HOPF SYMMETRY

AND ITS BREAKING

Braid Statistics and Confinement in Planar Physics

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Contents

Introduction

A crucial step in understanding any physical system is a study of its symmetries. In theories of fields or particles, the properties that distinguish different kinds of particles, such as mass, spin, electric charge, isospin and color, are linked to the way these particles' fields or wave functions transform under various symmetry operations. These symmetry operations can be quite diverse in nature; one may think of spacetime symmetries, such as Lorentz boosts, translations, rotations, reflections and time reversal, but also of many types of gauge transformations and of charge conjugation. Despite this diversity, it has been possible to give an efficient mathematical description of all these symmetries in terms of groups. As a consequence, group theory has become one of the standard tools of the theoretical physicist. Nevertheless, even the applicability of group theory has its limits and one does encounter situations, especially in low dimensional systems, where different methods are required to efficiently describe all symmetries. In such cases, a generalization of group theory, the theory of Hopf algebras, or quantum groups, may come to the rescue.

Hopf symmetry does not just generalize group symmetry, it also unifies the description of symmetry with the description of the exchange properties of particles. In other words, the Hopf symmetry in a system tells us not only what the natural quantum numbers of the particles are and how these quantum numbers behave under fusion, but it also describes the non-local part of the interactions. Thus, Hopf symmetry comes into its own especially in the description of particles with non-trivial exchange properties, that is, particles which are not bosons or fermions. Such "anyons" do not seem to occur as elementary particles in our (3+1)-dimensional world, but they feature naturally in many models of lower dimensional systems.

Among physical phenomena which are believed to involve anyonic excitations, the fractional quantum Hall effect is probably the best understood. A fractional quantum Hall state forms when electrons caught at the interface between a semiconductor and an insulator are cooled to very low temperatures ($\sim 10\mathrm{mK}$) and subjected to a strong magnetic field ($\sim 20\mathrm{T}$). Under these conditions, there is a Hall effect which differs spectacularly from the classical Hall effect. The Hall conductance does not rise linearly with the applied B-field, but instead exhibits plateaus at values which are integer [1] and fractional [2] multiples of the fundamental unit $\frac{e^2}{h}$ (see figure 1). At each plateau, the diagonal elements of the conductance tensor vanish; the current is perpendicular to the applied voltage. The electrons form a fluid state which has localized excitations. These are called quasiparticle s when they correspond to a local peak in the electron density and quasiholes when they correspond to a local dip. Quantum Hall quasiparticles and quasiholes exhibit many exotic properties. For example, their charge is typically a fraction of the charge of an electron. Also they are believed to be neither bosons nor fermions. Their exchanges are governed by the braid group and are even predicted to be non-Abelian in some cases. We may thus safely say that the states of matter at the quantum Hall plateaus are crying out for a quantum group theoretic treatment. We describe such a treatment in chapter 2. We ar-

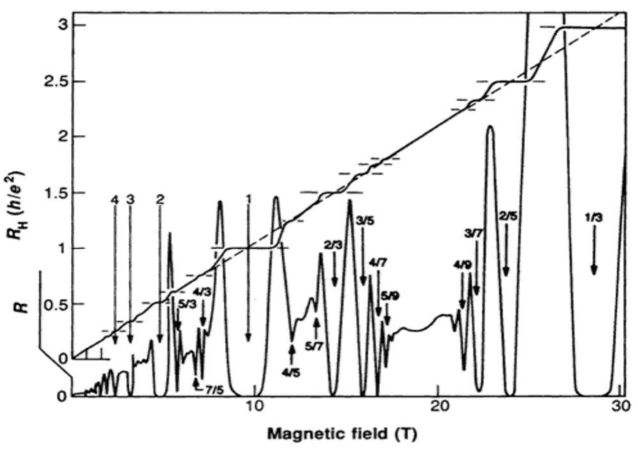

Figure 1: Plot of the Hall resistance for a typical sample. Taken from [3]

gue that the fusion and braiding properties of the excitations may always be elegantly described by means of a quantum group and we show this explicitly for a series of states proposed by Read and Rezayi. For this series, we also use the Hopf symmetry to give a complete description of the non-Abelian braiding of the quasiholes.

In chapter 3, we turn to an issue which succeeds the discovery of any new symmetry, namely the determination of the generic types of physical behavior which may arise as that symmetry is broken spontaneously. This issue is of central importance in areas of physics ranging from crystallography to string theory. For symmetries described by groups it is a textbook subject, but for symmetries described by Hopf algebras, no theory of symmetry breaking has to date been available. We propose a theory of Hopf symmetry breaking that deals efficiently with symmetries described by finite-dimensional Hopf algebras (these include all symmetries described by finite groups) and we explore its physical consequences. We apply our theory to discrete gauge theories: planar gauge theories whose gauge symmetry has been broken down to a discrete group through the Higgs effect. These theories are probably the simplest of all gauge theories. Nevertheless, they have a rich spectrum of fundamental and topological excitations, which exhibit non-trivial fusion and Aharonov-Bohm interactions (braiding). They also have a Hopf symmetry which describes the particle spectrum, fusion and braiding completely and which deals with fundamental and topological excitations on equal footing. Thus, these theories provide a unique opportunity to study questions that involve an interplay between fundamental and topological excitations, such as the problem of confinement of electric charges due to a condensation of topological fluxes. We study what happens when the Hopf symmetry is broken by the formation of a condensate and find Higgs and confinement phenomena similar to those in continuous gauge theories in 3+1 dimensions. We hope that the approach to the study of these phenomena that we develop here may in the long run also prove useful in more realistic models, both of elementary particles and of excitations in condensed matter systems.

The structure of this thesis is as follows. In chapter 1, we give a brief review of some aspects of Hopf algebra theory that are relevant to us, emphasizing the relationship to two-dimensional physics. Chapters 2 and 3 both use the material in chapter 1 to some extent, but they can be read independently of each other. Also, both chapter 2 and chapter 3 start with introductory material of their own, which gives an overview of the relevant physics and mathematics. In particular, section 2.2 gives a short introduction to the bulk theory of the quantum Hall effect and section 3.2 introduces discrete gauge theories. A summary in Dutch can be found at the end of this thesis.

Chapter 1

Hopf symmetry in planar physics

In this chapter, we will explain what Hopf algebras are, how they generalize groups, and why they are suitable for the description of particles, especially in two space dimensions. We also establish notation and collect some formulae for reference. For much more information on Hopf algebras and quantum groups, one may consult for example [4, 5, 6, 7, 8].

1.1 Definitions and philosophy

Definition 1 *A* Hopf algebra *is an associative algebra $\mathcal{A}$ with multiplication μ and unit 1, that has extra structures ϵ, S and Δ called counit, antipode and coproduct.The* coproduct *or* comultiplication Δ *is an algebra map from $\mathcal{A}$ to $\mathcal{A} \otimes \mathcal{A}$ with the following property, called coassociativity:*

$$(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta. \tag{1.1}$$

Here, id *is the identity map on $\mathcal{A}$. The* counit ϵ *of $\mathcal{A}$ is an algebra map from $\mathcal{A}$ to $\mathbb{C}$, or equivalently, a one-dimensional representation of $\mathcal{A}$, which satisfies*

$$(\epsilon \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \epsilon)\Delta = \mathrm{id}. \tag{1.2}$$

Finally, the antipode S *of $\mathcal{A}$ is a linear map from $\mathcal{A}$ to $\mathcal{A}$ which satisfies*

$$\mu(S \otimes \mathrm{id})\Delta(a) = \mu(\mathrm{id} \otimes S)\Delta(a) = \epsilon(a)1 \tag{1.3}$$

for each $a \in \mathcal{A}$.

Some types of Hopf algebras are also called *quantum groups*, but there is no agreement on exactly which types. We will use the term quantum group to refer to quasitriangular Hopf algebras (see definition 4)

Our basic philosophy will be that the one particle Hilbert space for each particle in a physical theory must carry an irreducible representation (irrep) of the Hopf algebra which describes the symmetry that is present in that theory. The spectrum of particles in an $\mathcal{A}$-theory thus corresponds to the spectrum of irreps of the Hopf algebra $\mathcal{A}$. The defining properties of a Hopf algebra are such that its spectrum of irreps has some properties that mimic those of the particle spectrum of a physical theory.

Given two representations π^1, π^2 of $\mathcal{A}$, one can define a tensor product representation $\pi^1 \otimes \pi^2$ by the formula

$$\pi^1 \otimes \pi^2 : \ x \to (\pi^1 \otimes \pi^2)(\Delta(x)). \tag{1.4}$$

Thus, Δ provides us with a way of describing the action of $\mathcal{A}$ on multi-particle states. The decomposition of the tensor product representations defined by means of Δ gives the fusion rules of the theory. From this it should be clear that, in physical applications, it is very desirable that all $\mathcal{A}$-modules are fully reducible. For finite dimensional Hopf algebras, this property is equivalent to semisimplicity. The coassociativity of Δ ensures that this tensor product is associative, so that the order in which particles are fused together does not matter. A graphical representation of the coassociativity of Δ is given in figure 1.1

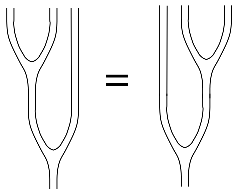

Figure 1.1: Graphical representation of the coassociativity relation (1.1). The order in which particles are fused does not matter.

In calculations that involve the coproduct, one often uses *Sweedler notation*:

$$\Delta(a) = \sum a_{(1)} \otimes a_{(2)}. \tag{1.5}$$

In this notation, coassociativity can be expressed as

$$\sum (a_{(1)})_{(1)} \otimes (a_{(1)})_{(2)} \otimes a_{(2)} = \sum a_{(1)} \otimes (a_{(2)})_{(1)} \otimes (a_{(2)})_{(2)}. \tag{1.6}$$

The counit can be seen as the trivial or vacuum representation of $\mathcal{A}$. It follows from (1.2) that we have

$$\epsilon \otimes \pi \cong \pi \otimes \epsilon \cong \pi \tag{1.7}$$

for any representation π of $\mathcal{A}$. Thus, if we assume that the vacuum (or an $\mathcal{A}$-neutral particle) transforms in the representation ϵ of $\mathcal{A}$, then we get the fusion properties that one would expect.

If we are given a representation π of $\mathcal{A}$, then the antipode makes it possible to define the representation $\bar{\pi}$ conjugate to π by the formula

$$\bar{\pi}(a) = \pi^{\mathrm{t}}(S(a)), \tag{1.8}$$

where the t denotes matrix transposition. To see that this is indeed a representation of $\mathcal{A}$, note that it follows from (1.3) that S is an antihomomorphism, that is $S(ab) = S(b)S(a)$ for any $a, b \in \mathcal{A}$. The properties (1.3) also ensure that the tensor product representations $\pi \otimes \bar{\pi}$ and $\bar{\pi} \otimes \pi$ contain the trivial representation ϵ in their decomposition. Thus, a particle which carries the representation π and its antiparticle, which carries the representation $\bar{\pi}$ may indeed annihilate.

We now recall the notion of a Hopf subalgebra and of the dual of a Hopf algebra, as they will turn out to be important in our discussion of Hopf symmetry breaking.

Definition 2 *For any Hopf algebra $\mathcal{A}$, we may define a* Hopf subalgebra *to be a subalgebra $\mathcal{B}$ of $\mathcal{A}$ which satisfies*

$$1 \in \mathcal{B}, \quad S(\mathcal{B}) \subset \mathcal{B}, \quad \Delta(\mathcal{B}) \subset \mathcal{B} \otimes \mathcal{B} \tag{1.9}$$

This implies that $\mathcal{B}$ is itself a Hopf algebra, with "the same" structures as $\mathcal{A}$.

Definition 3 *For any finite dimensional Hopf algebra $\mathcal{A}$, the* dual Hopf algebra *is the vector space $\mathcal{A}^*$ of linear functionals from $\mathcal{A}$ to $\mathbb{C}$ with Hopf algebra structures $1^*, \mu^*, \Delta^*, \epsilon^*$ and S^* defined by:*

$$\begin{array}{rlrl} 1^*: & a \mapsto \epsilon(a) & \mu^*(f_1, f_2): & a \mapsto f_1 \otimes f_2 \circ \Delta(a) \\ \epsilon^*: & f \mapsto f(1) & \Delta^*(f): & (a_1, a_2) \mapsto f(\mu(a_1, a_2)) \\ S^*(f): & a \mapsto f(S(a)) & & \end{array} \tag{1.10}$$

Here, a, a_1, a_2 are arbitrary elements of $\mathcal{A}$ and f, f_1, f_2 are arbitrary elements of $\mathcal{A}^*$.

It is often possible to define duals for infinite dimensional Hopf algebras analogously, but the technical details (how to complete the dual tensor product, etc.) vary from case to case.

1.2 Braiding and spin

The particle exchanges in a system of n identical particles are governed by the fundamental group of the configuration space for this system [9]. In 3+1 or more dimensions, this fundamental group is the permutation group S_n, but in 2+1 dimensions, it is the braid group B_n. The elements of this group (the braids) are all the topologically inequivalent ways in which one may connect n points in a plain to the corresponding n points in a parallel plain using n strings. Multiplication is given by concatenation of braids. Figure 1.2 illustrates how a braid corresponds to a path through configuration space.

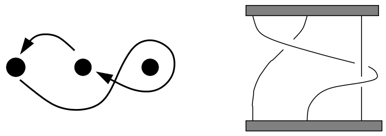

Figure 1.2: *On the left:* a closed path in configuration space. Particles are indicated by black dots. They move along the arrows, so that the original particle positions are once again occupied after the movement.
On the right: the corresponding braid. The direction of the movement on the left corresponds to the downward direction on the right.

B_n is generated by $n-1$ elementary exchanges $\tau_1, \ldots, \tau_n$ subject to the relations

$$\begin{aligned} \tau_i \tau_j &= \tau_j \tau_i \quad (|i-j| \geq 2) \\ \tau_i \tau_{i+1} \tau_i &= \tau_{i+1} \tau_i \tau_{i+1}, \end{aligned} \tag{1.11}$$

Braid diagrams for the generators and relations are given in figure 1.3.

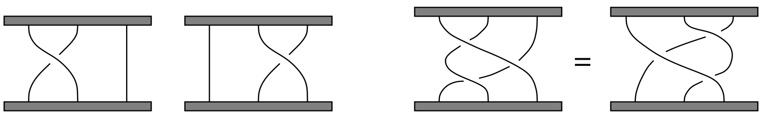

Figure 1.3: Braid diagrams for the generators τ_1 and τ_2 of the Braid group B_3 and for the single relation between them: $\tau_1 \tau_2 \tau_1 = \tau_2 \tau_1 \tau_2$.

The permutation group S_n is the quotient of B_n that is obtained when we add the relations

$$(\tau_i)^2 = e. \tag{1.12}$$

Suppose that we have two particles that carry the $\mathcal{A}$-representation π, with module V_π. The total internal state of the system can then be represented by a state $|\, s\,\rangle$ in the tensor product $V_\pi \otimes V_\pi$. If we exchange the particles, how does the state of the system change? Usually, we would describe the exchange simply by interchanging the tensor factors in the state $|\, s\,\rangle$. We call this exchange of the factors in the tensor product σ. An exchange of two adjacent particles in a system of N identical particles may then be described by the action of σ on the corresponding factors of the N-fold tensor product. For example, in a 4-particle system, the exchange of the second and third particles is accomplished by the operation $1 \otimes \sigma \otimes 1$. Such exchanges generate a representation of the permutation group S_N and are thus adequate for systems in 3+1 or more dimensions. However, in 2+1 dimensions, we need a more general exchange recipe if we want to have braid group representations that are not permutation group representations. Such a recipe is included in the Hopf algebraic framework when the Hopf algebra in question is quasitriangular.

Definition 4 *A Hopf algebra $\mathcal{A}$ is* quasitriangular *if there is an invertible element $R \in \mathcal{A} \otimes \mathcal{A}$ which has the properties*

$$\Delta^{\mathrm{op}} R = R\Delta \tag{1.13}$$
$$(\Delta \otimes \mathrm{id})R = R_{13}R_{23} \tag{1.14}$$
$$(\mathrm{id} \otimes \Delta)R = R_{13}R_{12}. \tag{1.15}$$

Here, Δ^{op} is the comultiplication, followed by an exchange of the tensor factors in $\mathcal{A} \otimes \mathcal{A}$ and R_{ij} is an abbreviation for the action of R on the factors i and j of $\mathcal{A}^{\otimes 3}$, so for example $R_{12} = R \otimes 1$. The element R is called the universal R-matrix *of $\mathcal{A}$.*

To exchange two adjacent particles, we now let R act before σ in the appropriate tensor factors. For example, in a system of three particles, all of which carry the representation π of $\mathcal{A}$ the exchange of the first and second particles will be effected by $(\sigma \circ (\pi \otimes \pi)(R)) \otimes 1$. Note that, in any tensor product of two representations, the universal R-matrix does indeed act as a matrix, but the matrix in question depends on the representations.

The defining properties of the R-matrix make sure that exchanging particles by means of σR makes physical sense. The properties (1.14) and (1.15) guarantee that braiding two particles around a third one and then fusing them together gives the same result as fusing the two particles first and then braiding the result around the third one (see figure 1.4). The property (1.13) ensures that the exchanges commute with the action of the quantum group. Hence, it also makes sure that the tensor products $\pi_1 \otimes \pi_2$ and $\pi_2 \otimes \pi_1$ of representations π_1 and π_2 of $\mathcal{A}$ are isomorphic, with the isomorphism from the module of $\pi_1 \otimes \pi_2$ to that of $\pi_2 \otimes \pi_1$ given by the exchange map $\sigma \circ (\pi_1 \otimes \pi_2)(R)$. Using either (1.13) and (1.14) or (1.13) and (1.15), one may also prove that

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \tag{1.16}$$

This is the so called quantum Yang-Baxter equation. It implies that, in any representation of $\mathcal{A}$, we have the identity

$$(\sigma R \otimes 1)(1 \otimes \sigma R)(\sigma R \otimes 1) = (1 \otimes \sigma R)(\sigma R \otimes 1)(1 \otimes \sigma R) \tag{1.17}$$

from which we see that, for a system of n identical particles that carry a representation of $\mathcal{A}$, the exchanges of adjacent particles, as performed using σR, satisfy the relations (1.11) of the braid group. Hence, since R is invertible, they generate a representation of this group. Since the

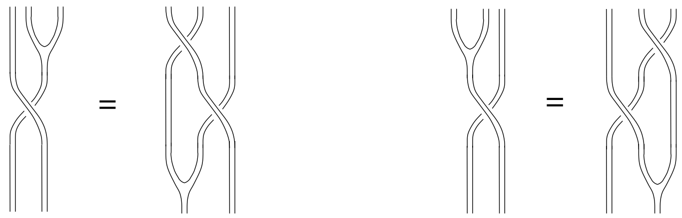

Figure 1.4: Graphical representation of the relations (1.13) and (1.14). Fusion and braiding commute.

exchanges commute with the action of the quantum group $\mathcal{A}$, it follows that the system carries a representation of $\mathcal{A} \times B_n$.

When the particles do not all carry the same quantum group representation, and are hence not identical, the R-matrix no longer gives us a representation of the braid group on the Hilbert space of the system, because the exchanges now act between different vector spaces; the flip operator σ sends $V_{\pi_1} \otimes V_{\pi_2}$ into $V_{\pi_2} \otimes V_{\pi_1}$. This is not a problem, because exchanges of non-identical particles are not symmetries of the system. What we do still get from the R-matrix is a representation of a so called colored braid group, which consists of the braids for which the final position of any particle is the original position of a particle of the same kind (or “color”). The action of this colored braid group on the Hilbert space of the theory describes the topological interactions between the different kinds of particles in the theory. In connection with this, one should note that the coloring restriction leaves plenty of room for non-trivial and even non-Abelian monodromies between distinguishable particles. The colored braidings will still commute with the quantum group action and they may still be generated by elementary exchanges of adjacent particles, although some of these exchanges will no longer have a clear physical meaning and should be called half-monodromies rather than braidings. We will be a bit sloppy about this in the sequel, but we hope that this will not cause confusion.

A description of the spin of the particles in a two dimensional theory can also be incorporated into a Hopf algebraic treatment when the Hopf algebra that describes the system is a ribbon Hopf algebra.

Definition 5 *A* ribbon Hopf algebra *is a quasitriangular Hopf algebra* $(\mathcal{A}, R)$ *with an invertible central element* c *that satisfies the equations*

$$\begin{aligned} c^2 = uS(u), \quad S(c) = c, \quad \epsilon(c) = 1 \\ \Delta(c) = (R_{21}R_{12})^{-1}(c \otimes c), \end{aligned} \tag{1.18}$$

where $u = \mu(S \otimes \mathrm{id})(R_{21})$. *The element* c *is called the* ribbon element

The action of the ribbon element on the physical Hilbert space is interpreted as the action of a rotation of the system over 2π. In particular, the action of c on an irrep of $\mathcal{A}$ describes the effect of rotating the particle that carries this irrep. Because c is central, the action of c on an irrep may always be described by a scalar factor, which is called the *spin factor* of the irrep and of the corresponding particle. The equations that c has to satisfy make sure that the spin of the vacuum is trivial, that the spin of a particle and its antiparticle are equal and that rotating a system of two particles over an angle of 2π may be accomplished both by acting with c on the two-particle system (making use of Δ) and by braiding the two particles around each other and then rotating them separately. This last property is illustrated in figure 1.5

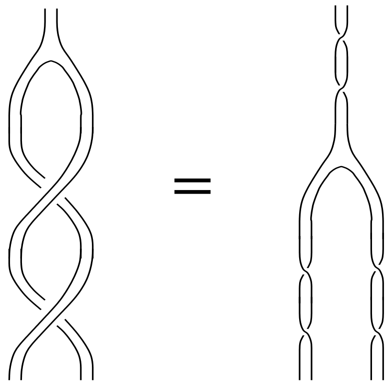

Figure 1.5: Graphical representation of the relation $(R_{21}R_{12})^{-1} = \Delta(c)(c \otimes c)^{-1} =$. The twists in the ribbons represent the action of c. The term "ribbon Hopf algebra" is inspired by pictures such as this one.

1.3 Hopf algebras and groups

If the excitation spectrum and fusion properties of a physical system can be described by means of a group H, then they can also be described by means of a Hopf algebra. When the group H is finite, the corresponding Hopf algebra is its group algebra $\mathbb{C}H$. This is the vector space generated by the elements of H, with the multiplication induced by the multiplication of H. The unit of this algebra is just the unit e of H. The comultiplication, antipode and counit of $\mathbb{C}H$ are given on the basis of group elements $h \in H$ by the formulae

$$\Delta(h) = h \otimes h \quad S(h) = *(h) = h^{-1} \quad \epsilon(h) = 1 \tag{1.19}$$

and one may check easily that the defining properties of a Hopf algebra are satisfied. Representations of $\mathbb{C}H$ are in one to one correspondence to representations of H, the tensor product defined by (1.4) is just the ordinary tensor product of group representations and from the formulae for ϵ and S, we see that the trivial representation and the conjugate of a representation are also defined in the usual way. $\mathbb{C}H$ is quasitriangular with the trivial R-matrix $e \otimes e$ and in fact, it is a ribbon Hopf algebra, with the trivial ribbon element $c = e$.

To a group with infinitely many elements one may usually associate several different Hopf algebras. For Lie groups, one usually takes the universal enveloping algebra of the Lie algebra of the group. This is the free algebra generated by the unit element 1 and a basis of the Lie algebra, modulo the relations given by the Lie bracket. For the Lie algebra $sl(2)$, for instance, we get the algebra $U(sl(2))$ generated by 1 and the three elements H, L^+ and L^-, subject to the relations

$$\begin{aligned} [H, L^\pm] &= \pm 2L^\pm \\ [L^+, L^-] &= H. \end{aligned} \tag{1.20}$$

The difference with $sl(2)$ itself is that elements like H^2L^+ are also in this algebra. The coproduct, counit and antipode are given by

$$\begin{aligned} \Delta(H) &= 1 \otimes H + H \otimes 1 \\ \Delta(L^\pm) &= 1 \otimes L^\pm + L^\pm \otimes 1 \\ \Delta(1) &= 1 \otimes 1 \\ \epsilon(1) &= 1, \quad \epsilon(H) = 0, \quad \epsilon(L^\pm) = 0 \\ S(1) &= 1, \quad S(H) = -H, \quad S(L^\pm) = -L^\pm \end{aligned} \tag{1.21}$$

and we see that these structures are just the infinitesimal versions of the structures we gave for group elements. The representations of the enveloping algebra are given by the representations of the corresponding Lie algebra and the tensor product and conjugation defined by (1.4) and (1.8) correspond to the usual tensor product and conjugation for representations of Lie algebras. Also, universal enveloping algebras are ribbon Hopf algebras with trivial R-matrix $1 \otimes 1$ and trivial ribbon element 1.

If one wants to associate a Hopf algebra to a real Lie group, it is often useful to take the universal enveloping algebra of the complexification of its Lie algebra, supplemented with a star structure. This is an antilinear algebra anti-automorphism $*$ which squares to the unit. Using this star, one may then define a unitary representation or $*$-representation of the algebra as a representation π which satisfies

$$\forall x : \quad \pi(*(x)) = \pi(x)^\dagger. \tag{1.22}$$

where the dagger indicates Hermitian conjugation. One may prove that every $*$-representation decomposes orthogonally into irreps, by noting that the orthogonal complement of a $*$-submodule of the representation's module is itself a $*$-submodule. $U(sl(2))$ has two star structures, one for each real form of $SL(2,\mathbb{C})$. We will only use the star structure that corresponds to $SU(2)$, which is given by

$$*(L^\pm) = L^\mp, \quad *(H) = H. \tag{1.23}$$

This star has the property

$$\forall x : \quad (* \otimes *)\Delta(x) = \Delta(*(x)), \tag{1.24}$$

from which it follows that every tensor product of $*$-representations is itself a $*$-representation with respect to the standard inner product on the tensor product space. As a consequence, the decomposition of tensor products of $*$-irreps is orthogonal.

A typical characteristic of Hopf algebras associated to groups is that they are *cocommutative*, that is, $\Delta^{\mathrm{op}} = \Delta$. One may check from the formulae above that this is indeed the case for group algebras and universal enveloping algebras. A well known theorem in Hopf algebra theory (see for example [4], section 5.6 for a proof and references) asserts that any cocommutative Hopf algebra over the complex numbers, defined with the ordinary (algebraic) tensor product, is in fact isomorphic to a crossed product of a group algebra and a universal enveloping algebra. In other words, any such Hopf algebra corresponds to a crossed product of a Lie group and a finite group. We may also associate commutative Hopf algebras with groups, via their duals, which are cocommutative. Hence, the name "quantum group" is usually reserved for Hopf algebras which are neither commutative nor non-cocommutative.

1.4 Introduction to $U_q(sl(2))$

After the previous section, we should obviously provide at least one example of a non-commutative and non-cocommutative quasitriangular Hopf algebra. The standard non-trivial example of such an algebra is a deformation of the universal enveloping algebra of $sl(2)$ which depends on a parameter q and is denoted $U_q(sl(2))$. In this section, we give a review of some of the representation theory of this algebra. The material presented here serves as a basis for our discussion in chapter 2 of the braid group representations associated with $U_q(sl(2))$ at roots of unity.

1.4.1 The algebra and its unitary representations

$U_q(sl(2))$ can be viewed[1] as the algebra generated by a unit 1 and the three elements H, L^+ and L^-. These satisfy the relations

$$\begin{aligned} [H, L^{\pm}] &= \pm 2L^{\pm} \\ [L^+, L^-] &= \frac{q^{H/2} - q^{-H/2}}{q^{1/2} - q^{-1/2}}, \end{aligned} \tag{1.25}$$

where q may be set to any non-zero complex value. One may check that these relations reduce to those of the Lie algebra $sl(2)$ when q goes to one. Hence, $U_1(sl(2))$ is just the universal enveloping algebra $U(sl(2))$ (cf. (1.20)) and we say that $U_q(sl(2))$ is a q-deformation of $U(sl(2))$. The coproduct of $U_q(sl(2))$ is given by

$$\begin{aligned} \Delta(H) &= 1 \otimes H + H \otimes 1 \\ \Delta(L^{\pm}) &= L^{\pm} \otimes q^{H/4} + q^{-H/4} \otimes L^{\pm} \\ \Delta(1) &= 1 \otimes 1 \end{aligned} \tag{1.26}$$

and we see from the coproduct of $L^{\pm}$ that $U_q(sl(2))$ is not cocommutative, except of course if $q = 1$. The Hopf structure is completed by the counit and the antipode:

$$\begin{array}{lll} \epsilon(1) = 1, & \epsilon(H) = 0, & \epsilon(L^{\pm}) = 0, \\ S(1) = 1, & S(H) = -H, & S(L^{\pm}) = -q^{\frac{\mp 1}{4}} L^{\pm}. \end{array} \tag{1.27}$$

If q is not a root of unity, the representation theory of $U_q(sl(2))$ is very similar to that of $U(sl(2))$. For each non-negative $j \in \frac{1}{2}\mathbb{Z}$ there is an irreducible highest weight representation of dimension $d = 2j + 1$. We will denote this representation by π^{Λ}, where $\Lambda = d - 1 = 2j$ is the highest weight. The modules V^{Λ} of these representations have an orthonormal basis that consists of kets $|\, j, m\,\rangle$, with $m = -j, -j+1, \ldots, j$ and the generating elements H, L^+ and L^- act on this basis as follows

$$\begin{aligned} H|\, j, m\,\rangle &= 2m|\, j, m\,\rangle \\ L^{\pm}|\, j, m\,\rangle &= \sqrt{\lfloor j \mp m \rfloor_q \lfloor j \pm m + 1 \rfloor_q}\ \ |\, j, m \pm 1\,\rangle. \end{aligned} \tag{1.28}$$

Here, the q-number $\lfloor m \rfloor_q$ is defined as

$$\lfloor m \rfloor_q = \frac{q^{m/2} - q^{-m/2}}{q^{1/2} - q^{-1/2}}. \tag{1.29}$$

These q-numbers enter the formulae for the representations through the commutation relation of L^+ and L^-, the right hand side of which can be written $\lfloor H \rfloor_q$. The q-number $\lfloor m \rfloor_q$ approaches m when q goes to one and hence we see that the representations given above reduce to the usual $U(sl(2))$ representations for $q = 1$.

