$, signalling that the symmetry is restored to $U(1)/\mathbb{Z}_n$.

Local internal abelian symmetry: defect condensates in superconductors

In the case of a system with local $U(1)$ symmetry, such as a superconductor the analysis is exactly the same as the analysis above. In a superconductor, the fermion field is denoted by Ψ, its conjugate by $\overline{\Psi}$. The unbroken phase has $U(1)$ as a local symmetry group, and the local symmetry transformations are given by

$$\begin{aligned} &\Psi \mapsto e^{i\alpha(x)}, \overline{\Psi} \mapsto e^{-i\alpha(x)} \\ &A_j(\vec{x}) \mapsto A_j(\vec{x}) + \partial_j \alpha(\vec{x}). \end{aligned} \tag{5.71}$$

The Cooper pair condensate $< \Psi\Psi >$ breaks the $U(1)$ symmetry to $\mathbb{Z}_2$. The defects in this theory are labelled by $\Pi_1(U(1)/\mathbb{Z}_2) = \mathbb{Z}$, and all charges of the original theory are present. Condensing $|1>$ restores $U(1)/\mathbb{Z}_2 \simeq U(1)$, thus partially restoring the gauge group, and destroying superconductivity.

Global external abelian symmetry: dislocation condensate in a hexagonal crystal

We neglected external symmetries in the above discussion, and the analysis is slightly different in the case of external symmetries. To illustrate this, consider the translational group in two dimensions, $\mathbb{R}^2$. The representations $\rho_{\vec{k}}$are labelled by a momentum vector $\vec{k} = (k_x, k_y) \in \mathbb{R}^2$:

$$\rho_{\vec{k}}(x, y) = e^{i(k_x x + k_y y)}. \tag{5.72}$$

Imagine a density wave in the y-direction $(0, a)$ condenses, giving a smectic with $\mathbb{R}\times\mathbb{Z}$ symmetry. The defects are labelled by $\Pi_1(\mathbb{R}^2/\mathbb{R}\times\mathbb{Z}) = \Pi_0(\mathbb{R}\times\mathbb{Z}) = \mathbb{Z}$. The electric irreps are irreps of $\mathbb{R}\times\mathbb{Z}$, and are labelled by (k_x, k_y), with $k_x \in \mathbb{R},\ k_y \in U(1)$. In the discussion of the XY model above, we considered the irreps of the isotropic phase in our ordered phase. In this case, we cannot, because the irreps of $\mathbb{R}$ that are not irreps of $U(1)$ correspond to momenta k_y in the y-direction that are larger than the ultraviolet cutoff. Namely, in our smectic we have a lattice spacing a in the y-direction, and we only allow for electric irreps $\rho_{(k_x,k_y)}$ with $k_y < \frac{2\pi}{a}$, i.e. we require k_y to be in the first Brillouin zone. We have to introduce the lattice spacing to make the theory physically sensible, namely we need the cutoff to make correlation functions converge. Note that momenta outside the first Brillouin zone correspond to irreps of $\mathbb{R}\times\mathbb{Z}$ that are equivalent to some irrep with momentum inside the first Brillouin zone, so we only need charges in the first Brillouin zone. But we make the stronger statement that momenta outside the first Brillouin zone are forbidden. So the electric irreps are labelled by $\vec{k} \in \mathbb{R}\times SO(2)$, with $SO(2) = [0, \frac{2\pi}{a})$. The phonons of the crystal are electric irreps whose momentum $\vec{k} = (k_x, k_y)$ satisfies $k_x = 0$.

Now condense the $|1>$ defect. The residual theory has no defects, and the unconfined electric irreps satisfy $k_y = 0$. Thus, all phonons are confined, since their charge now satisfies $k_x = k_y = 0$.

So the residual theory has no defects, and all the phonons are confined. This signals a return to the isotropic theory. Note the difference with the analysis of the XY model. In the XY model, we studied the fields that transformed under a representation of the isotropic theory. The Goldstone mode was not confined, but the symmetry was restored to $SO(2)$. This means that the original electric condensate was destroyed, and even though the Goldstone mode wasn't confined the symmetry was restored. In the case of translational symmetry, the ultraviolet cutoff imposes a restriction on the possible momenta. After condensation of the $|1>$, all the Goldstone modes are confined, signalling a return to the isotropic theory.

Now that we've seen how the symmetry is restored, we can explain what happens in the Abrikosov lattice, for example. The symmetry is first fully restored by the defect condensate, and then it is broken to a lattice, by the condensation of density waves, just as in the formation of the smectic.

Global external nonabelian symmetry: defect condensates in hexagonal crystals

We still have to explain the hexatic phase. We start with a hexagonal crystal whose Hopf symmetry is

$$\mathcal{A} = F(\mathbb{Z}^2 \rtimes (\mathbb{Z}_6 \times \mathbb{Z})) \otimes \mathbb{C}((\mathbb{Z}^2 \rtimes D_6) \times \mathbb{Z}). \tag{5.73}$$

Now consider a class sum defect condensate

$$|(1,0)> + |(0,1)> + |(-1,1)> + |(-1,0)> + |(0,-1)> + |(1,-1)> . \tag{5.74}$$

By drawing the condensate, we can "see" what $\mathcal{T}_{\mathcal{R}}$ and $\mathcal{U}$ will be (see fig. 5.4)

$$\mathcal{T}_{\mathcal{R}} = F(\mathbb{Z}^2 \rtimes (\mathbb{Z}_6 \times \mathbb{Z})/\mathbb{Z}^2) \otimes \mathbb{C}((\mathbb{Z}^2 \rtimes D_6) \times \mathbb{Z}) \tag{5.75}$$

$$\mathcal{U} = F(\mathbb{Z}_6 \times \mathbb{Z}) \otimes \mathbb{C}(D_6 \times \mathbb{Z}) \tag{5.76}$$

$\mathbb{Z}_6 \times \mathbb{Z}$ corresponds to disclinations, and $D_6 \times \mathbb{Z}$ corresponds to rotations and inversions. The $\mathbb{Z}$ part of $D_6 \times \mathbb{Z}$ corresponds to multiples of the 2π rotation.

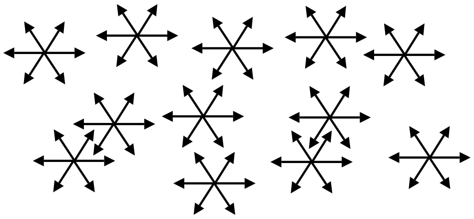


Figure 5.4: The defect condensate $|(1,0)> + |(0,1)> + |(-1,1)> + |(-1,0)> + |(0,-1)> + |(1,-1)>$ with hexagonal symmetry.

Note that the translational symmetries have completely disappeared. There are no more dislocations, and the translational phonons are confined. As we discussed earlier, this signals the restoration of translational symmetries, thus the unconfined and residual symmetry algebras of the hexatic phase are actually

$$\mathcal{T}_{\mathcal{R}} = F(\mathbb{Z}^2 \rtimes (\mathbb{Z}_6 \times \mathbb{Z})/\mathbb{Z}^2) \otimes \mathbb{C}((\mathbb{R}^2 \rtimes D_6) \times \mathbb{Z}) \tag{5.77}$$

$$\mathcal{U} = F(\mathbb{Z}_6 \times \mathbb{Z}) \otimes \mathbb{C}(\mathbb{R}^2 \rtimes (D_6 \times \mathbb{Z})). \tag{5.78}$$

Thus we've understood the phase transition from the hexagonal crystal to the hexatic phase. We now wish to comment on a phase described in [73]: the *topological nematic*. This is a two dimensional phase obtained from an isotropic crystal, where dislocations have condensed, but rotational symmetry is completely unbroken. In our defect condensate that led to the hexatic phase, the residual rotational symmetry group was $\mathbb{Z}_6$, but in the topological nematic the rotational symmetry group is $U(1)$. In our discussion on crystals in chapter 3, we came to the conclusion that an isotropic crystal only makes sense in the continuum limit, because by definition a crystal is not isotropic. However, in the continuum limit one can definitely have a Lagrangian of a crystal in terms of the displacement field $\vec{u}(\vec{x})$, that is invariant under $ISO(2) = \mathbb{R}^2 \rtimes SO(2)$. It turns out (see chapter 3) that the two dimensional hexagonal and isotropic crystals have exactly the same Lagrangians, so we could consider this isotropic crystal to be the continuum limit of a hexagonal crystal.

Now the topological nematic is a phase obtained by a defect condensation in the isotropic crystal. To obtain a defect condensate that fully restores the rotational symmetry group $SO(2)$, we must first realize that in the continuum limit the dislocations carry a continuous label. The Burgers vector of a dislocation in the continuum limit is characterized by a two-dimensional vector $|(a,b)>$, with $a, b \in \mathbb{R}^2$. Thus a dislocation condensate that leads to restoration of the full rotational symmetry group is given by

$$|\phi_{\mathcal{R}}> = \int_0^{2\pi} d\theta |(cos\theta, sin\theta)> . \tag{5.79}$$

In a sense, this condensate is the continuum limit of the class sum defect condensate we considered in the hexatic phase. It is indeed a class sum defect condensate of the isotropic crystal. Thus our analysis naturally incorporates the topological nematic. Furthermore, our analysis allows for partial restoration of the symmetry group, by considering defect condensates that are not class sum.

We now return to the hexatic phase. We will analyze the phase transition from the hexatic phase to the isotropic phase. The hexatic phase is a phase described by the Hopf algebra

$$\mathcal{A} = F(\mathbb{Z}_6 \times \mathbb{Z}) \otimes \mathbb{C}(\mathbb{R}^2 \rtimes (D_6 \times \mathbb{Z})).$$

The condensate is $|r>$, the $\frac{2\pi}{6}$ rotation. The outcome is

$$\mathcal{T}_{\mathcal{R}} = F(e) \otimes \mathbb{C}(\mathbb{R}^2 \rtimes (\mathbb{Z}_6 \times \mathbb{Z})) \tag{5.80}$$

$$\mathcal{U} = \mathbb{C}\mathbb{R}^2. \tag{5.81}$$

Our formalism also allows for other phase transitions. Consider, for example, the defect-antidefect condensate

$$|(1,0)> + |(-1,0)> .$$

The residual and unconfined symmetry algebras are

$$\mathcal{T}_{\mathcal{R}} = F((\mathbb{Z}^2 \rtimes \mathbb{Z}_6) \times \mathbb{Z})/0 \times \mathbb{Z}) \otimes \mathbb{C}(\mathbb{Z}^2 \rtimes (\mathbb{Z}_2 \rtimes \mathbb{Z}_2)) \qquad (5.82)$$
$$\mathcal{U} = F((\mathbb{Z} \rtimes \mathbb{Z}_2) \times \mathbb{Z}) \otimes \mathbb{C}\mathbb{Z} \rtimes (\mathbb{Z}_2 \rtimes \mathbb{Z}_2). \qquad (5.83)$$

The dislocations along the $(1,0)$ direction are condensed, and the phonons with momentum in that direction are confined. Thus the symmetry is restored in that direction, so that

$$\mathcal{T}_{\mathcal{R}} = F((\mathbb{Z}^2 \rtimes \mathbb{Z}_6) \times \mathbb{Z})/0 \times \mathbb{Z}) \otimes \mathbb{C}(\mathbb{Z}^2 \rtimes (\mathbb{Z}_2 \rtimes \mathbb{Z}_2)) \qquad (5.84)$$
$$\mathcal{U} = F((\mathbb{Z} \rtimes \mathbb{Z}_2) \times \mathbb{Z}) \otimes \mathbb{C}((\mathbb{R} \times \mathbb{Z}) \rtimes (\mathbb{Z}_2 \rtimes \mathbb{Z}_2)). \qquad (5.85)$$

This is a 2D smectic. Remember that in the smectics we discussed in chapter 4 the Goldstone modes corresponding to oscillations of the rods within a plane were massive, because of the analog of the Higgs phenomenon. It was crucial that the symmetry group of the rods was not internal, but coupled to external symmetries. In the 2D smectic we have uncovered now, there are no internal symmetries. However, the defect-antidefect pairs in our condensate behave like rods in a conventional smectic. Note that they are coupled to external symmetries, in fact they arise due to the breaking of external symmetries. Thus we also expect the Goldstone modes associated with the oscillation of our "rods" to be massive, and only the longitudinal oscillation of the planes to be massless.

We note that our analysis has naturally led us to the quantum smectic discussed in [73].

5.6 Defining a coproduct for $\mathcal{T}_{\mathcal{R}}$

$\mathcal{T}_{\mathcal{R}}$ is not always a Hopf algebra, because the coproduct of $\mathcal{A}$ does not necessarily satisfy $\Delta(\mathcal{T}_{\mathcal{R}}) \subset \mathcal{T}_{\mathcal{R}} \otimes \mathcal{T}_{\mathcal{R}}$. It does satisfy $\Delta(\mathcal{T}_{\mathcal{R}}) \subset \mathcal{A} \otimes \mathcal{T}_{\mathcal{R}}$. This means that we cannot take the tensor product of irreps of $\mathcal{T}_{\mathcal{R}}$, in other words we cannot fuse two particles in $\mathcal{T}_{\mathcal{R}}$. We can only fuse particles in $\mathcal{T}_{\mathcal{R}}$ with particles of $\mathcal{A}$ coming in from the left.

The reason we can't fuse particles of $\mathcal{T}_{\mathcal{R}}$ is that some excitations of the condensate $|\phi_{\mathcal{R}}>$ are confined. The condensate takes on the value $|\phi_{\mathcal{R}}>$ to the right, and $|\phi_{\mathcal{L}}> \neq |\phi_{\mathcal{R}}>$ to the left of a confined excitation, in an irrep Ω of $\mathcal{T}_{\mathcal{R}}$. Thus particles coming in from the left see the condensate $|\phi_{\mathcal{L}}>$. Thus particles to the left of are excitations of $|\phi_{\mathcal{L}}>$, and they transform under irreducible representations of the right residual symmetry algebra of $|\phi_{\mathcal{L}}>$.

Matters are complicated further by the observation that given two states $|v_1>$ and $|v_2>$ in a confined irrep Ω, the value of the condensate to the left of $|v_1>$ need not be equal to the value the condensate takes to the left of $|v_2>$.

Thus the condensate does not take a well defined value to the left of the irrep Ω.

One possible interpretation of a configuration with a confined excitation Ω, is that the condensate has only condensed to the right of Ω. To the left of Ω the system is in the unbroken phase, with $\mathcal{A}$ symmetry. Thus particles to the left of Ω are irreps of $\mathcal{A}$. We can fuse particles in irreps Π of $\mathcal{A}$ with Ω, and look at the fusion rules. The outcome of this fusion tells us what the particle in the irrep $\mathcal{A}$ can become when it enters the ordered phase.

This interpretation of a confined excitation is reasonable, but it is unsatisfactory. Namely, we still want to be a able to fuse the irreps of the residual symmetry algebra. There should be a finite set of quantum numbers in the broken phase, and we should be able to tell which of these quantum numbers can form hadronic composites. We want to be able to talk about the fusion of the quantum numbers in the broken phase without bringing in the quantum numbers of $\mathcal{A}$. We know that if $\mathcal{T}_{\mathcal{R}}$ is a Hopf algebra, we can fuse irreps of $\mathcal{T}_{\mathcal{R}}$. We will now discuss how to fuse irreps of $\mathcal{T}_{\mathcal{R}}$ when $\mathcal{T}_{\mathcal{R}}$ is not a Hopf algebra.

A purely magnetic phase

Let us first study a simple system that only has magnetic degrees of freedom. Its symmetry algebra is $F(H)$, with H some finite group. A basis of H is given by $\{P_h : h \in H\}$, where P_h is a projection operator. It measures the flux of a configuration : if the flux of a state $|v>$ is h, then $P_h \cdot |v>=|v>$. If the flux of $|v>$ isn't h, then $P_h|v>=0$.

Now consider a single defect condensate $|\phi_{\mathcal{R}}>=|g>$. Then using (5.48) (with $N_A=\{e\}$) we find for the residual symmetry algebra:

$$\mathcal{T}_{\mathcal{R}} = F(H/(g)), \tag{5.86}$$

where (g) is the subgroup of H generated by $|g>$. Let us assume that (g) is not a normal subgroup of H, so that $\mathcal{T}_{\mathcal{R}}$ is not a Hopf algebra. A basis of $\mathcal{T}_{\mathcal{R}}$ is given by $\{P_{h_i(g)} : h_i \in F\}$, where F consists of a set $\{h_i\}$ of representatives of the left (g) cosets in H. These projection operators measure defects modulo the condensed defect $|g>$. This is a consequence of our definition of operators in $\mathcal{T}_{\mathcal{R}}$: the operators in $\mathcal{T}_{\mathcal{R}}$ are the operators that do not notice when a particle fuses with the condensate $|g>$.

If we apply the coproduct Δ on a basis element $P_{h_k(g)}$ of $\mathcal{T}_{\mathcal{R}}$, we obtain

$$\Delta(P_{h_k(g)}) = \sum_{h_i\in F} P_{h_k(g)h_i^{-1}} \otimes P_{h_i(g)} = \sum_{h_i\in F} P_{h_k h_i^{-1}(h_i g h_i^{-1})} \otimes P_{h_i(g)}. \tag{5.87}$$

This coproduct has a natural interpretation: if a projection operator $P_{h_i(g)}$ measures a defect on the right, then the projection operators on the left measure defects modulo $h_i(g)h_i^{-1}$. Thus $\phi_{\mathcal{L}} = |h_i g h_i^{-1}>$: the condensate on the left is conjugated by h_i. Note that the defect of the particle that $P_{h_i(g)}$ measured is only defined modulo g, but this doesn't affect $|\phi_{\mathcal{L}}>$, since $|h_i g^n g(h_i g^n)^{-1}>=|h_i g h_i^{-1}>\ \forall n \in \mathbb{Z}$.

We assume that there is a confined excitation $|v>$ in an Ω of $\mathcal{T}_{\mathcal{R}}$, and that $P_{h_i(g)}|v>=|v>$. We have a condensate $|\phi_{\mathcal{L}}>$ to the left of the confined excitation measured by $P_{h_i(g)}$ that is different from the condensate $|\phi_{\mathcal{R}}>$ to the right. We can, however, *redefine our projection operators to the left of the confined excitation.* The projection operators to the left of $|v>$ are defined modulo $|h_i g h_i^{-1}>$. A basis of these projection operators is $\{P_{h'_k(h_i g h_i^{-1})} : h'_k \in J\}$, where J consists of a set $\{h'_k\}$ of representatives of the left $(h_i g h_i^{-1})$ cosets in H.

To every projection operator $P_{h'_k(h_i g h_i^{-1})}$, we associate a projection operator $P'_{h_i(g),h'_k(g)}$ in $\mathcal{T}_{\mathcal{R}}$:

$$P'_{h_i(g),h'_k(g)} = P_{h_i^{-1}h'_k(h_i g h_i^{-1})h_i} = P_{h_i^{-1}h'_k h_i(g)}. \tag{5.88}$$

Using this definition, we can define a coproduct Δ' for $\mathcal{T}_{\mathcal{R}}$:

$$\Delta'(P_{h_k(g)}) = \sum_{h_i\in F} P'_{h_i(g),h'_k(g)} \otimes P_{h_i(g)} = \sum_{h_i\in F} P_{h_i^{-1}h'_k h_i(g)} \otimes P_{h_i(g)}. \tag{5.89}$$

Δ' is a map from $\mathcal{T}_{\mathcal{R}}$ to $\mathcal{T}_{\mathcal{R}} \otimes \mathcal{T}_{\mathcal{R}}$. One can check that Δ' is an algebra morphism, i.e.

$$\Delta'(P_{h_j(g)}P_{h_k(g)}) = \Delta'(P_{h_j(g)})\Delta'(P_{h_k(g)}). \tag{5.90}$$

We can use Δ' to fuse two irreps Ω_2 and Ω_1 of $\mathcal{T}_{\mathcal{R}}$:

$$\Omega_2 \otimes \Omega_1(a) = (\Omega_2 \otimes \Omega_1) \circ \Delta'(a). \tag{5.91}$$

Δ' is not coassociative, i.e.

$$(\Delta' \otimes id) \circ \Delta' \neq (id \otimes \Delta') \circ \Delta'. \tag{5.92}$$

This implies that the tensor product of three irreps Ω_3, Ω_2 and Ω_1 of $\mathcal{T}_{\mathcal{R}}$ is not associative:

$$(\Omega_3 \otimes \Omega_2) \otimes \Omega_1 \neq \Omega_3 \otimes (\Omega_2 \otimes \Omega_1). \tag{5.93}$$

The interpretation of this non coassociativity is as follows: when we take the tensor product $\Omega_2 \otimes \Omega_1$, Ω_2 is defined *with respect to the condensate to the right of* Ω_1. If a third particle Ω_3 comes in from the left, then it becomes defined with respect to the condensate *between* Ω_2 *and* Ω_1. So we have a natural ordering for the tensor product, namely we must take $(\Omega_3 \otimes \Omega_2) \otimes \Omega_1$, which corresponds to having Ω_1 in the system, then bringing in Ω_2 from the left, and then bringing in Ω_3. $\Omega_3 \otimes (\Omega_2 \otimes \Omega_1)$ is unphysical, because it isn't clear how the particles were put in the system.

The general case

In general, if we bring in $\Omega_1, \Omega_2, \ldots, \Omega_n$ from the left in that order, the resulting configuration is $((\ldots(\Omega_n \otimes \Omega_{n-1}) \otimes \Omega_{n-2}) \otimes \ldots) \otimes \Omega_2) \otimes \Omega_1$. We have been forced to introduce an ordering in our fusion. This ordering corresponds to an ordering in the coproduct:

$$(\Delta' \otimes id \otimes \ldots \otimes id) \circ \ldots \circ (\Delta' \otimes id \otimes id) \circ (\Delta' \otimes id) \circ \Delta'(a). \tag{5.94}$$

Once we have defined a non coassociative coproduct Δ', we can fuse confined excitations $\mathcal{T}_{\mathcal{R}}$, and study the possible hadronic composites. To define a Δ', we need a linear map γ with the following properties:

$$\begin{aligned}
&\gamma : \mathcal{A} \otimes \mathcal{T}_{\mathcal{R}} \mapsto \mathcal{A} \otimes \mathcal{T}_{\mathcal{R}} \\
&\forall a, b, \in \mathcal{T}_{\mathcal{R}} : \gamma(\Delta(ab)) = \gamma(\Delta(a))\gamma(\Delta(b)) \\
&\gamma|_{\Delta(\mathcal{T}_{\mathcal{R}})} : \Delta(\mathcal{T}_{\mathcal{R}}) \mapsto \mathcal{T}_{\mathcal{R}} \otimes \mathcal{T}_{\mathcal{R}} \text{ is injective} \\
&\gamma(\Delta(amod\mathcal{I})) = \Delta(a)mod(\mathcal{I} \otimes \mathcal{I})
\end{aligned}$$

where $\mathcal{I}$ was a subideal of $\mathcal{T}_{\mathcal{R}}$ such that $\mathcal{U} = \mathcal{T}_{\mathcal{R}}/\mathcal{I}$. The last demand is equivalent to demanding that we do not alter Δ at the level of $\mathcal{U}$. Then we define

$$\Delta'(a) = \gamma \circ \Delta(a). \tag{5.95}$$

We can be more explicit in the case of a condensate in a phase described by a generalized quantum double $F(H_m) \otimes \mathbb{C}H_{el}$, when the right residual symmetry algebra $\mathcal{T}_{\mathcal{R}}$ is of the form (5.29) : $\mathcal{T}_{\mathcal{R}} = F(H_m/B) \otimes N$. We've seen that $\mathcal{T}_{\mathcal{R}}$ is of this form for electric condensates, and defect condensates. The action of N on B is well defined, because N acts trivially on B. For every orbit A_i in G/B under the action of N, pick a preferred element g_i. Now define a map $\sigma : H_m/B \mapsto H_m$ such that $\sigma(g_iB) \in g_iB$. In other words, σ sends every left B coset g_iB into a chosen element in g_iB. We demand that for all $n \in N$

$$\sigma(n \cdot g_iB) = n \cdot \sigma(g_iB). \tag{5.96}$$

Using σ, we define

$$\begin{aligned}
&\gamma : (F(H_m) \otimes \mathbb{C}H_{el}) \otimes (F(H_m/B) \otimes N) \to (F(H_m) \otimes \mathbb{C}H_{el}) \otimes (F(H_m/B) \otimes N) \\
&\gamma(P_g n \otimes P_{g'B} n') = P_{\sigma(g'B)^{-1} g \sigma(g'B)} n \otimes P_{g'B} n'.
\end{aligned} \tag{5.97}$$

It is straightforward but lengthy to verify that $\gamma|_{\Delta(\mathcal{T}_{\mathcal{R}})}$ is an algebra morphism[10]. Thus we have a coproduct

$$\Delta'(P_{gB}n) = \gamma \circ \Delta(P_{gB}n) = \sum_{g_i} P_{\sigma(g_iB)^{-1}gB} n \otimes P_{g_iB} n. \tag{5.98}$$

An example: $\mathcal{A} = D(\overline{D_{2n}})$

As an example, we consider a phase described by the quantum double of the double cover of an even dihedral group: $\mathcal{A} = D(\overline{D_{2n}})$. The group structure of $\overline{D_{2n}}$ is discussed in appendix A:

$$\overline{D_{2n}} = \{r^k s^m : k = 0, 1, \ldots, 2n-1; m = 0, 1\} \tag{5.99}$$

---

[10] γ is not an algebra morphism of $D(H) \otimes \mathcal{T}_{\mathcal{R}}$, it is only an algebra morphism when restricted to $\Delta(\mathcal{T}_{\mathcal{R}})$.

with $r^{2n} = s^2 = -e$.

Now condense a single defect[11] $|s>$. The right residual and unconfined symmetry algebras are

$$\mathcal{T}_{\mathcal{R}} = F(\overline{D_{2n}}/(s)) \otimes \mathbb{C}\overline{\mathbb{Z}_2 \rtimes \mathbb{Z}_2} \tag{5.100}$$
$$\mathcal{U} = D(\mathbb{Z}_2), \tag{5.101}$$

where $\overline{\mathbb{Z}_2 \rtimes \mathbb{Z}_2} = \{r^{kn}s^m : k = 0,1,2,3; m = 0,1\}$.

The irreps of $\mathcal{T}_{\mathcal{R}}$ are given in table 5.5, and the irreps of $\overline{\mathbb{Z}_2 \rtimes \mathbb{Z}_2}$, which occurs as a centralizer of two orbits in $\overline{D_{2n}}/(s)$, are given in table 5.6.

We write the basis of the irrep Ω^i_k as $|i>, |-i>$, where $|i>$ corresponds to the defect $|r^i(s)>$ (remember that the defects are defined modulo the condensate $<s>$). The action of $\mathcal{T}_{\mathcal{R}}$ on this basis is set by

$$\begin{aligned}
&P_{r^j<s>} \cdot |i> = \delta_{j,i} \\
&r^{tn} \cdot |i> = (-1)^{tk}|i> \quad t = 0,1,2,3 \\
&s|i> = |-i> .
\end{aligned}$$

We can write this compactly in one equation:

$$P_{r^j<s>} r^{tn} s^l |i> = \delta_{j,(1-2l)i}(-1)^{tk}|(1-2l)i> \quad t \in \mathbb{Z}_4, l \in \mathbb{Z}_2. \tag{5.102}$$

The left (s) cosets in $\overline{D_{2n}}/(s)$ are $r^i(s) = \{r^i, -r^i, r^i s, -r^i s : i = 0,1,\ldots,2n-1\}$. To define the coproduct, we must choose $\sigma(r^i(s)) \in r^i(s)$ for every i. If we choose $\sigma(r^i(s)) = r^i$, then the coproduct Δ' is coassociative. This can be traced back to the fact that the r^i form a group. Note that in general, it is not possible to choose representatives of the cosets so that they form a group.

Using Δ', we can determine the following fusion rule:

$$\Omega^i_k \otimes \Omega^j_l = \Omega^{i+j}_{k+l} \oplus \Omega^{i-j}_{k+l} \tag{5.103}$$

The unconfined irreps are given in table 5.6. This fusion rules implies, for example, that Ω^i_1 and Ω^{n-i}_1 can fuse to Ω^n_2, which is unconfined. Thus we have made a hadronic composite.

If we choose $\sigma(r^i(s)) = r^i s$, the fusion rules are the same, even though the coproduct is not coassociative. However, in this case the definition of the particles is altered. Namely, if an irrep Ω^i_k is present to the right, then $|j>$ in the irrep Ω^j_l coming in from the left should be interpreted as $|-j>$. This can be checked by applying the projection operators:

$$P_{r^l<s>} \cdot (|j> \otimes |i>) = \delta_{l,-j+i}|j> \otimes |i> \tag{5.104}$$

We have therefore discovered that our choice of coproduct Δ' alters the meaning of the labels of our irreps, when they are to the right of a confined excitation.

---

[11] We take even dihedral groups, because in $D(\overline{D_{2n+1}})$, condensing $|s>$ yields $\mathcal{U} = \mathbb{C}e$, which is slightly less interesting, but note that we can still study hadrons! Only the hadron must be in the trivial irrep, since that is the only unconfined irrep.

Preferred element g_A	e	r^n	r^i $i = 0, 1, \ldots, n-1$
Orbit A in $\overline{D_{2n}}/ < s >$	$< s >= \{e, s, -e, -s\}$	$r^n < s >$	$\{r^i < s >, r^{-i} < s >\}$
Normalizer N_A	$\overline{\mathbb{Z}_2 \rtimes \mathbb{Z}_2}=$ $\{r^{tn}s^k :$ $t \in \mathbb{Z}_4, k \in \mathbb{Z}_2\}$	$\overline{\mathbb{Z}_2 \rtimes \mathbb{Z}_2}=$ $\{r^{tn}s^k :$ $t \in \mathbb{Z}_4, k \in \mathbb{Z}_2\}$	$\overline{\mathbb{Z}_2}=$ $\{e, s, -e, -s\}$
Irrep of N_A	$\alpha_{k,l}$ $k \in \mathbb{Z}_4, l \in \mathbb{Z}_2$	$\alpha_{k,l}$ $k \in \mathbb{Z}_4, l \in \mathbb{Z}_2$	β_m $m \in \mathbb{Z}_4$
Irrep of Ω	$\Omega^0_{k,l}$	$\Omega^n_{k,l}$	Ω^i_m
Unconfined irreps	$\Omega^0_{k,0}$ $k = 0, 2$	$\Omega^n_{k,0}$ $k = 0, 2$	

Table 5.5: The irreps of $\mathcal{T}_{\mathcal{R}} = F(\overline{D_{2n}}/ < s >) \otimes \mathbb{C}\overline{\mathbb{Z}_2 \rtimes \mathbb{Z}_2}$.

The reason for this is the following: we can determine the irreps of $\mathcal{T}_{\mathcal{R}}$, and that gives us a finite set of labels. When we have one particle in an irrep of $\mathcal{T}_{\mathcal{R}}$ in the system, its meaning is unambiguous. Now if a confined excitation is present, then we know that the residual symmetry algebra of the condensate $\phi_{\mathcal{L}}$ to the left of the excitation may be different from the right residual symmetry algebra $\mathcal{T}_{\mathcal{R}}$. However, we still want to use the same labels for particles to the left of the confined excitations, because they are excitations of a symmetry algebra isomorphic to $\mathcal{T}_{\mathcal{R}}$. Thus we are adding information to the labels of irreps of $\mathcal{T}_{\mathcal{R}}$, namely we are defining their meaning when they appear to the left of a confined excitation. They should not be considered as the same particle: for example, $|i > |i >$ should not be interpreted as the fusion of $|i >$ with itself. The correct interpretation is that we have the confined excitation $|i >$ of $\mathcal{T}_{\mathcal{R}}$, and we brought in a particle from the left, which under our choice of coproduct is labelled by $|i >$.