We will be interested in $U_q(sl(2))$ with a star-structure defined as in (1.23), i.e.

$$*(L^{\pm}) = L^{\mp}, \quad *(H) = H. \tag{1.30}$$

[1] We write "viewed" rather than "defined", because we are skipping some subtleties which occur in the definition. These subtleties are not very important for our purposes. For full details, one may consult [7]

$U_q(sl(2))$ with this star structure is also called $U_q(su(2))$. Let us check when the representations defined above are unitary with respect to this star. First note from (1.28) that we always have $\pi^\Lambda(H)^\dagger = \pi^\Lambda(H)$ and $\pi^\Lambda(L^\pm)^\dagger = (\pi^\Lambda)^{\mathrm{t}}(L^\mp)$, where the t denotes matrix transposition. Hence, π^Λ is unitary when the matrix elements of $L^\pm$ are real, i.e. when the square root in (1.28) is real for all admissible values of m. This will be the case if q is real and positive and also if $q = e^{i\phi}$ with $\phi \in \mathbb{R}$, $|\phi| \leq \frac{2\pi}{2j+1}$, since for these values of q the q-numbers in the square root are real and positive. Thus we see that, for real q, all the representations above are $*$-representations, while for $q = e^{i\phi}$, the representations π^{2j} with $|\phi| \leq \frac{2\pi}{2j+1}$ are $*$-representations.

More generally, many aspects of the representation theory of $U_q(sl(2))$ depend on the properties of q-numbers. Two simple examples of q-number identities which are useful in representation theoretic calculations and which hold for all $q \in \mathbb{C}$ are

$$\begin{aligned} q^{n/2}\lfloor m \rfloor_q + q^{-m/2}\lfloor n \rfloor_q &= \lfloor m+n \rfloor_q \\ \lfloor n+m \rfloor_q \lfloor n-m \rfloor_q &= \lfloor n \rfloor_q^2 - \lfloor m \rfloor_q^2. \end{aligned} \tag{1.31}$$

1.4.2 Tensor products and Clebsch-Gordan coefficients

Using the coproduct (1.26), one may define tensor products of $U_q(sl(2))$-representations in the usual way (cf. (1.4)). Tensor product decompositions and even Clebsch-Gordan coefficients for tensor products of unitary representations of $U_q(sl(2))$ may then be calculated similarly as for $U(sl(2))$. The highest weight state $|\, j, j\,\rangle$ of each the irreducible representations in the tensor product may be found by solving the equations $L^+|\, j,j\,\rangle = 0$ and $H|\, j,j\,\rangle = 2j|\, j,j\,\rangle$. The other states are produced by repeatedly acting with L^-. In the calculations, the following formula for the coproduct of $(L^-)^n$ is a great help:

$$\Delta((L^-)^n) = (\Delta(L^-))^n = \sum_{m=0}^{n} \left\lfloor \begin{array}{c} n \\ m \end{array} \right\rfloor_q (L^{-1})^m q^{-(n-m)H/4} \otimes (L^-)^{n-m} q^{m/4}. \tag{1.32}$$

The q-binomial $\left\lfloor \begin{array}{c} n \\ m \end{array} \right\rfloor_q$ in this formula is defined by

$$\left\lfloor \begin{array}{c} n \\ m \end{array} \right\rfloor_q := \frac{\lfloor n \rfloor_q!}{\lfloor m \rfloor_q! \lfloor n-m \rfloor_q!}, \quad \text{with} \quad \lfloor n \rfloor_q! := \prod_{m=1}^{n} \lfloor m \rfloor_q. \tag{1.33}$$

When q is not a root of unity, the tensor product representation $\pi^\Lambda \otimes \pi^{\Lambda'}$ has the same decomposition into irreps as for $q = 1$, i.e.

$$\pi^\Lambda \otimes \pi^{\Lambda'} = \bigoplus_{\Lambda''=|\Lambda-\Lambda'|}^{\Lambda+\Lambda'} \pi^{\Lambda''}, \tag{1.34}$$

where Λ'' increases in steps of 2. It follows from this decomposition that the irreps of $U_q(sl(2))$ are all self-conjugate.

Explicit Clebsch-Gordan coefficients may be calculated for any tensor product of irreducibles, using (1.32) and (1.28). One writes

$$|\, j,m\,\rangle = \sum_{m_1,m_2} \left[\begin{array}{ccc} j_1 & j_2 & j \\ m_1 & m_2 & m \end{array} \right]_q |\, j_1,m_1\,\rangle |\, j_2,m_2\,\rangle \tag{1.35}$$

for the vector with H-eigenvalue $2m$ in the irrep π^{2j} in the decomposition of the tensor product $\pi^{2j_1}\otimes\pi^{2j_2}$. The above formula is only meant to introduce the notation for the q-Clebsch-Gordan coefficients. Several general formulae for these coefficients are proved in [10] and [11] and collected in [12]. We will not give these explicit (complicated) formulae here, but we do give the coefficients for the case $j_2=\frac{1}{2}$, as an illustration and because this case is of special interest to us later. For $j_1>0$, one has $j=j_1\pm\frac{1}{2}$ and

$$\begin{aligned}
|\,j_1+\tfrac{1}{2},j_1+\tfrac{1}{2}-p\,\rangle &= q^{p/4}\sqrt{\lfloor 2j_1+1-p\rfloor_q/\lfloor 2j_1+1\rfloor_q}\;\;|\,j_1,j_1-p\,\rangle|\,\tfrac{1}{2},\tfrac{1}{2}\,\rangle \\
&\quad +q^{(p-2j_1-1)/4}\sqrt{\lfloor p\rfloor_q\lfloor 2j_1+1\rfloor_q}\;\;|\,j_1,j_1-p+1\,\rangle|\,\tfrac{1}{2},-\tfrac{1}{2}\,\rangle \\
|\,j_1-\tfrac{1}{2},j_1-\tfrac{1}{2}-p\,\rangle &= q^{(p-2j_1)/4}\sqrt{\lfloor p+1\rfloor_q\lfloor 2j_1+1\rfloor_q}\;\;|\,j_1,j_1-p-1\,\rangle|\,\tfrac{1}{2},\tfrac{1}{2}\,\rangle \\
&\quad -q^{(p+1)/4}\sqrt{\lfloor 2j_1-p\rfloor_q\lfloor 2j_1+1\rfloor_q}\;\;|\,j_1,j_1-p\,\rangle|\,\tfrac{1}{2},-\tfrac{1}{2}\,\rangle. \qquad (1.36)
\end{aligned}$$

The coefficient for $j_1=0$ is the same as for $U(sl(2))$. In making a decomposition such as the one above, one has the freedom to multiply all the states in any given summand irrep by a constant phase factor. Here, the phases are chosen in such a way that, when q goes to one, the coefficients reduce to the usual Clebsch-Gordan coefficients for $U(sl(2))$. One may check directly (for example using (1.31)) that, when q is a positive real number, the tensor product vectors on the right hand side are orthonormal. One may also see it as a consequence of the fact that star an coproduct commute (i.e. (1.24) is satisfied) when q is real and positive. This implies that tensor product decompositions of $*$-irreps are always orthogonal for $q\in\mathbb{R}_+$, a fact which is reflected in the following identity for the q-Clebsch-Gordan coefficients:

$$\sum_{m_1,m_2}\begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}_q \begin{bmatrix} j_1 & j_2 & j' \\ m_1 & m_2 & m' \end{bmatrix}_q = \delta_{j,j'}\delta_{m,m'}. \qquad (1.37)$$

Although tensor product decomposition is orthogonal with respect to the standard inner product only when q is real and positive, this equation holds by analytic continuation for all q where the summands are not singular.

Another useful identity (taken from ([12])), which relates the coefficients for the tensor product $\pi^{2j_1}\otimes\pi^{2j_2}$ with those for the opposite tensor product is:

$$\begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}_q = (-1)^{j_1+j_2-j}\begin{bmatrix} j_2 & j_1 & j \\ m_2 & m_1 & m \end{bmatrix}_{q^{-1}} \qquad (1.38)$$

In particular, this allows one to write down the Clebsch-Gordan coefficients for $\pi^1\otimes\pi^{2j}$ using formula (1.36).

1.4.3 R-matrix and braiding

The universal R-matrix for $U_q(sl(2))$ is given by

$$R = q^{\frac{H\otimes H}{4}}\sum_{n=0}^{\infty}\frac{(1-q^{-1})^n}{\lfloor n\rfloor_q!}q^{n(1-n)/4}(q^{nH/4}(L^+)^n)\otimes(q^{-nH/4}(L^-)^n). \qquad (1.39)$$

We see that, when q approaches one, only the $n=0$ term in (1.39) contributes and we get $R=1\otimes 1$, as expected.

The action of the universal R-matrix on the module $V^{\Lambda_1} \otimes V^{\Lambda}$ of the tensor product representation $\pi^{\Lambda_1} \otimes \pi^{\Lambda}$ is given by

$$\begin{aligned} R|\, j_1, m_1 \rangle |\, j_2, m_2 \rangle \;&=\; \textstyle\sum_{n\geq 0} \sqrt{\left\lfloor \begin{array}{c} j_1 - m_1 \\ n \end{array} \right\rfloor_q \left\lfloor \begin{array}{c} j_2 + m_2 \\ n \end{array} \right\rfloor_q \frac{\lfloor j_1+m_1+n \rfloor_q! \lfloor j_2-m_2+n \rfloor_q!}{\lfloor j_1+m_1 \rfloor_q! \lfloor j_2-m_2 \rfloor_q!}}\, q^{\frac{n(1-n)}{4}} \\ &\quad \times q^{\frac{1}{2}(m_2 n - m_1 n + 2m_1 m_2)} (1-q^{-1})^n |\, j_1, m_1+n \rangle |\, j_2, m_2 - n \rangle, \end{aligned} \tag{1.40}$$

where the sum extends over all n for which the kets on the right hand side are well defined. Using this formula, one may easily find the exchange matrix σR in any tensor product module. For example, in the tensor product $\pi^1 \otimes \pi^1$ of two two-dimensional modules, we have

$$\sigma R^{1,1} := q^{-1/4} \begin{pmatrix} q^{1/2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & q^{1/2} - q^{-1/2} & 0 \\ 0 & 0 & 0 & q^{1/2} \end{pmatrix}. \tag{1.41}$$

Note that, if $q \neq 1$, this is not a unitary matrix, which is not good if it is supposed to represent a symmetry transformation on a physical system. Still, we could hope to make σR unitary by choosing a suitable inner product on the module $V^1 \otimes V^1$. This will certainly not succeed unless $|q| = 1$. To see this, note that the eigenvalues of $R^{1,1}$ are $q^{1/4}$ (with multiplicity 3) and $-q^{-3/4}$ and these will only have norm 1 (as required for the eigenvalues of a unitary transformation) if q does. From this, one may already guess that the most interesting values of q for applications to (2+1)-dimensional physics are the roots of unity that we have avoided up to now. A treatment of $U_q(sl(2))$ at roots of unity is given in chapter 2.

1.5 Some Features of 2D CFT

While group symmetries may be described in terms of Hopf algebras, the converse is not necessarily true; there are physical systems whose symmetry algebra is a non-commutative and non-cocommutative Hopf algebra. Examples of (2+1) dimensional systems with a quantum group symmetry are the discrete gauge theories of chapter 3 and the Hall states of chapter 2, but also (2+1)-dimensional gravity [13, 14]. Systems which can be described by means of two-dimensional conformal field theory (CFT) also provide a large class of examples. Therefore, we will devote this section to a brief and sketchy description of some features of two-dimensional CFT, particularly the ones which are relevant to the connection with quantum groups. For a thorough introduction to conformal field theory and references, see [15].

A conformal field theory is a field theory whose action is invariant under the conformal group. The conformal group is not really a group, but rather the monoid of all locally well-defined conformal (i.e. angle-preserving) transformations of spacetime, with multiplication given by composition. Nevertheless, infinitesimal conformal transformations give a well defined Lie algebra. In three or more spacetime dimensions, the conformal group is basically the Poincare group with dilations and the spacetime inversion $x^\mu \mapsto x^\mu/|x|^2$ thrown in. In two dimensions, it is customary to introduce complex coordinates $z = (x_0 + ix_1)$ and $\overline{z} = (x_0 - ix_1)$ and then the conformal group corresponds to the set of all complex-differentiable maps from (part of) the complex plain into itself. This set is clearly infinite dimensional and as a consequence, conformal symmetry is especially powerful in two dimensions.

Two dimensional CFTs play an important role in the description of the critical behaviour of planar systems in (classical) statistical physics and they are also ubiquitous in string theory, where the fields live on the string's ((1+1)-dimensional) worldsheet. We will not pursue either of these applications, but rather use CFT as a kind of machine to produce wave functions for ((2+1)-dimensional) quantum Hall systems (see chapter 2).

One may show that in any non-trivial unitary two dimensional CFT, the conformal symmetry has to be anomalous. Thus, the symmetry of the theory is not described by the Lie algebra of the conformal group, but rather by a central extension of this algebra. This extension is spanned by elements L_n $(n \in \mathbb{Z})$ and the central element C and its non-trivial brackets are given by

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{C}{12}(n^3 - n)\delta_{n+m,0} \tag{1.42}$$

In a physical CFT, the action of C on the Hilbert space is just multiplication with a constant real factor c, which is called the *central charge* and which is determined by the anomaly. If we forget about the generator C and replace it with c in the equation above, then the resulting algebra is called the *Virasoro algebra* (note that this is not a Lie algebra). Any two dimensional CFT has at least the Virasoro algebra as a symmetry algebra, but there are CFTs that have extra symmetries (such as gauge symmetries) and these possess larger symmetry algebras that include the Virasoro algebra. Such algebras are called *chiral algebras*. If one treats the coordinates z and $\bar{z}$ as independent complex variables, then the full symmetry algebra of a CFT is actually the tensor product of two copies of a chiral algebra $\mathcal{A}$; one copy for holomorphic transformations of z and one for antiholomorphic transformations of $\bar{z}$. These copies of $\mathcal{A}$ are called the left and the right algebra (hence the name "chiral algebra"). The right algebra is usually denoted by $\overline{\mathcal{A}}$ and its modes are denoted by $\overline{L}_n$.

The fields of a CFT fall into highest weight representations of the left and right algebra. The fields that correspond to the highest weight vector of such a representation are called *primary fields*, while the other fields are called *descendant fields*, or simply *descendants*. In a *rational* conformal field theory, that is, a CFT with $c \in \mathbb{Q}$, there can only be finitely many primary fields Φ_i. These primary fields are characterized by their *conformal weights* h_i and $\overline{h}_i$. These are the L_0 and $\overline{L_0}$ eigenvalues of the highest weight state of the corresponding chiral algebra representation. *(Left) chiral primaries* are primary fields Φ_i which transform trivially under the right algebra $\overline{\mathcal{A}}$. This is equivalent to having $\overline{h}_i = 0$. In many applications, it is assumed that primary fields may be factorized into left and right chiral primaries. These two halves of the theory may then be considered separately when calculating correlation functions.

A very important tool in calculating correlations functions for the fields of a CFT is the *operator product expansion (OPE)*. Given two chiral primaries Φ_i, Φ_j, we may use the conformal symmetry to write the following formula for their product.

$$\Phi_i(z_1)\Phi_j(z_2) = \sum_k a_{ij}^{k,d}(z_1 - z_2)^{h_{k,d} - h_i - h_j}\Phi_{k,d}. \tag{1.43}$$

This formula holds as z_1 approaches z_2. The index d labels descendants which may possibly occur on the right hand side and the $a_{ij}^{k,d}$ are structure constants. The requirement that the operator algebra defined in this way is associative is very restrictive and in some cases it enables one to calculate the spectrum of conformal weights and all correlation functions of a theory with a given chiral algebra exactly.

From the OPE of two primary fields, one may define their *fusion rules*. One writes

$$\Phi_i \times \Phi_j = \sum_k N_{ij}^k \Phi_k, \tag{1.44}$$

where N_{ij}^k counts the number of times that the field Φ_k or its descendants appear in the OPE of Φ_i and Φ_j. One may now define a *chiral correlator* as the vacuum expectation value of a radially ordered product of chiral primary fields, written as follows

$$\langle \mathcal{R} \left(\Phi_1(z_1) \ldots \Phi_n(z_n) \right) \rangle \tag{1.45}$$

Here, the $\mathcal{R}$ indicates radial ordering, defined by

$$\mathcal{R}\Phi_1(z_1)\Phi_n(z_2) = \begin{cases} \Phi_1(z_1)\Phi_n(z_2) & \text{if} \quad |z_1| > |z_2| \\ \Phi_n(z_2)\Phi_1(z_1) & \text{if} \quad |z_2| > |z_1| \end{cases} \tag{1.46}$$

Chiral correlators are in principle completely determined by the fusion rules and the conformal weights of the fields involved. In particular, the correlator will clearly be zero if the fields $\Phi_1, \ldots, \Phi_n$ cannot fuse to the vacuum sector. In practice, chiral correlators are very difficult to calculate and closed expressions are known only in some very special cases. Also, the correlator above is typically not single valued in the coordinates $(z_1, \ldots, z_n)$; when one of these coordinates is taken around another and their labels are subsequently exchanged, the correlator may pick up a phase or it may even transform into a different function of $(z_1, \ldots, z_n)$ which is linearly independent of the original function. In general, such coordinate exchanges generate a finite dimensional representation of the braid group B_n. A special basis for the finite dimensional space of functions of $(z_1, \ldots, z_n)$ on which this braid group acts, is given by the *conformal blocks* of the correlator. There is one of these blocks for every fusion channel through which the fields $\Phi_1, \ldots, \Phi_n$ may fuse to the vacuum sector. If there is only one such channel, then the conformal block is proportional to the correlator itself and the braid group representation that is associated with it will be Abelian. However, if there are m blocks, then the braid group representation is m-dimensional and it can be non-Abelian.

Some aspects of the description of CFT that we have given are very reminiscent of quantum group theory. In particular, for each CFT, there is a finite set of objects (the primaries) which have fusion rules and representations of the braid group associated with them. On the other hand, there are also some differences. The fusion of primary fields is not defined through tensor product decomposition of chiral algebra representations, but through the operator product. Braiding is defined through analytic continuation, rather than by algebraic means. Nevertheless, given a CFT, one could hope to find a quantum group whose finite dimensional irreps are in one to one correspondence with the primary fields and whose fusion and braiding, as defined through the coproduct and the R-matrix, are the same as those of the corresponding CFT primaries. If this could be achieved, it would yield a much simpler description of fusion and braiding and it would also be a very strong indication that the CFT in question has this Hopf algebra as a symmetry.

As it turns out, the class of quantum groups that we have described in this chapter is not large enough to reproduce the fusion and braiding of all CFTs. One may see this already by the example of the simplest non trivial CFT, the $c = \frac{1}{2}$ CFT which describes the critical point of the two-dimensional Ising model. This model has three chiral primary fields, 1,ψ and σ with conformal weights $h_1 = 0$, $h_\psi = \frac{1}{2}$ and $h_\sigma = \frac{1}{16}$. The fusion rules for the trivial field 1 are as one would expect and the fusion rules for σ and ψ are given by

$$\begin{array}{l} \psi \times \psi = 1 \\ \psi \times \sigma = \sigma \quad \sigma \times \sigma = 1 + \psi. \end{array} \tag{1.47}$$

If 1,ψ and σ are to correspond to the finite dimensional irreps of a Hopf algebra, then we must be able to assign integer dimensions d_1, d_ψ and d_σ to them which are consistent with the fusion rules. From $1 \times 1 = 1$, we see that $d_1 = 1$ and it then follows from $\psi \times \psi = 1$ that $d_\psi = 1$. However, using $\sigma \times \sigma = 1 + \psi$, this leads to the conclusion that d_σ must equal $\sqrt{2}$, which is not an integer…

Nevertheless, there is a class of quantum group-like algebras that generalizes the class of quasitriangular Hopf algebras presented here and that will reproduce the fusion rules of the Ising model and in fact, the fusion rules of all CFTs. We will present much more information about this in sections 2.4 and 2.5.

Chapter 2

Quantum groups and non-Abelian braiding in quantum Hall states

Wave functions describing quasiholes and electrons in non-Abelian quantum Hall states are well known to correspond to conformal blocks of certain coset conformal field theories. In this chapter we explicitly analyze the algebraic structure underlying the braiding properties of these conformal blocks. We treat the electrons and the quasihole excitations as localized particles carrying charges related to a quantum group that is determined explicitly for the cases of interest. The quantum group description naturally allows one to analyze the braid group representations carried by the multi-particle wave functions. As an application, we construct the non-Abelian braid group representations which govern the exchange of quasiholes in the fractional quantum Hall effect states that have been proposed by N. Read and E. Rezayi [16], recovering the results of C. Nayak and F. Wilczek [17] for the Pfaffian state as a special case.

2.1 Introduction

In a (2+1)-dimensional setting, quantum mechanics leaves room for particles with exchange properties other than those of bosons and fermions and the exchanges of n such particles are governed by a representation of the braid group B_n. These representations may be Abelian, or, more excitingly, non-Abelian. Quasihole excitations of fractional quantum Hall plateaus have already provided us with examples of the former possibility and may possibly reveal the latter as well. Several (series of) candidate non-Abelian states have been proposed in the literature. Examples are the Pfaffian state [18], the spin singlet states of Ardonne and Schoutens [19], the states proposed in [20], which exhibit spin-charge separation, and the parafermionic generalizations of the Pfaffian state proposed by Read and Rezayi [16]. It is the last series of states that we will focus on, although the methods we use will also be applicable to the other cases.

It has been suggested that the Read-Rezayi states should give a good description of quantum Hall plateaus which occur at several filling fractions [21, 22, 16]. In particular, the Pfaffian is thought to describe the plateau observed [23, 24, 25] at filling fraction $\nu = \frac{5}{2}$. Numerical support for these claims has been provided in [26, 16, 27], where it was shown that some of the RR-states (among which the Pfaffian state) have large overlaps with the exact ground states for electrons with Coulomb interactions at the same filling fractions. Also, a link has recently been made between fractional quantum Hall systems and rotating Bose-Einstein condensates [28, 29] and the Read-Rezayi states are thought to be relevant to the description of such condensates when the rotation frequency is sufficiently high [30]. Many aspects of the Read-Rezayi states have already been well-studied. For example, one may show (see [18, 16, 31]) that they are exact ground states of certain ultra-local Hamiltonians with $k+1$-body interactions, which gives hope that they will indeed represent new universality classes of two dimensional physical systems. Also, the zero modes of these Hamiltonians have been counted and in some cases explicit bases for the spaces of these zero modes have been obtained [31, 32]. Finally, there is recent work which explains how the RR-states may be obtained as projections of Abelian theories [33, 34, 35]. Still, before the appearance of the paper [36] on which most of this chapter is based, the braiding of the quasiholes had been described explicitly only for the Pfaffian state [17].

One of our general aims here is to analyze some of the properties of Hall systems, not by studying the explicit form of the wave functions but rather by exploiting the underlying algebraic structure, which in turn derives from the associated conformal field theories. This allows us for example to give an explicit description of the braid group representations that govern the exchange properties of the quasiholes for all of the RR-states. In order to do this, we first describe the electrons and quasiholes of the RR-states as particles that carry a representation of a certain quantum group. That such a description is possible is a logical consequence of the well known relation between quantum groups and conformal field theories and in fact, we expect that a similar description is possible for all the non-Abelian quantum Hall states that have been proposed. We believe that the quantum group description of quantum Hall states will prove a useful complement to the existing conformal field theory and wave function methods, both technically, because it makes braiding calculations much easier, and conceptually. The reason that braiding calculations are so much simplified, is that the quantum group picture allows one to deal with quasiholes and electrons without dealing with their exact spatial coordinates. Exchanging two particles becomes a purely algebraic operation, simple enough to be carried out explicitly for large numbers of particles.

The material is organized as follows. In section 2.2, we give a very brief introduction to the bulk theory of the quantum Hall effect. In particular, we motivate the use of conformal field theory in the construction of trial wave functions for fractional quantum Hall states in an elementary way. In section 2.3, we review the description of the Read-Rezayi states in terms of conformal blocks of parafermionic conformal field theories. We also count the number of independent states with a fixed number of quasiholes in fixed positions and we review the results of Nayak and Wilczek for the braiding of the quasiholes of the Pfaffian state. In section 2.4, we give motivation for the use of quantum groups in the description of non-Abelian quantum Hall states and provide the necessary background. In particular we describe the braid group representations that describe the exchanges in a system of localized particles with a hidden quantum group symmetry. In section 2.5, we recall the connection between quantum groups and conformal field theories and we obtain the quantum groups which can be used to describe the braiding of the parafermion CFTs which are important for the Read-Rezayi states. In section 2.6 we describe the RR-states as systems of point particles with a hidden quantum group symmetry and give the explicit form of the associated braid group representations. We also check that the results of Nayak and Wilczek for the case of the Pfaffian are recovered. A discussion of the results, including questions for future research can be found in section 2.7.

2.2 Hall states and CFT

This section gives a very brief introduction to some aspects of the quantum Hall effect, especially to the way conformal field theory enters into the description of the bulk properties of quantum Hall states. For a much fuller introduction, one may for instance consult the books [37, 38, 39]

2.2.1 The integer effect

As was mentioned in the introduction, the most striking characteristic of the quantum Hall effect is the occurrence of plateaus in the conductance at values $\nu\frac{e^2}{h}$, where ν is an integer or a simple fraction. The integer quantum Hall effect ($\nu \in \mathbb{N}$) may be understood in terms of a system of non-interacting electrons in a magnetic field, which scatter on impurities. In order to introduce some of the basic concepts in the quantum Hall literature, it is useful to first have a brief look at the system without even the impurities. This is the problem of free particles of charge $-e$ and mass m in two dimensions, under the influence of a magnetic field $\mathbf{B} = (0, 0, B)$. It was solved by Landau in 1930 (see for instance [40] for a treatment). In terms of a dimensionless complex coordinate $z = (x + iy)/\ell$, where $\ell = \sqrt{hc/(eB)}$ is the magnetic length, the one-particle Hamiltonian is given by

$$H = (-i\,\nabla - e\mathbf{A})^2 = \frac{1}{2}\hbar\omega_c(4\partial_z\partial_{\bar{z}} + z\partial_z - \bar{z}\partial_{\bar{z}} - \frac{1}{4}z\bar{z}). \tag{2.1}$$

Here $\omega_c = \frac{eB}{mc}$ is the cyclotron frequency. Also, here and in the sequel, we work in symmetric or central gauge, which means that

$$\mathbf{A} = (\frac{B}{2}x, -\frac{B}{2}y, 0). \tag{2.2}$$

For simplicity, we have neglected the spin of the electrons. In many Hall systems, this is actually a good way to proceed, since only one spin direction occurs, due to the large Zeeman splitting. A basis of eigenstates of the Hamiltonian is given by

$$\psi_{m,n}(z) = (\partial_z - \frac{\bar{z}}{4})^m(\partial_{\bar{z}} - \frac{z}{4})^n e^{-z\bar{z}/4} = e^{z\bar{z}/4}\partial_{\bar{z}}^m\partial_z^n e^{-z\bar{z}/2}. \tag{2.3}$$

The corresponding energy levels are called Landau levels. they are independent of m and hence infinitely degenerate,

$$E_n = \hbar\omega_c(n + \frac{1}{2}). \tag{2.4}$$

The first Landau level is of particular importance to us, as it is the only level that plays a role in the physics at very high magnetic fields. From (2.3), we see that the wave functions in this Landau level are exactly all functions which are a product of the Gaussian factor $e^{-z\bar{z}/4}$ and a holomorphic function. The action of the angular momentum operator on these states takes a very simple form; it just multiplies each term $z^m e^{-z\bar{z}/4}$ by a factor $m\hbar$. The main effect of confining the particles to a finite region in the plane (the sample) is that the Landau levels are no longer infinitely degenerate. Wave functions of exceedingly high angular momentum would place their electron outside the sample. Effectively, each single-particle state takes a surface area $\frac{hc}{eB} = \ell^2$ so that the Landau levels now contain eBA/hc states each, where A is the surface area of the sample. The number of states in a Landau level thus equals the number of fundamental flux quanta $\frac{e}{hc}$ that pierce the sample. These results are really independent of the sample geometry, but for convenience, we will always take the sample to be circular and centered at the origin.

The quotient of the number of electrons in the sample by the number of states in a Landau level is called the filling factor or filling fraction. In a system of free electrons, it is just the number of filled Landau levels, hence the term. It is seen experimentally that the conductance plateau at conductance $\nu\frac{e^2}{h}$ occurs at filling fraction ν. Hence, one speaks of the plateau at filling fraction ν.[1]

Integer filling "fractions" are special, since a system at integer filling fraction has a gap of $\hbar\omega_c$ to the next unoccupied single electron state. This suggests that scattering of electrons should be inhibited at these filling fractions and hence provides an explanation for the dips in the longitudinal resistance of the system at these values of ν. To explain the fact that there is a plateau in the resistance around integer filling fractions, one has to go beyond free electrons and introduce impurities. These impurities localize some of the states in each Landau level and shift their energies away from the quantized values (2.4). The states which remain extended also don't have their energies shifted by much. Now the crucial idea is that, at low temperatures, only the extended states contribute to the transport of electric charge across the system. Thus, when the B-field is varied and the Fermi level of the system sweeps through the energy levels, the conductivity remains constant as long as the Fermi level is in a band of localized states and changes rapidly as it moves through a band of extended states. In other words, the plateaus correspond to bands of localized states between the Landau levels.

Of course, in the model with impurities there is no longer a real gap, but there is still a gap between the bands of extended states, a mobility gap. Another aspect of the addition of impurities seems more problematic at first. The number of extended states in the system depends on the number and nature of the impurities and therefore it would seem that the conductance would also depend on these. However an argument of Laughlin's [41] which was later refined by Halperin [42], shows that the contribution of each band of extended states to the conductance is actually independent of the number of states in that band.

2.2.2 The fractional effect

The explanation of the integer quantum Hall effect which we have so sparsely sketched above does not provide an understanding of the fractional quantum Hall effect; there seems no reason why there should be a gap or a mobility gap at fractional ν. In order to understand the fractional effect, one has to take the interactions between the electrons into account. Crucial stepping stones in the theory of the fractional effect were Laughlin's variational wave functions for a system of N electrons on a disc [43]. In terms of the complex coordinates z_k for the electrons, the ground state wave functions he proposed are

$$\Psi_N^m(z_1,\ldots,z_N)=\prod_{i<j}(z_i-z_j)^{2m+1}\,e^{-\left(\frac{1}{4}\sum_i z_i\bar{z}_i\right)} \tag{2.5}$$

where m is an integer. One may arrive at these wave functions in the following way. First, one restricts to the space of functions of the Jastrow form:

$$\Psi(z_1,\ldots,z_N)=\prod_{i<j}f(z_i-z_j). \tag{2.6}$$

[1] Note that the "location" of a plateau is much less accurately determined than the conductance at the plateau, so that when people speak of "the plateau at $\nu=\frac{1}{3}$", this is a reference to the value of the conductance, rather than to the filling fraction.

The choice of this form for the variational wave function is really where the repulsive interactions between the electrons are included; any f with $f(0) = 0$ will tend to keep the particles apart. After the assumption of the Jastrow form, the wave functions (2.5) are determined by three physical requirements.

1. The wave function must be totally antisymmetric, since the electrons are fermions. Hence f must be odd.

2. In order to minimize energy, the wave function must be built up from single electron wave functions in the lowest Landau level. That is, it must be holomorphic up to a factor of $e^{-z\bar{z}/4}$ for each electron. This requirement is reasonable if the energy scale for the interaction is small compared to $\hbar\omega_c$

3. The ground state must be an eigenstate of angular momentum. This means that the holomorphic function multiplying the Gaussian factors must be a homogeneous polynomial in the z_k. Since angular momentum commutes with the Hamiltonian, this condition is certainly satisfied if the ground state is non-degenerate (i.e. if there is a gap).

The wave functions (2.5) are the only wave functions of the Jastrow form which satisfy these three requirements. Therefore, this argumentation predicts a discrete series of ground states, corresponding to different filling fractions.

By employing a plasma analogy, that is, by reinterpreting the probability density for the wave function Ψ_N^m as the Boltzmann weight for a plasma of mutually repelling particles of charge m Laughlin found that Ψ_m represents a liquid state of constant density at filling fraction $\nu = \frac{1}{2m+1}$. For small numbers of electrons (~ 10), one may also, by numerical methods, check that Ψ_N^m has very good overlap with the exact ground state of the system at $\nu = \frac{1}{2m+1}$. A simple way to find the filling fraction straight from the expression (2.5) is the following. Since the electrons fill the sample, the highest occupied single particle angular momentum state will always be the highest state in the first Landau level. On the other hand, we may read off the maximal angular momentum for a single particle from (2.5); it is just the maximal power of any single z_k, which is $(2m + 1)(N - 1)$. This means that the first Landau level contains $\sim (2m+1)N$ states, while there are only N electrons and hence the filling fraction is $\nu = \frac{1}{2m+1}$.

The Hall system at $\nu = \frac{1}{2m+1}$ has gapped quasihole and quasiparticle excitations, which carry a single flux quantum and which have charge $\frac{\pm e}{2m+1}$. This fractional charge has been confirmed by shot noise measurements in [44, 45, 46], but was already expected much earlier on theoretical grounds. We will concentrate on the quasiholes. A trial wave function for the system with n quasihole excitations at locations $w_1, \ldots, w_n$ is obtained from (2.5) by adiabatically inserting a flux quantum at each of these positions. This leads to the expression

$$\Psi_{N,n}^m = \Psi_N^m \prod_{i=1}^{n} \prod_{j=1}^{N} (z_j - w_i). \tag{2.7}$$

We see that the electrons are all kept away from the quasiholes by the factors $z_j - w_i$ and in fact, one may show that, at the locations w_i, there are dips in the electron density of typical size given by the magnetic length. One way to see that the quasiholes must have charge $\frac{\pm e}{2m+1}$ is to note that, if $2m + 1$ quasiholes are inserted at the same location w, then the fluid has a hole the size of an electron at w, which at constant positive background charge density corresponds to a charge $+e$. The quasiholes also have braid statistics. When two quasiholes are taken to each other's position, this leads to a factor of $e^{i\pi/(2m+1)}$ in the wave function, as may be shown by a Berry phase calculation [47]. At this point, there does not seem to be direct experimental evidence for or against the braiding properties of the quasiholes, but all models of the quantum Hall effect predict them and they are generally held to be correct.

Since the advent of the Laughlin states, much progress has been made in the theory of the fractional quantum Hall effect. We cannot hope to give a fair representation of this here, but we will mention some salient points. An important step forward, both conceptually and in terms of explaining observed phenomena, was the introduction of composite fermions by Jain [48]. The idea is basically that the interaction between electrons can be taken into account by assuming that each electron "grabs" an even number of flux quanta, which subsequently become "invisible" to the other electrons. The composite fermions which are constructed this way can then be viewed as free charged particles, which once again fill up Landau levels, but in a reduced (or enhanced) external magnetic field. Adding disorder, we obtain the usual picture of the integer Hall effect, but now at non-integer filling fraction. To calculate the filling fractions which may arise in this way, assume we have N_e electrons and N_f flux quanta, that is $B = N_f \frac{hc}{e}$. Then $\nu = |N_e/N_f|$. However, the composite fermions see only $N_f \pm 2mN_e$ flux quanta, since $\pm 2mN_e$ flux quanta have been "grabbed". Thus, the composite fermions have an effective filling fraction $\nu^* = |N_e/(N_f \pm 2mN_e)|$. This is assumed to be an integer. Expressing ν in terms of ν^* and m, one gets

$$\nu = \frac{\nu^*}{2m\nu^* \pm 1}. \tag{2.8}$$

The plus sign is obtained when the grabbed flux quanta are parallel to the external magnetic field. Note that the filling fractions $\nu = \frac{1}{2m+1}$ are reproduced for $\nu^* = 1$, i.e. one filled Landau level of composite fermions. Also note that the denominators of all the obtained filling fractions are odd. A different series of filling fractions (also with odd denominators) is obtained through a scheme proposed by Haldane and Halperin [49, 50] in which successive Hall plateaus are built up by the condensation of quasihole or quasiparticle excitations into a Laughlin-like state. The methods of chapter 3 could, after sufficient generalization, be used to study such quasihole or quasiparticle liquids. Unfortunately for us, however, it seems that the Haldane-Halperin hierarchy scheme is not very relevant to experimental situations, since even for the most stable observed fractions, it can take many "layers" of quasihole liquids to reproduce the right filling fractions. The Jain hierarchy on the other hand reproduces the most stable states for low values of ν^* and m.

Another important feat in the theory of quantum Hall systems was the construction of field theories which describe the states at the plateaus of the Jain hierarchy (early references are [51, 52]). Such a construction starts from a field theory description of the integer effect at filling ν^* and then replaces the gauge field $\mathbf{A}$ which represents the external magnetic field by the sum of $\mathbf{A}$ and a new gauge field $\mathbf{a}$, whose dynamics are governed by a Chern-Simons term,

$$\mathcal{L} = \frac{1}{2p}\frac{1}{4\pi}\epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho. \tag{2.9}$$

The $\mathbf{a}$-field couples to the electron's field through the covariant derivative $D_\mu = \partial_\mu - e(A_\mu + a_\mu)$. Since the Chern-Simons term is topological (does not depend on the metric), the $\mathbf{a}$-field does not represent any propagating degrees of freedom (no new particles are introduced). Nevertheless, the coupling of $\mathbf{a}$ to the electron's field has an important effect. It forces the densities of electric charge and magnetic ($\nabla \times \mathbf{a}$) flux to be proportional, in such a way that each electron grabs $2p$ flux quanta. This way, Jain's picture of composite fermions is implemented. The field theories constructed this way reproduce all features of fractional Hall states that we have described in this section at the semiclassical level. In the mean time, more complicated field theories, which describe plateaus at different filling fractions, have also appeared (refs ?). However, the Lagrangians of all these theories feature generalizations of the Chern-Simons term presented above. One may even argue that Chern-Simons terms should always appear in the description of the low-energy degrees of freedom of quantum Hall plateaus, since they are the most relevant terms (at large scales) that one can write down for a gauge field in 2+1 dimensions.

2.2.3 CFT and trial wave functions

The Laughlin-Jain picture of the fractional quantum Hall effect explains many of the plateaus observed at fractional ν, but not all. Most notably, two plateaus observed at $\nu = \frac{5}{2}$ and $\nu = \frac{7}{2}$ [24, 53] do not fit into the hierarchy, because their filling factors have even denominator. A logical first step in the study of the states of matter at these and other "exotic" plateaus is the construction of trial wave functions for the ground state and for the states with localized bulk excitations. In a seminal paper [18], Moore and Read argued that such trial wave functions could be conveniently constructed using conformal field theory correlators. The basic recipe is as follows. Take a conformal field theory with chiral primary fields ϕ_i and associate one of these to the electron and another to the quasihole. Now write down the following correlator

$$\tilde{\Psi}_{N,n} := \left\langle \phi_{bg}(z_\infty) \prod_{i=1}^{n} \phi_{qh}(w_i) \prod_{j=1}^{N} \phi_e(z_j) \right\rangle . \tag{2.10}$$

Here, ϕ_{qh} is the chiral field associated to the quasihole and ϕ_e is the chiral field associated to the electron. The field ϕ_{bg} represents a positive background charge which is needed to make the correlator non-vanishing and which is conveniently inserted at infinity. The conformal blocks of this correlator, multiplied by the usual Gaussian factors $e^{-z_j \bar{z}_j/4}$, are trial wave functions for a system with N electrons with complex coordinates z_j, which has n quasihole excitations inserted at positions w_i[2]. One may also consider states with several types of quasiholes and electrons, for example spin up and spin down electrons. To find trial wave functions for such states, one simply introduces a field for each type of electron or quasihole.

Let us give a simple argument as to why it is reasonable to construct trial wave functions in this way. Remember that Laughlin's ground state wave functions followed uniquely from four requirements: It should be of the Jastrow form, totally antisymmetric in the electron coordinates, an eigenstate of angular momentum and built up from lowest Landau level wave functions. If we want to find more general trial wave functions, it makes most sense to relax only the first of these requirements; we will no longer to require the wave function to be of the Jastrow form. Nevertheless, we still want it to keep pairs of particles well separated, thus implementing the repulsive interactions. The conformal blocks above automatically have this property, if the operator that represents the electron is chosen appropriately. In fact, from the operator product expansion (1.43), we see that the blocks will behave as

$$(z_i - z_j)^{h_f - 2h_e} \tag{2.11}$$

[2] In the original scheme proposed by Moore and Read, the background charge was not located at infinity, but homogeneously spread over "spacetime" (the sample). This had the advantage that the Gaussian factors could be absorbed in the correlator, but it is inconvenient for calculational purposes.

when two electrons approach one another. Here h_e is the conformal weight of the electron operator and h_f is the conformal weight of the fusion product of two electron operators. If the electron operator is chosen so that $h_f > 2h_e$, then we see that the electrons are indeed kept apart. Now let us look at the other three requirements on the ground state wave function.

1. To make the wave function totally antisymmetric, the difference $h_f - 2h_e$ must be an odd integer, fixing the eigenvalue for electron exchange to -1. If the electron operator is also chosen to be a simple current, that is, a field whose fusion with any primary is once again a single primary, then the spaces of conformal blocks for the correlators with only electron operators (no quasiholes) are all one-dimensional and we get a single fully antisymmetric wave function for any given number of electrons.

2. The requirement that the wave function is built up from single electron wave functions in the lowest Landau level is automatically satisfied, since the conformal blocks are by definition holomorphic functions.

3. It follows from the conformal Ward identity (see for example [15], section 5.2.2) that the wave function is also an eigenstate of angular momentum.

The last point needs some clarification. The angular momentum operator M is given by

$$M = \sum_i z_i \partial_{z_i} - \bar{z}_i \partial_{\bar{z}_i} \tag{2.12}$$

and it follows from the fact that $\tilde{\Psi}_{N,0}$ is holomorphic in the z_j that we have

$$M\tilde{\Psi}_{N,0} e^{-\sum_k z_k \bar{z}_k} = \left(\sum_i z_i \partial_{z_i} \tilde{\Psi}_{N,0} \right) e^{-\sum_k z_k \bar{z}_k}. \tag{2.13}$$

On the other hand, one of the conformal Ward identities for $\tilde{\Psi}_{N,0}$ is

$$\left(z_\infty \partial_{z_\infty} + \sum_i z_i \partial_{z_i} \right) \tilde{\Psi}_{N,0} = -(h_{bg} + N h_e) \tilde{\Psi}_{N,0}, \tag{2.14}$$

where h_{bg} and h_e are the conformal weights of the operators ϕ_{bg} and ϕ_e. Now we know that the N fields ϕ_e at the locations z_j fuse together to the conjugate ϕ^c_{bg} of ϕ_{bg} when we bring the z_j together in one point z_0. We are then left with a two point correlator, which is fixed by conformal invariance,

$$\left\langle \phi_{bg}(z_\infty) \phi^c_{bg}(z_0) \right\rangle \sim (z_\infty - z_0)^{-2h_{bg}}. \tag{2.15}$$

This also describes the behavior of $\tilde{\Psi}_{N,0}$ as a function of z_∞ when we take z_∞ to infinity and confine the z_j to the sample. Hence, on the sample, $\tilde{\Psi}_{N,0}$ is an eigenfunction of $z_\infty \partial_{z_\infty}$ with eigenvalue $-2h_{bg}$. But this implies that it is also an eigenfunction of $\sum_i z_i \partial_{z_i}$ with eigenvalue $h_{bg} - N h_e$. It follows that

$$M\tilde{\Psi}_{N,0} e^{-\sum_k z_k \bar{z}_k} = (h_{bg} - N h_e) \tilde{\Psi}_{N,0} e^{-\sum_k z_k \bar{z}_k}. \tag{2.16}$$

In view of the supposed repulsive nature of the interactions between the electrons, excitations over the ground state given by $\Psi_{N,0}$ should correspond to localized dips (or peaks) in the electron density. The insertion of operators ϕ_{qh} at points w_i is a nice way of creating such dips, since it conserves three out of the four properties we required of the ground state; the electrons are still well-separated and the wave function is still totally antisymmetric in the electron's coordinates and holomorphic (up to the Gaussians). Unsurprisingly, states with localized excitations created in this way are typically not eigenstates of the total angular momentum. From the operator product expansion 1.43, we read off that, when an electron coordinate z_i approaches the location w_j of a quasihole, $\tilde{\Psi}_{N,n}$ has the following behavior:

$$\tilde{\Psi}_{N,n}(z_i) \sim (z_i - w_j)^{h_f - h_e - h_{qh}}. \tag{2.17}$$

Here, h_f denotes the conformal weight of the fusion product of ϕ_e and ψ_{qh} and if we want the wave function to be single valued in the electron's coordinates then it is clear that we should choose ϕ_{qh} so that $h_f - h_e - h_{qh}$ is an integer (we say that ϕ_e and ϕ_{qh} are mutually local). Moreover, if there is to be a dip in the electron density at w_j then this integer should be positive and if the energy associated with this dip should be as small as possible then it is logical to require

$$h_f - h_e - h_{qh} = 1. \tag{2.18}$$

We have given a completely elementary motivation for the use of conformal field theory in the construction of trial wave functions and at the same time found some requirements on the CFTs that can be used for this purpose. For more information on such requirements, one may see for instance [54]. Clearly, the arguments in this section can also be applied to systems of bosonic particles, such as the rotating Bose-Einstein condensates of [29, 28, 30]. For such systems, one should of course require the wave function to be totally symmetric, rather than antisymmetric. The motivation for the use of CFT that we have given here is quite different from the original motivation given in [18]. There, the starting point was a deep connection, found by Witten [55], between conformal field theory an Chern-Simons theory. Witten showed that the Hilbert space of a Chern-Simons theory defined on a Riemann surface Σ with n punctures (plus time) can be identified with the space of conformal blocks associated to a CFT-correlator of n fields inserted at these punctures. The punctures in the CS-theory may be interpreted as the worldlines of particles moving through the Chern-Simons medium and the idea is that fusion and braiding of these particles corresponds to fusion and braiding of vertex operators in the associated CFT. Assuming that each Hall plateau has a description in terms of CS-theory, it is thus natural to conjecture that the wave functions for the electrons at any plateau may be obtained as the conformal blocks of correlators in some CFT.

As an example of the use of CFT, we reconstruct Laughlin's wave functions $\Psi^m_{N,n}$. Consider the theory of a chiral boson on a circle of radius $\sqrt{2m+1}$. This is a CFT, whose chiral primary fields may be written $e^{ip\phi/\sqrt{2m+1}}$, where ϕ is the field that describes the boson. The conformal weight of the field $e^{ip\phi/\sqrt{2m+1}}$ is $\frac{p^2}{2(2m+1)}$ and the fusion is given by

$$e^{ip\phi/\sqrt{2m+1}} \times e^{iq\phi/\sqrt{2m+1}} = e^{i(p+q)\phi/\sqrt{2m+1}}. \tag{2.19}$$

Now we associate the operator $e^{i\sqrt{2m+1}\phi}$ with the electron and the operator $e^{i\phi/\sqrt{2m+1}}$ with the quasihole and we calculate

$$\begin{aligned} \lim_{z_\infty \to \infty} \left\langle e^{-i((2m+1)N+n)\phi/\sqrt{2m+1}}(z_\infty) \prod_{i=1}^n e^{i\phi/\sqrt{2m+1}}(w_i) \prod_{j=1}^N e^{i\sqrt{2m+1}\phi}(z_j) \right\rangle \\ \sim \prod_{i<j}(z_i - z_j)^{2m+1} \prod_{i,j}(z_j - w_i) \prod_{i<j}(w_i - w_j)^{1/(2m+1)}, \end{aligned} \tag{2.20}$$

reproducing the Laughlin wave functions. The factors $(w_i - w_j)^{1/(2m+1)}$ above are in principle just constants which can be absorbed in the normalization, but as they stand, they conveniently reproduce the statistics of the quasiholes by analytic continuation. Similarly, any trial wave function obtained from a CFT in the way we have described comes with a Braid group representation defined by analytic continuation. It is stressed by Nayak and Wilczek [17] that the braiding of quasiholes should in principle always be obtained from a Berry phase calculation. However, with these authors, we will assume that the braiding that is given by analytic continuation coincides with the braiding which would be obtained from such a calculation[3].

An interesting possibility, suggested in [18], is to construct trial states using a quasihole operator ϕ_{qh} for which the fusion $\phi_{qh} \times \phi_{qh}$ has multiple channels. In this case, the spaces of conformal blocks corresponding to the correlators $\tilde{\Psi}_{N,n}$ will increase in dimension as quasiholes are added, introducing the possibility of non-Abelian braiding between the quasiholes. Using the operators σ and ψ of the Ising model (see section 1.5), Moore and Read constructed a trial wave function which is now the leading candidate for the description of the plateau at $\nu = \frac{5}{2}$ [25] and which does indeed exhibit non-Abelian braiding [17]. This state is now called the Pfaffian or Moore-Read state and it is the simplest of the Read-Rezayi series of states, which is described in section 2.3.

2.3 The CFT description of the Read-Rezayi states

2.3.1 The Parafermionic CFT

The Read-Rezayi states are constructed using a conformal field theory in the way we have described in section 2.2.3. The CFT in question is the tensor product of the theory of a chiral boson on a circle with the $\mathbb{Z}_k$-parafermionic theory of Zamolodchikov and Fateev [57, 58]. Before we write down any explicit expression for the RR-states, we recall some well known facts about the parafermionic CFT. The $\mathbb{Z}_k$-parafermionic CFT has central charge $c = \frac{2(k-1)}{k+2}$ and may be described completely in terms of a chiral algebra generated by the modes of k parafermionic currents (see [57, 58] and also [59] for some more recent work in this vein). For $k = 2$, the central charge is $\frac{1}{2}$, the parafermions are just ordinary fermions and we have the Ising model (cf. section 1.5). For general k, the theory has two different coset descriptions and it is these descriptions that we will use here. The cosets involved are $\widehat{sl(2)}_k/\widehat{U(1)}_k$ and $\widehat{sl(k)}_1 \times \widehat{sl(k)}_1/\widehat{sl(k)}_2$. The first of these descriptions was used extensively already in [57, 58], to determine fusion rules, characters and partition functions for the parafermions. The treatment of the parafermions in most of the literature on the RR-states has been influenced by this description. The second coset was introduced by Bais, Bouwknegt, Surridge and Schoutens in [60, 61] and used in [62] to construct a Coulomb gas representation of the theory which led to alternative character formulae [63]. This coset description has recently also been used in the work of Cappelli, Georgiev and Todorov on the RR-states [34]. In the rest of this section, we will give a quick description of both pictures and indicate how they are connected.

The coset $\widehat{sl(2)}_k/\widehat{U(1)}_k$.

For the coset $\widehat{sl(2)}_k/\widehat{U(1)}_k$, we have more information about the primary fields than usual. In particular, it is known that one can decompose certain fields of the parent $\widehat{sl(2)}$ WZW-theory as

[3] It seems to be difficult to check this equality for braidings that involve more than two particles. Results for two-particle braidings are given in [56].

a product of a coset primary field and a $U(1)$ primary field (see formula (2.26) below). In order to describe this decomposition, it is convenient to start with a short description of the fields and fusion rules of the $\widehat{sl(2)}$ theory before moving on to the parafermions (for much more detail on WZW-theories, see for example [15]). Note that when we speak of primary fields in the sequel, we will always mean chiral primary fields.

Recall that the spectrum generating algebra of the $\widehat{sl(2)}_k$ model is the affine Lie algebra $\widehat{sl(2)}$ at level k. The Virasoro algebra is embedded in the enveloping algebra of the affine algebra through the Sugawara construction.When discussing primary fields of the $\widehat{sl(2)}_k$ model, we need to distinguish between primary fields of the affine algebra (affine primaries) and primary fields of the Virasoro algebra (Virasoro primaries). Each affine primary field is necessarily also a Virasoro primary, but not vice versa. In fact, one can always find infinitely many Virasoro primaries among the affine descendants of an affine primary.

Let us be more explicit. If θ is the highest root of a simple Lie algebra g, then the affine primaries of the $\hat{g}_k$-model are labeled by the dominant integral weights Λ of g for which $(\Lambda, \theta) \leq k$ For $g = sl(2)$ this just means $0 \leq \Lambda \leq k$. We will call the $\widehat{sl(2)}_k$ primary fields G^Λ. The conformal dimension h_Λ of G^Λ is given by

$$h_\Lambda = \frac{\Lambda(\Lambda+2)}{4(k+2)} \tag{2.21}$$

The fusion rules of the G^Λ are

$$G^\Lambda \times G^{\Lambda'} = \bigoplus_{\Lambda''=|\Lambda-\Lambda'|}^{\min\{\Lambda+\Lambda', 2k-\Lambda-\Lambda'\}} G^{\Lambda''}, \tag{2.22}$$

There is an affine descendant field of G^Λ for each of the states in the $\widehat{sl(2)}$ module with highest weight Λ. Among these descendants, there are infinitely many Virasoro primaries, which we may name G^Λ_λ. The field G^Λ_λ is by definition the field of lowest conformal dimension among the affine descendants of G^Λ which carry $sl(2)$-weight λ. Naturally, we have $G^\Lambda = G^\Lambda_\Lambda$. Also, we have to demand that $(\Lambda - \lambda) = 0 \pmod 2$, otherwise the weight λ will not appear in the representation with highest weight Λ. One may check (see for instance [15]) that all the G^Λ_λ. defined this way are indeed Virasoro primary. Their conformal weights are given by

$$h^\Lambda_\lambda = \frac{\Lambda(\Lambda+2)}{4(k+2)} + n_{\Lambda,\lambda} \tag{2.23}$$

where $n_{\Lambda,\lambda}$ is the lowest grade at which the weight λ appears in the affine Lie algebra representation of highest weight Λ. If λ is a weight in the (ordinary) Lie algebra representation of highest weight Λ, then $n_{\Lambda,\lambda}$ will be zero and we will have $h_\Lambda = h^\Lambda_\lambda$. The fusion rules of the G^Λ_λ are easily obtained from (2.22) and the sum rule for weights in operator products. They are

$$G^\Lambda_\lambda \times G^{\Lambda'}_{\lambda'} = \bigoplus_{\Lambda''=|\Lambda-\Lambda'|}^{\min\{\Lambda+\Lambda', 2k-\Lambda-\Lambda'\}} G^{\Lambda''}_{\lambda+\lambda'}, \tag{2.24}$$

Now we turn to the $\mathbb{Z}_k$-parafermionic theory, as described by the coset $\widehat{sl(2)}_k/\widehat{U(1)}_k$. As usual for cosets, the Virasoro primary fields of the parafermion CFT may be labeled by a highest weight Λ of the horizontal algebra of the parent theory ($sl(2)$) and by a similar weight $\mathcal{P}\lambda$ of the embedded theory ($U(1)$), which is obtained by a projection matrix $\mathcal{P}$ from a weight λ of the parent theory. These weights moreover have to satisfy a branching condition, which ensures that the representation $\mathcal{P}\lambda$ of the embedded algebra can occur as a summand in the decomposition of the representation Λ of the parent algebra into representations of the embedded algebra. If we denote by M the root lattice of the horizontal algebra of the parent algebra, then this branching condition is

$$\mathcal{P}\Lambda - \mathcal{P}\lambda \in \mathcal{P}M. \tag{2.25}$$

In the case of $\widehat{sl(2)}_k/\widehat{U(1)}_k$, the projection matrix is trivial and the branching rule just says that the difference of the weights Λ and λ has to be an element of the root lattice of $sl(2)$ i.e. the difference of Λ and λ has to be an even number. Thus, the parafermion theory has Virasoro primaries Φ_λ^Λ labeled by a highest weight $0 \leq \Lambda \leq k$ of $sl(2)$ and a weight λ of $sl(2)$ for which we have $\Lambda - \lambda = 0 \pmod 2$.