In summary, for the $|s >$ condensate in $D(\overline{D_{2n}})$, the fusion rules are independent of the choice of coproduct. We do not expect this to be a general result (although in the cases we've worked out the fusion rules don't depend on the choice of coproduct). The *physics* of the phase, on the other hand, should not depend on our choice of coproduct, since this choice boils down to a definition of our labels. An interesting follow up on this research would be to study the influence of the choice of coproduct on the fusion rules.

5.7 Dyonic condensates

We have applied our symmetry breaking analysis to electric condensates, and defect condensates, with remarkable success. We will now study an example of a

Element of $\overline{\mathbb{Z}_2 \rtimes \mathbb{Z}_2}$	e	$-e$	r^n	r^{-n}	s	$-s$	sr^n	sr^{-n}
Irrep $\alpha_{k,l}$ $k \in \mathbb{Z}_4,\ l \in \mathbb{Z}_2$	1	$(-1)^k$	i^k	i^{-k}	$(-1)^l i^k$	$(-1)^l i^{-k}$	$(-1)^{l+k}$	$(-1)^m$

Table 5.6: The irreps of $\overline{\mathbb{Z}_2 \rtimes \mathbb{Z}_2}$.

dyonic condensate. In the case of dyonic condensates, $\mathcal{T}_\mathcal{R}$ and $\mathcal{U}$ are considerably harder to extract. In particular, $\mathcal{T}_\mathcal{R}$ is in general a complicated algebra, and it is quite a challenge to determine its irreps. The study of dyonic condensates is a good test of our claims, such as the claim that there is a Hopf algebra $\mathcal{U}$ whose irreps are the unconfined irreps. We will purposely seek out cases where $\mathcal{T}_\mathcal{R}$ is not a Hopf algebra, because we've seen that if $\mathcal{T}_\mathcal{R}$ is a Hopf algebra, then the right residual symmetry algebra $\mathcal{T}_\mathcal{R}$ is equal to the residual symmetry algebra $\mathcal{T}$ studied in [61], where it is proven that if $\mathcal{T}$ is the residual symmetry algebra, then the unconfined irreps are indeed irreps of a Hopf algebra $\mathcal{U}$.

We consider a phase described by a generalized quantum double $F(H_m) \otimes \mathbb{C}H_{el}$, and take a vector in a dyonic irrep Π^A_α, with $A \neq \{e\}$, and $\alpha \neq id$. g_A is a chosen preferred element of A, N_A is the normalizer of g_A in H_{el}, $x_i : i = 0, 1, \dots, n-1$ are representatives of left N_A cosets with $x_1 = e$, so that $A = \{g_i = x_i g_A x_i^{-1}\}$ with $g_1 = g_A$.

We condense a single defect dyon. i.e. a vector of the form $|g_j, v>$. The condition of trivial self braiding gives the restriction that the irrep Π^A_α in which $|g_j, v>$ lives must satisfy $\alpha(g_A) = \mathbb{1}$.

Without loss of generality, we can take $g_i = g_A$, since g_A was an arbitrarily chosen element of A. Thus our single defect dyon condensate is $|g_A, v>$. In function notation,

$$|\phi_\mathcal{R}(x)>= 1_{N_A}(x)\alpha(x^{-1})|v> . \tag{5.105}$$

$\mathcal{T}_\mathcal{R}$ is made up of all the functions $f \in F(H_m \times H_{el})$ that satisfy

$$f(x_1(x \cdot g_A), y_1)|\phi_\mathcal{R}(y_1^{-1}x)>= f(x_1, y_1)|\phi_\mathcal{R}(x)> \ \forall x_1 \in G; x, y_1 \in H. \tag{5.106}$$

Filling in $x = e$ we get

$$\begin{aligned} & f(x_1 g_A, y_1)|\phi_\mathcal{R}(y_1^{-1})>= f(x_1, y_1)|\phi_\mathcal{R}(e)> \\ \Rightarrow\ & f(x_1 g_A, y_1)1_{N_A}(y_1^{-1})\alpha(y_1)|v>= f(x_1, y_1)1_{N_A}(e)\alpha(e)|v> \\ \Rightarrow\ & \left\{ \begin{array}{l} \text{if } y_1 \notin N_A \text{, then } 0 = f(x_1, y_1) \Rightarrow \mathcal{T}_\mathcal{R} \subset F(H_m \times N_A) \\ \text{if } y_1 \in N_A \text{, then } f(x_1 g_A, y_1)\alpha(y_1)|v>= f(x_1, y_1)|v> \end{array} \right. \end{aligned} \tag{5.107}$$

This last equation leads to the following conclusions: if $f(x, y) \neq 0$, then $f(xg_A, y) \neq 0$, and $|v>$ is an eigenvector of $\alpha(y)$, with an eigenvalue we call λ_y:

$$\lambda_y = \frac{f(x, y)}{f(xg_A, y)}. \tag{5.108}$$

This equation must hold for all $x \in H_m$, thus $\frac{f(x,y)}{f(xg_A,y)}$ is independent of x.

We filled in $x = e$ into (5.106). One can check that filling in $x \neq 0$ leads to the same equations we just derived. $\mathcal{T}_{\mathcal{R}}$ is spanned by the functions $f \in F(H_m \times H_{el})$ that satisfy

1. $f \in F(H_m \times N_A)$
2. If $f(x,y) \neq 0$, then $f(xg_A, y) \neq 0$, and $|v>$ is an eigenvector of $\alpha(y)$ with eigenvalue $\lambda_y = \frac{f(x,y)}{f(xg_A,y)}$.

A basis of $\mathcal{T}_{\mathcal{R}}$ is given by functions $f_{x_\zeta,y}$, labelled by a representative x_ζ of a left (g_A) coset in $G/(g_A)$, and an element y of H such that $|v>$ is an eigenvector of $\alpha(y)$ with eigenvalue λ_y:

$$f_{x_\zeta,y} \sum_{k=0}^{n-1} P_{x_\zeta g_A^k} y \lambda_y^{-k}, \tag{5.109}$$

where n is the smallest integer such that $g_A^n = e$.

These right residual symmetry operators are a beautiful manifestation of the dyonic nature of the condensate. Our attention now turns to whether $\mathcal{T}_{\mathcal{R}}$ is a Hopf algebra. Using our results at the beginning of this chapter, $\mathcal{T}_{\mathcal{R}}$ is a Hopf algebra $\iff$ $\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{L}}$. Thus if we determine $\mathcal{T}_{\mathcal{L}}$, we can compare it with $\mathcal{T}_{\mathcal{R}}$ to find out if $\mathcal{T}_{\mathcal{R}}$ is a Hopf algebra.

The derivation of $\mathcal{T}_{\mathcal{L}}$ is analogous to that of $\mathcal{T}_{\mathcal{R}}$. A basis of $\mathcal{T}_{\mathcal{L}}$ is given by functions $g_{x_\eta,y}$, labelled by a representative x_η of a right $< g_A >$ coset in $< g_A > \ H_m$, and an element y of H_{el} such that $|v>$ is an eigenvector of $\alpha(y)$ with eigenvalue λ_y:

$$g_{x_\eta,y} = \sum_{k=0}^{n-1} P_{x_\eta g_A^k} y \lambda_y^{-k}. \tag{5.110}$$

As an explicit example, let us study single defect dyons of the form $|r^k, \beta_l>$ in $D(\overline{D_n})$. β_l is an irrep of the centralizer N_{r^k} of r^k. If we assume that n is odd, then $N_{r^k} = \{r^m : m = 0, 1, \ldots, 2n-1\}$. The irreps β_l of N_{r^k} are labelled by an integer $l \in \mathbb{Z}_{2n-1}$:

$$\beta_l(r^m) = exp(i\frac{2\pi ml}{2n}). \tag{5.111}$$

Now condense $|r^k, \beta_l>$ with $k \neq \frac{1}{2}n, n, \frac{3n}{2}$. Since we require the dyon to have trivial self braiding, l and k satisfy $lk = 0(mod 2n)$.

The right residual symmetry algebra $\mathcal{T}_{\mathcal{R}}$ is

$$\mathcal{T}_{\mathcal{R}} = F(\mathbb{Z}_{gcd(k,2n)}) \otimes \mathbb{C}y_{k,l}\mathbb{Z}_{2n} \oplus A \tag{5.112}$$

where $y_{k,l} = \frac{gcd(k,2n)}{gcd(gcd(k,2n),gcd(l,2n))}$, and A is some algebra with complicated multiplication, whose irreps are all confined. The unconfined symmetry algebra is

$$\begin{aligned}\mathcal{U} &= F(x_l\mathbb{Z}_{gcd(k,2n)}) \otimes \frac{2n}{gcd(gcd(l,2n),gcd(k,2n))}\mathbb{Z}_{2n} \\ &\simeq F(\mathbb{Z}_{\frac{gcd(k,2n)gcd(l,2n)}{2n}}) \otimes \mathbb{C}\mathbb{Z}_{gcd(k,gcd(l,2n))} \qquad (5.113)\end{aligned}$$

where $x_l = \frac{2n}{gcd(l,2n)}$.

The proof of (5.112) and (5.113) lengthy, though there are some steps that simplify matters considerably. The greatest common divisor appears because $gcd(k,2n)$ is the smallest number in the series $\{tkmod(2n) : t \in \mathbb{N}\}$[12]. A basis of $\mathcal{T}_{\mathcal{R}}$ is given by functions $f_{p,m}$ and $g_{p,m}$:

$$f_{p,m} = \sum_{t=0}^{x_k} P_{r^{p+kt}} r^m j^{-mt} \quad p = 0,1,\ldots,gcd(k,2n); m = 0,1,\ldots,2n-1$$

$$g_{p,m} = \sum_{t=0}^{x_k} P_{sr^{p+kt}} r^m j^{-mt} \quad p = 0,1,\ldots,gcd(k,2n); m = 0,1,\ldots,2n-1$$

where $x_k = \frac{2n}{gcd(k,2n)}$, $j = exp(\frac{i2\pi l}{2n})$. For $f_{p,m}$ (and $g_{p,m}$) to be non-zero, we must have $j^{-mx_k} = 1$ (if $j \neq 1$). This means that $mx_k = 0(modx_l)$, so that m is a multiple of $\frac{gcd(k,2n)}{gcd(gcd(l,2n),gcd(k,2n))}$.

The $f_{p,m}$ therefore span the first part of $\mathcal{T}_{\mathcal{R}}$ we wrote down before, and the $g_{p,m}$ span A.

Now for the irreps that braid trivially with the vacuum. Some intermediate steps are

$$P(P_{s^m r^p} s^n r_q) = \delta_{n,0} j^{\frac{p-(p \ mod \ gcd(k,2n))}{gcd(k,2n)} z_k q} (\delta_{m,0} f_{p \ mod \ gcd(k,2n),q} + \delta_{m,1} g_{p \ mod \ gcd(k,2n),q}) \qquad (5.114)$$

where z_k is the smallest positive integer such that $z_k k = gcd(k,2n)(mod2n)$.

$$\epsilon(P(P_{s^n r^p} s^m r^q)) = \delta_{n,0}\delta_{m,0}\delta_{p,0 \ modgcd(k,2n)} j^{\frac{p-(p \ mod \ gcd(k,2n))}{gcd(k,2n)} z_k q} \qquad (5.115)$$

Equation (5.26) gives for an irrep Ω of $\mathcal{T}_{\mathcal{R}}$:

$$\Omega(\sum_{q=0}^{gcd(k,2n)-1} f_{q,0} j^q) = \Omega(1) \text{ and } \Omega(\sum_{q=0}^{gcd(k,2n)-1} g_{q,0} j^q) = 0 \qquad (5.116)$$

Let us call the irreps of the first part of $\mathcal{T}_{\mathcal{R}}$ (the part spanned by the $f_{p,m}$) ρ_f, and the irreps of A will be called ρ_g. The ρ_f are zero on all of A, while the ρ_g are zero on all $f_{p,m}$. The first of the two equations we just wrote down leads to the fact that all ρ_g are confined, because $1 = \sum_{q=0}^{gcd(k,2n)-1} f_{q,0} + g_{q,0}$, so that

$$\rho_g(\sum_{q=0}^{gcd(k,2n)-1} f_{q,0} j^q) = \rho_g(1) \Rightarrow 0 = \rho_g(1) \Rightarrow \rho_g \equiv 0 \qquad (5.117)$$

---

[12] This follows from a basic lemma in algebra theory, that $gcd(k,n)$ is the smallest positive integer in the set $\{xk + yn : x, y \in \ Z\}$, and from the fact that $(2n-1)k = -k(mod2n)$, so that we have $\{(xk+yn)mod2n : \ x, y \in \mathbb{Z}\} = \{(xk+yn)mod2n : \ x \in \mathbb{N}, y \in \mathbb{Z}\}$.

Equation (5.27) tells us that

$$\sum_{q=0}^{gcd(k,2n)-1} \Omega(f_{q,-k} + g_{q,-k}) = \Omega(1) \tag{5.118}$$

We know that the irreps of the first part of $\mathcal{T}_{\mathcal{R}}$ are labelled by two integers, so we call the irreps $\rho_{a,b}; a = 0, 1, \ldots, gcd(k, 2n) - 1; b = 0, x_k, 2x_k, \ldots, -x_k(mod(2n))$. The trivial braiding conditions give

$$j^a = 1 = e^{-\frac{i2\pi bk}{2n}} \tag{5.119}$$

which leads to $a = 0, x_l, 2x_l, \ldots$ and $b = 0, x_k, 2x_k, \ldots$ This proves (5.113).

To obtain the case of a pure defect condensate, fill in $l = 2n$ (which is equivalent to $l = 0$, but in the equations it only makes sense to fill in $l = 2n$):

$$\mathcal{U} \simeq D(\mathbb{Z}_{gcd(k,2n)}). \tag{5.120}$$

5.8 Conclusions and outlook

The Hopf symmetry description of phases with spontaneously broken symmetries exposes the duality between electric modes and topological defects in a very explicit and beautiful way. Furthermore, it encodes the braiding of modes algebraically, with a braid operator acting on the vector space of the modes. We have studied the recent discovery that symmetry breaking can be studied by studying the representation theory of the Hopf algebra. We've seen that dropping the restriction that the residual symmetry be a Hopf algebra leads to a novel criterion that gives physically sensible answers. It allows for a unified description of symmetry breaking induced by electric, magnetic and dyonic condensates. We've shown that our analysis reproduces the conventional theory of electric phase transitions, and that it gives answers for defect condensates that are consistent with the defect-mediated phase transitions discussed in the literature. As an example, we've analyzed all electric and defect-mediated phase transitions in achiral tetrahedral, octahedral and icosahedral nematic. The spectrum in the residual phase consists of two types of excitations: unconfined excitations, which are well excitations around which the condensate is singly valued, and hadronic composites, which are combinations of confined excitations that are unconfined as a whole. In the tetrahedral nematic, we've found that different condensates lead to a different spectrum, either in the unconfined excitations or the hadronic composites, so that in principle the phases obtained by the different condensates are distinguishable. For the octahedral and icosahedral nematics there are defect condensates that given the same spectrum according to our analysis. Finally we worked out a simple example of a dyonic condensate.

There are many open questions which should be addressed. Firstly, we have made certain mathematical conjectures plausible, but we haven't proved them. We hope that mathematicians will take up the challenge of proving

Conjecture 1. *$\mathcal{T}_\mathcal{R}$ is a semisimple algebra*[13]*, and there is a Hopf algebra $\mathcal{U}$ whose irreps are precisely the irreps of $\mathcal{T}_\mathcal{R}$ that braid trivially with the condensate. $\mathcal{U}$ has a natural braiding, which follows from the braiding of the original Hopf symmetry $\mathcal{A}$.*

Conjecture 2. *Choosing different coproducts Δ' for $\mathcal{T}_\mathcal{R}$ gives the same physical answers, such as the possible hadronic composites.*

Our criterion for symmetry breaking was defined for a general Hopf algebra. It didn't make explicit use of the coassociativity of the Hopf algebra, so it can also be applied to quasi-Hopf algebras, for example (see [26] for the definition). The Hopf symmetry approach has been applied to many physical systems, to which it would be very interesting to apply our symmetry breaking formalism:

- Go back to the gauge theory in (2.59) (the non relativistic version of the planar Abelian Higgs model). If we add the Chern-Simons term[19] $q\vec{J} \cdot \vec{A} - \mu \vec{A} \cdot (\nabla \times \vec{A})$ to the Lagrangian (where $\vec{J}$ is the current of the charged particles), and the symmetry is spontaneously broken to a discrete H, then the electrically charged particles become automatically magnetically charged[19]. There is a Hopf algebra, called a quasi-quantum double $D^\omega(H)$, that captures the fusion rules and braiding properties of the excitations of this system. $D^\omega(H)$ is very similar in structure to the quantum double $D(H)$, but the representation theory is more complex. Our symmetry breaking formalism is definitely applicable to $D^\omega(H)$, and the study of Hopf symmetry breaking in $D^\omega(H)$ would be a very natural extension of this work.