Since the parafermion fields Φ_λ^Λ are now labeled in the same way as the Virasoro primary fields G_λ^Λ of the $\widehat{sl(2)}_k$ theory, one might hope that there is a simple relation between these fields. In fact, it was pointed out already in [57] that each of the fields G_λ^Λ may be written as the product of a field Φ_λ^Λ from the parafermion theory and a vertex operator of the $\widehat{U(1)}_k$ theory, which is just the theory of a free boson on a circle of radius $\sqrt{2k}$. This was further clarified in [58], using the results of [64]. One has

$$G_\lambda^\Lambda = \Phi_\lambda^\Lambda e^{i\lambda\phi/\sqrt{2k}} \tag{2.26}$$

From this relation, one immediately reads off that the field Φ_λ^Λ must have conformal weight $(h')_\lambda^\Lambda$ given by

$$(h')_\lambda^\Lambda = h_\lambda^\Lambda - \frac{\lambda^2}{4k} = \frac{\Lambda(\Lambda+2)}{4(k+2)} - \frac{\lambda^2}{4k} + n_{\Lambda,\lambda} \tag{2.27}$$

As in other coset theories, the labeling of the fields Φ_λ^Λ as we introduced it above is redundant. First of all, the $U(1)$ label λ is usually taken to be defined modulo $2k$, since the (extended) $\widehat{U(1)}_k$ characters χ_λ and $\chi_{\lambda+2k}$, that correspond to the vertex operators $e^{i\lambda\phi/\sqrt{2k}}$ and $e^{i(\lambda+2k)\phi/\sqrt{2k}}$, are equal (see for example [15]). Because of this and because of the fusion rules (2.29) below, the label λ is called the $\mathbb{Z}_{2k}$ charge of the field Φ_λ^Λ. [4] Also, in order to get proper behavior of the fields' characters under modular transformations, one has to identify fields whose labels are sent onto each other by an external automorphism of the parent algebra [65]. In the case at hand, this means that we have to identify Φ_λ^Λ with $\Phi_{\lambda-k}^{k-\Lambda}$. Collecting, we get the field identifications

$$\begin{aligned} \Phi_\lambda^\Lambda &\equiv \Phi_{\lambda+2k}^\Lambda \\ \Phi_\lambda^\Lambda &\equiv \Phi_{\lambda-k}^{k-\Lambda}. \end{aligned} \tag{2.28}$$

Using these identifications, we can choose a labeling of the primaries such that λ is a weight in the representation Λ of $sl(2)$, i.e. $-\Lambda \leq \lambda \leq \Lambda$. In fact, we may require $-\Lambda < \lambda \leq \Lambda$ and if we do this then every set of labels corresponds uniquely to a Virasoro primary. Thus, the number of Virasoro primaries is $\frac{1}{2}k(k+1)$ (note: there are only k primaries of the full parafermion algebra: the fields Φ_Λ^Λ).

[4] Note that in the original parafermion theory of [57], there was a $\mathbb{Z}_k \times \tilde{\mathbb{Z}}_k$ symmetry. The $\mathbb{Z}_k \times \tilde{\mathbb{Z}}_k$ charge $(l, \tilde{l})$ of the field $\Phi_\lambda^\Lambda(z)\Phi_{\bar{\lambda}}^{\bar{\Lambda}}(\bar{z})$ was given by $l = \frac{1}{2}(\Lambda + \bar{\Lambda}), \tilde{l} = \frac{1}{2}(\Lambda - \bar{\Lambda})$, so that clearly in this theory, one needed $\Lambda + \bar{\Lambda}$ to be even. Here, we will not require this and thus allow chiral fields like Φ_1^1.

One may check that the conformal weights given in (2.27) are equal for identified fields. Also, note that the grade $n_{\Lambda,\lambda}$ in (2.27) is zero if the labels (Λ, λ) are in the range chosen above. Using the factorization (2.26) and the field identifications, we may now also write down the fusion rules for the parafermion fields. They are

$$\Phi_{\lambda}^{\Lambda} \times \Phi_{\lambda'}^{\Lambda'} = \bigoplus_{\Lambda''=|\Lambda-\Lambda'|}^{\min\{\Lambda+\Lambda',2k-\Lambda-\Lambda'\}} \Phi_{\lambda+\lambda'}^{\Lambda''}, \tag{2.29}$$

In other words, they are the same as the fusion rules for the G_{λ}^{Λ}, except that the labels on the right hand side have to be brought back into the set chosen above, using the field identifications (2.28).

The coset $\widehat{sl(k)}_1 \times \widehat{sl(k)}_1/\widehat{sl(k)}_2$

The coset $\widehat{sl(k)}_1 \times \widehat{sl(k)}_1/\widehat{sl(k)}_2$ is a special case of the general class considered in [60, 61]. Its current algebra is a so called W-algebra and much is known about such algebras. In the quantum Hall application however, the parafermion analysis seems to be more directly relevant and applicable [34]. Nevertheless we expect that our discussion of the braid group representations that feature in the parafermionic models (see section 2.6), will readily extend to all the W-theories.

The Virasoro primaries of $\widehat{sl(k)}_1 \times \widehat{sl(k)}_1/\widehat{sl(k)}_2$. may be labeled by an $\widehat{sl(k)}_1 \times \widehat{sl(k)}_1$ weight (or, equivalently, two $\widehat{sl(k)}_1$ weights) and an $\widehat{sl(k)}_2$ weight. Let us call the $\widehat{sl(k)}_1$ weights μ_1 and μ_2 and the $\widehat{sl(k)}_2$ weight μ, then we can write $\Phi_{\mu}^{\mu_1,\mu_2}$. The weights μ_1, μ_2 and μ once again have to satisfy the branching condition (2.25). In this case, the projection $\mathcal{P}$ maps (μ_1, μ_2) onto $\mu_1 + \mu_2$ and it maps the root lattice of $sl(k) \times sl(k)$ onto the root lattice of $sl(k)$. Hence we have the following requirement

$$\mu_1 + \mu_2 - \mu \in M_{sl(k)} \tag{2.30}$$

where $M_{sl(k)}$ is the root lattice of $sl(k)$. In other words, the weights $\mu_1 + \mu_2$ and μ should be in the same conjugacy class (for details on this concept see for example [66, 15]). In terms of the Dynkin labels of the weights, this means that one has

$$\sum_{j=1}^{k-1} j(\mu_1^{(j)} + \mu_2^{(j)} - \mu^{(j)}) = 0 \bmod \mathrm{k} \tag{2.31}$$

Now denote by e_i the $sl(k)$ weight whose Dynkin labels $e_i^{(j)}$ are given by $e_i^{(j)} = \delta_{ij}$ (These correspond to the fundamental representations of $sl(k)$). Then μ_1 is either zero or equal to one of the e_i, since it is a level one weight. The same goes for μ_2. For the level two weight μ, there are three possibilities. It can be zero, equal to one of the e_i or equal to the sum of two of the e_i (which may be the same). If we define $e_0 = 0$, then we may simplify this description and say that μ_1 and μ_2 will equal one of the e_i and μ will equal the sum of two of the e_i (where $i \in \{0, \ldots, k-1\}$) The branching rule above then states that only triples (μ_1, μ_2, μ_3) of the form $(e_l, e_{m+n-l \bmod \mathrm{k}}, e_m + e_n)$ are admissible. This leaves $\frac{1}{2}k^2(k+1)$ admissible triples. However, there are also field identifications, induced by the external automorphisms of $\widehat{sl(k)}_1 \times \widehat{sl(k)}_1$. These identifications take the form

$$\Phi_{e_l+e_m}^{e_i,e_j} \equiv \Phi_{e_{l+s}+e_{m+s}}^{e_{i+s},e_{j+s}} \tag{2.32}$$

for $s \in \{1, \ldots, k-1\}$. The sums in the indices on the right hand side have to be taken modulo k. Using these identifications, we can choose to set either μ_1 or μ_2 to zero. Say we set μ_1 to zero. Then we are left with the triples $(0, e_{m+n \text{ modk}}, e_m + e_n)$. Clearly, μ_2 is now uniquely determined by μ and we may choose to label the fields by the $\widehat{sl(k)}_2$ weight only: Φ_μ. Every $\widehat{sl(k)}_2$ weight is admissible and we are left with as many Virasoro primary fields as there are $\widehat{sl(k)}_2$ weights: $\frac{1}{2}k(k+1)$. This is just a reduction of the number of fields before identification by a factor of k, as was to be expected. Also, we get the same number of fields that we got in the other coset description of the parafermionic CFT.

The fractional part of the conformal weight of the field $\Phi_\mu^{\mu_1,\mu_2}$ can be calculated directly from the coset description; it is the same as the fractional part of the difference between the conformal weight of the field with labels (μ_1, μ_2) in the parent theory and the conformal weight of the field with label μ in the embedded theory. One may show that this recipe always yields the same fractional part, independently of the labels μ_1, μ_2, μ that are chosen to represent a certain field (i.e. labels that are identified through (2.32) yield the same fractional part). Let us look at the field $\Phi_{e_m+e_n}$, with $m \leq n$. A particularly convenient choice of labels for this field, made in [34], is $(e_{k-n}, e_m, e_{k+m-n})$. The conformal dimension of the WZW-field labeled by the weight e_m is given by

$$h_p(e_m) = \frac{(e_m, e_m + 2\rho)}{2(p+k)} = \frac{m(k-m)(k+1)}{2k(k+p)}, \tag{2.33}$$

where ρ is the Weyl-vector of $sl(k)$ and p is the level (here, we have p=1 or p=2). From this, we find

$$\begin{aligned} h_1(e_{k-n}) + h_1(e_m) - h_2(e_{k+m-n}) &= \frac{m(k-n)}{k} + \frac{(n-m)(k+m-n)}{2k(k+2)} \\ &= \frac{(k+m-n)(k+m-n+2)}{4(k+2)} - \frac{(m+n-k)^2}{4k} \end{aligned} \tag{2.34}$$

The middle expression is the one given in [34] and from the last expression, we see that it is equal to the weight of the field Φ_λ^Λ with $\Lambda = k+m-n$ and $\lambda = m+n-k$ (cf. formula (2.27)). Thus, we have the correspondence

$$\Phi_{m+n-k}^{k+m-n} \equiv \Phi_{e_m+e_n} \Longleftrightarrow \Phi_\lambda^\Lambda \equiv \Phi_{e_{\frac{\Lambda+\lambda}{2}} + e_{\frac{2k-\Lambda+\lambda}{2}}} \tag{2.35}$$

which is further supported by the fact that these fields have the same fusion rules. [5] In fact, the Φ_μ fusion rules are the same as the fusion rules for the corresponding $\widehat{sl(k)}_2$ representations and these are the same as the fusion rules of the $\widehat{sl(2)}_k/\widehat{U(1)}_k$ coset as a consequence of level-rank duality (see [15] and references therein). One may also find the equality of the fusion rules directly by looking at the fusion rules of the field $\Phi_1^1 \equiv \Phi_{e_1}$ with an arbitrary field. These fusion rules are easily seen to be the same and since the field Φ_1^1 generates all the fields in the theory by repeated fusion, it follows that the fusion rules of all the fields that are identified through (2.35) are the same in both cosets.

[5] Note that we could also identify the field $\Phi_{e_m+e_n}$ with the field Φ_{k-m-n}^{k+m-n}, which is the conjugate of the field Φ_{m+n-k}^{k+m-n}. It is impossible to decide between these identifications on the level of conformal weights and fusion rules.

2.3.2 Definition of the Read-Rezayi states

The CFT which is used to define the Rezayi states is the tensor product of the parafermionic CFT and the theory of a chiral boson which is also used in the reconstruction of the Laughlin states (see section 2.2.3). The chiral primary fields of this tensor product theory are just products of a primary of the parafermionic theory and a primary of the bosonic theory. Let us give the operators corresponding to the electron and the quasihole. The electron operator is the product of the operator $\Phi_2^0 = \Phi_{2e_1}$ from the parafermionic theory with the operator $e^{i\sqrt{\frac{kM+2}{k}}\zeta}$ from the bosonic theory. Here M is an odd integer and we have denoted the bosonic field by ζ, to avoid confusion with the bosonic field ϕ in the factorization formula (2.26). Similarly, the quasihole operator is the product of the operator $\Phi_1^1 = \Phi_{e_1}$ with the bosonic vertex operator $e^{\frac{i\zeta}{\sqrt{k(kM+2)}}}$. When $k = 2$, the electron and quasihole operators are just the operators ψ and σ (respectively) from the Ising model (cf. section 1.5). Extending this notation to general k, we may write

$$\begin{aligned} \text{electron} &\equiv \psi e^{i\sqrt{\frac{kM+2}{k}}\zeta} \\ \text{quasihole} &\equiv \sigma e^{\frac{i\zeta}{\sqrt{k(kM+2)}}} \end{aligned} \tag{2.36}$$

These combinations of bosonic and parafermionic fields satisfy all the requirements given in section 2.2.3. In fact, if the parafermionic factors are given, then the bosonic factors are fixed by these requirements. The bosonic factor for the electron follows by requiring that electrons are mutually local (that is, the OPE of two electron operators does not have a branch cut). This makes sure that the wave functions defined below are single valued in the electrons' coordinates. The extra requirement that M must be odd is needed to make the wave function antisymmetric in the electrons' coordinates. The exponent of the bosonic factor for the quasihole is fixed up to integer times $\sqrt{\frac{k}{kM+2}}\zeta$ by the requirement that the quasihole and the electron are mutually local. It is fixed uniquely if we require (2.18)

The linear space of RR-states $\Psi^k_{N,n}$ which have N electrons with coordinates $z_1, \dots z_N$ and n quasiholes located at positions $w_1, \dots w_n$ is now generated by the conformal blocks of a correlator of N electron fields and n quasihole fields inserted at these positions and supplemented by a positive background charge, which ensures overall charge neutrality [16]. This correlator may be factorized into parafermionic and bosonic correlators, the latter of which may be evaluated explicitly, after which one obtains

$$\begin{aligned} \Psi^k_{N,n}(z_1,\dots,z_N,w_1,\dots w_n) &= \langle \sigma(w_1)\dots\sigma(w_n)\psi(z_1)\dots\psi(z_N)\rangle \\ &\times \prod_{i<j}(z_i-z_j)^{M+2/k}\prod_{i=1}^{N}\prod_{j=1}^{n}(z_i-w_j)^{1/k} \\ &\times \prod_{i<j}(w_i-w_j)^{\frac{1}{k(kM+2)}}F_g(z_1,\dots,z_N,w_1,\dots w_n). \end{aligned} \tag{2.37}$$

Here, the z_i and w_i are complex coordinates which parametrize the sample. F_g is a factor which depends on the geometry of the sample. If the sample is a disc, then this factor just implements the usual Gaussian factors which confine the electrons to the disc[6].

[6] As before, it depends on the treatment of the background charge if the factor F_g comes directly from the conformal block. If one treats the background charge the way we did in section 2.2.3, then the factor F_g has to be added by hand.

The filling fractions of the Read-Rezayi states may be read off as the quotient of the highest factor of any single z_j by N, in the limit of large N. Noting that the contribution of the parafermionic factor is negligible for large N, one finds $\nu = \frac{k}{kM+2}$. For $k = 2$ and $M = 1$, the above wave function reduces to the Pfaffian or Moore-Read state [18] with N electrons and n quasiholes. This state has $\nu = \frac{1}{2}$. Adding two completely filled Landau levels, one with spin up and one with spin down electrons, we arrive at a filling fraction of $\frac{5}{2}$, which is the experimentally relevant value.

2.3.3 Fusion of quasiholes and the Bratteli diagram

It is interesting to know the number of independent states which the formula (2.37) encodes, i.e. the number of independent states with N electrons that have n quasiholes at fixed positions $w_1, \ldots, w_n$. This interest is twofold. First of all, we want to know which combinations (N, n) are allowed. Second, the number of independent states is also the dimension of the braid group representation that governs the exchanges of electrons and quasiholes. Hence a necessary condition for non-Abelian braiding is that it be larger than one. A basis for the space of states that we are looking for is given by the states we obtain if we replace the parafermion correlator in (2.37) by its respective conformal blocks. The number of such blocks is equal to to the number of fusion channels that make the correlator in (2.37) non-vanishing. Hence, the number we are looking for is just the number of ways in which N electron fields ψ and n quasihole σ fields may fuse into the vacuum.

Now the fusion of the ψ fields is very simple; it just corresponds to addition of the $\mathbb{Z}_{2k}$ charges. Hence the N electron fields fuse into the $\Phi^0_{2N} = \Phi_{2e_N}$ sector. The fusion rules of the sigma fields, as given in equation (2.29), are a bit more complicated, but they have a nice graphical description in terms of a Bratteli diagram (see figures 2.1,2.2):

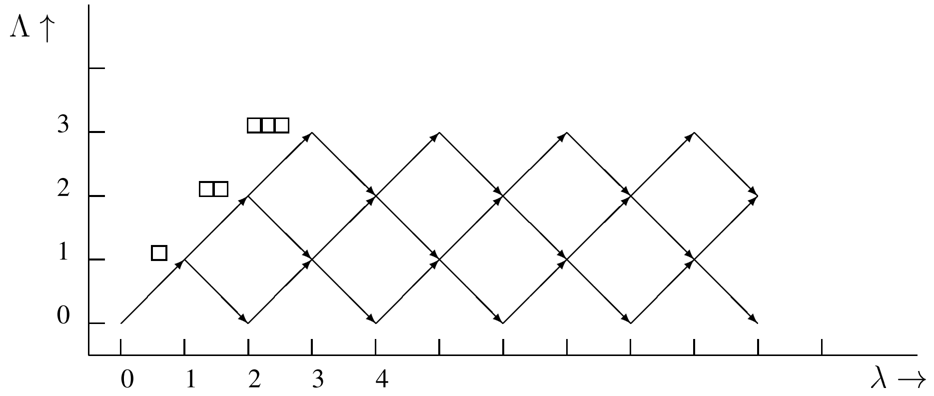

Figure 2.1: fusion diagram for the field σ. The diagram must be thought extended indefinitely in the λ-direction and up to $\Lambda = k$ in the Λ-direction (the case $k = 3$ is as drawn here). On each line, we have drawn the Young diagram of the $sl(2)$ representation that resides on that line.

These diagrams must be read as follows. Each starting point or end point of an arrow has coordinates (Λ, λ) and represents the Φ^Λ_λ sector of the parafermion CFT. Note that this means that coordinates related by the identifications (2.28) represent the same sector. In figure 2.2, one may see this explicitly for $k = 3$. Here, we have at each node of the diagram inserted the Young diagram for the $\widehat{sl(3)}_2$ weight of the field which resides there. The correspondence between the

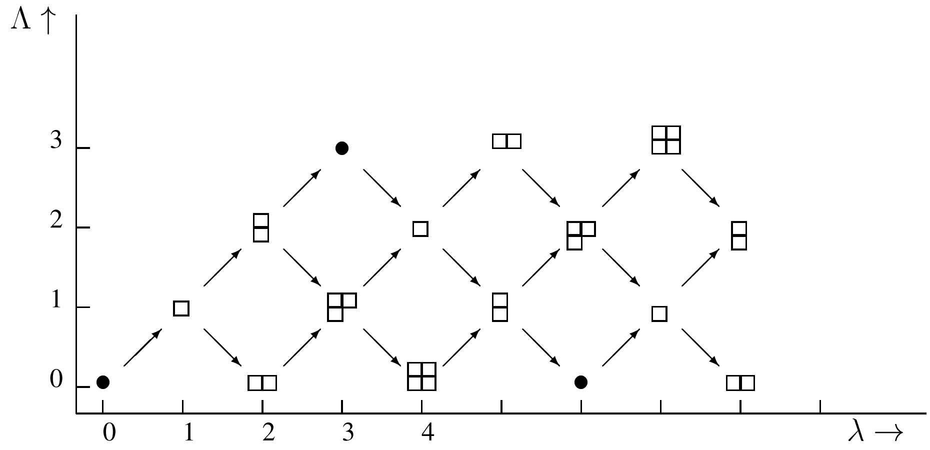

Figure 2.2: The same diagram as in figure 2.1, but this time each site in the diagram is labeled by the Young diagram for the $\widehat{sl(k=3)}_2$ weight of the field that resides there. The dot represents the empty diagram. Again, generalization to arbitrary k is straightforward. Note that in this picture, the weights label the fields unambiguously, whereas in figure 2.1, one still has to take the field identifications (2.28) into account

fields of the parafermionic theory and such weights or diagrams is one to one and we see that the same diagram appears in different places. The fusion rules of the sigma field are encoded in the arrows; we start in the lower left corner, that is, in the Φ_0^0 sector, which is the vacuum sector of the theory. Then we take the operator product expansion with the field $\sigma = \Phi_1^1$, which naturally, following the arrow, lands us in the Φ_1^1 sector. Once more taking the OPE with σ, we end up, following the arrows, in the Φ_2^2 or in the Φ_2^0 sector. In this way, each path of length n through the diagram represents a fusion channel for n σ-fields.

To make the parafermionic correlator in the wave function (2.37) non-vanishing, the parafermionic parts of all the quasihole and electron fields need to fuse into the vacuum sector. Now since the electron fields $\psi(z_1), \ldots, \psi(z_N)$ in the correlator fuse to Φ_{2N}^0, it follows that the quasihole fields $\sigma(w_1), \ldots, \sigma(z_n)$ have to fuse to the field $\Phi^0{}_{-2N} = \Phi_{k-2N}^k$. The number of ways to do this is just the number of paths of length n through the diagram of figure 2.1 which end up at a point whose coordinates (Λ, λ) satisfy either $(\Lambda, \lambda) = (0, -2N \bmod 2\mathrm{k})$ or $(\Lambda, \lambda) = (k, k - 2N \bmod 2\mathrm{k})$. Clearly, for fixed N, such paths occur only for values of n which are a multiple of k apart, so quasiholes can only be created in multiples of k at a time (maybe with the exception of the first few quasiholes if N is not a multiple of k). Note that, although the same fields (or sectors) occur at different heights in the diagram, the same field never occurs more than once at given λ and hence different paths are never identified by the field identifications. Thus, the number of fusion channels for the parafermion CFTs is the same as that for the corresponding WZW-theories.

2.3.4 Counting the independent n-quasihole states

Let us denote the number of paths through the Bratteli diagram which end up at the point (Λ, n) by $D(\Lambda, n)$. Also, let us define $D(\Lambda, n) = 0$ if there is no point with coordinates (Λ, n). The number of independent n-quasihole states encoded by (2.37) is then $D(0, n)$ in case $2N + n = 0 \pmod{2k}$, $D(k, n)$ in case $2N + n = k \pmod{2k}$, and zero otherwise. It should be obvious from looking at the Bratteli diagram that the $D(\Lambda, n)$ satisfy the following recursion relation:

$$D(\Lambda, n) = D(\Lambda - 1, n - 1) + D(\Lambda + 1, n - 1). \tag{2.38}$$

Using this relation and the fact that $D(1,1)$ equals one, $D(\Lambda,n)$ can be easily calculated in each particular case. At least for low k, the recursion relation can also be used to prove simple closed expressions for the $D(\Lambda,n)$. In particular, we find for $k=2, k=3$ and $k=4$

$$\begin{aligned}
D_2(0,2n) &= D_2(1,2n-1) = 2^{n-1}\\
D_3(0,2n) &= D_3(1,2n-1) = \mathrm{Fib}(2n-2)\\
D_3(2,2n) &= D_3(3,2n+1) = \mathrm{Fib}(2n-1)\\
D_4(0,2n) &= D_4(1,2n-1) = \frac{3^{n-1}+1}{2}\\
D_4(2,2n) &= 3^{n-1}\\
D_4(3,2n+1) &= D_4(4,2n+2) = \frac{3^{n-1}-1}{2}.
\end{aligned} \tag{2.39}$$

In these equations, we have written D_k instead of D for clarity and we have used the notation $\mathrm{Fib}(n)$ to denote the n^{th} Fibonacci number, defined by

$$\begin{aligned}
\mathrm{Fib}(0) &= \mathrm{Fib}(1) = 1\\
\mathrm{Fib}(n+1) &= \mathrm{Fib}(n) + \mathrm{Fib}(n-1).
\end{aligned} \tag{2.40}$$

It is also not that difficult to find and prove a closed formula for infinite k. We have

$$D_\infty(\Lambda,n) = \frac{\Lambda+1}{n+1}\begin{pmatrix} n+1 \\ \frac{n-\Lambda}{2}\end{pmatrix} \quad (n+\Lambda = 0 \pmod 2) \tag{2.41}$$

Of course this formula is valid for all k as long as $n+\Lambda \leq 2k$.

To get formulae for other values of k it is more convenient to rewrite the recursion relation (2.38) in matrix form. We consider the $D(\Lambda,n)$ at a fixed n together as a k-vector and write the step from n to $n+1$ as multiplication with a $(k+1)\times(k+1)$ matrix M_k. that is, we have

$$\begin{pmatrix} D(0,n+1) \\ \vdots \\ D(k,n+1)\end{pmatrix} = M_k \begin{pmatrix} D(0,n) \\ \vdots \\ D(k,n)\end{pmatrix} \tag{2.42}$$

where M_k is given by

$$(M_k)_{ij} = \delta_{i,j+1} + \delta_{i+1,j} \tag{2.43}$$

The asymptotic behavior of the $D(\Lambda,n)$ for large n will be related to the largest eigenvalue of the matrix M_k. The eigenvalues of the M_i are just the zeros of their characteristic polynomials P_k. For these, we can easily deduce a recursion relation and "initial conditions:"

$$\begin{aligned}
P_2(\lambda) &= \lambda^2 - 1\\
P_3(\lambda) &= \lambda^3 - 2\lambda\\
P_{i+1}(\lambda) &= \lambda P_i(\lambda) - P_{i-1}(\lambda)
\end{aligned} \tag{2.44}$$

but these are just the defining relations for the Chebyshev polynomials, whose zeros are given by (see for example [67])

$$\lambda_{k,m} = 2\cos\left(\frac{(m+1)\pi}{k+2}\right) \tag{2.45}$$

Since we know all the eigenvalues of M_k, we can now in principle solve for the eigenvectors and using the solution, give explicit formulae for the $D_k(\Lambda, n)$ for any k. We will however content ourselves with giving the asymptotic behavior of the $D_k(\Lambda, n)$ at large n. The largest eigenvalues (in absolute value) of the matrix M_k are clearly λ_0 and $\lambda_k = -\lambda_0$. Hence, the asymptotic behavior of the $D_k(\Lambda, n)$ is given by

$$\begin{aligned} D_k(\Lambda, n) &\sim \left(2\cos\left(\frac{\pi}{k+2}\right)\right)^n \quad (\Lambda + n \text{ even}). \\ D_k(\Lambda, n) &= 0 \quad (\Lambda + n \text{ odd}). \end{aligned} \tag{2.46}$$

This conforms with the closed formulae we gave for $k = 2, 3, 4$.

2.3.5 Braiding for $k = 2$

In the previous section, we have calculated the dimensions of the braid group representations that govern the exchanges of the electrons and the quasiholes of the RR-states. We have seen that these dimensions increase with the number of quasiholes, which is an indication for non-Abelian braiding. However, this indication is not conclusive evidence. To be sure, one needs to calculate the actual matrices that describe the braiding of the σ-fields in the conformal block in formula (2.37) above. Nayak and Wilczek [17] have done this calculation for the case $k = 2$ (the Pfaffian state). The method they used was basically to compute the conformal block for four quasihole fields explicitly and then to extend the resulting braid group representation to a braid group representation for any even number of quasiholes[7]. For general k, it is quite difficult to calculate conformal blocks for four, let alone for arbitrary numbers of quasiholes. Fortunately it turns out that we can circumvent this problem by using the known duality between conformal field theory and quantum groups and using this, we will give a nice description of the braiding for arbitrary k. However, we will first briefly recall the results of Nayak and Wilczek for $k = 2$, for later reference.

The braid group representation for $n = 2m$ quasiholes has dimension 2^{m-1} (cf. (2.39)). Nayak and Wilczek describe this space as a subspace of a tensor product of m two dimensional spaces. Each of the two dimensional spaces has basis vectors $\{|+\rangle, |-\rangle\}$ and the physical subspace of the tensor product is the space generated by the vectors whose overall sign is positive (so for $m = 2$, $|--\rangle$ is physical, but $|+-\rangle$ is not). On the tensor product space, there is a spinor representation of $SO(2m) \times U(1)$. The $U(1)$ acts as a multiplicative factor, while the generators σ_{ij} of the $SO(2m)$ may be written in terms of the Pauli matrices σ_i. We have

$$\sigma_{ij} = \frac{1}{4} i[\gamma_i, \gamma_j] \tag{2.47}$$

with

$$\begin{aligned} \gamma_1 &= \sigma_1 \otimes \sigma_3 \otimes \ldots \otimes \sigma_3 \\ \gamma_2 &= \sigma_2 \otimes \sigma_3 \otimes \ldots \otimes \sigma_3 \\ \gamma_3 &= 1 \otimes \sigma_1 \otimes \sigma_3 \otimes \ldots \otimes \sigma_3 \\ \gamma_4 &= 1 \otimes \sigma_2 \otimes \sigma_3 \otimes \ldots \otimes \sigma_3 \\ &\vdots \\ \gamma_{2m} &= 1 \otimes \ldots \otimes 1 \otimes \sigma_2 \end{aligned} \tag{2.48}$$

[7] Note that the four point blocks in the case $k = 2$ are just the four point blocks for the chiral Ising model, which have, within a different context, been known for a long time (see for instance [68] for explicit expressions). The same is true for the corresponding braid group representations. However, the embedding of the resulting braid group representation into a rotation group, as given by Nayak and Wilczek (see below) seems to be new.

Here the states $|+\rangle$ and $|-\rangle$ are the spin up and spin down states for the Pauli matrices.