  It has been proven that condensates of particles in theories with Chern-Simons terms can become superconducting[72]. The idea that this might explain High-Tc superconductivity has recently been overthrown, because it conflicts with experiments.

- There is a quantum group description of the Fractional Quantum Hall effect[61]. The quantum group is $U_q(sl_2)$, the quantum deformation of sl_2. It is infinite dimensional, but the infinite dimensionality is not a daunting one. The basic ingredient, sl_2, is a finite dimensional Lie algebra, and we can take formal tensor products of elements of the Lie algebra, which is why $U_q(sl_2)$ is infinite dimensional. It would be very interesting to study condensates in such systems, especially since experimentalists have extensively studied phases where anyons have condensed.

- Finally, 2+1D gravity, and 3D crystals, can be described as a Chern-Simons gauge theory. It has been proved[45] that there is a continuous quantum double that captures the fusion rules and braiding properties of the excitations of these systems. In the case of 2+1D gravity, it is denoted

---

[13] Requiring that $\mathcal{T}_\mathcal{R}$ is semisimple is equivalent to requiring that all representations are fully reducible. This means that there are a finite number of irreps out of which all representations are built. Thus we have a finite set of quantum numbers.

by $D(SU(1,1))$. For 3D crystals the quantum double is $D(SU(2))$. We strongly believe that our symmetry breaking formalism is applicable to continuous quantum double.

Bibliography

[1] Alejandro Adem and R. James Milgram. *Cohomology of finite groups.* Springer-Verlag, 1994.

[2] Y. Aharonov and D. Bohm. Significance of Electromagnetic Potentials in the Quantum Theory. *Phys. Rev.*, 115(3):485–491, 1959.

[3] M.G. Alford, K. Benson, S. Coleman, J. March-Russell, and F. Wilczek. Interactions and excitations of non-abelian vortices. *Phys. Rev. Lett.*, 64(14):1632–1635, 1990.

[4] F.A. Bais. Flux metamorphosis. *Nucl. Phys. B*, 170:32–43, 1980.

[5] F.A. Bais and M. de Wild Propitius. Discrete gauge theories. In *The III International Conference on Mathematical Physics, String Theory and Quantum Gravity, Theor. Math. Phys.,*, volume 98, page 509. Proceedings of the Conference, Alushta, 1993, 1994.

[6] F.A. Bais, B.J. Schroers, and J.K. Slingerland. Broken quantum symmetry and confinement phases in planar physics. *Phys. Rev. Lett.*, 89:181601, 2002.

[7] F.A. Bais, P. van Driel, and M. de Wild Propitius. Quantum symmetries in discrete gauge theories. *Phys. Lett. B*, 280:63–70, 1992.

[8] A.P. Balachandran, F. Lizzi, and V.G.J. Rodgers. Topological symmetry breakdown in cholesterics, nematics and helium 3. *Phys. Rev. Lett.*, 52(20):1818–1821, 1984.

[9] L. Balents and D.R. Nelson. Quantum smectic and supersolid order in helium films and vortex arrays. *Phys. Rev. B*, 52(17):12951–12968, 1995.

[10] K.M. Benson. Charge violation and alice behavior in global and textured strings. *Phys. Rev. D*, 64(085002):1–11, 2001.

[11] M. Berry. Quantal phase factors accompanying adiabatic changes. *Proc. R. Soc. Lond. A*, 392:45–47, 1984.

[12] Z. Bradac, S. Kralj, M. Svetec, and S. Zumer. Annihilation of nematic point defects: Postcollision scenarios. *Phys. Rev. E*, 67:050702–1/4, 2003.

[13] M. Bucher, H.K. Lo, and J. Preskill. Topological approach to Alice Electrodynamics. *Nucl. Phys.*, B386:3–26, 1992.

[14] P.H. Butler. *Point group symmetry applications.* Plenum Press, New York, 1981.

[15] P.M. Chaikin and T.C. Lubensky. *Principles of condensed matter physics.* Cambridge University Press, 1995.

[16] I. Chuang, R. Durrer, N. Turok, and B. Yurke. Cosmology in the laboratory: defect dynamics in liquid crystals. *Science*, 251:1336–1342, March 1991.

[17] C.W. Curtis and I. Reiner. *Representation Theory of Finite Groups and Associative Algebras.* John Wiley and Sons, 1988.

[18] P.G. de Gennes and P.J. Prost. *The Physics of Liquid Crystals.* Oxford Science Publications, second edition edition, 1993.

[19] M. de Wild Propitius. *Topological interactions in broken gauge theories.* PhD thesis, Universiteit van Amsterdam, 1995.

[20] R. Dijkgraaf, V. Pasquier, and P. Roche. Quasi hopf algebras, group cohomology and orbifold models. *Nucl. Phys. B (Proc. Suppl.)*, 18B:60–72, 1990.

[21] E. Fradkin and S.A. Kivelson. Liquid-crystal phases of quantum hall systems. *Phys. Rev. B*, 59(12):8065–8072, 1999.

[22] M. Greiner, C. A. Regal, and D. S. Jin. Fermionic condensates. In *arXiv:cond-mat/0502539.* International Conference on Atomic Physics, 2005.

[23] A. Hatcher. *Algebraic topology.* Self-published notes, 2001.

[24] H.K.-Lo and J. Preskill. Non-abelian vortices and non-abelian statistics. *Phys. Rev. D*, 48(10):4821–4835, 1993.

[25] P. John. Core deformations of topological defects. Master's thesis, Universiteit van Amsterdam, 1990.

[26] C. Kassel. *Quantum groups.* Springer-Verlag, Berlin, 1995.

[27] M.O. Katanaev. Geometric theory of defects. *arXiv: cond-mat/0407469*, November 2004.

[28] M.V. Khazan. Analog of the Aharonov-Bohm effect superfluid $^3He - A$. *JETP Lett.*, 41:486, 1985.

[29] H. Kleinert. *Gauge fields in Condensed Matter, Vol. I: Superflow and Vortex Lines, Disorder Fields, Phase Transitions.* world scientific, Singapore, 1989.

[30] H. Kleinert. *Gauge fields in Condensed Matter, Vol. II: Stresses amnd Defects, Differential Geometry, Crystal Defects.* world scientific, Singapore, 1989.

[31] H. Kleinert. Vortex, defect, and monopole gauge fields for nambu-goldstone systems and their phase transitions. *arXiv:cond-mat/9503030*, 1995.

[32] T.H. Koornwinder, F.A. Bais, and N.M. Muller. Tensor product representations of the quantum double of a compact group. *Comm. in Math. Phys.*, 198:157–186, 1998.

[33] T.H. Koornwinder and N.M. Muller. The quantum double of a (locally) compact group. *J. of Lie Theory*, 7:101–120, 1997.

[34] J M Kosterlitz. The critical properties of the two-dimensional xy model. *J. Phys. C*, 7:1046–1060, 1974.

[35] M. Kosterlitz and D.J. Thouless. Ordering, metastability and phase transitions in two-dimensions. *J. Phys. C*, 6:1181–1203, 1973.

[36] L.D. Landau and E.M. Lifshitz. *Theory of Elasticity.* Pergamon Press, 1959.

[37] L.D. Landau and E.M. Lifshitz. *Statistical physics.* Pergamon Press, 1980.

[38] H.-K. Lo and J. Preskill. Non-abelian vortices and non-abelian statistics. *Phys. Rev. D*, 48(10):4821–4833, 1993.

[39] T.C. Lubensky and L. Radzihovsky. Theory of bent-core liquid-crystal phases and phase transitions. *Phys. Rev. E 66, 031704*, 2002.

[40] J. March-Russell, J. Preskill, and F. Wilczek. Internal frame dragging and a global analogue of the aharonov-bohm effect. *Phys. Rev. Lett.*, 68:2567, 1992.

[41] P. McGraw. Global analogue of Cheshire charge. *Phys. Rev. D*, 50:952–961, 1994.

[42] N.D. Mermin. Topological theory of defects in ordered media. *Rev. Mod. Phys. 51, 591*, 1979.

[43] N.D. Mermin and H. Wagner. *Phys. Rev. Lett.*, 17:1133–1136, 1966.

[44] K. Mullen, H.T.C. Stoof, M. Wallin, and S.M. Girvin. Hexatically ordered superfluids. *Phys. Rev. Lett.*, 72(25):4013–4016, 1994.

[45] N. Muller. *Topological interactions and quantum double symmetries.* PhD thesis, Universiteit van Amsterdam, 1998.

[46] D.R. Nelson. *Defects and Geometry in Condensed Matter Physics.* Cambridge University Press, 2002.

[47] D.R. Nelson and B.I. Halperin. Dislocation-mediated melting in two dimensions. *Phys. Rev. B*, 19(5):2457–2484, 1979.

[48] T. Niori, T. Sekine, J. Watanabe, T. Furukawa, and H. Takezoe. *J. Mater. Chem.*, 6:1231, 1996.

[49] B. Ovrut. Isotropy subgroups of SO(3) and Higgs potentials. *J. Math. Phys 19 (2)*, February 1978.

[50] M.E. Peskin and D.V. Schroeder. *An Introduction to Quantum Field Theory.* Addison-Wesley, 1995.

[51] R. Pierce. *Associative Algebras.* Springer, 1980.

[52] M.M.H. Postma. Alice electrodynamics. Master's thesis, Universiteit van Amsterdam, 1997.

[53] L. Radzihovsky and A.T. Dorsey. Theory of quantum hall nematics. *Phys. Rev. Lett.*, 88(21):216802, 2002.

[54] L. Radzihovsky and T.C. Lubensky. Fluctuation-driven 1st-order isotropic-to-tetrahedratic phase transition. *arXiv:cond-mat/0008325*, 2000.

[55] R. Rajaraman. *Solitons and instantons.* North-Holland Publishing Company, 1982.

[56] M. Salle. On the global analogue of the Aharonov-Bohm effect. Master's thesis, Universiteit van Amsterdam, December 1997.

[57] J.A. Salthouse and M.J. Ware. *Point group character tables and related data.* Cambridge university press, 1972.

[58] A. Schwarz. *Quantum Field Theory and Topology.* Springer-Verlag, 1993.

[59] A. Schwarz. *Topology for physicists.* Springer-Verlag, 1994.

[60] J. Serre. *Linear Representations of Finite Groups.* Graduate Texts in Mathematics, Vol. 42, 1996.

[61] J.K. Slingerland. *Hopf symmetry and breaking.* PhD thesis, Universiteit van Amsterdam, 2002.

[62] A. Sonnet, A. Kilian, and S. Hess. Alignment tensor versus director: Description of defects in nematic liquid crystals. *Phys. Rev. E*, 52(1):718–722, 1995.

[63] J. Striet. *Alice electrodynamics.* PhD thesis, Universiteit van Amsterdam, 2003.

[64] J. Striet and F.A. Bais. Simple models with alice fluxes. *Phys. Lett. B*, 497:172–180, 2000.

[65] M. Tinkham. *Introduction to superconductivity.* McGraw-Hill, 1996.

[66] J. Toner and D.R. Nelson. *Phys. Rev.*, B23:363, 1981.

[67] G. Toth, C. Denniston, and J.M. Yeomans. Hydrodynamics of topological defects in nematic liquid crystals. *Phys. Rev. Lett.*, 88(10):105504–1/4, 2002.

[68] E. van Meer. Magnetic monopoles in gauge field theories. Master's thesis, Universiteit van Amsterdam, 1997.

[69] E. Verlinde. A note on braid statistics and the non-abelian aharonov-bohm effect. International Colloquium on Modern Quantum Field Theory, Bombay, 1990, World Scientific, Singapore, 1991.

[70] X.G. Wen. *Quantum field theory of many-body systems.* Oxford University Press, 2004.

[71] F. Wilczek. Quantum mechanics of fractional-spin particles. *Phys. Rev. Lett.*, 49(14):957–959, 1982.

[72] F. Wilczek. *Fractional Statistics and Anyon Superconductivity.* World Scientific, 1991.

[73] J. Zaanen and Z. Nussinov. Duality in 2+1d quantum elasticity: superconductivity and quantum nematic order. *arXiv:cond-mat/0309397*, Sep 2003.

Appendix A

SO(3) and SU(2)

A.1 The subgroups of SO(3)

There are only a handful of types of finite subgroups of SO(3). These are

- $\mathbb{Z}_n$, denoted by C_n in crystallographic notation
- D_n, the dihedral group of order n
- T, the tetrahedral group
- O, the octahedral group
- I, the icosahedral group, also denoted by Y or K.

All finite groups are isomorphic to a subgroup of a permutation group S_n from some n. This follows from the observation that when a group G acts on itself by left multiplication, it permutes its own elements. This left action establishes a homomorphism from the group G to the group S_n of permutation of n elements, for a group with n elements. Therefore, G is isomorphic to a subgroup of S_n. Note that G could also be isomorphic to a subgroup of S_k from some $k < n$. The finite subgroups above are isomorphic to the following:

$$T \simeq A_4 \tag{A.1}$$

$$O \simeq S_4 \tag{A.2}$$

$$I \simeq A_5 \tag{A.3}$$

A_n, the alternative group of order n, is the group of even permutations of n elements.

These isomorphisms have a natural interpretation:

$T \simeq A_4$: even permutations of the 4 vertices of a tetrahedron

$O \simeq S_4$: all permutations of the 4 diagonals of a cube

$I \simeq A_5$: even permutations of the 5 cubes inscribed inside an icosahedron.

The icosahedral group is the symmetry group of the icosahedron, as well as the dodecahedron, which is the 'dual' of the icosahedron. By dual, we mean we put a point on the center of every face of the icosahedron, connect these points, and obtain a dodecahedron. The tetrahedron is self dual, and the dual of the cube is the octahedron.

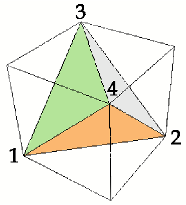


Figure A.1: The tetrahedron, with labelled vertices. It is self dual.

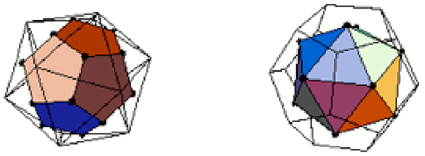


Figure A.2: The icosahedron and the dodecahedron are dual.

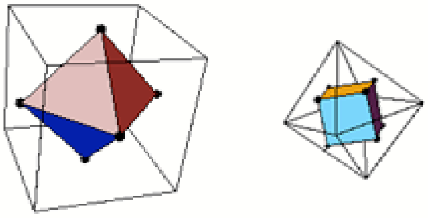


Figure A.3: The octahedron and the cube are dual.

A.2 Double cover groups

$SU(2)$ is a double cover of $SO(3)$. There is a two-to-one homomorphism

$$\rho : SU(2) \to SO(3) \tag{A.4}$$

To specify a rotation in $SO(3)$, one must specify an axis around which the rotation takes place, and a rotation angle $\theta \in [0, 2\pi)$. Denote by $\hat{n}$ a unit vector

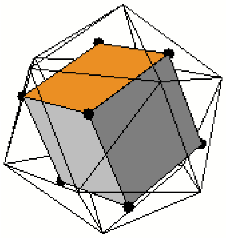


Figure A.4: The icosahedron, with one of the five inscribed cubes.

along the axis of rotation, and define positive θ to correspond to counterclockwise rotation with respect to $\hat{n}$. Then this rotation is denoted by $R(\hat{n},\theta)$.

We can parametrize matrices in $SU(2)$ in a very similar way: take any unit vector $\hat{n}$, and any angle $\theta \in [0, 4\pi)$. Notice how θ runs over a larger range than in the $SO(3)$ case. Now associate to $\hat{n}$ and θ the following matrix in $SU(2)$:

$$u(\hat{n},\theta) = exp(i\frac{\theta}{2}\hat{n}\cdot\vec{\sigma}) = cos(\frac{\theta}{2})\mathbb{1} + isin(\frac{\theta}{2})\hat{n}\cdot\vec{\sigma} \tag{A.5}$$

where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is a vector containing the three Pauli matrices, and $\mathbb{1}$ is the unit matrix.

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.6}$$

The homomorphism is now easily accomplished: just replace u by R.

$$\begin{aligned} &\rho : SU(2) \to SO(3) \\ &u(\hat{n},\theta) \mapsto R(\hat{n},\theta) \end{aligned} \tag{A.7}$$

The kernel of this homomorphism is $\{+1,-1\}$. The irreps of SO(3) are precisely the irreps of SU(2) that factor over ρ. A irrep of $SU(2)$ that factors over ρ is called bosonic, otherwise it is fermionic.

The homomorphism from $SU(2)$ to the quaternions is just as elegant: replace $-i\sigma_x$ by I, $-i\sigma_y$ by J, $-i\sigma_z$ by K.

Armed with these identifications, double cover groups become more transparent. The double cover of a subgroup G of $SO(3)$ is the inverse of this group under ρ. It has twice as many elements, but note that it is not simply $G \times \mathbb{Z}_2$. The double cover of G is denoted by $\overline{G}$.

Let us work out the group structure of $\overline{D_n}$ as an example. The easiest presentation is as elements of $\mathbb{H}$. Take the z axis as the rotation axis. Then

$$r^k = u(\hat{z}, \frac{\pi}{n}k)$$

$$s = u(\hat{x}, \pi)$$

One easily checks that $r^k = s^2 = -1$. All elements of $\overline{D_n}$ are of the form $\{\pm r^k s^m; k = 0, 1, \ldots, n-1; m = 0, 1\}$.
We can also write the following presentation, which is used in this work:

$$\begin{aligned}\overline{D_n} = \{ \quad & r^k s^i : k = 0, 1, \ldots, n-1; i = 0, 1, 2, 3 : s^2 = r^n, \\ & s^4 = e, r^k r^j = r^{k+j}, s r^k s^{-1} = r^{n-k}\} \end{aligned} \tag{A.8}$$

Here r^n is a 2π rotation (just like s^2), which commutes with all others elements.

Appendix B

Hopf algebra's

In this appendix we will develop the mathematical theory behind Hopf algebra's. We present all the structures of a Hopf algebra, and explain their physical interpretation. We give these structures for the quantum double of a finite group, and finally we discuss generalized quantum doubles in the main text.

B.1 Quasitriangular Hopf algebras

Algebra's are an ubiquitous mathematical structure. Famous examples are $\mathbb{R}$, $\mathbb{C}$, matrices, polynomials, functions from a manifold to $\mathbb{R}$, etc. Another mathematical structure which is less widespread is a coalgebra, a structure that is in a precise sense a dual object to an algebra. A Hopf algebra is simultaneously an algebra and a coalgebra, with certain compatibility conditions between the algebra and coalgebra structures. Let us work through the definitions, and important examples.

B.1.1 Algebra's

An algebra $\mathcal{A}$ is vector space over a field $\mathbb{F}$(which we will take to be $\mathbb{R}$ or $\mathbb{C}$), with a bilinear multiplication. Since it is bilinear, we can think of this multiplication as a map $\mu : \mathcal{A} \otimes \mathcal{A} \mapsto \mathcal{A}$. Our algebras will all have a unit element, i.e. an element 1 which satisfies $1a = a1 = a \; \forall a \in \mathcal{A}$. One can also assign a map to this unit, which embeds the field $\mathbb{F}$ into the center of the algebra. We write $\eta : \mathbb{F} \mapsto \mathcal{A}$. We require that η be an algebra morphism (the field is also an algebra, with itself as ground field). All the algebra's we will consider will be associative with respect to multiplication.
A very important example is the *group algebra* of a finite group. Label the groups elements by g_i. To get an algebra, choose a field $\mathbb{F}$, and put the group elements in "kets", writing them like $|g_i>$. The group algebra is then the set of objects of the form $\sum_i \lambda_i |g_i>$. This structure is very important, as it contains all the information about the group. For example, irreducible representations of the

group algebra are in one-to-one correspondence with irreducible representations of the group.
Electric modes transform under an irrep of a group, which is equivalent to saying that they transform under an irrep of the corresponding group algebra.

B.1.2 Coalgebra's

A coalgebra $\mathcal{C}$ is a vector space, and it comes equipped with a comultiplication and counit. The comultiplication is a linear map $\Delta : \mathcal{A} \mapsto \mathcal{A} \otimes \mathcal{A}$. The counit is a linear map $\epsilon : \mathcal{A} \mapsto \mathbb{F}$. They must satisfy

$$\begin{aligned} \text{Coassociativity :} \qquad & (\Delta \otimes id) \circ (\Delta) = (id \otimes \Delta) \circ \Delta \\ \forall c \in \qquad & (\epsilon \otimes id) \circ \Delta(c) = c = (id \otimes \epsilon) \circ \Delta(c) \end{aligned}$$

As an example, take all the linear functions from a finite group H to $\mathbb{F}$, which we denote by $F(H)$. They form a vector space, which is spanned by the functions $\{\delta_{g_i} : g_i \in H\}$. A coalgebra structure is then given by

$$\begin{aligned} \epsilon(\delta_{g_i}) &= \delta_{g_i,e} \\ \Delta(\delta_{g_i}) &= \sum_{g_k \in H} \delta_{g_i g_k^{-1}} \otimes \delta_{g_k} \end{aligned}$$

We need only define these functions on a basis of $F(H)$ since they are linear.
Given an algebra $\mathcal{A}$, its dual $\mathcal{A}^{*}$[1] can be given a coalgebra structure, that depends explicitly on the algebra structure. It is achieved in a rather natural way. The structures on the coalgebra are:

$$\begin{aligned} \Delta(f)(a_1, a_2) &= f(a_1 a_2) \\ \epsilon(f) &= f(1) \end{aligned}$$

This definition is the reason for the denominations of coproduct and counit.
The coalgebra $F(H)$ we defined above is obtained from the group algebra $\mathbb{C}H$ by these definitions, and is therefore the dual. It just so happens that $F(H)$ is also an algebra with pointwise multiplication: $f_1 \times f_2(x) := f_1(x) f_2(x)$. As we will see, it is in fact a Hopf algebra, just as $\mathbb{C}H$ is.
Given a finite dimensional coalgebra, its dual can be given an algebra structure, but this is in a slightly less natural way, and we will not discuss it (it is reminiscent of the isomorphism $\mathcal{A}^{**} \simeq \mathcal{A}$ for a finite dimensional vector space).
This is why we say that the coalgebra is a structure dual to the algebra.

B.1.3 Quasitriangular Hopf algebra's

A Hopf algebra is simultaneously an algebra and a coalgebra, with certain compatibility conditions. Namely, we demand that ϵ and Δ be algebra morphisms.

---

[1] The dual of an algebra $\mathcal{A}$ is defined as $F(\mathcal{A}) = \mathcal{A}^*$, the set of linear functions from $\mathcal{A}$ to $\mathbb{C}$.

This is equivalent to demanding that η and μ be coalgebra morphisms (see [26]). It also contains an antipode map S, which is the unique map from $\mathcal{A}$ to $\mathcal{A}$ that satisfies

$$\mu(S \otimes id)\Delta(a) = \mu(id \otimes S)\Delta(a) \tag{B.1}$$

From this definition, one can derive the following relations:

$$\begin{aligned} \forall a, b \in \mathcal{A}: \quad & S(ab) = S(b)S(a) && \text{(B.2)} \\ & S(1) = 1 && \text{(B.3)} \\ & (S \otimes S)\Delta = \Delta^{op} S && \text{(B.4)} \\ & \epsilon(S(h)) = \epsilon(h) && \text{(B.5)} \end{aligned}$$

Physically, the antipode map is used to construct antiparticles out of particles. More specifically, given an irrep Π, define the *antiparticle* or *conjugate* irrep $\overline{\Pi}(a) = \Pi^t S(a)$. It's an irrep because S is an antimorphism, and so it transposition. When we fuse particle and antiparticle irreps, we don't necessarily get the vacuum, but it is guaranteed that ϵ is present in the fusion rules.

Lemma 5. $\Pi \otimes \overline{\Pi}$ *can decay into the vacuum channel* ϵ.

Proof. Call V the representation space of Π (which is also the representation space of $\overline{\Pi}$). Decompose $V \otimes V$ into a traceless part, and the trace:

$$v, w \in V : v \otimes w = (v \otimes w - \frac{1}{3} v \cdot w \mathbb{1}) + \frac{1}{3} v \cdot w \mathbb{1}$$

These two subspaces of $V \otimes V$ are invariant under the action of $\Pi \otimes \overline{\Pi}$, because (using the results above)

$$\begin{aligned} tr((\Pi \otimes \overline{\Pi})(v \otimes w - \frac{1}{3} v.w\mathbb{1})) &= \sum_{(a),i,j,k} a^{(1)}_{ik} S(a^{(2)})_{li} (v_k w_l - \frac{1}{3} v \cdot w \delta_{kl} \\ &= \epsilon(S(a)) \sum_{kl} (v_k w_l - \frac{1}{3} \delta_{kl}) \\ &= 0 \end{aligned}$$

In the penultimate line we used $\mu(S \otimes id)\Delta^{op}(a) = \epsilon(S(a))$. This proves that the traceless part gets mapped into the traceless part. The trace is also left invariant: $\Pi \otimes \overline{\Pi}(a)\mathbb{1} = \epsilon(a)\mathbb{1}$. □

The standard example of a Hopf algebra is the *group algebra*. We have already discussed the algebra structure. It is turned into a Hopf algebra with the following definitions:

$$\begin{aligned} &\Delta(g_i) = g_i \otimes g_i \\ &\epsilon(g_i) = 1 \\ &S(g_i) = g_i^{-1} \end{aligned}$$

Given a finite dimensional Hopf algebra $\mathcal{A}$, one can define a Hopf algebra structure on $\mathcal{A}^*$, and $\mathcal{A}^*$ is then called the dual Hopf algebra. The definitions are quite natural, and explain the names coproduct and counit:

$$\begin{aligned}
&\text{unit } 1^* : 1^*(a) = \epsilon(a) \\
&\text{Counit } \epsilon^* : \epsilon^*(f) = f(1) \\
&\text{Multiplication } : f_1 \cdot f_2(a) := (f_1 \otimes f_2)\Delta(a) \\
&\text{Coproduct } \Delta^* : \Delta^*(f)(a_1, a_2) = f(a_1 a_2)
\end{aligned}$$

We've seen that $F(H)$ has an algebra structure, and a coalgebra structure. It turns out that these are compatible in the sense defined earlier. Defining the antipode as $S(P_h) = \delta_{h,e}$, $F(H)$ turns into a Hopf algebra. And, as the reader may have guessed, $F(H) = H^*$ as a Hopf algebra, i.e. $F(H)$ is the Hopf algebra dual to $\mathbb{C}H$.

And now we are finally ready to explain the statement "defects are dual particles to electric modes". Namely, the physical measurement operators on the defects are projection operators P_h, which have the Hopf algebra structure of $F(H)$ (given that the defects live in H). This is not a trivial statement, it will become clear when we discuss the fusion rules, and notice that they are exactly the fusion rules of defects. For now, note that the coproduct has an immediate interpretation:

$$\Delta(P_h) = \sum_{h'} P_{h'} \otimes P_{h'^{-1}h}. \tag{B.6}$$

It makes sure that when two defects fuse, the total "flux" (the homotopy class of a loop surrounding both defects) is equal to the product of the fluxes (the composition of the loops defining each defect). Therefore, the defects transform under irreps of $F(H)$, and are therefore dual particles to irreps of H.

A Hopf algebra $\mathcal{A}$ is called quasitriangular when there is an invertible element R of $\mathcal{A} \otimes \mathcal{A}$ that satisfies:

$$\begin{aligned}
(\Delta \otimes id)R = R_{13}R_{23} \quad & (id \otimes \Delta)R = R_{13}R_{12} \\
\forall a \in \mathcal{A} : \quad & R\Delta(a)R^{-1} = \Delta(a)
\end{aligned}$$

where if we write $R = \sum_{(R)} R^{(1)} \otimes R^{(2)}$, then

$$R_{ij} = \sum_{(R)} 1 \otimes \cdots \otimes R^{(1)} \otimes \cdots \otimes R^{(2)} \otimes \cdots \otimes 1$$

where $R^{(1)}$ is in the i-th, and $R^{(2)}$ in the j-th position.

Given a Hopf algebra $\mathcal{A}$, there is a natural to "double" it, creating a new Hopf algebra $D(\mathcal{A})$ called Drinfeld's quantum double of $\mathcal{A}$. As a vector space, $D(\mathcal{A}) = \mathcal{A}^* \otimes \mathcal{A}$, so it's a tensor product of $\mathcal{A}$ and its dual. For the discussion of the Hopf algebra structure on $D(\mathcal{A})$, see [26]. For our purposes, we need only know what the structure is like for H a discrete group. We also specify a braid matrix, making it a quasitriangular Hopf algebra.

As vector space, $D(H)$ is $F(H) \otimes \mathbb{C}H$, which is the same as $F(H \times H)$ for finite H. Denote a basis of $D(H)$ as $P_h g$ (that's the notation used in [**??**]), where $g \in H$, and $P_h \in F(H)$ is defined by $P_h(x) = \delta_{h,x}$ $(x \in H)$. If we see it as an element of $F(H \times H)$, the function is $P_h g(x, y) = \delta_{h,x}\delta_{g,y}$. We can also define $g \equiv 1g = \sum_{h \in H} P_h g$, and $P_h \equiv P_h e$. We've therefore embedded H and $F(H)$ into $D(H)$, and their embeddings form an algebra basis of $D(H)$ (meaning that any element can be written as a sum of products of these basis elements). The structure is determined by the following formulae:

$$P_h g P_{h'} g' = \delta_{h,gh'g^{-1}} P_h g g' \tag{B.7}$$

$$\Delta(P_h g) = \sum_{h' \in H} P_{hh'^{-1}} g \otimes P_{h'} g \tag{B.8}$$

$$\epsilon(P_h g) = \delta_{h,e} \tag{B.9}$$

$$S(P_h g) = P_{g^{-1}h^{-1}g} g^{-1} \tag{B.10}$$

$$R = \sum_{g \in H} P_g e \otimes g \tag{B.11}$$

These relations can also be given in the language of functions in $F(H \times H)$, and this notation is convenient in several cases, so we give it now:

$$(f_1 f_2)(x, y) = \sum_{h \in H} f_1(x, h) f_2(h^{-1}xh, h^{-1}y) \tag{B.12}$$

$$\Delta(f)(x_1, y_1; x_2, y_2) = f(x_1 x_2, y_1)\delta_{y_1}(y_2) \tag{B.13}$$

$$\epsilon(f) = \sum_{z \in H} f(e, z) dz \tag{B.14}$$

$$S(f)(x, y) = f(y^{-1}x^{-1}y, y^{-1}) \tag{B.15}$$

$$R(x_1, y_1; x_2, y_2) = \delta_e(y_1)\delta_{x_1}(y_2) \tag{B.16}$$

This notation is also important because it generalizes to the case where H is compact, or locally compact. For results on such cases, see [32] and [33].

The irreducible representations of $D(H)$ follow from the fact that it's a transformation group algebra (see [61] for details). Since we will come across other transformation group algebras in the main text, we give the general definition. As a vector space, the transformation group algebra is $F(X) \otimes \mathbb{C}H$, where X is a finite set and H is a finite group. A basis is given by

$$\{P_x h : x \in X, h \in H\}. \tag{B.17}$$

Just as in the quantum double case, we can consider it to be the vector space $F(X \times H)$ of functions

$$f : X \times H \to \mathbb{C}. \tag{B.18}$$

The element $P_x h \in F(X) \otimes \mathbb{C}H$ corresponds to the function

$$f(y, z) = \delta_{x,y}\delta_{h,z} \ \forall y \in X, z \in H. \tag{B.19}$$

Furthermore, there is an action of the group H on X. This means that the elements $h \in H$ act as bijections of X, in a manner that is consistent with the group structure (i.e. we have a homomorphism from H to bijections of X). Denote the action of $h \in H$ on $x \in X$ by $h \cdot x$. We now turn this vector into an algebra, by introducing a multiplication.