Now let τ_i represent the exchange of quasihole i and quasihole $i+1$, then the action of the braid group (cf. (1.11)) on the n-quasihole space is embedded in the action of $SO(2n)\times U(1)$ as follows:

$$\tau_i \equiv e^{i\frac{\pi}{4}} e^{i\frac{\pi}{2}\sigma_{i,i+1}} \tag{2.49}$$

The $SO(2m)$ generators $\sigma_{i,i+1}$ which appear in this equation are given by

$$\begin{aligned} \sigma_{1,2} &= \tfrac{1}{2}\,\sigma_3 \otimes 1 \otimes \ldots \otimes 1 \\ \sigma_{2,3} &= \tfrac{1}{2}\,\sigma_2 \otimes \sigma_2 \otimes 1 \otimes \ldots \otimes 1 \\ \sigma_{3,4} &= \tfrac{1}{2}\,1 \otimes \sigma_3 \otimes 1 \otimes \ldots \otimes 1 \\ \sigma_{4,5} &= \tfrac{1}{2}\,1 \otimes \sigma_2 \otimes \sigma_2 \otimes 1 \otimes \ldots \otimes 1, \text{ etc.} \end{aligned} \tag{2.50}$$

So we see that, for odd i, τ_i acts only on the i^{th} tensor factor, whereas for even i, τ_i acts only on the $(i-1)^{\text{th}}$ and i^{th} tensor factors. Moreover, the 2×2-matrix which describes the action for even i and the 4×4-matrix which describes it for odd i do not vary with i. Explicitly, they are given by

$$\tau_{2i+1} \equiv \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \quad \tau_{2i} \equiv \frac{1}{2}\begin{pmatrix} 1+i & 0 & 0 & -1+i \\ 0 & 1+i & 1-i & 0 \\ 0 & 1-i & 1+i & 0 \\ -1+i & 0 & 0 & 1+i \end{pmatrix} \tag{2.51}$$

2.4 The quantum group picture

In this extensive section, we give a description of the braiding for a system of n particles with a hidden quantum group symmetry. We expect that the braiding properties of a quantum Hall state with n quasiholes are conveniently described in terms of such a system. In the first subsection, we motivate the quantum group theoretic approach and mention some general features. In the remaining subsections, we work these ideas out in detail for the quantum group $U_q(sl(2))$. In particular, we give a fairly detailed description of the relevant representation theory of $U_q(sl(2))$ for q a root of unity, which culminates in an explicit description of the associated braid group representations. We are well aware of the fact that most of the material treated in this section is not new, but since it came from quite a variety of sources, it seemed useful to give a self-contained treatment here.

2.4.1 Using a quantum group rather than the full CFT

One may always describe a quantum system in terms of its explicit wave functions, but it can be extremely profitable to exploit its operator algebra, in particular its symmetries. These allow one to extract many of the physical features without reference to the explicit realization in terms of wave functions. Quite similarly one could in the present context remark that there is an aspect of the description of the Read-Rezayi states that is less than satisfactory: one has to use the full machinery of a (conformal) field theory to calculate wave functions or even just braiding properties for a finite number of quasiholes and electrons. There are many questions one may want to answer for which this seems like overkill: for example one would hope to be able to describe the braiding of finitely many particles by means of a theory with only finitely many degrees of freedom. Indeed, there is such an alternative description and we pursue it here. It is well known that conformal field theories possess a hidden quantum group symmetry (see section 2.5 for details and references). What we propose is to describe the electrons and quasiholes of a quantum Hall state that would usually be described by a certain CFT as localized particles that carry representations of the quantum group that is associated with this CFT. Such a description has several advantages.

- It avoids the introduction of a field theory to describe a system with only a finite number of particle degrees of freedom.

- It provides a conceptual understanding of a phenomenon which emerges in the usual CFT description. This is the fact that, while a state with a low number of indistinguishable quasiholes can be described with a one component wave function, a system with a higher number of these quasiholes may need a wave function with several components. Clearly, it should only be possible to distinguish between these components by making a measurement that involves several holes (otherwise the holes would not be indistinguishable). Hence, there should be operators in the many hole Hilbert space that distinguish states that cannot be distinguished by operators that act only on the state of one of the particles. The quantum group picture provides these in a natural way. They are the operators that correspond to the global quantum group charges of groups of quasiholes. Even though all individual quasiholes have the same quantum group charge, a group of n such holes can occur in different representations leading to distinguishable n-hole states. As a simple example, suppose that the quasiholes carried the two dimensional representation of $SU(2)$ (or of $U(sl(2))$). In that case a two quasihole state could be either in the singlet or in the triplet representation and the singlet states could be distinguished from the triplet states by measuring the global charge.

- The quantum group picture allows for an elegant description of the braiding properties of the n-quasihole states; all braiding properties are encoded into a single algebraic object: the quantum group's R-matrix (cf. section 1.4.3). Starting from the R-matrix, braiding calculations can be done in a purely algebraic way and often a detailed picture of the braid group representation that governs the exchanges of particles can be constructed. In a CFT description, the information contained in the R-matrix of the quantum group would be much less manifest. In fact, to extract it from this description of the system, one would have to calculate the braiding and fusion matrices starting from the conformal blocks of the CFT, which is usually quite hard.

Of course the description we propose also has its disadvantages when compared to the CFT description. For instance, it seems much harder to describe dynamical aspects of the quantum Hall states in this framework. Still, we like to emphasize that the quantum group picture we propose is a useful complementary way of thinking about non-Abelian quantum Hall states.

2.4.2 Return to $U_q(sl(2))$; representations at roots of unity

In section 1.4, we have given an introduction to the representation theory of the quantum group $U_q(sl(2))$ for the case that q is not a root of unity. When q is a root of unity, the properties of most of the representations defined by the formulae (1.28) in section 1.4 change quite drastically. Specifically, at $q = e^{2i\pi/(k+2)}$, the representations π^{2j} with $j > \frac{k+1}{2}$ will no longer be irreducible. This can be traced back to the fact that for $q = e^{2\pi i/(k+2)}$, one has the identity

$$\lfloor k+2 \rfloor_q = 0. \tag{2.52}$$

Because of this, $(L^+)^{k+2}$ and $(L^-)^{k+2}$ are mapped to zero in all the representations defined by (1.28). Of course, in the representations with $j < \frac{k+2}{2}$, this was already the case and for these representations, nothing essential changes. In particular, they are still irreducible. However, in the representations with $j \geq \frac{k+2}{2}$, there will now be extra highest and lowest weight states, which are annihilated by L^+ resp. L^-. For example, the state $(L^+)^{k+1}|\, j, -j\,\rangle$ in the module V^{2j} of the representation π^{2j}, $(j \geq \frac{k+2}{2})$ will now be an extra highest weight state, since $(L^+)^{k+2} = 0$ in this representation. The descendants of this highest weight state (that is, the states which can be obtained from it by applying powers of L^-) now span an invariant subspace W of V^{2j}, so π^{2j} is no longer irreducible. Figure 2.4.2 illustrates this situation in a simple case. Although the module π^{2j} is now reducible, it can not be written as a direct sum of irreducibles.

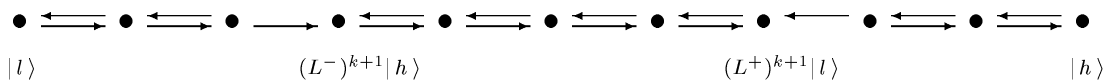

Figure 2.3: Diagram of an indecomposable representation as defined by (1.28). The dots represent the basis states $|\, j, m\,\rangle$, in particular, we have written $|\, h\,\rangle$ for the highest weight state and $|\, l\,\rangle$ for the lowest weight state. The arrows $\rightarrow$ and $\leftarrow$ indicate the action of L^+ and L^- resp.

One says that it is *indecomposable*. This indecomposability is directly related to the fact that π^{2j} is not a $*$-representation. For a $*$-representation, the orthogonal complement of an invariant submodule of the representation module is itself invariant and this guarantees that any finite dimensional representation has an orthogonal decomposition into irreducibles. The fact that π^{2j} does not have a decomposition into irreducibles shows that it is not just non unitary, but even non unitarisable. That is, it is impossible to choose an inner product such that π^{2j} is unitary with respect to it.

Summarizing, for $q = e^{2i\pi/(k+2)}$, we are left with only $k+2$ irreducibles out of the infinitude that we would usually get from (1.28)[8]. These are the unitary representations π^{2j} with $j < \frac{k+2}{2}$. The other representations defined by (1.28) are no longer irreducible. They have become indecomposable, and therefore they are non unitarisable.

2.4.3 Tensor products

Tensor product decomposition at roots of unity

In section 1.4.2, we described the tensor product of representations of $U_q(sl(2))$ for the case that q is not a root of unity. In that case, many of the usual properties of tensor products at $q = 1$ could be recovered. For example, the tensor product of two irreps could be decomposed into a direct sum of irreps (see (1.34)). When q is a root of unity, say $q = e^{i2\pi/(k+2)}$, the situation is quite different. In this case, tensor products of two irreps will not split into a direct sum of irreps, but will contain indecomposable summands. This is not in itself surprising, because the representations π^Λ with $\Lambda > k+1$ that would occur in the usual decomposition (1.34) become indecomposable for $q = e^{i2\pi/(k+2)}$. However, what really happens is a bit more complicated.

[8]Note that these irreps are by no means all the irreps at $q = e^{i2\pi/(k+2)}$. In fact, there are more irreps of dimensions $1, \ldots, k+1$ (how many more depends on the precise definition of $U_q(sl(2))$, see e.g. [7, 69]) and there is a family of inequivalent representations of dimension $k+2$, parameterized by a complex number z. However, these representations will not concern us here.

As an example, let us look at the decomposition of the tensor product of the spin $\frac{1}{2}$ and the spin $\frac{k+1}{2}$ module. As usual, the tensor product space may be decomposed into eigenspaces of the operator H. These eigenspaces will be one dimensional for the extremal eigenvalues $H = k+1$ and $H = -(k+1)$ and two dimensional for the other eigenvalues. If q were not a root of unity, then we would have two highest weight states in the tensor product module. The $H = k+1$ state $|\, \frac{k+1}{2}, \frac{k+1}{2}\, \rangle |\, \frac{1}{2}, \frac{1}{2}\, \rangle$ and the $H = k-1$ state $|\, \frac{k-1}{2}, \frac{k-1}{2}\, \rangle$ given in (1.36). At $q = e^{i2\pi/(k+2)}$, the coefficients of this second state diverge, but if we multiply the state by $\lfloor k+2 \rfloor_q$, then this no longer happens and we still have two good highest weight states. However, we have a third candidate highest weight state, which is the $H = k-1$ state one gets when one lets $(L^+)^{k+1}$ act on the lowest weight state $|\, \frac{k+1}{2}, -\frac{k+1}{2}\, \rangle |\, \frac{1}{2}, -\frac{1}{2}\, \rangle$ (remember $(L^+)^{k+2}$ gives zero for this value of q). This new highest weight state is just proportional to the state $|\, \frac{k+1}{2}, \frac{k-1}{2}\, \rangle$ given in (1.36). Comparing this state with the other highest weight state at $H = k-1$, we see that although they would be linearly independent for any arbitrary q, they are actually proportional to each other for $q = e^{2\pi i/(k+2)}$. It follows that the irreducible spin $\frac{k-1}{2}$-module has become a submodule of the module generated by the highest weight state at $H = k+1$. Also, since we have only one highest weight state in the $H = k-1$ eigenspace and since this space is two-dimensional, there must also be a non-highest weight state in this eigenspace. The two dimensional H-eigenspaces of the tensor product module will then be spanned by a descendant of the highest weight state at $H = k-1$ and a descendant of the non-highest weight state at $H = k-1$. We see thus that, at $q = e^{2\pi i/(k+2)}$, the modules π^{k+2} and π^k have disappeared from the decomposition of $\pi^1 \otimes \pi^{k+1}$ and instead there is one indecomposable module, which has the module π^k as an irreducible submodule.

This general picture extends to all tensor products of irreps; in general, all the modules π^Λ with $\Lambda > k$ and all the corresponding modules $\pi^{2k-\Lambda}$ will disappear from the decomposition (1.34) and instead, there will be indecomposable modules with the modules $\pi^{2k-\Lambda}$ as irreducible submodule. The structure of these indecomposable modules is analogous to the structure of the module we described above and is illustrated in figure 2.4.3. For more detail on tensor product decomposition when q is a root of unity, one can consult for example [70, 69, 7].

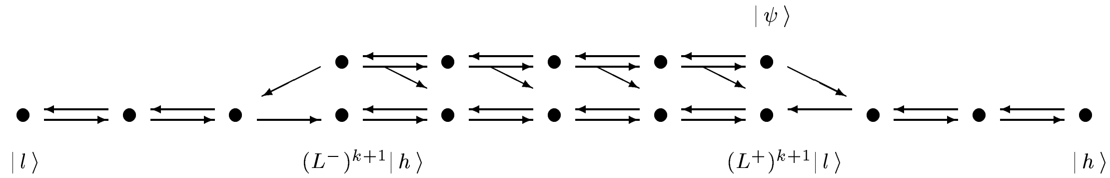

Figure 2.4: Diagram of an indecomposable representation which can occur in the tensor product of two $U_q(sl(2))$-irreps at $q = e^{2\pi i/(k+2)}$. The dots represent the basis states in the module, the arrows $\rightarrow$ and $\leftarrow$ indicate the action of L^+ and L^- resp. The split arrows are meant to indicate that the descendants of the state $|\, \psi \,\rangle$ are mapped onto linear combinations of descendants of $|\, \psi \,\rangle$ and $(L^+)^{k+1} |\, l \,\rangle$

Clearly, the indecomposable representations which occur in the tensor products are non unitarisable; this follows from the indecomposability, but one can also see easily that any "inner product" that would make these representations unitary would give the states in the irreducible submodule zero norm.

Truncated tensor products

The indecomposable representations that turn up in tensor product decompositions at roots of unity are non-physical. Thus, one needs to define a new "tensor product" $\hat{\otimes}$ in which the indecomposable modules are somehow projected out. However, one cannot just take the old tensor product and project out the indecomposable modules, since the the tensor product obtained in this way would not be associative. One would have for example $(\pi^1\hat{\otimes}\pi^k)\hat{\otimes}\pi^{k+1} = 2\pi^{k+1}$ and $\pi^1\hat{\otimes}(\pi^k\hat{\otimes}\pi^{k+1}) = \{0\}$ for odd k. (For even k, there are similar problems). Also, the fusion rule $\pi^\Lambda \otimes \pi^{k+1} = \{0\}$ (Λ even) is clearly unphysical; after adding a particle in the representation π^{k+1} we would be left with a zero-dimensional Hilbert space! These problems can be solved both at once by projecting out not just the indecomposable modules, but also any modules of type π^{k+1} that may occur. The resulting tensor product is called the *truncated tensor product.*

The truncated tensor product decomposition at $q = e^{i2\pi/(k+2)}$ is given by the following formula, which is identical to the formula (2.22) for the fusion rules of $\widehat{sl(2)}_k$ chiral primaries:

$$\pi^\Lambda\hat{\otimes}\pi^{\Lambda'} = \bigoplus_{\Lambda''=|\Lambda-\Lambda'|}^{\min\{\Lambda+\Lambda',2k-\Lambda-\Lambda'\}} \pi^{\Lambda''}. \tag{2.53}$$

From this formula, one may check easily that the truncated tensor product is indeed associative, that is, the tensor product modules $(\pi^{\Lambda_1}\hat{\otimes}\pi^{\Lambda_2})\hat{\otimes}\pi^{\Lambda_3}$ and $\pi^{\Lambda_1}\hat{\otimes}(\pi^{\Lambda_2}\hat{\otimes}\pi^{\Lambda_3})$ are isomorphic. Note however that these two modules are different subspaces of the ordinary tensor product, so we might say that the truncated tensor product is associative at the level of $U_q(sl(2))$-modules, but not associative at the level of states.

As an illustration, let us take a closer look at the truncated tensor product of the two-dimensional irrep π^1 with the unitary irreps $\pi^0, \pi^1, \ldots, \pi^k$. For this case, the truncated tensor product decomposition is given by

$$\begin{aligned} \pi^0\hat{\otimes}\pi^1 &= \pi^1 \\ \pi^\Lambda\hat{\otimes}\pi^1 &= \pi^{\Lambda+1} \oplus \pi^{\Lambda-1} \quad (\Lambda \in \{1, \ldots, k-1\}) \\ \pi^k\hat{\otimes}\pi^1 &= \pi^{k-1} \end{aligned} \tag{2.54}$$

As one can see, the only difference with the ordinary tensor product occurs in the last line. The decomposition on the level of states can be read off from (1.36). Using this formula, we can also give an example of the non-associativity at the level of states that we were talking about: At $k = 1$ (or $q = e^{2\pi i/3}$), the truncated tensor products $V_1 = (\pi^1\hat{\otimes}\pi^1)\hat{\otimes}\pi^1$ and $V_2 = \pi^1\hat{\otimes}(\pi^1\hat{\otimes}\pi^1)$ are both isomorphic to π^1 as $U_q(sl(2))$-modules, but any state in V_1 may be written as

$$\left(q^{-1/4}|\,\tfrac{1}{2},-\tfrac{1}{2}\,\rangle|\,\tfrac{1}{2},\tfrac{1}{2}\,\rangle - q^{1/4}|\,\tfrac{1}{2},\tfrac{1}{2}\,\rangle|\,\tfrac{1}{2},-\tfrac{1}{2}\,\rangle\right)\left(\alpha_1|\,\tfrac{1}{2},\tfrac{1}{2}\,\rangle + \alpha_2|\,\tfrac{1}{2},-\tfrac{1}{2}\,\rangle\right), \tag{2.55}$$

while any state in V_2 may be written as

$$\left(\beta_1|\,\tfrac{1}{2},\tfrac{1}{2}\,\rangle + \beta_2|\,\tfrac{1}{2},-\tfrac{1}{2}\,\rangle\right)\left(q^{-1/4}|\,\tfrac{1}{2},-\tfrac{1}{2}\,\rangle|\,\tfrac{1}{2},\tfrac{1}{2}\,\rangle - q^{1/4}|\,\tfrac{1}{2},\tfrac{1}{2}\,\rangle|\,\tfrac{1}{2},-\tfrac{1}{2}\,\rangle\right). \tag{2.56}$$

From this, we see that a vector in V_1 can only equal a vector in V_2 if it is zero. Hence, V_1 and V_2 are different subspaces of $\pi^1 \otimes \pi^1 \otimes \pi^1$.

The non-associativity of the truncated tensor product might seem like a problem at first sight, because we want to have a unique three-particle Hilbert space, but this problem disappears if we can find a canonical $U_q(sl(2))$-isomorphism between the two three-particle spaces which preserves the inner product. We will say more about this in section (2.4.6).

Before ending this section, let us write down two useful identities for truncated tensor decomposition which are related to the external automorphism of $\widehat{sl(2)}_k$ that we discussed in relation to the field identifications (2.28). If we define

$$\hat{\Lambda} := k - \Lambda, \tag{2.57}$$

then we have

$$\hat{\Lambda}\hat{\otimes}\Lambda' = \bigoplus_{\Lambda''=|\Lambda-\Lambda'|}^{\min\{\Lambda+\Lambda',2k-\Lambda-\Lambda'\}} \hat{\Lambda}'' \quad \text{and} \quad \hat{\Lambda}\hat{\otimes}\hat{\Lambda}' = \bigoplus_{\Lambda''=|\Lambda-\Lambda'|}^{\min\{\Lambda+\Lambda',2k-\Lambda-\Lambda'\}} \Lambda''. \tag{2.58}$$

Here, we have written Λ instead of π^Λ to avoid overloading the notation. These identities tell us that the truncated fusion rules of $U_q(sl(2))$ do not allow us to make a distinction between a particle that carries the representation Λ and a particle that carries the representation $\hat{\Lambda}$.

2.4.4 Quantum trace and quantum dimensions

Using the coproduct and the antipode, one may define the adjoint action of a quantum group $\mathcal{A}$ on the space of linear operators on an $\mathcal{A}$-module V by

$$(a \cdot \hat{O})|\, v\,\rangle = \sum a_{(1)}\hat{O}S(a_{(2)})|\, v\,\rangle. \tag{2.59}$$

Here we have used Sweedler notation for the coproduct. From the fact that S is an antihomomorphism, one can see that (2.59) defines a representation of $\mathcal{A}$, while using the property (1.3), one can see that $\mathcal{A}$ acts trivially on operators that commute with the action of $\mathcal{A}$ on V.

The action of $U_q(sl(2))$ on an operator $\hat{O}$ is given explicitly by

$$\begin{aligned} H \cdot \hat{O} &= [H, \hat{O}] \\ L^{\pm} \cdot \hat{O} &= L^{\pm}\hat{O}q^{-H/4} - q^{-(H\pm1)/4}\hat{O}L^{\pm}, \end{aligned} \tag{2.60}$$

which reduces to the usual commutator for $q \to 1$.

One can define a kind of trace on operators, which has the property that it transforms trivially under $U_q(sl(2))$ when the operator is transformed. For $q = 1$, the ordinary trace has this property, since $\mathrm{Tr}([a, \hat{O}]) = 0 = \epsilon(a)\mathrm{Tr}(\hat{O})$ for all $a \in sl(2)$ and for arbitrary $\hat{O}$. However, for $q \neq 1$, we have to use a modified trace to get this property. This trace is usually called the quantum trace and we will denote it Tr_q. Of course, the quantum trace is supposed to preserve some nice properties of the ordinary trace. Most importantly, the trace of a tensor product of operators should be the product of the traces of the tensor factors, that is

$$\mathrm{Tr}_q(\hat{O}_1 \otimes \hat{O}_2) = \mathrm{Tr}_q(\hat{O}_1)\mathrm{Tr}_q(\hat{O}_2). \tag{2.61}$$

A quantum trace with this property can be defined for a large class of quantum groups (see cf. [7]). For $U_q(sl(2))$, it is given by

$$\mathrm{Tr}_q(\hat{O}) = \mathrm{Tr}(q^{H/2}\hat{O}). \tag{2.62}$$

One may verify readily that $\mathrm{Tr}_q(a \cdot \hat{O}) = \epsilon(a)\mathrm{Tr}_q(\hat{O})$. The fact that (2.61) is satisfied follows from the comultiplication $\Delta(q^{H/2}) = q^{H/2} \otimes q^{H/2}$.

Using the quantum trace, one may define the quantum dimension $\dim_q(\pi)$ of a representation of $U_q(sl(2))$ as the quantum trace of the unit operator in that representation. For the representations π^Λ, this yields $\dim_q(\pi^\Lambda) = \lfloor \Lambda + 1 \rfloor_q = \lfloor \dim(\pi^\Lambda) \rfloor_q$. In particular, the quantum dimension of π^{k+1} is zero. The quantum dimensions of all the indecomposable modules of dimension $2k+4$ that appeared in the (untruncated) tensor products of the π^Λ are also zero, since these modules were a (non-direct) sum of two modules of dimensions $k+2-d$ and $k+2+d$ and we have $\lfloor k+2-d \rfloor_q + \lfloor k+2+d \rfloor_q = \lfloor d \rfloor_q + \lfloor -d \rfloor_q = 0$. Since the quantum dimensions of the modules $\pi^1, \ldots, \pi^k$ are non zero, we see that we might also have defined the truncated tensor product of two modules in this set as the ordinary tensor product with the modules of quantum dimension zero projected out. With this definition, the truncated tensor product is automatically associative and the module π^{k+1} does not need separate treatment.

The quantum dimensions of an irrep of a quantum group are not just useful in defining the truncated tensor product, they also have a physical meaning. The quantum dimension of an irrep can be seen as the effective number of internal degrees of freedom associated with a particle tat carries that irrep. More precisely, the dimension of the n-fold truncated tensor product of an irrep with quantum dimension d_q is proportional to $(d_q)^n$ at large n. In connection with this, one should note that the number $2\cos\left(\frac{\pi}{k+2}\right)$ which plays the same role for the number of n-quasihole states (cf. (2.46)) can be written as $\lfloor 2 \rfloor_q$, where $q = e^{2i\pi/(k+2)}$. Of course, quantum dimensions are usually not integers. This brings us back to a point mentioned in section 1.5, where we mentioned that the only "dimension" that could be associated the field σ of the Ising model was $\sqrt{2}$. Truncated tensor products allow for such non-integer dimensions and in fact, for $k = 2$, we have $\lfloor 2 \rfloor_q = \sqrt{2}$.

Quantum traces may also be used to construct knot invariants (see for example [71],[7] and references therein). For $U_q(sl(2))$, one of the knot invariants which can be constructed this way is the famous Jones polynomial [72].

2.4.5 Braidings for two particles

When we use the truncated tensor product, the process of braiding is a bit more complicated than in our discussion in section 1.4. The R-matrix (1.39) still describes the braiding of two particles[9], but if we go to three or more particles, then we can get problems. For example, three particles in the representation π^Λ may be described by a state in the truncated tensor product space $(V^\Lambda \hat{\otimes} V^\Lambda) \hat{\otimes} V^\Lambda$ and we can exchange the two leftmost particles by means of $\sigma(\pi^\Lambda \hat{\otimes} \pi^\Lambda)(R) \otimes 1$, which gives us a state in $(V^\Lambda \hat{\otimes} V^\Lambda) \hat{\otimes} V^\Lambda$, as it should. However, if we want to exchange the two rightmost particles, then we can leave the space $(V^\Lambda \hat{\otimes} V^\Lambda) \hat{\otimes} V^\Lambda$ if we just apply $1 \otimes \sigma(\pi^\Lambda \hat{\otimes} \pi^\Lambda)(R)$. One may see this explicitly in the example we gave in formula (2.55); exchanging the last two particles in this state by means of the exchange matrix given in (1.41), we get a state which can clearly not be written in the same form and hence does not belong to $(V^1 \hat{\otimes} V^1) \hat{\otimes} V^1$. If we use the other bracketing of the truncated tensor product (i.e. $V^\Lambda \hat{\otimes} (V^\Lambda \hat{\otimes} V^\Lambda)$), then we can exchange the last two particles in the expected way, but then the problem occurs in the exchange of the first two. In this way, we can always expect problems when we try to exchange two particles over a bracket. Thus, we will not get a representation of the braid group on the truncated tensor product, unless we modify the way in which we exchange particles. We will explain the modification that is needed in some detail in section 2.4.6. In the mean time, we give a description of the braidings for two particles.

Let us look at the braiding in a tensor product of two irreps π^{Λ_1} and π^{Λ_2}. We can decompose this tensor product into irreps as in equation (1.34) or (2.53). From these formulae, we see that any irrep can occur at most once in this decomposition; we say that the tensor product decomposition is multiplicity-free. It follows from this, using Schur's lemma, that any map from the

[9]Note that the R-matrix (1.39) is not well defined if $q = e^{i2\pi/(k+2)}$, since the q-factorial $\lfloor n \rfloor_q!$ which appears in the $(L^+)^n) \otimes (L^-)^n)$ term becomes zero for $n \geq k+2$. This problem can be resolved by adding the relations $(L^+)^{k+2} = (L^-)^{k+2} = 0$ to the algebra for this value of q. To us, this subtlety is not very important, since these relations already hold in the unitary representations we are interested in.

tensor product module $V^{\Lambda_1} \otimes V^{\Lambda_2}$ to the tensor product module $V^{\Lambda_2} \otimes V^{\Lambda_1}$ that commutes with the quantum group action on these modules, is a constant on each of the irreducible summands of $V^{\Lambda_1} \otimes V^{\Lambda_2}$. The exchange matrix $\sigma R \equiv \sigma(\pi^{\Lambda_1} \otimes \pi^{\Lambda_2})(R)$ is such a map. Hence, we can choose bases for $V^{\Lambda_1} \otimes V^{\Lambda_2}$ and $V^{\Lambda_2} \otimes V^{\Lambda_1}$ such that the action of σR is described by a diagonal matrix with respect to these bases. Of course, the basis vectors in each case are just the basis vectors $|\, \frac{\Lambda}{2}, m\,\rangle$ of each irreducible summand π^{Λ} and the action of σR on these will depend on Λ_1, Λ_2 and Λ and not on m. Explicitly, one has

$$\sigma(\pi^{\Lambda_1} \otimes \pi^{\Lambda_2})(R)|_{V^{\Lambda}} = (-1)^{\frac{\Lambda_1}{2} + \frac{\Lambda_2}{2} - \frac{\Lambda}{2}} q^{\frac{1}{2}(c_{\Lambda} - c_{\Lambda_1} - c_{\Lambda_2})} \tag{2.63}$$

where $c_{\Lambda_i} = \frac{\Lambda_i}{2}(\frac{\Lambda_i}{2} + 1)$ is the value of the undeformed Casimir for the representation π^{Λ_i}. This can be derived from the formula (1.40) for the elements of the R-matrix, using the formulae for the Clebsch-Gordan coefficients given in [12]. For the case $\Lambda_2 = 2$, one may also check it from (1.36), using (1.38). Note that the eigenvalues of σR are all roots of unity when q is a root of unity. Therefore, if we use the inner products on the tensor product spaces that makes the bases described above orthonormal, then σR is a unitary operator.