Definition 1. *$F(X \times H)$ is a transformation group algebra if the multiplication of $P_x h$ and $P_{x'} h'$ is given by:*

$$P_x h P_{x'} h' = \delta_{x,h\cdot x'} P_x h h' \tag{B.20}$$

In function notation:

$$(f_1 f_2)(x,y) = \sum_{h \in H} f_1(x,h) f_2(h^{-1}\cdot x, h^{-1} y) \tag{B.21}$$

We define an *inner product* on $F(X \times H)$. We give it in both notations: in terms of elements $P_x h \in F(X) \otimes \mathbb{C}H$, and in terms of functions $f_i \in F(X \times H)$.

$$\begin{aligned} (P_x h, P_{x'} h') &= \delta_{x,x'} \delta_{h,h'} \\ (f_1, f_2) &= \sum_{x \in X, h \in H} f_1(x,h)^* f_2(x,h). \end{aligned} \tag{B.22}$$

We can split X up into orbits under the action of H. The orbit of an $x_0 \in X$ is given by $\{h \cdot x_0 : h \in H\}$. Call $\{A\}$ the collection of orbits. For each orbit A, choose a preferred element x_1^A, and define the *normalizer* N_A to be the subgroup of $h \in H$ that satisfy $h \cdot x_1^A = x_1^A$. The O_A and N_A play a central role in the determination of the irreps.

Theorem 2. *Choose an orbit A in X, a preferred element x_1^A of A, and an irrep α of N_A. The orbit $A = \{x_1^A, x_2^A, \ldots, x_n^A : x_i^A \in X\}$. Let h_i^A be any element in H such that $h_i^A \cdot x_1^A = x_i^A$. Then the h_i^A form representatives of left N_A cosets in H. Further call e_j the basis vectors of the vector space V_α on which the irrep α acts.*
An irreducible unitary representation Π_α^A of $F(X \times H)$ is given by inducing the irrep α. A basis of the vector space is $\{|x_i^A, e_j^\alpha >\}$, and the action of $P_x g \in F(X \times H)$ is given by

$$\Pi_\alpha^A(P_x g)|x_i^A, e_j^\alpha >= \delta_{x, g h_i^A \cdot x_1^A} |x_j^A, \alpha(n) e_j^\alpha >$$

where x_j^A is defined by $g h_i^A = h_j^A n, n \in N_A$. This is possible because $g h_i^A$ sits in a particular coset of H/N_A, and the h_i^A are representatives of the left N_A cosets.
Furthermore, all unitary irreducible representations are equivalent to some Π_α^A, and Π_α^A is equivalent to Π_β^B iff $O_A = O_B$ and α is equivalent to β.

Thus irreps of a transformation group algebra are labelled by an orbit A in X under the action of H, and an irrep α of the normalizer N_A of a chosen preferred element x_1^A of A.

The notation in terms of basis elements makes the action of the irreps transparent. An alternate notation for the Hilbert space is $h_i^A \otimes |e_j^\alpha> \equiv |x_i^A, e_j^\alpha>$. Then the action of a global symmetry transformation $g \equiv 1g$ is simply

$$\Pi_\alpha^A(g) h_i^A \otimes |e_j^\alpha> = g h_i^A \otimes |e_j^\alpha> = h_j^A n \otimes |e_j^\alpha> = h_j^A \otimes |\alpha(n) e_j^\alpha> \equiv |x_i^A, \alpha(n) e_j^\alpha>$$

In words, the part of g that "shoots through" the defect acts on the electric part.

The function notation is rather opaque, but extremely useful in calculations in the main text. The Hilbert space of the irrep Π_α^A is given by:

$$F_\alpha(H, V_\alpha) = \{|\phi>: H \to V_\alpha || \phi(hn) >= \alpha(n^{-1}) |\phi(h)>, \forall h \in H, \forall n \in N\}$$

To make contact with the notation above, $|\phi(h)>$ corresponds to the vector attached to the "flux" $h \cdot x_1^A$. For example, the function $|\phi>$ associated with $|x_i^A, e_j^\alpha>$ is $|\phi(h_i^A)> = 1_{h_i^A N_A}(h) |e_j^\alpha>$. To explain the rest of the definition, note that $\forall n \in N$

$$|x_i^A, e_j^\alpha> \equiv h_j^A \otimes |e_j^\alpha> = h_j^A n n^{-1} \otimes |e_j^\alpha> = h_j^A n \otimes |\alpha(n^{-1}) e_j^\alpha>$$

which explains why $|\phi(xn)> = \alpha(n^{-1})|\phi(x)>$.
Then the action of $f \in F(X \times H)$ on $|\phi>$ under the irrep Π_α^A gives a new function $\Pi_\alpha^A(f)\phi \in F_\alpha(H, V_\alpha)$, defined by

$$(\Pi_\alpha^A(f)\phi)(h) = \sum_{z \in H} f(h \cdot x_1^A, z) |\phi(z^{-1}h)>$$

One easily checks that this is equivalent to the definition given above.

The quantum double $D(H)$ is a special case of a transformation group algebra, with $X = H$ and $h \cdot x = hxh^{-1}$. Thus in $D(H)$ the orbits of $X = H$ under the action of H are conjugacy class of $X = H$. N_A is the centralizer of the preferred element g_A of A.

We proved earlier on that the antipode can be used to create an antiparticle irrep from any irrep Π_α^A. In the case of $D(H)$, the antiparticle irrep of Π_α^A is $\Pi_{\overline{\alpha}}^{A^{-1}}$, where A^{-1} is the conjugacy class of g_A^{-1}, and $\overline{\alpha}(n) = \alpha^t(n^{-1})$. In particle, for an electric irrep Π_α^e the antiparticle irrep is $\Pi_{\overline{\alpha}}^e$. An example of this is the **3** irrep of $SU(3)$, which is the antiparticle irrep of **3**. The quarks transform under the **3** irrep, while the antiquarks transform under the **3** irrep of $SU(3)$.

Two chapters back, we analyzed the classical fusion rules of defects, and of electric modes. The fusion of two electric modes was tantamount to taking the tensor product of their representations, and analyzing the decomposition of this tensor product into irreps. For magnetic modes, the fusion rules of two defects corresponded to the multiplication of their conjugacy classes. This multiplication is commutative, and gives a union of conjugacy classes (some of which may appear more than once in the decomposition):

$$C^A \times C^B = \sum_C N_C^{AB} C^C. \tag{B.23}$$

We would like the fusion rules of the Hopf algebra to reproduce these fusion rules. But in the decomposition $\Pi_1^A \otimes \Pi_1^B = \sum_{D,\gamma} N_{11C}^{AB\gamma}\Pi_\gamma^C$, there will be dyons! The defect fusion rules are still present in this decomposition, though. We will now prove that if we "sum over" the electric part of the irreps (i.e. we neglect the electric part), then the defect fusion rules are reproduced.

Lemma 6. *For two flux irreps of D(H), we have*

$$\Pi_1^A \otimes \Pi_1^B = \sum_{D,\gamma} N_{11C}^{AB\gamma}\Pi_\gamma^C.$$

By summing over the electric irreps γ and their dimensions d_γ, we get the classical fusion rules:

$$C^A \times C^B = \sum_C (\sum_\gamma d_\gamma N_{11C}^{AB\gamma}) C^C$$

Proof. The best way to see this is to study the representation space,$\{|g_i > |h_j >, g_i \in A, h_j \in B\}$. The global symmetry transformations g act as $g \cdot |g_i > |h_j >= |g g_i g^{-1} > |g h_j g{-}1 >$. Let all the $g \in H$ act on these vectors (which transforms them into other vectors of the same form), and decompose the vector space into invariant subspaces V_i. These are obviously the conjugacy classes in the multiplication of C^A and C^B. These subspaces are also invariant under the action of the projection operators, which act as $P_h \cdot |g_i > |h_j >= \delta_{h,g_i h_j}|g_i > |h_j >$. Now write $V_i = \sum_{h\in H} P_h V_i$. $P_h V_i$ is invariant under the action of N_h, so it's a representation of N_h. Decompose this into irreps $\gamma_{h,i}^k$ of N_h, writing $P_h V_i = \sum_k W_{\gamma_{h,i}^k}$. Acting by global symmetry transformations g, we get $g \cdot W_{\gamma_{h,i}^k} = W_{\gamma_{ghg^{-1},i}^k}$.
These are exactly the irreps γ in the decomposition above. Obviously, for a given $P_h V_i$, the irreps $\gamma_{h,i}$ satisfy $\sum_k d_{\gamma_{h,i}^k} = dim(P_h V_i)$. We've explicitly constructed the decomposition into irreps, since if $P_h V_i \neq 0$, acting on $P_h V_i$ with global symmetry transformations generates all of V_i, and the action is invertible. Thus, summing over the dyons gives the class multiplication. ☐

B.2 Generalized quantum doubles

In the main text we came across phases whose Hopf symmetry was a variation on a quantum double, which we called a *generalized quantum double*. An example is $F(T) \otimes \mathbb{C}T_d$. We will now give the general definition of a generalized quantum double[2].

As a vector space a generalized quantum double is $F(H_m) \otimes \mathbb{C}H_{el}$, where H_m and H_{el} are two groups. It is also a transformation group algebra, so there

---

[2] We came up with this structure to deal with the phases with inversion or reflection symmetries.

is an action of H_{el} on H_m. We denote the action of $h \in H_{el}$ on $g \in H_m$ by $h \cdot g$. We require this action to satisfy the following relation:

$$\forall h \in H_{el},\ \forall g, g' \in H_m : h \cdot (gg') = (h \cdot g)(h \cdot g'). \tag{B.24}$$

A basis of $F(H_m) \otimes \mathbb{C}H_{el}$ is given by $\{P_g h : g \in H_m, h \in H_{el}\}$. $F(H_m) \otimes \mathbb{C}H_{el}$ is a Hopf algebra with the following structure:

$$\begin{aligned} \forall g, g' \in H_m, \forall h, h' \in H_{el} : \qquad & P_g h P_{g'} h' = \delta_{g, h\cdot g'} P_g h h' && \text{(B.25)} \\ & \Delta(P_g h) = \sum_{g' \in H_m} P_{g'} h \otimes P_{g'^{-1} g} h && \text{(B.26)} \\ & \epsilon(P_g h) = \delta_{g,e} && \text{(B.27)} \\ & S(P_g h) = P_{h^{-1} \cdot g^{-1}} h^{-1} && \text{(B.28)} \end{aligned}$$

These structures satisfy all the axioms of a Hopf algebra, thus $F(H_m) \otimes \mathbb{C}H_{el}$ is a Hopf algebra. Just as in the quantum double case, we can write elements of $F(H_m) \otimes \mathbb{C}H_{el}$ as functions in $F(H_m \times H_{el})$. An element of $F(H_m \times H_{el})$ is a function

$$f : H_m \times H_{el} \to \mathbb{C}. \tag{B.29}$$

The element $P_g h \in F(H_m) \otimes \mathbb{C}H_{el}$ corresponds to the function

$$f(y, z) = \delta_{g,y} \delta_{h,z} \ \forall y \in H_m, z \in H_{el}. \tag{B.30}$$

The structures of $F(H_m \times H_{el})$ are

$$\begin{aligned} & \forall f, f_1, f_2 \in F(H_m \times H_{el}); x, x_1, x_2 \in H_m; y, y_1, y_2 \in H_{el} : \\ & (f_1 \times f_2)(x, y) = \sum_{h \in H_{el}} f_1(x, h) f_2(h^{-1} \cdot x, h^{-1} y) && \text{(B.31)} \\ & \Delta(f_1)(x_1, y_1; x_2, y_2) = f(x_1 x_2, y_1) \delta_{y_1}(y_2) && \text{(B.32)} \\ & \epsilon(f) = \sum_{h \in H_{el}} f(e, h) dh && \text{(B.33)} \\ & S(f)(x, y) = f(y^{-1} \cdot x^{-1}, y^{-1}) && \text{(B.34)} \end{aligned}$$

Making $F(H_m) \otimes \mathbb{C}H_{el}$ quasitriangular, i.e. introducing a braid matrix, is not that easy. We have found a way to do it that applies to all cases in the main text. We need a homomorphism:

$$\Gamma : H_m \to H_{el}, \tag{B.35}$$

that satisfies the following relations:

$$\begin{aligned} & \forall g, g_1, g_2 \in H_m, h \in H_{el} : && \text{(B.36)} \\ & \Gamma(g_1) \cdot g_2 = g_1 g_2 g_1^{-1} && \text{(B.37)} \\ & \Gamma(h \cdot g) = h \Gamma(g) h^{-1}. && \text{(B.38)} \end{aligned}$$

$F(H_m) \otimes \mathbb{C}H_{el}$ is then a quasitriangular Hopf algebra with the following braid matrix:

$$R = \sum_{g \in H_m} P_g e \otimes \Gamma(g). \tag{B.39}$$

The inverse of R is

$$R^{-1} = \sum_{g \in G} P_g e \otimes \Gamma(g^{-1}). \tag{B.40}$$

The quantum double $D(H)$ is a special case of a generalized quantum double, with $H_m = H_{el} = H$, $h \cdot g = hgh^{-1}\ \forall h, g \in H$, and $\Gamma \equiv id$.

Armed with our knowledge of generalized quantum doubles, we can study an achiral tetrahedral nematic, for example. The elements of T_d are denoted as permutations of the four vertices of a tetrahedron, see fig. B.1.

The defects are labelled by

$$\Pi_1(O(3)/T_d) = \Pi_1(SO(3)/T) = \Pi_1(SU(2)/\overline{T}) = \overline{T}. \tag{B.41}$$

Global symmetry transformations conjugate the defects, so these defects are grouped together in orbits under the action of the residual symmetry group, in this case T_d. This is not the same as the conjugacy classes of $\overline{T}$! To see this, we must first set a notation for the defects. Write an element of $\overline{T}$ as

$$u(\hat{n}, \theta) = sign(cos\frac{\theta}{2})u'(\hat{n}, \theta') \quad -\pi < \theta' \leq \pi \quad \theta' = \theta(mod 2\pi), \tag{B.42}$$

where $u(\hat{n}, \theta) \in SU(2)$ is defined in Appendix A. Every defect corresponds to a $u(\hat{n}, \theta)$. The element of $u(\hat{n}, \theta)$ that corresponds to a defect is not uniquely defined, we have some choice. We denote the defects as cycles of T with square brackets with a plus or minus sign, e.g. ±[123]. The minus sign corresponds to the 2π defect, i.e. the nontrivial loop in $SO(3)$. The defect [123] corresponds to the $u'(\hat{n}, \theta')$ with $-\pi < \theta' \leq \pi$ such that

$$R(\hat{n}, \theta') = (123).$$

We need to know the axis $\hat{n_1}$ in fig. B.1, such that (123) corresponds to a $\frac{2\pi}{3}$ around $\hat{n_1}$.

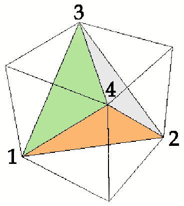


Figure B.1: The tetrahedron, with labelled vertices.

From the figure we see that $\hat{n_1} = \frac{1}{\sqrt{3}}(1, 1, 1)$. Thus we define $[123] = u(\hat{n_1}, \frac{2\pi}{3})$

The trivial defect is denoted by 1, and the 2π defect by -1.