2.4.6 q-$6j$-symbols and their properties

In this section, we introduce $6j$-symbols and truncated $6j$-symbols for $U_q(sl(2))$. In the first subsection, we deal with the q-$6j$-symbols which are associated to the ordinary tensor product of $U_q(sl(2))$-irreps. In the second subsection, we restrict to the case where q is a root of unity and introduce the $6j$-symbols for the truncated tensor product. We also describe how these truncated $6j$-symbols allow one to deal with the non-associativity of the truncated tensor product.

$6j$-symbols for the ordinary tensor product

If we take a tensor product of three $U_q(sl(2))$ modules $\pi^{\Lambda_1}, \pi^{\Lambda_2}$ and π^{Λ_3}, then there are two different ways to decompose this tensor product into irreducibles. We may either first decompose the product $\pi^{\Lambda_1} \otimes \pi^{\Lambda_2}$ and then the resulting modules $\pi^{\Lambda'} \otimes \pi^{\Lambda_3}$, or we may first decompose the product $\pi^{\Lambda_2} \otimes \pi^{\Lambda_3}$ and then the resulting modules $\pi^{\Lambda_1} \otimes \pi^{\Lambda''}$. These two procedures yield two different natural bases for the vector space $V^{\Lambda_1} \otimes V^{\Lambda_2} \otimes V^{\Lambda_3}$. In each case, the basis vectors are labeled by their H-eigenvalue, the label of their overall fusion channel and the label of their intermediate fusion channel (which is the representation into which π^{Λ_1} and π^{Λ_2} fuse in the first case and the representation into which π^{Λ_2} and π^{Λ_3} fuse in the second case). Let us call the vectors in the first basis $e^{j_1,j_2,j_3}_{j_{12},j,m}$ and the vectors in the second basis $f^{j_1,j_2,j_3}_{j_{23},j',m'}$. Here, j_1, j_2 and j_3 correspond to Λ_1, Λ_2 and Λ_3, m and m' give the H-eigenvalues, j and j' give the overall fusion channels and j_{12} and j_{23} represent the intermediate fusion channels. The vectors $e^{j_1,j_2,j_3}_{j_{12},j,m}$ and $f^{j_1,j_2,j_3}_{j_{23},j',m'}$ may be written in terms of the standard (product) basis for the tensor product by means of the Clebsch-Gordan coefficients. We have

$$\begin{aligned} e^{j_1,j_2,j_3}_{j_{12},j,m} &= \sum_{m_1,m_2,m_3} \begin{bmatrix} j_1 & j_2 & j_{12} \\ m_1 & m_2 & m_{12} \end{bmatrix}_q \begin{bmatrix} j_{12} & j_3 & j \\ m_{12} & m_3 & m \end{bmatrix}_q |\, j_1, m_1\,\rangle |\, j_2, m_2\,\rangle |\, j_3, m_3\,\rangle \\ f^{j_1,j_2,j_3}_{j_{23},j,m} &= \sum_{m_1,m_2,m_3} \begin{bmatrix} j_2 & j_3 & j_{23} \\ m_2 & m_3 & m_{23} \end{bmatrix}_q \begin{bmatrix} j_1 & j_{23} & j \\ m_1 & m_{23} & m \end{bmatrix}_q |\, j_1, m_1\,\rangle |\, j_2, m_2\,\rangle |\, j_3, m_3\,\rangle \end{aligned} \tag{2.64}$$

where $m_{12} = m_1 + m_2$ and $m_{23} = m_2 + m_3$. The vectors in the e-basis may also be expressed in terms of the f-basis vectors and this expression takes the following form:

$$e^{j_1,j_2,j_3}_{j_{12},j,m} = \sum_{j_{23},j',m'} \delta_{jj'}\delta_{mm'} \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array} \right\} f^{j_1,j_2,j_3}_{j_{23},j',m'}. \tag{2.65}$$

The coefficients represented by the curly brackets are called the $6j$-symbols of $U_q(sl(2))$. By definition, these q-$6j$-symbols equal the $6j$-symbols for $SU(2)$ when q equals one. The $6j$-symbol in the formula above will clearly be zero unless the representation j_{12} occurs in the tensor product of the representations j_1 and j_2, the representation j occurs in the tensor product of the representations j_{12} and j_3, etcetera. It follows that the $6j$-symbol will be zero unless its arguments satisfy the following requirements:

$$\begin{array}{ll} |j_1 - j_2| \leq j_{12} \leq j_1 + j_2, & j_1 + j_2 + j_{12} \in \mathbb{Z} \\ |j_2 - j_3| \leq j_{23} \leq j_2 + j_3, & j_2 + j_3 + j_{23} \in \mathbb{Z} \\ |j_{12} - j_3| \leq j \leq j_{12} + j_3, & j_{12} + j_3 + j \in \mathbb{Z} \\ |j_1 - j_{23}| \leq j \leq j_1 + j_{23}, & j_1 + j_{23} + j \in \mathbb{Z}. \end{array} \tag{2.66}$$

If these requirements are met, then the $6j$-symbol may be written in terms of Clebsch-Gordan coefficients; using (2.64) and the relations (1.37), one easily finds that

$$\left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array} \right\} = \frac{\sum_{m_2,m_3} \left[\begin{array}{ccc} j_1 & j_2 & j_{12} \\ m_1 & m_2 & m_{12} \end{array} \right]_q \left[\begin{array}{ccc} j_{12} & j_3 & j \\ m_{12} & m_3 & m \end{array} \right]_q \left[\begin{array}{ccc} j_2 & j_3 & j_{23} \\ m_1 & m_2 & m_{23} \end{array} \right]_q}{\left[\begin{array}{ccc} j_1 & j_{23} & j \\ m_1 & m_{23} & m \end{array} \right]_q} \tag{2.67}$$

From this formula, one may obtain explicit formulae for the $6j$-symbols. We will not do the (long) computations here, but just give one of the possible explicit answers, as given in [12](see also [73]).

$$\begin{aligned} &\left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array} \right\} = \\ &\quad \sqrt{\lfloor 2j_{12}+1 \rfloor_q \lfloor 2j_{23}+1 \rfloor_q}\, \Delta(j_1,j_2,j_{12})\Delta(j_{12},j_3,j)\Delta(j_2,j_3,j_{23})\Delta(j_1,j_{23},j) \\ &\quad \times \sum_z \left\{ \frac{(-1)^z \lfloor z+1 \rfloor_q!}{\lfloor z-j_1-j_2-j_{12} \rfloor_q! \lfloor z-j_{12}-j_3-j \rfloor_q! \lfloor z-j_2-j_3-j_{23} \rfloor_q! \lfloor z-j_1-j_{23}-j \rfloor_q!} \right. \\ &\qquad \left. \times \frac{1}{\lfloor j_1+j_2+j_3+j-z \rfloor_q! \lfloor j_1+j_{12}+j_3+j_{23}-z \rfloor_q! \lfloor j_2+j_{12}+j+j_{23}-z \rfloor_q!} \right\}, \end{aligned} \tag{2.68}$$

where

$$\Delta(a,b,c) := \sqrt{\frac{\lfloor -a+b+c \rfloor_q! \lfloor a-b+c \rfloor_q! \lfloor a+b-c \rfloor_q!}{\lfloor a+b+c+1 \rfloor_q!}}. \tag{2.69}$$

The sum in (2.68) is taken over all z for which all the q-factorials in the summands are well-defined.

The q-$6j$-symbols are invariant under many symmetries (described in [12, 73]) which are analogues of the symmetries of the $6j$-symbols of $SU(2)$ (see [74]). For us, the most important of these are the so called classical symmetries. These symmetries can be treated slightly more elegantly if one works with the q-Racah coefficients instead of the q-$6j$-symbols. The Racah coefficients are just the $6j$-symbols with a different normalization; they are given by the formula for the $6j$-symbols above with the first square root factor left out. Invariance under the classical symmetries means that the Racah coefficients remain unchanged under permutations of the columns and under exchanging the upper and lower entry in two columns simultaneously. In effect, this means that we have the following identities for the $6j$-symbols

$$\begin{aligned} \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array} \right\} &= \left\{ \begin{array}{ccc} j_2 & j_1 & j_{12} \\ j & j_3 & j_{23} \end{array} \right\} = \sqrt{\tfrac{\lfloor 2j_{12}+1\rfloor_q \lfloor 2j_{23}+1\rfloor_q}{\lfloor 2j_2+1\rfloor_q \lfloor 2j+1\rfloor_q}} \left\{ \begin{array}{ccc} j_1 & j_{12} & j_2 \\ j_3 & j_{23} & j \end{array} \right\} \\ \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array} \right\} &= \left\{ \begin{array}{ccc} j_1 & j & j_{23} \\ j_3 & j_2 & j_{12} \end{array} \right\} \end{aligned} \tag{2.70}$$

and all the identities generated by these. The other symmetries of the $6j$-symbols are analogues of the Regge and reflection symmetries.

When $q \in \mathbb{R}_+$, the bases for the three-fold tensor product given in (2.64) are orthonormal and hence the basis transformation between these bases is unitary. As a consequence, the $6j$-symbols satisfy the following orthogonality relation (see cf. [12])

$$\sum_{j_{12}} \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{23} \end{array} \right\} \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j'_{23} \end{array} \right\} = \delta_{j_{23} j'_{23}}. \tag{2.71}$$

Here, we have used the fact that the $6j$-symbols are real for $q \in \mathbb{R}_+$. When q is not a positive real number, the above relation for the $6j$-symbols remains valid by analytic continuation, as long as the summands are not singular, but it does not tell us that the matrix for the basis transformation we mentioned is orthogonal unless all the $6j$-symbols that appear are real. For $|q| = 1$, these $6j$-symbols will be real as long as $|\arg(q)|$ is small enough to make sure that all the q-numbers that appear in these $6j$-symbols are positive. This will be the case(cf. formula(2.68)) when

$$|\arg(q)| < \min_{j_{12}} \{ \frac{2\pi}{j_1 + j_2 + j_{12} + 1}, \frac{2\pi}{j_2 + j_3 + j_{23} + 1}, \frac{2\pi}{j_{12} + j_3 + j_4 + 1}, \frac{2\pi}{j_1 + j_{23} + j_4 + 1} \}, \tag{2.72}$$

where the minimum is over all j_{12} that appear in (2.71). Hence we see that also for $|q| = 1$, $|\arg(q)|$ small enough, the matrix of the coordinate transformation from the e to the f basis of the charge j subspace of the space $V^{2j_1} \otimes V^{2j_2} \otimes V^{2j_3}$ is real-orthogonal.

The fact that the transformation from the e to the f basis is orthogonal for $|q| = 1$, $|\arg(q)|$ small enough, can be used in the construction of an interesting q-deformed inner product on the N-fold tensor product of irreducible $U_q(sl(2))$-modules. The definition of this inner product is simple: we declare the set of basis vectors for the N-fold tensor product that is obtained by iterative tensoring of irreps from the right, using the Clebsch-Gordan formula (1.35), to be orthonormal. This inner product clearly makes the tensor product decomposition orthogonal. Also, for the case $N = 2$, it coincides with the inner product we mentioned at the end of section 2.4.5 and which makes the braiding unitary. However, the definition of the inner product which we have just given is not very satisfying, since we might as well have defined a similar inner product by declaring a basis obtained by tensoring from the left or by tensoring sometimes from the right and sometimes from the left to be orthonormal. Fortunately, iterative use of (2.71) shows that all the candidate orthonormal bases are sent onto each other by orthogonal matrices, so that declaring one of these bases orthonormal is equivalent to declaring another orthonormal. Of course, all this is only true when $|\arg(q)|$ is small enough. For any fixed N, a value of q whose argument is small enough may be found, but on the other hand for any fixed value of

$|\arg(q)|$, it will not be difficult to construct tensor product representations in which the inner product does depend on the order of the tensoring. In fact, we can expect this to happen as soon as the decomposition of the tensor product module contains non-unitary irreps (if this does not happen, then the tensor product representation itself is actually $*$-representation). As we will see, this problem disappears when we work with truncated tensor products.

Truncated $6j$-symbols

When q is a root of unity ($q = e^{\pi i/(k+2)}$), we can define truncated $6j$-symbols, related to the truncated tensor product. For these to be non-zero, the conditions (2.66) have to be changed in such a way that they require that j_{12} be not just in the tensor product, but even in the truncated tensor product of j_1 and j_2, etcetera. This means that the upper bounds $j_1 + j_2, \ldots, j_1 + j_{23}$ in (2.66) are sharpened to $\min\{j_1 + j_2, k - j_1 - j_2\}, \ldots, \min\{j_1 + j_{23}, k - j_1 - j_{23}\}$. When the arguments satisfy these sharpened conditions, the truncated $6j$-symbols are still given by the formula (2.68). The truncated $6j$-symbols defined in this way give a canonical isomorphism between the truncated tensor product modules $(V^{2j_1}\hat{\otimes}V^{2j_2})\hat{\otimes}V^{2j_3}$ and $V^{2j_1}\hat{\otimes}(V^{2j_2}\hat{\otimes}V^{2j_3})$. This isomorphism also intertwines the q-deformed inner products which are defined by declaring the natural bases of the truncated tensor product spaces orthogonal (compare the discussion at the end of the previous subsection). To see this, note first of all that the truncated $6j$-symbols are real (this follows easily from (2.68)). Also, it is known that the truncated $6j$-symbols satisfy an analogue of the orthogonality relations (2.71). We have

$$\sum_{j_{12}} \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{23} \end{array} \right\} \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{23}' \end{array} \right\} = \delta_{j_{23}j_{23}'}, \tag{2.73}$$

where the sum is now restricted to the j_{12} that are allowed by the truncated tensor product. It follows that the matrix of the mapping between $(V^{2j_1}\hat{\otimes}V^{2j_2})\hat{\otimes}V^{2j_3}$ and $V^{2j_1}\hat{\otimes}(V^{2j_2}\hat{\otimes}V^{2j_3})$ is real-orthogonal and hence that the mapping preserves the inner product.

The proof of the relations (2.73) uses the usual orthogonality relations (2.71) and also a symmetry of the truncated $6j$-symbols which does not have an analogue at $q = 1$ (the truncated $6j$-symbols also still satisfy the usual symmetries which are present at $q = 1$). This symmetry is part of a set of symmetries mentioned already in [12]. If we define $\overline{j} := k + 1 - j$, then the symmetries in this set can all be generated from the untruncated symmetries (such as the classical symmetries (2.70)) and the identity

$$\left\{ \begin{array}{ccc} \overline{j_1} & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array} \right\} = (-1)^{j_2+j_{23}-j-j_{12}+2j_1+1} \left\{ \begin{array}{ccc} \overline{j_1} & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array} \right\} \tag{2.74}$$

In particular, we get from this that

$$\left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array} \right\} = i(-1)^{j_2+j_3-j_1-j+2j_{12}+1} \left\{ \begin{array}{ccc} j_1 & j_2 & \overline{j_{12}} \\ j_3 & j & j_{23} \end{array} \right\} \tag{2.75}$$

To prove the truncated orthogonality relations (2.73), we now start from the the untruncated orthogonality relations (2.71). We split the sum in (2.71) into three parts as in

$$\sum\nolimits_{j_{12}=\max\{|j_1-j_2|,|j_3-j_4|\}}^{\min\{j_1+j_2,j_3+j_4\}} =$$

$$\sum\nolimits_{j_{12}=\max\{|j_1-j_2|,|j_3-j_4|\}}^{\min\{j_1+j_2,j_3+j_4,k-j_1-j_2,k-j_3-j_4\}} + \sum\nolimits_{\min\{j_1+j_2,j_3+j_4,k-j_1-j_2,k-j_3-j_4\}+1}^{k-j_1-j_2} + \sum\nolimits_{k-j_1-j_2+1}^{\min\{j_1+j_2,j_3+j_4\}}. \tag{2.76}$$

Now if $\min\{j_1+j_2, j_3+j_4, k-j_1-j_2, k-j_3-j_4\}$ equals j_1+j_2 or j_3+j_4, then all the $6j$-symbols in the last two summations are zero, because their arguments don't satisfy the conditions (2.66). If $\min\{j_1+j_2, j_3+j_4, k-j_1-j_2, k-j_3-j_4\}$ equals $k-j_1-j_2$, then the second summation on the right hand side is empty and the third is zero because the j_{12} and $\overline{j_{12}}$ terms cancel each other using (2.75) (if there is a middle term in the summation then this also vanishes using (2.75)). Finally, if $\min\{j_1+j_2, j_3+j_4, k-j_1-j_2, k-j_3-j_4\}$ equals $k-j_3-j_4$, then one can use the explicit formula (2.68) for the $6j$-symbols to show that all the terms of the middle summation vanish, while (2.75) still makes sure that the last summation vanishes because of pairwise cancellation of terms. In any case, the summation on the left, which is the summation in (2.71), equals the first summation on the right, which is the summation in (2.73) and this shows the validity of the truncated orthogonality relations.

Thus, using the isomorphism given by the truncated $6j$-symbols, we can identify the spaces $(V^{2j_1}\hat{\otimes}V^{2j_2})\hat{\otimes}V^{2j_3}$ and $V^{2j_1}\hat{\otimes}(V^{2j_2}\hat{\otimes}V^{2j_3})$ and their inner products, so that we have a well-defined three-particle Hilbert space. The isomorphism may also be used to define braiding transformations on truncated tensor products. Recall from section 2.4.5 that we could use the R-matrix to define braiding of two particles, but that there were difficulties if we wanted to braid particles "over a single bracket" in a multi-particle Hilbert space. These difficulties can now be resolved using the mappings given by the truncated $6j$-symbols. For example, if we want to exchange the two rightmost particles in the representation $(\pi^{2j}\hat{\otimes}\pi^{2j})\hat{\otimes}\pi^{2j}$, then we can first use the $6j$-symbols to map the representation space onto that for $\pi^{2j}\hat{\otimes}(\pi^{2j}\hat{\otimes}\pi^{2j})$, then use the R-matrix to exchange the particles and finally use the inverse of the mapping given by the $6j$-symbols to get back to the representation space of $(\pi^{2j}\hat{\otimes}\pi^{2j})\hat{\otimes}\pi^{2j}$. Similarly, any braiding in a multiple truncated tensor product may now be achieved by using the $6j$-symbols to move the brackets around before and after the actual braiding.

Next to the symmetries (2.75), the truncated $6j$-symbols have another set of symmetries that do not have an analogue at $q=1$. These symmetries are related to the identities (2.58) for the truncated fusion rules. For the case of even k, they were noted already in [75]. They are generated by the following identities

$$\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix} = (-1)^{k+j_1+j_3+j_{12}+j_{23}} \begin{Bmatrix} \hat{j}_1 & j_2 & \hat{j}_{12} \\ \hat{j}_3 & j & \hat{j}_{23} \end{Bmatrix} = (-1)^{k+j_{12}+j_3+j} \begin{Bmatrix} \hat{j}_1 & \hat{j}_2 & j_{12} \\ j_3 & j & \hat{j}_{23} \end{Bmatrix}. \tag{2.77}$$

Here, we have defined $\hat{j} := \frac{k}{2} - j$, in accordance with the definition of $\hat{\Lambda}$ in (2.57). Of course all the identities related to these by the classical, Regge and reflection symmetries are also symmetries. The above identities may be proved in the following way. First notice that the replacements of spins are made in a way that is consistent with the truncated tensor product decomposition. Hence, the arguments of the $6j$-symbol on the left satisfy the truncated version of the conditions (2.66) exactly if the arguments in the other two $6j$-symbols do. This means we can fill in formula (2.68) in all three cases. To show that the results are equal, one needs an identity which holds for q-factorials at $q=e^{2\pi i/(k+2)}$. We have

$$\lfloor k+1-a\rfloor_q! = \frac{\lfloor k+1\rfloor_q!}{\lfloor a\rfloor_q!} \tag{2.78}$$

Using this identity, it is easy to show that the Δ-factors are equal for all three $6j$-symbols in (2.77). For the middle $6j$-symbol in (2.77), we can now see that the sum over z in (2.68) is equal to that for the untransformed $6j$-symbol by making the substitution $z \to z+k-(j_1+j_3+j_{12}+j_{23})$ and using the q-factorial identity above twice. The proof for the rightmost $6j$-symbol in (2.77) is similar, but uses the substitution $z \to -z+k+j_3+j+j_{12}$.

2.4.7 Weak quasitriangular quasi-Hopf algebras

The whole procedure of truncating the tensor product so that it is no longer associative and then defining braiding by identification mappings may be elegantly formalized and brought to the level of the algebra, at the cost of making the connection with non-truncated $U_q(sl(2))$ somewhat less apparent. This has been done in [76] and the resulting structure is called a weak quasitriangular quasi-Hopf algebra, or a weak quasi quantum group. Let us call this $\mathcal{Q}$. Some important features of the resulting picture are the following. The coproduct is modified in such a way that it has the truncation built in. As a result, one no longer has $\Delta(1) = 1 \otimes 1$ and one also loses coassociativity. A so-called coassociator is introduced to compensate for this loss. This coassociator is an element $\phi = \sum_k \phi_k^1 \otimes \phi_k^2 \otimes \phi_k^3$ of $\mathcal{Q}^{\otimes 3}$ which is not invertible in $\mathcal{Q}$, but which has a quasi-inverse called ϕ^{-1} which is the inverse in all the representations that one is interested in and which has the following important property for all $a \in \mathcal{Q}$:

$$\phi(\Delta \otimes 1)\Delta(a) = (1 \otimes \Delta)\Delta(a)\phi \tag{2.79}$$

This ensures that the representations $(\pi^{2j_1} \hat{\otimes} \pi^{2j_2}) \hat{\otimes} \pi^{2j_3}$ and $\pi^{2j_1} \hat{\otimes} (\pi^{2j_2} \hat{\otimes} \pi^{2j_3})$ are isomorphic, with the isomorphism given by $\pi^{2j_1} \hat{\otimes} \pi^{2j_2} \hat{\otimes} \pi^{2j_3}(\phi)$. Of course, this isomorphism is just the one given by the truncated $6j$-symbols. Clearly, one would like to be able to go from one bracketing of a multiple tensor product to another, using ϕ, in such a way that it does not matter which individual steps are taken on the way. This will be the case if the diagram in figure 2.4.7 commutes.

$$\begin{tikzcd}
 & & (V^{2j_1} \otimes V^{2j_2}) \otimes (V^{2j_3} \otimes V^{2j_4}) \arrow[drr] & & \\
((V^{2j_1} \otimes V^{2j_2}) \otimes V^{2j_3}) \otimes V^{2j_4} \arrow[urr] \arrow[dr] & & & & V^{2j_1} \otimes (V^{2j_2} \otimes (V^{2j_3} \otimes V^{2j_4})) \\
 & (V^{2j_1} \otimes (V^{2j_2} \otimes V^{2j_3})) \otimes V^{2j_4} \arrow[rr] & & V^{2j_1} \otimes ((V^{2j_2} \otimes V^{2j_3}) \otimes V^{2j_4}) \arrow[ur] &
\end{tikzcd}$$

Figure 2.5: This diagram shows two ways of going from one bracketing of a fourfold tensor product to another. The arrows denote the canonical isomorphisms given by the coassociator (or the truncated $6j$-symbols). The diagram will commute if the condition (2.80) on the coassociator is satisfied

To make this diagram commute, we need to impose the following condition on the coassociator [77]:

$$(1 \otimes 1 \otimes \Delta)(\phi)(\Delta \otimes 1 \otimes 1)(\phi) = (1 \otimes \phi)(1 \otimes \Delta \otimes 1)(\phi)(\phi \otimes 1) \tag{2.80}$$

In terms of $6j$-symbols, this condition becomes

$$\begin{gathered}
\left\{ \begin{matrix} j_{12} & j_3 & j_{123} \\ j_4 & j & j_{34} \end{matrix} \right\} \left\{ \begin{matrix} j_1 & j_2 & j_{12} \\ j_{34} & j & j_{234} \end{matrix} \right\} = \\
\sum_{j_{23}} \left\{ \begin{matrix} j_1 & j_2 & j_{12} \\ j_3 & j_{123} & j_{23} \end{matrix} \right\} \left\{ \begin{matrix} j_1 & j_{23} & j_{123} \\ j_4 & j & j_{234} \end{matrix} \right\} \left\{ \begin{matrix} j_2 & j_3 & j_{23} \\ j_4 & j_{234} & j_{34} \end{matrix} \right\}
\end{gathered} \tag{2.81}$$

This condition will clearly be satisfied for non-truncated $6j$-symbols, since the sides of the equation just correspond to two ways of doing the same basis transformation in that case. For non-truncated $6j$-symbols the coordinate transformations change to mappings that really do something, but one may show that the equation above still holds.

When there is a non-trivial coassociator, the conditions (1.14) and (1.15), which guaranteed the compatibility of fusion and braiding, change to

$$\begin{aligned}(\Delta \otimes 1)(R) &= \phi_{312} R_{13} \phi_{132}^{-1} R_{23} \phi_{123} \\ (1 \otimes \Delta)(R) &= \phi_{231}^{-1} R_{13} \phi_{231} R_{12} \phi_{123}^{-1}\end{aligned} \tag{2.82}$$

and these in turn imply the following quasi-Yang-Baxter equation [77], which is the analogue of (1.16):

$$R_{12}\phi_{132}R_{13}\phi_{132}^{-1}R_{23}\phi_{123} = \phi_{321}R_{23}\phi_{231}^{-1}R_{13}\phi_{213}R_{12} \tag{2.83}$$

This relation ensures that the recipe that we gave for performing braidings does indeed give a representation of the braid group.

2.4.8 Braiding and $6j$-symbols

In this section, we will give a systematic description of the braid group representations that are associated with (truncated) tensor products of $U_q(sl(2))$ representations. Let us look at a tensor product of n quantum group irreps $\pi^{\Lambda_1}, \ldots, \pi^{\Lambda_n}$. In such a tensor product, there are a number of natural bases which reflect the structure of the tensor product. More precisely, there is one such basis for each way in which the tensor product can be built up by adding subsequent factors. We have already described the situation for three tensor factors in detail in section 2.4.6. In this case, there were two of these natural bases and the transformation that related these was described by the $6j$-symbols. In the case of n factors, we will choose to work with the natural basis one gets by adding subsequent tensor factors on the right, i.e. the basis induced by the following "bracketing" of the tensor product:

$$\pi^{\Lambda_1} \otimes \pi^{\Lambda_2} \otimes \ldots \otimes \pi^{\Lambda_n} = (\ldots(\pi^{\Lambda_1} \otimes \pi^{\Lambda_2}) \otimes \pi^{\Lambda_3}) \ldots \otimes \pi^{\Lambda_{n-1}}) \otimes \pi^{\Lambda_n}). \tag{2.84}$$

The elements of this basis can be labeled by their overall H eigenvalue m, their overall fusion channel j_n and and $n-1$ intermediate fusion channels $j_1, \ldots, j_{n-1}$. We may thus write these basis elements as $e^{\mathcal{J}_1,\ldots,\mathcal{J}_n}_{j_1,\ldots,j_n,m}$, where we have defined $\mathcal{J}_i = \frac{\Lambda_i}{2}$. Clearly, $j_1 = \mathcal{J}_1$ and j_i is one of the summands in $j_{i-1}\hat{\otimes}\mathcal{J}_i$ for $i > 1$. If there is no cause for confusion, we will suppress the upper indices and write $e_{j_1,\ldots,j_n,m}$. It is easy to show that the set of $e_{j_1,\ldots,j_n,m}$ for which all the j's are held fixed, forms a basis for an irrep of $U_q(sl(2))$ of type π^{2j_n}, i.e. the action on this set corresponds to the action given in formula (1.28). Hence, it follows that the tensor product representation becomes a $*$-representation if we take the inner product which makes the $e_{j_1,\ldots,j_n,m}$ orthonormal and if each of the possible π^{2j_n} is itself a $*$-representation. Note that if we are working with a truncated tensor product, then there will be a different truncated tensor product space for each bracketing, because of the non-associativity of this tensor product. The bases we have described here then provide canonical bases for the different subspaces of the ordinary tensor product that one gets from the different bracketings.

The basis of $e_{j_1,\ldots,j_n,m}$ is very suited to a description of the braiding. Suppose we want to exchange particles i and $i+1$, i.e. we want to calculate the action of the exchange τ_i on $e_{j_1,\ldots,j_n,m}$. We can do this in three steps:

1. Move particle i completely to the left, using right-over-left exchanges. Since the representations $\pi^{\Lambda_1}, \ldots, \pi^{\Lambda_{i-1}}$ fuse together to the representation $\pi^{2j_{i-1}}$ and since the fusion of this $\pi^{2j_{i-1}}$ with π^{Λ_i} gives π^{2j_i}, this operation gives us just a constant factor. We have
$$e^{\mathcal{J}_1,\ldots,\mathcal{J}_n}_{j_1,\ldots,j_n,m} \to (-1)^{j_i-\mathcal{J}_i-j_{i-1}} q^{\frac{1}{2}(c_{j_{i-1}}+c_{\mathcal{J}_i}-c_{j_i})} f^{\mathcal{J}_i,\mathcal{J}_1,\ldots,\mathcal{J}_n}_{j_1,\ldots,j_n,m}. \tag{2.85}$$
Here we have defined $c_j = j(j+1)$, in accordance with the definition of c_Λ above. Also, the vector $f^{\mathcal{J}_i,\mathcal{J}_1,\ldots,\mathcal{J}_n}_{j_1,\ldots,j_n,m}$ is an element of the natural basis for the tensor product that one gets by first tensoring together $\pi^{\Lambda_1}, \ldots, \pi^{\Lambda_{i-1}}$, adding successive factors on the right, then tensoring on π^{Λ_i} from the left and finally tensoring on the remaining factors from the right. To get the result (2.85), one uses (2.63) and, repeatedly, (1.14) and (1.15) or, for truncated tensor products, (2.82).

2. Now change the bracketing, using the $6j$-symbols, so that we end up in a basis in which the representations π^{Λ_i} and $\pi^{\Lambda_1}, \ldots, \pi^{\Lambda_{i-1}}$ no longer fuse to a fixed representation, but the representations $\pi^{\Lambda_1}, \ldots, \pi^{\Lambda_{i-1}}$ and $\pi^{\Lambda_{i+1}}$ do. The new basis is the natural basis for the tensor product which one gets by first tensoring together $\pi^{\Lambda_1}, \ldots, \pi^{\Lambda_{i-1}}$, adding successive factors on the right, then tensoring on $\pi^{\Lambda_{i+1}}$ from the right, then tensoring on π^{Λ_i} from the left and finally tensoring on the remaining factors from the right. In the new basis, the label j_i (which gave the overall quantum group charge of particles 1 to i) is replaced by a new label j', which gives the overall quantum group charge of particles $1, 2, \ldots, i-1, i+1$. All the other labels are as before. If we denote the elements of the new basis by $g^{\mathcal{J}_i,\mathcal{J}_1,\ldots,\mathcal{J}_n}_{j_1,\ldots,j',\ldots,j_n,m}$, then the f-basis can be written in terms of the g's as
$$f^{\mathcal{J}_i,\mathcal{J}_1,\ldots,\mathcal{J}_n}_{j_1,\ldots,j_i,\ldots,j_n,m} = \sum_{j'} \left\{ \begin{array}{ccc} \mathcal{J}_i & j_{i-1} & j_i \\ \mathcal{J}_{i+1} & j_{i+1} & j' \end{array} \right\} g^{\mathcal{J}_i,\mathcal{J}_1,\ldots,\mathcal{J}_n}_{j_1,\ldots,j',\ldots,j_n,m}, \tag{2.86}$$
where we have used the fact that the representations carried by the particles $1, \ldots, i-1$ fuse to $\pi^{2j_{i-1}}$ and that these particles can thus be treated as one particle that carries the representation $\pi^{2j_{i-1}}$.

3. Now we move particle i to the right, using left-over-right exchanges, until it has reached the position to the right of particle $i+1$. At the end of this process, we have effectively only produced a left-over-right exchange of the particles i and $i+1$, as we wanted. In the g basis, the process of exchanging particle i past particles $1, \ldots, i-1$ and $i+1$ is described once again by a simple phase factor (compare the first step of the calculation), since the representations on particles $1, \ldots, i-1, i+1$ fuse to $\pi^{2j'}$ and this fuses with $\pi^{2\mathcal{J}_i}$ into the fixed fusion channel $\pi^{2j_{i+1}}$. We get
$$g^{\mathcal{J}_i,\mathcal{J}_1,\ldots,\mathcal{J}_n}_{j_1,\ldots,j',\ldots,j_n,m} \to (-1)^{j'+\mathcal{J}_i-j_{i+1}} q^{\frac{1}{2}(c_{j_{i+1}}-c_{j'}-c_{\mathcal{J}_i})} e^{\mathcal{J}_1,\ldots,\mathcal{J}_{i+1},\mathcal{J}_i,\ldots,\mathcal{J}_n}_{j_1,\ldots,j',\ldots,j_n,m}, \tag{2.87}$$
where $e^{\mathcal{J}_1,\ldots,\mathcal{J}_{i+1},\mathcal{J}_i,\ldots,\mathcal{J}_n}_{j_1,\ldots,j',\ldots,j_n,m}$ is an element of the basis we started with.

We may now write down the action of the elementary exchange τ_i on the e-basis as the cumulative effect of these three steps. We have
$$\tau_i e^{\mathcal{J}_1,\ldots,\mathcal{J}_i,\mathcal{J}_{i+1},\ldots,\mathcal{J}_n}_{j_1,\ldots,j_i,\ldots,j_n,m} = \sum_{j'} (-1)^{j_i-j_{i-1}+j'-j_{i+1}} q^{\frac{1}{2}(c_{j_{i-1}}-c_{j_i}+c_{j_{i+1}}-c_{j'})} \left\{ \begin{array}{ccc} \mathcal{J}_i & j_{i-1} & j_i \\ \mathcal{J}_{i+1} & j_{i+1} & j' \end{array} \right\} e^{\mathcal{J}_1,\ldots,\mathcal{J}_{i+1},\mathcal{J}_i,\ldots,\mathcal{J}_n}_{j_1,\ldots,j',\ldots,j_n,m}. \tag{2.88}$$

Using equation (2.73), one may check easily that the matrix that describes this transformation is unitary if q is a root of unity, which is the case we are interested in. Hence, if we take the inner product which makes the $e_{j_1,\ldots,j_n,m}$ orthonormal, then the braid group representation which governs the exchanges of particles with $U_q(sl(2))$-charges is unitary, as it should be. If either $\mathcal{J}_i = \mathcal{J}_{i+1}$ or $j_{i-1} = j_{i+1}$, then it follows from the classical symmetries (2.70) that the matrix for τ_i is also symmetric.

2.4.9 Hidden quantum group symmetry

We will say that a quantum mechanical system has a hidden quantum group symmetry if there is an action of a quantum group $\mathcal{A}$ on the Hilbert space of the theory which has the property that it commutes with all the observables of the theory. For a system of particles which carry $U_q(sl(2))$-representations, this means in particular that the H-eigenvalues associated to the particles will not be observable, while on the other hand, one can allow observables which make it possible to determine the $U_q(sl(2))$-representation associated to each of the particles. In other words, the total "quantum spin" of each particle would be measurable, but the components of this quantum spin would not be measurable. The above definition of hidden quantum group symmetry is just what we have distilled from various sources in the literature that mention hidden quantum group symmetries (see section 2.5 for references). Note however that there does not seem to be a completely standard definition of this concept. Let us say more about what the above definition means within our context. Suppose we have a system of n particles that carry representations $\pi^{\Lambda_1}, \ldots, \pi^{\Lambda_n}$ of a quantum group $\mathcal{A}$. In that case the whole system will be in a state in the tensor product space $V^{\Lambda_1} \otimes \ldots \otimes V^{\Lambda_n}$. If this tensor product may be decomposed into irreducibles then the decomposition will take the form

$$V^{\Lambda_1} \otimes \ldots \otimes V^{\Lambda_n} = \bigoplus_{\Lambda} U^{\Lambda}_{\Lambda_1,\ldots,\Lambda_n} \otimes V^{\Lambda}. \tag{2.89}$$

Here, $U^{\Lambda}_{\Lambda_1,\ldots,\Lambda_n}$ is a vector space whose dimension equals the multiplicity of the irrep V^{Λ} of $\mathcal{A}$ in the tensor product. When no confusion seems possible, we will just write U^{Λ}. If the $\mathcal{A}$-symmetry of this system is a hidden symmetry, then it follows that all the observable operators act only on the spaces U^{Λ} without mixing these. That is, every observable $\hat{O}$ should take the form

$$\hat{O} = \sum_{\Lambda} \hat{O}^{\Lambda} \otimes I_{V^{\Lambda}}, \tag{2.90}$$

where each $\hat{O}^{\Lambda}$ is an operator acting on U^{Λ} and $I_{V^{\Lambda}}$ is the identity operator on V^{Λ}. Since all the observables have this structure, the state of the system can be uniquely characterized by a list of vectors, one for each of the spaces U^{Λ}. Usually, the overall quantum group charge(s) of the system will be well-defined. In other words, the state of the system will be described by a vector in one of the summands in the decomposition (2.89). In fact, there may be superselection rules which prevent superposition of states from different summands in (2.89). If the system as a whole is in the quantum group representation π^{Λ}, then the state of the system may be described by a vector in the space U^{Λ}. Now note that the braid group representation on the tensor product $V^{\Lambda_1} \otimes \ldots \otimes V^{\Lambda_n}$ which comes from the action of the R-matrix of $\mathcal{A}$ induces an action of the braid group on each of the U^{Λ}. This follows from the fact that the action of the braid group elements, like the action of any observable, commutes with the action of $\mathcal{A}$. Any operator that represents a braid group element will thus be of the general form given above for observables.

Thus, if one wants to describe only the monodromy or braid group representation that governs the statistics of the system at a fixed number of particles with given overall quantum group charge Λ, one can restrict oneself to the space U^Λ. It should be clear from the previous section what form such a representation would take for a system of n particles with a hidden $U_q(sl(2))$ symmetry. In this case we have the canonical basis of the $e^{\mathcal{J}_1,\ldots,\mathcal{J}_n}_{j_1,\ldots,j_n,m}$ for the tensor product of the n representations of $U_q(sl(2))$. Of these, we need only retain the ones whose overall charge j_n is equal to the fixed total charge of the system, say $j_n = j$. These vectors may then be written as tensor products,

$$e^{\mathcal{J}_1,\ldots,\mathcal{J}_n}_{j_1,\ldots,j,m} = e^{\mathcal{J}_1,\ldots,\mathcal{J}_n}_{j_1,\ldots,j} \otimes |\, j, m\,\rangle, \tag{2.91}$$

where $e^{\mathcal{J}_1,\ldots,\mathcal{J}_n}_{j_1,\ldots,j}$ now denotes a vector in the space U^j. The braid group representation on U^j may now be read off immediately from the formula (2.88) which gave the braiding for the full tensor product of $U_q(sl(2))$-representations. The matrix elements between the $e^{\mathcal{J}_1,\ldots,\mathcal{J}_n}_{j_1,\ldots,j_n,m}$ which are given in this formula can be used in unchanged form for the vectors $e^{\mathcal{J}_1,\ldots,\mathcal{J}_n}_{j_1,\ldots,j}$, since they already did not depend on m and did not mix different j_n. A similar treatment of braid group representations for systems with hidden quantum group symmetry is possible in any situation in which $6j$-symbols may be defined for the quantum group representations involved. This is the case if the tensor products of these representations have a multiplicity free decomposition into irreducibles.

2.4.10 Braiding of identical particles and fusion diagrams

In the previous subsections, we have described the braiding for a system of n particles with a hidden $U_q(sl(2))$-symmetry. Let us now look at the special case in which the particles are identical. This case is of interest for the description of the braiding of identical quasiholes in the RR-states. When the particles are identical, they all carry the same quantum group representation $\pi^{2\mathcal{J}}$ and hence the upper indices on the elements $e^{\mathcal{J}_1,\ldots,\mathcal{J}_n}_{j_1,\ldots,j}$ of the canonical basis for the space U^j are all equal to $\mathcal{J}$. Fixing $\mathcal{J}$, we may thus forget about the upper indices and write just $e_{j_1,\ldots,j_n}$. As in the previous section, we also fix $j_n = j$. Now the n-tuple $(j_1,\ldots,j_n)$ may be seen as a path of length n through the space of representation labels of $U_q(sl(2))$, which starts at the trivial representation $j_0 = 0$ and ends at j. Of course, not all paths through the space of representation labels will correspond to an element of the canonical basis. A path will represent a basis vector precisely if the representation at position m of the path may always be found in the tensor product of the representation at position $m-1$ with the representation $\pi^{2\mathcal{J}}$. This means precisely that the path lies on the fusion (or Bratteli) diagram for the representation $\pi^{2\mathcal{J}}$. Thus, the paths of length n on the Fusion diagram of the representation $\pi^{2\mathcal{J}}$ may be taken as a basis for the braid group representation that describes exchanges in a system of n particles that carry the quantum group representation $\pi^{2\mathcal{J}}$. From equation (2.88), one may now easily read off that the braid group generator τ_m will only mix paths that are identical everywhere except at position m.

As an example let us look at the case of n particles in the 2-dimensional representation of $U_q(sl(2))$. The fusion diagram for this representation is just the diagram drawn in figure 2.1. Let $p = (p^{(1)},\ldots,p^{(n)}) = ((\Lambda_p^{(1)},1),\ldots,(\Lambda_p^{(n)},n))$ be a path on this diagram which starts at $(0,0)$, then goes through $p^{(1)}, p^{(2)}$, etcetera and which ends at the point $p^{(n)} = (\Lambda_p^{(n)}, n) = (\Lambda, n)$. Then there is either no path which differs from p only at its m^{th} vertex or there is exactly one such path. If there is such a path, we will call it $\sigma_m(p)$. Let us write down the action of the exchange τ_m on a path p. We start with the cases in which p does not have a partner path. Using equation (2.88), we see that such paths just get a phase factor. There are four cases:

$$\begin{aligned}
\Lambda_p^{(m-1)} = 2j < \Lambda_p^{(m)} < \Lambda_p^{(m+1)} \quad &\Rightarrow \quad \tau_m p = q^{1/4}\, p \\
\Lambda_p^{(m-1)} = 2j > \Lambda_p^{(m)} > \Lambda_p^{(m+1)} \quad &\Rightarrow \quad \tau_m p = q^{1/4}\, p \\
\Lambda_p^{(m-1)} = \Lambda_p^{(m+1)} = 0,\ \Lambda_p^{(m)} = 1 \quad &\Rightarrow \quad \tau_m p = -q^{-3/4}\, p \\
\Lambda_p^{(m-1)} = \Lambda_p^{(m+1)} = k,\ \Lambda_p^{(m)} = k-1 \quad &\Rightarrow \quad \tau_m p = -q^{-3/4}\, p
\end{aligned} \tag{2.92}$$

These equations may be summarized by saying that the path p gets a factor of $q^{1/4}$ if it does not change direction at its m^{th} vertex, whereas it gets a factor $-q^{-3/4}$ if it does change direction (which can only happen at the boundary of the diagram). In obtaining the equations, we used the following values for the $6j$-symbols involved:

$$\begin{aligned}
\left\{ \begin{array}{ccc} \frac{1}{2} & j & j+\frac{1}{2} \\ \frac{1}{2} & j+1 & j+\frac{1}{2} \end{array} \right\} &= \left\{ \begin{array}{ccc} \frac{1}{2} & j & j-\frac{1}{2} \\ \frac{1}{2} & j-1 & j-\frac{1}{2} \end{array} \right\} = \left\{ \begin{array}{ccc} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{array} \right\} = 1 \\
\left\{ \begin{array}{ccc} \frac{1}{2} & \frac{k}{2} & \frac{k-1}{2} \\ \frac{1}{2} & \frac{k}{2} & \frac{k-1}{2} \end{array} \right\} &= -1
\end{aligned} \tag{2.93}$$

We are now left with the case in which p does have a partner path $\sigma(p)$. In this case, we will certainly have $\Lambda_p^{(m-1)} = \Lambda_p^{(m+1)} = \Lambda_{\sigma(p)}^{(m-1)} = \Lambda_{\sigma(p)}^{(m+1)} = 2j$ and, exchanging p with $\sigma(p)$ if needed, we can also make sure that $\Lambda_p^{(m)} > \Lambda_{\sigma(p)}$, so that $\Lambda_p^{(m)} = 2j+1,\ \Lambda_{\sigma(p)}^{(m)} = 2j-1$. The relevant $6j$-symbols for this case are given by

$$\begin{aligned}
\left\{ \begin{array}{ccc} \frac{1}{2} & j & j+\frac{1}{2} \\ \frac{1}{2} & j & j+\frac{1}{2} \end{array} \right\} &= \frac{1}{\lfloor 2j+1 \rfloor_q} \\
\left\{ \begin{array}{ccc} \frac{1}{2} & j & j-\frac{1}{2} \\ \frac{1}{2} & j & j-\frac{1}{2} \end{array} \right\} &= \frac{-1}{\lfloor 2j+1 \rfloor_q} \\
\left\{ \begin{array}{ccc} \frac{1}{2} & j & j+\frac{1}{2} \\ \frac{1}{2} & j & j-\frac{1}{2} \end{array} \right\} &= \left\{ \begin{array}{ccc} \frac{1}{2} & j & j-\frac{1}{2} \\ \frac{1}{2} & j & j+\frac{1}{2} \end{array} \right\} = -\frac{\sqrt{\lfloor 2j+2 \rfloor_q \lfloor 2j \rfloor_q}}{\lfloor 2j+1 \rfloor_q}
\end{aligned} \tag{2.94}$$

and combining this with the phase factors in (2.88), we see that, in the linear space with basis $\{p, \sigma(p)\}$, the exchange τ_m is represented by the matrix

$$\tau_m \equiv \frac{q^{-1/4}}{\lfloor d \rfloor_q} \begin{pmatrix} -q^{-d/2} & -\sqrt{\lfloor d+1 \rfloor_q \lfloor d-1 \rfloor_q} \\ -\sqrt{\lfloor d+1 \rfloor_q \lfloor d-1 \rfloor_q} & q^{d/2} \end{pmatrix} \tag{2.95}$$

where we have defined $d := 2j+1$. This matrix for τ_m is obviously symmetric. It is also unitary, as can be easily seen, using the fact that $\lfloor d+1 \rfloor_q \lfloor d-1 \rfloor_q$ equals $\lfloor d \rfloor_q^2 - 1$. We will denote the braid group representation on the paths which start from $(0,0)$ and end at (Λ, n) by ρ_n^Λ and the corresponding modules by U_n^Λ.

An induction argument taken from [78] shows that the ρ_n^Λ are all irreducible and that they are non-isomorphic for different Λ. The representation ρ_1^1 of the trivial group B_1 is irreducible because it is one dimensional and $\Lambda = 1$ is the only possibility at $n = 1$. Now suppose that, for all Λ and all $n < m$, all ρ_n^Λ are irreducible and non-isomorphic for different Λ. Then the representations ρ_m^Λ are irreducible and mutually non-isomorphic for all Λ. To see this look at U_m^Λ and suppose for convenience that Λ does not equal 0 or k. U_m^Λ has a unique decomposition into B_{m-1}-invariant submodules which is clearly given by $U_n^\Lambda = U_{m-1}^{\Lambda-1} \oplus U_{m-1}^{\Lambda+1}$ (just forget the last step in the paths). Because the ρ_{m-1}^{Λ} are non-isomorphic for different values of Λ (by the induction hypothesis), it follows immediately that the ρ_m^{Λ} are also non-isomorphic for different values of Λ; their modules have different decompositions into irreducible B_{m-1}-modules. Moreover, since $\rho_{m-1}^{\Lambda-1}$ and $\rho_{m-1}^{\Lambda+1}$ are irreducible and non-isomorphic, it follows that the only possible proper B_{m-1}-invariant submodules of U_m^{Λ} are $U_{m-1}^{\Lambda-1}$ and $U_{m-1}^{\Lambda+1}$. However, these will clearly be mixed by the exchange τ_{m-1}, so that U_m^{Λ} has no proper B_m-invariant subspaces. Hence ρ_m^{Λ} is irreducible. Of course if Λ equals 0 or k then the argument becomes even simpler and we need not repeat it.

2.5 Conformal field theory and quantum groups

In this section, we review the correspondence between conformal field theory and quantum groups. In section 2.5.1, we give a short general description of this correspondence, illustrated with the example of $U_q(sl(2))$ versus the $\widehat{sl(2)}_k$ WZW-theory. In the next section, we go on to describe the quantum group $U_q(sl(m))$ and its relation to the $\widehat{sl(m)}_k$ WZW-theory. In section 2.5.3, we describe representations of the braid group B_n which factor over the Hecke algebra $H_{n,q}$. These are important in the description of the braiding of a system of n particles with hidden $U_q(sl(m))$-symmetry. In section 2.5.4, we describe a quantum group for the chiral boson. Finally, in section 2.5.5, we indicate quantum groups which correspond to the parafermion theory that is used in the description of the Read-Rezayi states.

2.5.1 The CFT-QG relation

The relation between quantum groups and conformal field theories has been much studied over the years and it is believed that every conformal field theory has associated to it some quantum group (or generalization thereof) with the following properties:

- Each chiral primary field of the CFT (or equivalently: each irreducible representation of the chiral algebra) corresponds to an irreducible representation of the quantum group.

- The fusion algebra of the CFT is identical to the representation ring of the quantum group, i.e. fusion of chiral primaries corresponds to taking the tensor product of quantum group irreps.

- The braiding of the chiral primary fields in conformal blocks corresponds to the braiding in the tensor product of quantum group representations, as described by means of an R-matrix and, if needed, a coassociator.

The points above can be illustrated by the case of the $\widehat{sl(2)}_k$ WZW-theory, whose associated quantum group is $U_q(sl(2))$ at $q = e^{\frac{2\pi i}{k+2}}$. For this value of q, the unitary irreducible representations π^{Λ} of $U_q(sl(2))$ that have positive quantum dimension are indeed in one to one correspondence with the affine primary fields G^{Λ} of the WZW-theory. Moreover, comparing equations (2.22) and (2.53), we see that the fusion rules for the WZW-fields are the same as the decomposition rules for tensor products of $U_q(sl(2))$-representations. We described the braid group representations associated to the fundamental representation of $U_q(sl(2))$ in section 2.4.10. The braiding of the corresponding conformal blocks of the WZW-theory was calculated by Tsuchiya and Kanie [79, 80] and this braiding is indeed the same as that described in section 2.4.10, up to a renormalization of the blocks. In connection with this, the q-$6j$-symbols may be identified with the fusion matrix of the $\widehat{sl(2)}_k$ conformal field theory as defined by Moore and Seiberg [81, 82]. The pentagon equation for this fusion matrix then corresponds to the equation (2.81) (see also figure 2.4.7) and the hexagon equation is just the quasi-Yang-Baxter equation (2.83), written in terms of $6j$-symbols by means of (2.88).

Note that it is essential in the above, that the truncated tensor product of $U_q(sl(2))$-representations is used, rather than the ordinary one. In other words, we may say that it is essential that one uses a weak quasi-quantum group rather than an ordinary quantum group. This is not a very special situation; the fusion rules of many CFTs cannot be reproduced by those of ordinary quantum groups (or quantum groups with an ordinary tensor product). On the other hand, there is mathematical work [8, 83] in which it is shown that, given a CFT, one may always find weak quasi quantum groups that will reproduce its fusion and braiding properties. This does not mean that it is known for all conformal field theories how the quantum group generators can be represented in terms of operators in the conformal field theory. In fact, no general construction for these operators seems to be known, although several proposals have been made for CFTs that have a Coulomb gas description [84, 85, 86]. Through this work, much is known about the quantum groups for the WZW-models. In particular, it is well known that for any semisimple Lie algebra g, the $\hat{g}_k$ WZW-model and the quantum group $U_q(g)$ at $q = e^{2\pi i/(k+\hat{g})}$ are related in the way we have described above (here $\hat{g}$ is the dual Coxeter number of g). In the following, we shall be especially interested in the case $g = sl(m)$, because of the close relation between the parafermion theory that describes the RR-states and the $\widehat{sl(2)}_k$ and $\widehat{sl(k)}_2$ WZW-theories.

Before we go on, let us cite a few general references on the relation between CFTs and quantum groups. Books that include information on this are for example [71, 87] and a review article is [68]. An early description of the correspondence between $U_q(sl(2))$ and the $\widehat{sl(2)}_k$ WZW-theory can be found in [88].

2.5.2 $U_q(sl(m))$ and the $\widehat{sl(m)}_k$ WZW-theory

In this subsection, we recall some facts about the quantum group $U_q(sl(m))$ that is associated with the $\widehat{sl(m)}_k$ WZW-theory. $U_q(sl(m))$ is a q-deformation of the universal enveloping algebra $U(sl(m))$ of $sl(m)$. Such a q-deformation can be constructed for any simple Lie algebra g. If we denote the simple roots of g by α_i, then we can associate to each of these three generators H_i, L_i^+, L_i^- and these will generate $U_q(g)$ as an algebra, subject to the relations

$$\begin{aligned}
&[H_i, H_j] = 0\\
&[H_i, L_j^\pm] = \pm A_{ji} L_j^\pm\\
&[L_i^+, L_j^-] = \delta_{ij} \lfloor H_i \rfloor_q\\
&[L_i^\pm, L_j^\pm] = \begin{cases} 0 \;\text{ if } A_{ij} = 0 \text{ or } i = j \\ \displaystyle\sum_{s=0}^{1-A_{ij}} (-1)^s q^{(\alpha_i,\alpha_i)s(s+A_{ij}-1)/2} \left\lfloor \begin{matrix} 1-A_{ij} \\ s \end{matrix} \right\rfloor_q (L_i^\pm)^{1-A_{ij}-s} L_j^\pm (L_i^\pm)^s \;\; \text{otherwise} \end{cases}
\end{aligned} \tag{2.96}$$

Here, A is the Cartan matrix of g. When $q = 1$, these relations reduce to the relations for the Chevalley-Serre basis of $U(g)$ and when $g = sl(2)$, they reduce to the relations we gave in section 2.4.2. If q is not a root of unity, the irreducible representations of $U_q(g)$ are labeled by dominant integral weights of g and one may give formulae for the action of the generators which are similar to those given in (1.28). When q is a root of unity, one finds again that all these representations remain well-defined, but many are no longer irreducible and in particular there are indecomposable representations.

The coproduct Δ, counit ϵ and antipode S are given by

$$\begin{aligned}
\Delta(H_i) &= 1 \otimes H_i + H_i \otimes 1 \\
\Delta(L_i^{\pm}) &= L_i^{\pm} \otimes q^{H_i/4} + q^{-H_i/4} \otimes L_i^{\pm} \\
\epsilon(1) &= 1, \quad \epsilon(L_i^{\pm}) = \epsilon(H_i) = 0 \\
S(H_i) &= -H_i, \quad S(L_i^{\pm}) = -q^{\rho/2} L_i^{\pm} q^{-\rho/2}
\end{aligned} \tag{2.97}$$

Here, ρ is the Weyl-vector of g, which is equal to half the sum of the positive roots, or equivalently, to the sum of the fundamental weights. One may check easily that this comultiplication, counit and antipode satisfy the conditions given in section 2.4.2. As usual, one can define the tensor product of representations through the formula (1.4) and as in the case of $U_q(sl(2))$, this tensor product will usually not be fully decomposable if q is a root of unity. However, it is once more possible to define a truncated tensor product which involves only a finite set of unitary irreducible representations and which is fully decomposable. If $q = e^{2\pi i/(k+\hat{g})}$, where $\hat{g}$ is the dual Coxeter number of g, then the irreducible representations involved are each labeled by a dominant integral weight Λ such that $(\Lambda, \theta) \leq k$, where θ is the highest root of g. Hence, they are in one to one correspondence with the affine primary fields of the $\hat{g}_k$ WZW-theory. Moreover, as in the case of $sl(2)$, the decomposition rules of the truncated tensor product are identical to the fusion rules of the WZW-primaries. One may also define a quantum trace and a corresponding quantum dimension and one may then go from the ordinary to the truncated tensor product by projecting out modules of zero quantum dimension.

The R-matrix is also known (see for example chapter eight of [7] for details and references), but it is in general not so easy to obtain the exchange matrices in any given tensor product of representations from it. The reason for this is that, to calculate the action of the R-matrix on a tensor product of representations, one needs formulae for the action of the elements of $U_q(g)$ associated to the roots of g on both representations in the tensor product. Although it is quite easy to obtain formulae similar to (1.28) for the action of the raising and lowering operators $L^{\pm}_{\alpha_i}$ associated with the simple roots α_i,, the same does not go for the action of the raising and lowering operators that correspond to non-simple roots. Nevertheless, the exchange matrices have been calculated in special cases, one of which is important to us. This is the case of the tensor product of the fundamental m-dimensional representation of $U_q(sl(m))$ with itself (see [7] for a detailed calculation). Let us denote this fundamental representation by π_{e_1}, where e_1 denotes the highest weight of the representation as in section 2.3.1. We may then write

$$\sigma(\pi_{e_1} \otimes \pi_{e_1})(R) = q^{\frac{1}{2m}} \left(q^{1/2} \sum_{i=1}^{m} E_{ii} \otimes E_{ii} + \sum_{i \neq j} E_{ij} \otimes E_{ji} + (q^{1/2} - q^{-1/2}) \sum_{i<j} E_{jj} \otimes E_{ii} \right) \tag{2.98}$$

where E_{ij} denotes the matrix whose (i, j) entry is one and whose other entries are zero. One may check easily that this formula gives back the matrix (1.41) in the case of $sl(2)$.