To have a notation for all the defects in $\overline{T}$, we first define the following axes:

$$\hat{n_1} = \frac{1}{\sqrt{3}}(1,1,1) \quad \hat{n_2} = \frac{1}{\sqrt{3}}(1,1,-1) \quad \hat{n_3} = \frac{1}{\sqrt{3}}(-1,1,1) \quad \hat{n_4} = \frac{1}{\sqrt{3}}(-1,1,-1).$$

Then the defects are given

$$\begin{array}{ccc}
\pm e & & \\
\pm[123] = \pm u(\hat{n_1}, \frac{2\pi}{3}) & \pm[124] = \pm u(\hat{n_2}, \tfrac{2\pi}{3}) & \pm[124] = \pm u(\hat{n_2}, \frac{2\pi}{3}) \\
\pm[234] = \pm u(\hat{n_3}, \frac{2\pi}{3}) & \pm[134] = \pm u(\hat{n_4}, \tfrac{2\pi}{3}) & \pm[132] = \pm u(\hat{n_1}, -\frac{2\pi}{3}) \\
\pm[142] = \pm u(\hat{n_2}, -\frac{2\pi}{3}) & \pm[243] = \pm u(\hat{n_3}, -\tfrac{2\pi}{3}) & \pm[143] = \pm u(\hat{n_4}, -\frac{2\pi}{3}) \\
\pm[(12)(34)] = \pm u(\hat{z}, \pi) & \pm[(13)(24)] = \pm u(\hat{y}, \pi) & \pm[(14)(23)] = \pm u(\hat{x}, \pi).
\end{array}$$

Using our notation, we can determine the defect classes under the action of T_d. First consider an element $R(\hat{n}, \phi)$ of T. Its action on a defect $u(\hat{n_i}, \theta)$ gives $u(\hat{n}, \phi)u(\hat{n_i}, \theta)u(\hat{n}, \phi)^{-1}$.

Now consider transformations in T_d that are connected to the identity, such as (12). We can always write such an element as $Inv \times R(\hat{n}, \phi)$, where Inv is the inversion matrix:

$$Inv = -\mathbb{1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{B.43}$$

Inv acts trivially on all the defects, thus the action of $Inv \times R(\hat{n}, \phi)$ on a defect $u(\hat{n_i}, \theta)$ gives $u(\hat{n}, \phi)u(\hat{n_i}, \theta)u(\hat{n}, \phi)^{-1}$.

We will now write the large symmetry transformations as $Inv \times R(\hat{n}, \phi)$. First we define the following axes:

$$\begin{gathered}
\hat{m_1} = \frac{1}{\sqrt{2}}(1,-1,0) \quad \hat{m_2} = \frac{1}{\sqrt{2}}(-1,0,1) \quad \hat{m_3} = \frac{1}{\sqrt{2}}(0,1,1) \\
\hat{m_4} = \frac{1}{\sqrt{2}}(0,-1,1) \quad \hat{m_5} = \frac{1}{\sqrt{2}}(1,0,1) \quad \hat{m_6} = \frac{1}{\sqrt{2}}(1,1,0).
\end{gathered}$$

The inversions are given by

$$\begin{array}{lll}
(12) = Inv \times R(m_1, \pi) & (13) = Inv \times R(m_2, \pi) & (14) = Inv \times R(m_3, \pi) \\
(23) = Inv \times R(m_4, \pi) & (24) = Inv \times R(m_5, \pi) & (34) = Inv \times R(m_6, \pi).
\end{array}$$

We can now derive the multiplication table of $\overline{T}$ using the multiplication in $SU(2)$. For example,

$$\begin{aligned}
&[123][123] = -[132] \\
&[123][124] = [(14)(23)] \\
&(12)[(12)(34)](12) = -[(12)(34)] \\
&\text{etc.}
\end{aligned}$$

$F(\overline{T}) \otimes \mathbb{C}T_d$ is a generalized quantum double. We just defined the action of T_d on $\overline{T}$. We still need a Γ: it is given by

$$\Gamma(\pm[123]) = (123), \Gamma(\pm[(12)(34)]) = (12)(34), etc.$$

Thus Γ turns square brackets into round brackets and neglects the eventual minus sign.

Appendix C

Defect condensates in achiral nematics

In this appendix, we give all defect condensates from an achiral octahedral and an achiral icosahedral nematic.

C.1 Achiral octahedral nematic

Using the isomorphism $O \simeq S_4$ discussed in appendix A, we can write elements of O as cycles, e.g. $(1234), (123), (23), (13)(24)$, etc. In analogy with the achiral tetrahedral nematic discussed in appendix B, we can denote elements of the defect group $\overline{O}$ as cycles with square brackets, with an eventual minus sign. Examples are $\pm[123], \pm[(12)(34)], \pm[(12)]$, where the minus sign is the 2π defect. The notation is explained in appendix B.

The electric group is O_i, consisting of the octahedral group and inversions. Defining

$$Inv = -\mathbb{1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Then $O_i = \{O, Inv \times O\}$. The Hopf symmetry is $F(\overline{O}) \otimes \mathbb{C}O_i$. The defect classes in $\overline{O}$ under the action of O_i are given in table C.1. The inversion Inv acts trivially on the defects, so for the octahedral nematic the defect classes are in fact the conjugacy classes of $\overline{O}$. In general, when a group carries a sublabel i, it means that the group contains Inv.

The single defect condensates in $F(\overline{O}) \otimes \mathbb{C}O_i$ are given in table C.2. The class sum defect condensates are given in table C.3. Finally, the combined defect condensates are given in table C.4. We note that we are giving a very representative sample. The other defect condensates give the same $\mathcal{T}_{\mathcal{R}}$ and $\mathcal{U}$ as one the defect condensates shown here (except for a small difference: there

Preferred element g_A	Defect class A of $\overline{O}$
$\pm e$	$C_{\pm e} = \{\pm e\}$
$\pm[123]$	$C_{\pm[123]} = \{[123], [124], [142], [132], [134], [234], [142], [143], [243]\}$
$\pm[1234]$	$C_{\pm[1234]} = \{[1234], [1243], [1324], [1342], [1423], [1432]\}$
$[(12)(34)]$	$C_{[(12)(34)]} = \{[(12)(34)], [(13)(24)], [(14)(23)], -[(12)(34)], -[(13)(24)], -[(14)(23)]\}$
$[12]$	$C_{[12]} = \{[12], [13], [23], [14], [24], [34], -[12], -[13], -[23], -[14], -[24], -[34]\}$

Table C.1: The defect classes of $\overline{O}$ under the action O_i, and the preferred elements g_A. When we write ± in front of a class, we mean that there are two classes, one with plus signs in front of all the elements, and one with minus signs in front of all the elements.

may be condensates where K is actually the double of a K given here. That slightly changes the magnetic part of $\mathcal{T}_{\mathcal{R}}$, but doesn't affect $\mathcal{U}$).

Single defect condensate in $\mathcal{A} = F(\overline{O}) \otimes \mathbb{C}O_i$	K	$\mathcal{T}_{\mathcal{R}}$	$\mathcal{U}$
$\mid -e >$	$\overline{C_1}$	$F(O) \otimes \mathbb{C}O_i$	$F(O) \otimes \mathbb{C}O_i$
$\mid \pm [12] >$	$\overline{C_2}$	$F(O/C_2) \otimes \mathbb{C}D_{2i}$	$F(C_2) \otimes \mathbb{C}C_{2i}$
$\mid \pm [123] >$	$\overline{C_3}$	$F(O/C_3) \otimes \mathbb{C}C_{3i}$	$\mathbb{C}C_i$
$\mid \pm [1234] >$	$\overline{C_4}$	$F(O/C_4) \otimes \mathbb{C}C_{4i}$	$\mathbb{C}C_i$
$\mid \pm [(12)(34)] >$	$\overline{C_2}$	$F(O/C_2) \otimes \mathbb{C}D_{4i}$	$F(D_2) \otimes \mathbb{C}D_{2i}$

Table C.2: Single defect condensates in an octahedral nematic. When we write ± is front of a condensate, we mean that the condensate with or without the minus sign gives the same symmetry breaking analysis.

Class sum defect condensates of $\mathcal{A} = F(\overline{O}) \otimes \mathbb{C}O_i$	K	$\mathcal{T}_{\mathcal{R}}$	$\mathcal{U}$
$\lvert C_{[(12)(34)]} >$	$\overline{D_2}$	$F(D_3) \otimes \mathbb{C}O_i$	$F(D_3) \otimes \mathbb{C}D_{3i}$
$\lvert C_{\pm[123]} >$	$\overline{T}$	$F(C_2) \otimes \mathbb{C}O_i$	$F(C_2) \otimes \mathbb{C}C_{2i}$
$\lvert C_{\pm[1234]} >$	$\overline{O}$	$\mathbb{C}O_i$	$\mathbb{C}C_i$
$\lvert C_{[12]} >$	$\overline{O}$	$\mathbb{C}O_i$	$\mathbb{C}C_i$

Table C.3: Class sum defect condensates in an achiral octahedral nematic. When we write ± is front of a condensate, we mean that the condensate with or without the minus sign gives the same symmetry breaking analysis.

Combined defect condensate in $\mathcal{A} = F(\overline{O}) \otimes \mathbb{C}O_i$	K	$\mathcal{T}_{\mathcal{R}}$	$\mathcal{U}$
$\lvert[123] > + \lvert[132] >$	$\overline{C_3}$	$F(O/C_3) \otimes \mathbb{C}D_{3i}$	$\mathbb{C}C_{2i}$
$\lvert[12] > + \lvert[34] >$	$\overline{D_2}$	$F(O/D_2) \otimes \mathbb{C}D_{4i}$	$F(C_2) \otimes \mathbb{C}C_i$
$\lvert[(12)(34)] > + \lvert[(13)(24)] >$	$\overline{D_2}$	$F(O/D_2) \otimes \mathbb{C}D_{4i}$	$F(C_2) \otimes \mathbb{C}C_{2i}$
$\lvert[12] > + \lvert[13] > + \lvert[23] >$	$\overline{D_3}$	$F(O/D_3) \otimes \mathbb{C}D_{3i}$	$\mathbb{C}C_i$

Table C.4: Combined defect condensates in an achiral octahedral nematic. In this table, $\mathcal{A}$ is the original symmetry algebra, K is the smallest subgroup of H_m that contains all the defects g_i that are in the condensate, $\mathcal{T}_{\mathcal{R}}$ is the residual symmetry algebra, and $\mathcal{U}$ is the unconfined symmetry algebra.

C.2 Achiral icosahedral nematic

Using the isomorphism $O \simeq A_5$ discussed in appendix A, we can write elements of I as cycles, e.g. $(12345), (123), (145)(23), (13)(24)$, etc. In analogy with the achiral tetrahedral nematic discussed in appendix B, we can denote elements of the defect group $\overline{I}$ as cycles with square brackets, with an eventual minus sign. Examples are $\pm[123], \pm[(12)(345)], \pm[(15)]$, where the minus sign is the 2π defect. The notation is explained in appendix B.

The electric group is I_i, consisting of the icosahedral group and inversions: $I_i = \{I, Inv \times I\}$. The Hopf symmetry is $F(\overline{I}) \otimes \mathbb{C}I_i$. The defect classes in $\overline{I}$ under the action of I_i are given in table C.5. The inversion Inv acts trivially on the defects, so the defect classes are the conjugacy classes of $\overline{I}$.

The single defect condensates in $F(\overline{O}) \otimes \mathbb{C}O_i$ are given in table C.6. The class sum defect condensates are given in table C.7. Finally, the combined defect condensates are given in table C.8. We give a very representative sample.

Preferred element g_A	Defect class A of $\overline{I}$
e	$C_e = \{e\}$
$-e$	$C_{-e} = \{-e\}$
$\pm[123]$	$C_{\pm[123]} = \pm\{[123], [132], [124], [142], [125],$ $[152], [134], [143], [135], [153],$ $[145], [154], [234], [243], [235],$ $[253], [245], [254], [345], [354]\}$
$\pm[12345]$	$C_{\pm[12345]} = \pm\{[12345], [12453], [12534], [13254],$ $[13425], [13542], [14235], [14352],$ $[14523], [15243], [15324], [15432]\}$
$\pm[12354]$	$C_{\pm[12354]} = \pm\{[12354], [12435], [12543], [13245],$ $[13452], [13524], [14253], [14325],$ $[14532], [15234], [15342], [15423]\}$
$[(12)(34)]$	$C_{[(12)(34)]} =$ $\{\pm[(12)(34)], \pm[(12)(35)], \pm[(12)(45)], \pm[(13)(24)], \pm[(13)(25)],$ $\pm[(13)(45)], \pm[(23)(45)], \pm[(14)(23)], \pm[(14)(25)], \pm[(14)(35)],$ $\pm[(24)(35)], \pm[(15)(23)], \pm[(15)(24)], \pm[(15)(34)], \pm[(25)(34)]\}$
$\pm[123]$	$C_{\pm[123]} = \pm\{[123], [124], [125], [132], [134],$ $[234], [135], [235], [142], [143],$ $[243], [145], [245], [345], [152],$ $[153], [253], [154], [254], [354]\}$

Table C.5: The defect classes of $\overline{I}$ under the action I_i, and the preferred elements g_A. When we write ± in front of a class, we mean that there are two classes, one with plus signs in front of all the elements, and one with minus signs in front of all the elements. If we write ± in front of an element inside a class, we mean that the element with the plus and the minus sign are both present in the class.

Single defect condensate in $\mathcal{A} = F(\overline{I}) \otimes \mathbb{C}I_i$	K	$\mathcal{T}_{\mathcal{R}}$	$\mathcal{U}$
$\vert -e >$	$\overline{C_1}$	$F(I) \otimes \mathbb{C}I_i$	$F(I) \otimes \mathbb{C}I_i$
$\vert [123] >$	$\overline{C_3}$	$F(I/C_3) \otimes \mathbb{C}C_{3i}$	$\mathbb{C}C_i$
$\vert -[123] >$	C_3	$F(\overline{I}/C_3) \otimes \mathbb{C}C_{3i}$	$\mathbb{C}C_i$
$\vert \pm[(12)(34)] >$	$\overline{C_2}$	$F(I/C_2) \otimes \mathbb{C}D_{2i}$	$F(C_2) \otimes \mathbb{C}C_{2i}$
$\vert [12345] >$	$\overline{C_5}$	$F(I/C_5) \otimes \mathbb{C}C_{5i}$	$\mathbb{C}C_i$
$\vert -[12345] >$	C_5	$F(\overline{I}/C_5) \otimes \mathbb{C}C_{5i}$	$\mathbb{C}C_i$

Table C.6: Single defect condensates in an icosahedral nematic. When we write ± is front of a condensate, we mean that the condensate with or without the minus sign gives the same symmetry breaking analysis.

Class sum defect condensate in $\mathcal{A} = F(\overline{I}) \otimes \mathbb{C}I_i$	K	$\mathcal{T_R}$	$\mathcal{U}$
$\lvert C_{\pm[12345]} >$	$\overline{I}$	$\mathbb{C}I_i$	$\mathbb{C}C_i$
$\lvert C_{\pm[12354]} >$	$\overline{I}$	$\mathbb{C}I_i$	$\mathbb{C}C_i$
$\lvert C_{[(12)(34)]} >$	$\overline{I}$	$\mathbb{C}I_i$	$\mathbb{C}C_i$

Table C.7: Class sum defect condensates in an achiral icosahedral nematic. When we write ± is front of a condensate, we mean that the condensate with or without the minus sign gives the same symmetry breaking analysis.

Combined defect condensate in $\mathcal{A} = F(\overline{I}) \otimes \mathbb{C}I_i$	K	$\mathcal{T_R}$	$\mathcal{U}$
$\lvert[(12)(34)] > +\lvert[(13)(24)] >$	$\overline{D_2}$	$F(I/D_2) \otimes \mathbb{C}D_{2i}$	$\mathbb{C}C_i$
$\lvert[(12)(34)] + \lvert[(13)(24)] > +\lvert[(14)(23)] >$	$\overline{D_2}$	$F(I/D_2) \otimes \mathbb{C}T_i$	$\mathbb{C}C_{3i}$
$\lvert[(12)(34)] > +\lvert[(12)(35)] > +\lvert[(12)(45)] >$	$\overline{D_3}$	$F(I/D_3) \otimes \mathbb{C}D_{3i}$	$\mathbb{C}C_i$
$\{\lvert[(12)(34)] > +\lvert[(13)(25)] > +\lvert[(15)(24)] > +\lvert[(23)(45)] > +\lvert[(14)(35)] >\}$	$\overline{D_5}$	$F(I/D_5) \otimes \mathbb{C}D_{5i}$	$\mathbb{C}C_{2i}$
$\lvert[123] > +\lvert[132] >$	$\overline{C_3}$	$F(I/C_3) \otimes \mathbb{C}D_{3i}$	$\mathbb{C}C_{2i}$
$\lvert[123] > +\lvert[134] > +\lvert[142] > +\lvert[243] >$	$\overline{T}$	$F(I/T) \otimes \mathbb{C}T_i$	$\mathbb{C}C_i$
$\{\lvert[123] > +\lvert[124] > +\lvert[132] > +\lvert[134] > +\lvert[234] > +\lvert[142] > +\lvert[143] > +\lvert[243] >\}$	$\overline{T}$	$F(I/T) \otimes \mathbb{C}T_i$	$\mathbb{C}C_i$
$\{\lvert[123] > +\lvert[152] > +\lvert[135] > +\lvert[253] > +\lvert[142] > +\lvert[134] > +\lvert[243] >\}$	$\overline{I}$	$\mathbb{C}C_{3i}$	$\mathbb{C}C_i$
$\{\lvert[12345] > +\lvert[13524] > +\lvert[14253] > +\lvert[15432] >\}$	$\overline{C_5}$	$F(I/C_5) \otimes \mathbb{C}D_{5i}$	$\mathbb{C}C_{2i}$

Table C.8: Combined defect condensates in an achiral icosahedral nematic. In this table, $\mathcal{A}$ is the original symmetry algebra, K is the smallest subgroup of H_m that contains all the defects g_i that are in the condensate, $\mathcal{T_R}$ is the residual symmetry algebra, and $\mathcal{U}$ is the unconfined symmetry algebra.

Appendix D

Irreducible representations

D.1 Irreducible representations of semidirect product groups

D.1.1 Irreps of a finite group

The theory of irreducible representations of finite groups is based on the great orthogonality theorem for irreps of finite groups: given two irreps Π_a and Π_b, then the matrix elements satisfy the following orthogonality relation:

$$\sum_{g \in G} \Pi_a(g)_{ij} \Pi_b(g)_{kl} = \frac{|G|}{d_a} \delta_{i,l} \delta_{j,k} \delta_{a,b} \tag{D.1}$$

where $|G|$ is the order of the group, and d_a the dimension of the irrep Π_a. $\delta_{a,b}$ is one if the irreps are equivalent, and zero if they aren't.

Define the character χ_a of an irrep Π_a as the trace of the irrep: $\chi_a(g) = Tr\Pi_a(g)$. Actually, the trace is just an example of a character function. More generally, a character function is a function from the group to the complex numbers that is constant on the conjugacy classes. From the orthogonality theorem, the following corollaries are easily proven:

1. The number of inequivalent irreps is equal to the number of conjugacy classes,
2. $\sum_a d_a^2 = |G|$ where the sum runs over all irreps,
3. The characters are orthogonal with respect to the following inner product:

   $$< x_a, x_b >= \frac{1}{|G|} \sum_{g \in G} x_a(g)^* x_b(g)$$

   They form an orthogonal basis of the space of character functions.

These corollaries are precious when working out the character table of a group.

D.1.2 Subduced and induced representations

Given a group G, a subgroup H, and a representation α of G, we can restrict this representation to elements of H. This gives a representation of H, which is called the subduced representation, and denoted $\alpha|_H$. If α was an irrep of G, then $\alpha|_H$ may be reducible. We saw physical examples of this: when the symmetry is broken from G to H, degeneracies may be lifted.

Inducing a representation goes the other way around. Given a representation β of H, construct a representation β^G of G as follows: first choose a set of representatives of left H cosets, call then x_i. Choose a basis $|v_j>$ of the vector space V_β on which β acts. Then define a new vector space, with basis $|x_i, v_j>$, and define the following action of G on this vector space:

$$\beta^G(g)|x_i, v_j >= |x_k, \beta(h)v_j >$$

where $gx_i = x_k h$ for some k, and some $h \in H$. All elements of G are of the form $x_k h$, because the x_k are a complete list of representatives of left H cosets. A notation for this new module (or irrep) is

$$\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V_\beta.$$

In words, we stick G onto V_β, and let H shoot through the tensor product and act on V_β.

D.1.3 Semi-direct product groups

A semidirect product of groups is an example of a group extension (see [1]). The name of the game is, given two groups N and G, to find groups E such that N is a normal subgroup of E and $E = G/N$. The question is how many different E exist satisfying these criteria. For example, $SU(2)$ is an extension of $SO(3)$, where $N = \mathbb{Z}_2$.

A famous example is the semi-direct product of N and G. We need a homomorphism $\lambda : G \to Aut(N)$ (where $Aut(N)$ is the group of automorphisms of N). The semi-direct product of N and G, denoted $N \rtimes G$, is the set $N \times G$ with the following multiplication:

$$(n, g)(n', g') = (n\lambda(g)n', gg').$$

Now we turn the question around: given a group G, a normal subgroup N, and a subgroup H, when do we have $G = N \rtimes H$? Here we make a specific choice that $\lambda(g)$ must correspond to conjugation with g: $\lambda(g)n = gng^{-1}$. One can check that $G = N \rtimes H \iff G = NH$ and $N \cap H = e$.

Examples are the Galilean group $\mathbb{R}^4 \rtimes SO(3)$, the Poincaré group $\mathbb{R}^4 \rtimes SO(3,1)$, the symmetry group of a hexagonal crystal $\mathbb{Z}^2 \rtimes \mathbb{Z}_6$. Note that in all these cases the normal subgroup is a group of translations, therefore it is abelian. There is a general prescription for determining the irreducible representations of $N \rtimes H$ with N abelian, called the little group method. We will now quickly summarize the prescription.

D.1.4 Irreps of $N \rtimes H$, with N abelian

Use Mackey's induction theorem (see [60], 8.2). The steps are:

1. Find the irreps of N. Take one of them, call it δ.
2. For $g \in G$, define α_g as the automorphism of N given by conjugation by g: $\alpha_g(n) = gng^{-1}\,\forall n \in N$. Call $H_\delta = \{h \in H | \delta \circ \alpha_h = \delta\}$. Then $N \rtimes H_\delta$ is the subgroup of G whose elements g satisfy $\delta \circ \alpha_g = \delta$. It is called a little group of the second kind.
3. Find the irreps of H_δ. Take one of them, call it λ.
4. Now define a representation $\delta \otimes \lambda$ of $N \rtimes H_\delta$ as follows:

   $$\delta \otimes \lambda(n, h) := \delta(n) \otimes \lambda(h).$$

   It is crucial that $\delta \circ \alpha_g = \delta$ for this to be a representation:

   $$\begin{aligned} \delta \otimes \lambda((n,h),(n',h')) &= \delta \otimes \lambda(n\alpha_h(n'), hh') \\ &= \delta(n)\delta \circ \alpha_h(n') \otimes \lambda(h)\lambda(h') \\ &= \delta(n)\delta(n') \otimes \lambda(h)\lambda(h') \\ &= (\delta(n) \otimes \lambda(h))(\delta(n') \otimes \lambda(h')) \end{aligned}$$

5. Induce a representation of $N \rtimes H$ from the representation $\delta \otimes \lambda$ of $N \rtimes H_\delta$. The vector space of the representation is

   $$Ind_{N \rtimes H_\delta}^{N \rtimes H}(\delta \otimes \lambda) = \mathbb{C}[N \rtimes H] \otimes_{\mathbb{C}[N \rtimes H_\delta]} V_{\delta \otimes \lambda}.$$

   A consequence of Mackey's theorem is that all the irreps of $N \rtimes H$ are contained in the set $\{\Pi_{\delta,\lambda}\}$. Note that some of the irreps in this set may be equivalent.

If the groups N and H are infinite (locally compact), the theorem still holds, if we demand that the irreps are bounded.
For the case of $N \rtimes H$ with N a translation group and H a rotation group, this story can be drawn on a lattice (this is used in the main text). An irrep $\Pi_{\vec{k}}$ of N is given by a momentum vector $\vec{k}$. Given a translation $\vec{x}$, the irrep is defined by

$$\Pi_{\vec{k}}(\vec{x})e^{i\vec{k}\cdot\vec{x}}.$$

Now take a rotation $R \in H$. Then $\Pi_{\vec{k}} \circ \alpha_R = \Pi_{R^{-1}\vec{k}R}$, so the action of R on the vector $\Pi_{\vec{k}}$ is to turn $\vec{k}$ by R^{-1}. This follows from the multiplication in $N \rtimes H$. Write elements of $N \rtimes H$ as (t, R), with $t \in N, R \in H$. Then

$$(t', R') \times (t, R) = (t' + R't, R'R)$$

To derive this, write $\vec{x}' = Rx + t$, and then apply the second transformation: $\vec{x}'' = R'x' + t' = R'Rx + R't + t'$.
Now we compute:

$$\Pi_{\vec{k}} \circ \alpha_R((t, e)) = \Pi_{\vec{k}}(Rt, e) = e^{i\vec{k}\cdot Rt} = e^{i(R^{-1}\vec{k})\cdot\vec{t}} = \Pi_{R^{-1}\vec{k}}((t, e)),$$

because the rotations are orthogonal. This new irrep may be equivalent to the previous one, if $\vec{k}$ and $R^{-1}\vec{k}$ differ by a reciprocal lattice vector. We call the set of vectors $R^{-1}\vec{k}$ that represent inequivalent irreps the star of $\vec{k}$, also known as the orbit of $\vec{k}$.
What is the dimension of such an irrep? The irrep of N is one-dimensional because N is abelian. The irrep of the little group $N \rtimes H_\delta$ has a certain dimension, which we call d_δ. When inducing, we "stick on" left H_δ cosets, so we multiply the dimension by $[H, H_\delta]$, where $[G : H]$ is the number of left H cosets in G. The total dimension is therefore $d_\delta \times [H, H_\delta]$.

An example from the main text

We now illustrate the derivation of an irrep of $\mathbb{Z}^2 \rtimes D_6$ discussed in the main text. Take orbit 5 in fig. 4.3, with momentum $\vec{k} = (\frac{\pi}{2}, \frac{\pi}{2})$ (i.e., $(\frac{\pi}{2a}, \frac{\pi}{2a})$ with $a = 1$). A rotation of 180 degrees (r^3) gives an equivalent vector, i.e. a momentum that represents the same irrep of $\mathbb{Z}^2$. As a matter of fact, so do e, sr^2, sr^{-1}, so $H_{\Pi_{\vec{k}}}$ is isomorphic to $\mathbb{Z}^2 \rtimes \mathbb{Z}_2 = (r^3) \times (sr^2)$, where in general (g) is the group generated by g. It has four one-dimensional irreps. Now $[D_6 : \mathbb{Z}^2 \rtimes \mathbb{Z}_2] = 3$, so we get four three-dimensional irreps. Denote an irrep of $\mathbb{Z}^2 \rtimes \mathbb{Z}_2$ by $\rho_{m,n}$, with $m, n \in \mathbb{Z}_2$. Explicitly, a basis of the irreps is given by a set of representatives of left $\mathbb{Z}^2 \rtimes \mathbb{Z}_2$ cosets, which we take to be $\{|e>, |r>, |r^2>\}$ (the order is important when we write down the matrices explicitly). The matrices of the irrep are

$$(a, b) \mapsto \begin{pmatrix} e^{i\frac{\pi}{2}a}e^{i\frac{\pi}{2}b} & 0 & 0 \\ 0 & e^{i\frac{\pi}{2}(a+b)}e^{i\frac{\pi}{2}(-a)} & 0 \\ 0 & 0 & e^{i\frac{\pi}{2}b}e^{i\frac{\pi}{2}(-a-b)} \end{pmatrix},$$

$$r \mapsto \begin{pmatrix} 0 & 0 & e^{i\pi m} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, s \mapsto \begin{pmatrix} 0 & & e^{i\pi n} \\ 0 & e^{i\pi n} & 0 \\ e^{i\pi n} & 0 & 0 \end{pmatrix}$$

Let us illustrate the derivation of the second column of the matrix corresponding to s:

$$s|r >= |sr >= |r(sr^2) >= |r > e^{i\pi n}.$$

D.2 Irreducible representations of algebras

Physicists demand that their symmetry algebras be semisimple, meaning that the radical of the algebra (which is the largest solvable ideal) is trivial. There are many equivalent definitions of the radical $J(A)$ of the algebra A:

$$\begin{aligned} J(A) \quad &= \{x \in A | 1 + xy \in A^* \forall y \in A\} \\ &= \{x \in A | xy \text{ nilpotent } \forall y \in A\} \\ &= \text{the unique maximal ideal consisting of nilpotent elements} \\ &= \text{the unique solvable ideal} \\ &= \text{the unique maximal nilpotent ideal} \\ &= \bigcap \{M | \text{M maximal right ideal of A}\} \\ &= \bigcap \{M | \text{M maximal left ideal of A}\} \end{aligned}$$

This definition is actually also adequate for any finite dimensional algebra, and in fact $Irreps(A) = Irreps(A/J(A))$, meaning that the irreps of A are in one-to-one correspondence with the irreps of $A/J(A)$, and that they factor through the obvious map from A to $A/J(A)$ to sends a to its equivalence class. Obviously $A/J(A)$ is semisimple, since we have modded out the radical.

There are very important theorems that apply to semisimple algebras, which make them appealing for physicists. First of all, all its representations are either irreducible or fully reducible, meaning that there are a finite number of particles (irreps) from which one can build all representations. Actually a finite-dimensional algebra is semisimple if and only if all its reducible representations are fully reducible.

A simple algebra is an algebra with no non-trivial ideals. It is therefore also semisimple. Any semisimple algebra is a direct sum of simple algebras.

Weddenburm's theorem states that any simple algebra is isomorphic to a matrix algebra. Therefore all semisimple algebras are direct sums of matrix algebras.

In the chapter on condensation, we assumed that $\mathcal{T}_{\mathcal{R}}$ was a semisimple algebra, and we will not prove this. We then modded out the intersection of the kernel of a set of irreps (the unconfined irreps) which we called $\mathcal{I}$, and we can actually prove that $\mathcal{T}_{\mathcal{R}}/\mathcal{I}$ is semisimple. First we derive an alternate definition of J(A), for any finite-dimensional algebra A:

$$J(A) = \bigcap \{ker\phi | \phi \text{ irrep of } A\}$$

Since $A/J(A)$ is semisimple, we have $A/J(A) = \bigoplus A_i$, with A_i simple. Note that when we write $A/J(A) = \bigoplus A_i$, we actually mean that if $i \neq j$, then $a_i a_j = 0$, for any $a_i \in A_i, a_j \in A_j$. Now the kernel of an irrep is an ideal, so for an irrep ϕ_i of a simple algebra A_i we have $ker\phi_i = \{0\}$. Here ϕ_i is an irrep of A, so $ker\phi_i = \bigoplus_{j \neq i} A_j$, and obviously $\bigcap_i ker\phi_i = \{0\}$, since $A_i \cap A_j = \{0\}$ for $i \neq j$. So

$$\cap ker\phi_i = 0 \text{ in } A/J(A) \Rightarrow \cap ker\phi_i' \subset J(A) \quad \phi_i' \text{irreps of } A$$

We also have $J(A) \subset ker\phi_i'$ due to $Irreps(A) = Irreps(A/J(A))$. So $J(A) = ker\phi_i'$.

Now take a set of irreps of A, $\{\psi_i\}$. Defining $I = \cap_i ker\psi_i$, is A/I semisimple?

It is. This follows from the observation that $A/I = (A/J(A))/(I/J(A))$, since $J(A) \subset I$. $A/J(A)$ is semisimple, and any quotient of a semisimple algebra is semisimple. Namely, for a semisimple algebra $B = \bigoplus_i B_i$, all ideals of B are B_i and sums of B_i's. Now mod out an ideal $I = \bigoplus_{i \in X}$, for some indexing set X. The radical of B/I then satisfies $J(B/I) = J(B)\ I$, so if $J(B/I)$ is non trivial, then so is $J(B)$.

For more details on the theory behind associative algebras, see [51]. For the standard reference on the representation theory of groups and algebras, see [17].