In the following, we want to describe the braid group representation that is associated with an n-fold truncated tensor product $\pi_{e_1}^{\hat{\otimes} n}$. Since tensor products that involve π_{e_1} are multiplicity free, we can apply the methods described in sections 2.4.8 and onwards. That is, we may define $6j$-symbols for the tensor products involved and describe the braiding by formula (2.88) and finally graphically, in terms of paths on the fusion diagram of the representation π_{e_1}. In fact, we can already say quite a lot about the braiding just from the fusion diagram, without a detailed knowledge of the $6j$-symbols. So let us describe this fusion diagram. For $q = e^{2\pi i/(k+\hat{g})}$, each vertex of the fusion diagram may be labeled the number of fundamental representations that have been tensored up to that point, together with a dominant integral weight of $sl(m)$ which satisfies the requirement $(\Lambda, \theta) \leq k$. We may equivalently represent this weight by its Young diagram and if we do this, then the requirement that $(\Lambda, \theta) \leq k$ translates to the restriction that the diagrams should not have more than k columns. Fusion diagrams of this kind have already been drawn in figures 2.1 and 2.2. Instead of using the particle number and the Young diagram for the overall $U_q(sl(m))$-charge, one may also use just a Young diagram to represent each vertex. This Young diagram is then the diagram which reduces to the Young diagram for $U_q(sl(m))$-charge if columns of m boxes are removed and whose number of boxes is equal to the number of representations tensored up to that vertex. As an example, we show a diagram for $U_q(sl(3))$ in figure 2.6.

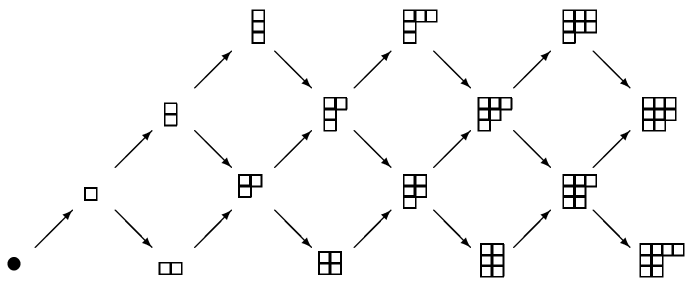

Figure 2.6: The Bratteli diagram for the fundamental representation of $U_q(sl(3))$ at $q = e^{2i\pi/5}$. This is in fact the same diagram as that shown in figure 2.2, but this time each site in the diagram is uniquely labeled by a Young diagram only. The diagrams in figure 2.2 may be recovered by removing columns of 3 boxes.

The connections between the different vertices are of course determined by the fusion rules for the fundamental representation. These can be elegantly described in terms of Young diagrams. The truncated tensor product of the fundamental representation with the representation that has Young diagram Y decomposes into the sum of the representations whose Young diagrams have at most m columns and may be formed by adding one box to Y and removing any columns of length m that result. If one keeps the columns of length m then one obtains the Young diagrams which label the vertices of the Bratteli diagram. The restriction on the number of columns then applies only to the number of columns of length less than m.

The representation of B_n that describes exchanges for a system of n particles with $U_q(sl(m))$ symmetry may now be described in terms of these Bratteli diagrams. In fact, given the overall $U_q(sl(m))$-charge of the system, we may find the Young diagram Y with n boxes which gives this overall charge and then the Braid group representation space is just the space of paths on the diagram which start at the empty diagram and end at Y. Moreover, each of the exchanges τ_i will only mix paths that are the same everywhere except possibly at the i^{th} vertex. Note that such paths occur at most in pairs, since there are no more than two orders in which one can place the two boxes that are added in going from vertex $i-1$ to vertex $i+1$ (if the boxes are added in the same row, for example, then there is only one admissible order and thus only one path). The paths which are mixed transform into each other by means of unitary matrices and since the diagrams all become periodic after a while, one needs only to find a finite number of such matrices (see also section 2.6.2 for this). To find these matrices exactly, one should calculate the $6j$-symbols of $U_q(sl(m))$. However, we will not do this here, but instead take a short cut by using the fact that all the braid group representations we need are related to representations of the Hecke algebra $H_{n,q}$, whose representation theory has been well studied.

2.5.3 The Hecke algebra $H_{n,q}$

In this paragraph, we give a short description of an algebra which plays an important role in our understanding of the braiding of $U_q(sl(m))$-representations: the Hecke algebra $H_{n,q}$. We will also describe the irreps of this algebra that are relevant to us.

$H_{n,q}$ may be defined as the complex algebra with generators $1, g_1, g_2, \ldots, g_{n-1}$, subject to the relations

$$\begin{aligned} g_i g_j &= g_j g_i \qquad (|i-j| \geq 2) \\ g_i g_{i+1} g_i &= g_{i+1} g_i g_{i+1} \\ g_i^2 &= (q-1) g^i + q \end{aligned} \tag{2.99}$$

From these relations, we see that $H_{n,q}$ is a q-deformation of the group algebra $\mathbb{C}S_n$ of the symmetric group, to which it reduces at $q = 1$. The reason that the Hecke algebra comes into play in the braiding of $U_q(sl(2))$ representations is that the exchange matrix for the fundamental representation of $U_q(sl(m))$, given in (2.98), satisfies the following extra relation next to the braiding relation given in (1.17):

$$(\sigma R^{e_1,e_1})^2 = (q^{\frac{m-1}{2m}} - q^{\frac{-m-1}{2m}}) \sigma R^{e_1,e_1} + q^{-1/m} (1 \otimes 1) \tag{2.100}$$

As a consequence of this relation, the braid group representation that can be constructed from the R-matrix also gives a representation of the Hecke algebra $H_{n,q}$. This representation is given by the prescription

$$g_i \mapsto q^{\frac{m+1}{2m}} (R^{e_1,e_1})_{i,i+1} \tag{2.101}$$

and one may easily verify that the defining properties of the R-matrix and (2.100) guarantee that the relations (2.99) for the g_i are satisfied.

The representations of the Hecke algebra which are induced by $U_q(sl(m))$ in this way all factor over a quotient of the Hecke algebra, the so called m-row quotient. This is because the exchange matrix (2.98) satisfies even further relations apart from the ones already given. For example, the 2-row quotient of the Hecke algebra (which is also called the Temperley-Lieb-Jones algebra) can be defined by adding the following relations to those given in (2.99):

$$1 + g_i + g_{i+1} + g_i g_{i+1} + g_{i+1} g_i + g_i g_{i+1} g_i = 0 \tag{2.102}$$

and one may check that the matrix $q^{3/4}\, R^{e_1,e_1}$ for $U_q(sl(2))$ satisfies the corresponding equation. Note that the situation we are describing already occurs in the $q = 1$ case. In that case, we are describing representations of $\mathbb{C}S_n$ in which the $m+1$-row antisymmetrizer vanishes (as it should do for exchanges in a tensor product of m-dimensional spaces). The Young tableaus for these representations thus have at most m rows. The equation above indeed just says that the 3-row antisymmetrizer vanishes, if one identifies the generators σ_i of the permutation group with the elements $-g_i$ of the Hecke algebra at $q = 1$. We have kept this minus sign for better compatibility with the literature on Hecke algebra representations (for example [78]). The relations that need to be added to the Hecke algebra relations to obtain the general m-row quotient may be written similarly as above; they are just the equations that say that the $m + 1$-row antisymmetrizers vanish, with each σ_i replaced by $-g_i$.

The representation theory of the Hecke algebra is analogous to that of the group algebra of the symmetric group as long as q is not a root of unity, but when q is a root of unity (and $q \neq 1$), there are complications, similar to those that arise in the representation theory of $U_q(sl(m))$. In particular, the representation ring of $H_{n,q}$ is no longer semisimple for these values of q. The same is true for the representation ring of the m-row quotients. However, one may restrict oneself to representations that factor over a certain subquotient of the m-row quotient and these representations do form a semisimple ring. We will now give a quick description of the irreducible representations of $H_{n,q}$ at $q = e^{2\pi i/(k+2)}$, that factor over this semisimple quotient. These representations (among many others) have been constructed by Wenzl [78] and, independently, by Ocneanu [89]. Another relevant early reference is [90]. They are q-deformations of Young's orthogonal representations of the symmetric group (see for example [91]). Each one of the representations we are interested in is characterized by a Young diagram Y that has at most m rows and at most k columns of length less than m. The module of the representation characterized by Y is the module generated by all paths on the Bratteli diagram of the fundamental representation of $U_q(sl(m))$ that start at the empty diagram and end at Y. Thus, we do indeed get the same representation modules that we described in section 2.5.2. Also, the action of the elementary exchanges is of the kind we described in section 2.5.2; g_i does not mix paths that differ in any place other than at their i^{th} vertex. Using the work of Wenzl and Ocneanu, we may now write down the matrices through which g_i acts on the spaces of paths that do differ only at this vertex. Let us denote the Young diagram at the i^{th} vertex of the path p as $Y_p^{(i)}$, so that $p = (Y_p^{(1)}, Y_p^{(2)}, \ldots, Y_p^{(n)} = Y)$. Then two paths p and p' can only be mapped into each other by g_i if one has $Y_p^{(n)} = Y_{p'}^{(n)}$ for all $n \neq i$. As we have already remarked, the spaces of paths that are mapped into each other by g_i are at most two dimensional, since the last two boxes that are added in going from the Young diagram at vertex $i-1$ to that at vertex $i+1$ can be added in at most two different orders. Suppose that there are indeed two admissible orders. These two orders then correspond to two paths p and p' that form a basis for the space that we are interested in. One may define the distance between the two boxes involved as the number of hops from box to box that one has to make if one walks from the first to the second box along the right hand side of the Young diagram $Y_p^{(i+1)}$. One may also define a signed version of this distance, which we will denote $d_{p,i}$. Suppose that the first box is added in row r_1 and column c_1 of the Young diagram $Y_p^{(i)}$ and that the second is added in row r_2 and column c_2 of the Young diagram $Y_p^{(i+1)}$, then this signed distance is given by

$$d_{p,i} = r_2 - r_1 + c_1 - c_2 \tag{2.103}$$

Clearly, this is equal to the ordinary distance if the first box is located higher up and more to the right than the second. In the opposite case the formula above gives minus the distance. Using this signed distance, we may now write the action of the exchange g_i on the path p as

$$g_i p = -\frac{q^{(1-d_{p,i})/2}}{\lfloor d_{p,i} \rfloor_q} p + \operatorname{sign}(d_{p,i}) \frac{\sqrt{\lfloor d_{p,i}+1 \rfloor_q \lfloor d_{p,i}-1 \rfloor_q}}{\lfloor d_{p,i} \rfloor_q} p' \tag{2.104}$$

Hence if $d_{p,i} > 0$ (which may be achieved by exchanging p and p' if necessary), then the matrix for the action of the exchange g_i on the basis $\{p, p'\}$ is given by

$$g_i \equiv -\frac{q^{1/2}}{\lfloor d_{p,i} \rfloor_q} \begin{pmatrix} q^{-d_{p,i}/2} & \sqrt{\lfloor d_{p,i}+1 \rfloor_q \lfloor d_{p,i}-1 \rfloor_q} \\ \sqrt{\lfloor d_{p,i}+1 \rfloor_q \lfloor d_{p,i}-1 \rfloor_q} & -q^{d_{p,i}/2} \end{pmatrix} \tag{2.105}$$

and one may check easily that this matrix is symmetric and unitary.

The action of the exchange g_i on a path p that does not have a partner path, i.e. for which there is no other path $p' \neq p$ such that $Y_p^{(n)} = Y_{p'}^{(n)}$ for all $n \neq i$, is just multiplication by a phase factor. This phase factor equals q if $d_{p,i} = -1$, which means that the two boxes were added in the same row, and it equals -1 if $d_{p,i} = 1$, which means the boxes were added to the same column, or if $d_{p,i} = k+1$, which happens at the "edges" of the fusion diagrams.

Clearly, the representations we have just described are the right ones for the description of the braiding of particles with a hidden $U_q(sl(m))$-symmetry. For a system of n particles with overall quantum group charge Λ, we need the representation labeled by the Young diagram that consists of n boxes and which reduces to the Young diagram of the charge Λ on removal of all columns with m boxes. With this correspondence, one may indeed check that the formulae given in this section are the same as the ones we gave for $U_q(sl(2))$ in section 2.4.10, up to the phase factor in (2.101). Using the explicit form of the exchange matrices, it is not difficult to prove some nice mathematical properties of the representations above. For example, Wenzl has proved [78] that they are irreducible and that representations labeled by different Young diagrams are non-isomorphic. The argument used in the proof is essentially the same as the one we described at the end of section 2.4.10 for the case of $U_q(sl(2))$.

2.5.4 A quantum group for the chiral boson

In this section, we will indicate how a quantum group may reproduce the fusion and braiding of the CFT that describes a chiral boson on a circle of radius $\sqrt{2p}$, where $p \in \mathbb{Z}$. This CFT has $2p$ chiral primary fields, which are the vertex operators $\nu_l = e^{il\phi/\sqrt{2p}}$, for $l \in \{-p+1, \ldots, p\}$. The vertex operator ν_l has conformal weight $\frac{l^2}{4p}$ and the fusion is very simple, one has

$$\nu_{l_1} \times \nu_{l_2} = \nu_{l_1+l_2 \bmod 2p}. \tag{2.106}$$

All conformal blocks of primaries may be calculated explicitly, giving

$$\langle \nu_{l_1}(z_1), \ldots, \nu_{l_n}(z_n) \rangle = \prod_{i<j} (z_i - z_j)^{l_i l_j/(2p)}. \tag{2.107}$$

It follows that the braiding of these blocks is Abelian; the half-monodromy which takes a ν_{l_1} around a ν_{l_2} gives the block a factor of $e^{il_1 l_2 \pi/(2p)}$.

We would like to reproduce the above data through a quantum group, that is we would like to find a quantum group which has $2p$ irreducible representations whose fusion rules and braiding are identical to those of the boson vertex operators. It turns out that this cannot be achieved by a quantum group whose coassociator is trivial. This can be most easily seen in the case $p = 1$. In this case, we would need a quantum group with two representations π_0 and π_1 satisfying

$$\begin{aligned} \pi_0 \otimes \pi_0 &= \pi_0 \\ \pi_0 \otimes \pi_1 &= \pi_1 \\ \pi_1 \otimes \pi_1 &= \pi_0. \end{aligned} \tag{2.108}$$

Now if we look at the threefold tensor product $\pi_1 \otimes \pi_1 \otimes \pi_1$, then we know on the one hand that braiding the left π_1 over the other two must give two factors of $e^{i\pi/2}$ yielding a total factor of $e^{i\pi} = -1$. On the other hand, the braiding factor F_b is also given by the following formula (which is only valid if the coassociator is trivial)

$$F_b = \pi_1 \otimes \pi_1 \otimes \pi_1((1 \otimes \Delta)(R)) = \pi_1 \otimes \pi_0(R). \tag{2.109}$$

Here we have used the information that the two rightmost representations must fuse to π_0. We are thus really just exchanging the left π_1 over a π_0 and this should give a factor of $+1$, which yields a contradiction. To describe the chiral boson, we should thus either use a quantum group with a non-trivial coassociator or relax the demands on the correspondence between quantum group and CFT. A good candidate for a non-coassociative quantum group would be $U_q(sl(2p))$ at $q = e^{2i\pi/3}$ (or $k = 1$). This weak quasi-quantum group does have $2p$ representations with the right fusion rules and the braiding for the fundamental representation does reproduce that for the vertex operator ν_1, up to a trivial scalar factor in every exchange. However, checking the correctness of the braiding for all the other representations of $U_q(sl(m))$ seems to be rather complicated and therefore we will choose a different approach.

If we relax the demands on our quantum group such that more than one quantum group representation may correspond to the same charge sector in the CFT, then we can reproduce the braiding of the chiral boson CFT by a very simple finite dimensional coassociative quantum group. As a Hopf algebra this quantum group is the group algebra of the cyclic group $\mathbb{Z}_{4p}$. A convenient basis for this algebra is given by the primitive idempotents $e_0, e_1, \ldots, e_{4p-1}$ which project on the isotypical components of the representations of Z_{4p} in the group algebra. In formulae: the e_i are elements that satisfy

$$e_i e_j = \delta_{ij} e_i \tag{2.110}$$

and the full set $\pi_0, \ldots, \pi_{2m-1}$ of irreps of $\mathbb{Z}_{4p}$ is given by

$$\pi_i(e_j) = \delta_{ij}. \tag{2.111}$$

The coproduct reads

$$\Delta(e_j) = \sum_i e_i \otimes e_{j-i \bmod 4p} \tag{2.112}$$

and one may easily check that this leads to the fusion rules

$$\pi_i \otimes \pi_j = \pi_{i+j \bmod 4p}. \tag{2.113}$$

Counit and antipode are given by

$$\epsilon(e_i) = \pi_0(e_i) = \delta_{0i}, \qquad S(e_i) = e_{4p-i}. \tag{2.114}$$

So far, we have just described the group algebra of $\mathbb{Z}_{4p}$ in terms of the e_i. Now let us introduce an R-matrix. One may easily check that the most general R-matrix which satisfies the requirements (1.13),(1.14) and (1.15) is of the form

$$R = \sum_{i,j} q^{ij} e_i \otimes e_j \tag{2.115}$$

where q is a $4p^{\text{th}}$ root of unity. If we take $q = 1$, then this is just the identity on $\mathbb{C}\mathbb{Z}_{4p} \otimes \mathbb{C}\mathbb{Z}_{4p}$ and the braiding is trivial. On the other hand, if $q \neq 1$ then we have non-trivial braiding and the braiding factor we get when taking a π_i around a π_j will be q^{ij}.

Let us call the quantum group we have just described $\mathbb{C}\mathbb{Z}_{4p,q}$. The correspondence between this simple quantum group and the chiral boson CFT can now be made as follows. The chiral sector corresponding to ν_l is represented by the two quantum group representations π_l and π_{l+2p}. Of these, the π_l with $0 \leq l < 2p$ represent the primary fields and their even conformal descendants, while the π_l with $2p \leq l < 4p$ represent all the odd descendants. The fusion rules of the quantum group are then consistent with those of the CFT. In particular, it is impossible to distinguish particles represented by the representations π^l and π^{l+2p} by means of the tensor product decomposition rules for these representations, just as it is impossible to distinguish the primary field ν_l from one of its descendants by means of the CFTs fusion rules. The braiding is also correct, if we choose $q = e^{i\pi/(2p)}$. The braiding factors we get for two different representatives of the same CFT-sector may now differ by a minus sign, but this is in fact just what we want. To clarify this, let us look once more at the example we gave for the case $p = 1$. For this case, the vacuum sector will now be represented by the representation π_0 and also by the representation π_2, but if we exchange a π_1 with a π_0 then we get a factor of 1, while if we exchange a π_1 with a π_2, we get factor of -1. We already know that if we have three ν_1 fields and we exchange the first over the last two, we will get a factor of -1. This is due to the fact that the correlator (2.107) will have a single zero at any point where two ν_1-fields are brought together. If one would just place a ν_0 at this point there would be no such zero and hence also no braiding factor. Hence we can think of π_0 as representing the vacuum sector, while we can think of π_2 as representing a charge-neutral bound state of two ν_1 fields.

2.5.5 Quantum groups for the parafermions

In this section, we shall describe quantum groups for the $\mathbb{Z}_k$-parafermion conformal field theory that is used in the description of the RR-states. Since there are two different coset descriptions of this CFT (cf sections 2.3.1 and 2.3.1), one can also expect to get two different quantum group theoretic descriptions.

The quantum group for $\widehat{sl(2)}_k/\widehat{U(1)}_k$

Let us start with the coset $\widehat{sl(2)}_k/\widehat{U(1)}_k$. For this coset, we have the factorisation formula (2.26), which describes a Virasoro primary field of the $\widehat{sl(2)}_k$ theory as a product of a parafermion field and a vertex operator for a chiral boson that lives on a circle of radius $\sqrt{2k}$. We already used this formula to explain the conformal weights and fusion rules of the parafermions and clearly, it can also be used to calculate braidings. To see this, look at the following equality of correlators which follows from (2.29):

$$\left\langle G^{\Lambda_1}_{\lambda_1}(z_1), \ldots, G^{\Lambda_n}_{\lambda_n}(z_n) \right\rangle = \left\langle \Phi^{\Lambda_1}_{\lambda_1}(z_1), \ldots, \Phi^{\Lambda_n}_{\lambda_n}(z_n) \right\rangle \left\langle e^{i\lambda_1\phi}(z_1), \ldots, e^{\lambda_n\phi}(z_n) \right\rangle \tag{2.116}$$

The braiding on the left hand side of this equation is just a braiding of $\widehat{sl(2)}_k$-fields and the matrices which describe this are known to be the same matrices that describe the braiding of $U_q(sl(2))$ representations at $q = e^{2i\pi/(k+2)}$ (the labels λ_i do not play a role in the braiding). The braiding on the right hand side will be described by matrices which are products of a matrix for braiding the parafermions and a known scalar factor for braiding the boson's vertex operators. Thus, we may obtain the braiding matrices for the parafermion fields by just bringing the scalar factor obtained from the bosonic correlator to the left.

The braiding matrices which are obtained from this recipe are the same braiding matrices that one gets for the quantum group $\mathcal{A}_{q_1,q_2} := U_{q_1}(sl(2)) \otimes \mathbb{C}\mathbb{Z}_{4k,q_2}$, where $q_1 = e^{2i\pi/(k+2)}$ and $q_2 = e^{-i\pi/2k}$. The irreducible representations of this quantum group are tensor products of $U_{q_1}(sl(2))$-irreps and $\mathbb{C}\mathbb{Z}_{4k,q_2}$-irreps and hence they are labeled by an $sl(2)$-weight $0 \leq \Lambda \leq k$ and an integer $0 \leq \lambda < 4k$. We will write these representations as π^{Λ}_{λ}. The representation π^{Λ}_{λ}

will represent the $\Phi^{\Lambda}_{\lambda \bmod 2k}$-sector of the parafermion CFT. As in the case of the chiral boson we thus have more than one quantum group representation that corresponds to the same sector of the CFT. In fact, we now have four quantum group representations for every sector of the CFT, since not only have we doubled the period of the label λ (as we did for the boson), but we have also not taken the second of the field identifications (2.28) into account. Looking at this identification, we see that the labels (Λ, λ) and $(k - \Lambda, \lambda - k)$, that should be identified, usually stand for quantum group representations of different dimensions ($\Lambda + 1$ and $k - \Lambda + 1$ respectively), although their quantum dimensions are equal ($\lfloor \Lambda + 1 \rfloor_q = \lfloor k + 2 - (\Lambda + 1) \rfloor_q = \lfloor k - \Lambda + 1 \rfloor_q$). Nevertheless, we believe that the quantum group $\mathcal{A}_{q_1,q_2}$ will give a good description of the CFTs braiding properties. To motivate this statement, let us first look at the tensor product decomposition of $\mathcal{A}_{q_1,q_2}$. This is given by

$$\pi^{\Lambda}_{\lambda} \hat{\otimes} \pi^{\Lambda'}_{\lambda'} = \bigoplus_{\Lambda''=|\Lambda-\Lambda'|}^{\min\{\Lambda+\Lambda', 2k-\Lambda-\Lambda'\}} \pi^{\Lambda''}_{\lambda+\lambda'}, \tag{2.117}$$

which is the same as the fusion rules (2.29) for the parafermions, except that the identifications (2.28) are not incorporated. Nevertheless, using the formulae (2.58) for the truncated tensor product of the $U_{q_1}(sl(2))$-representations, one can see that it is impossible to distinguish particles in representations that correspond to the same parafermion sector by means of these fusion rules alone. In other words, it is consistent with these fusion rules to declare that particles in the representations π^{Λ}_{λ}, $\pi^{\Lambda}_{\lambda+2k}$ and $\pi^{k-\Lambda}_{\lambda\pm k}$ are indistinguishable, just as it is consistent with the fusion rules of a CFT to declare all descendants of a field indistinguishable.

Now let us look at the braiding of $\mathcal{A}_{q_1,q_2}$-representations. To describe this braiding, we can use the bases that we introduced for $U_{q_1}(sl(2))$ in section (2.4.8), because the representations of $\mathbb{CZ}_{4k,q_2}$ are one-dimensional. The matrices that describe the braiding w.r.t. these bases will be the product of the matrices for $U_{q_1}(sl(2))$ that we gave in (2.88) with the powers of q_2 that we get from the R-matrix (2.115) for $\mathbb{CZ}_{4k,q_2}$. Using the symmetries (2.77) of the truncated $6j$-symbols, one may then check that, when one changes the representations which represent CFT-sectors in a way which is consistent with the quantum group's fusion rules, the elements of the braiding matrices will at most get minus signs. Again, this situation is similar to the situation for the chiral boson that we discussed in section 2.5.4.

We are now left with the difficulty of choosing the quantum group representations which should represent the fields Φ^0_2 and Φ^1_1, which are important for the description of electrons and quasiholes in the RR-states. We will use the representations π^0_2 and π^1_1 for this (rather than for example π^k_{k-2} and π^1_{k+1}). This choice keeps comparison to the CFT-picture easy and it gives good results. Also, it gives results which are consistent with those of the quantum group for the coset $\widehat{sl(k)}_1 \times \widehat{sl(k)}_1/\widehat{sl(k)}_2$, for which there is a one-to-one correspondence between quantum group representations and CFT-sectors.

The quantum group for $\widehat{sl(k)}_1 \times \widehat{sl(k)}_1/\widehat{sl(k)}_2$

For the coset $\widehat{sl(k)}_1 \times \widehat{sl(k)}_1/\widehat{sl(k)}_2$, we do not have a factorisation formula like (2.26) and therefore we cannot find the braiding matrices for this coset by the method we used for $\widehat{sl(2)}_k/\widehat{U(1)}_k$ in the previous section (cf. formula (2.116)). Still, the results of the previous section and also the fusion rules and modular properties [10] of $\widehat{sl(k)}_1 \times \widehat{sl(k)}_1/\widehat{sl(k)}_2$ suggest a natural candidate for a quantum group related to this coset: the quantum group $U_{q_3}(sl(k)) \otimes U_{q_3}(sl(k)) \otimes U_{q_4}(sl(k))$ with $q_3 = e^{i\pi/(k+1)}$ and $q_4 = e^{-i\pi/(k+2)} = (q_1)^{-1}$. The irreducible representations of this quantum group are tensor products of those of the factors and hence they are labeled by two $\widehat{sl(k)}_1$-weights and an $\widehat{sl(k)}_2$-weight, just like the fields $\Phi_\mu^{\mu_1,\mu_2}$ of section 2.3.1. Let us thus denote the quantum group irreps as $\pi_\mu^{\mu_1,\mu_2}$. We are now in the same situation that we encountered in the case of the coset $\widehat{sl(2)}_k/\widehat{U(1)}_k$; we have several quantum group representations per CFT-sector, because the quantum group does not take the identifications (2.32) into account. However, in this case, there is a subset of representations of the quantum group which closes under fusion and which contains exactly one representation for each CFT-sector. In fact, there are two such subsets: the set of $\pi_\mu^{\mu_1,\mu_2}$ with $\mu_1 = 0$ and the set of $\pi_\mu^{\mu_1,\mu_2}$ with $\mu_2 = 0$. If we restrict to one of these sets (clearly, it does not matter which of the two we use), then we are effectively forgetting about one of the $U_{q_3}(sl(k))$-factors of the quantum group and hence we may say that the quantum group for the coset $\widehat{sl(k)}_1 \times \widehat{sl(k)}_1/\widehat{sl(k)}_2$ is $\mathcal{B}_{q_3,q_4} := U_{q_3}(sl(k)) \otimes U_{q_4}(sl(k))$. The irreducible representations of this quantum group are labeled by an $\widehat{sl(k)}_1$-weight μ_1 and an $\widehat{sl(k)}_2$-weight μ. For the irreps that are relevant to the description of the coset CFT, the $\widehat{sl(k)}_1$-weight is uniquely determined by the $\widehat{sl(k)}_2$-weight through the branching rule (2.30), so that we may choose to label the relevant irreps by just a single $\widehat{sl(k)}_2$-weight μ. We may write these representations π_μ and they are in one-to one correspondence with the fields Φ_μ we defined in section (2.3.1). Clearly, the fusion rules of the π_μ are the same as those of the Φ_μ; they are identical to the fusion rules for the corresponding $\widehat{sl(k)}_2$-fields or $U_{q2}(sl(k))$-representations. We have not checked if all the braiding representations we get from the quantum group $U_{q_3}(sl(k)) \otimes U_{q_4}(sl(k))$ are equivalent to those one gets from the quantum group $\mathcal{A}_{q_1,q_2}$ of the previous section, but we do know this for the representations that are related to the quasiholes of the RR-states. In section 2.5.5, the quasihole was represented by the irrep π_1^1 of $\mathcal{A}_{q_1,q_2}$, whereas here, it must clearly be represented by the irrep π_{e_1} of $\mathcal{B}_{q_3,q_4}$ (for the notation e_1, see section 2.3.1). Explicit calculation of braiding matrices, using formulae (2.105), (2.101) and (2.115) shows that the braid group representations related to these irreps are indeed equivalent. Rather than writing out all these calculations in detail here, we will make some remarks which make this result very plausible. First of all, the braid group representations we get from the tensor products $(\pi_1^1)^{\otimes n}$ and $\pi_{e_1}^{\otimes n}$ have the same fusion diagram associated to them. This guarantees for example that the representations will have a similar structure (see the discussion at the end of section 2.5.2) and in particular that their dimensions are equal. Second, the eigenvalues of the matrices that represent the fundamental exchanges may be easily found if we note that the representation matrices of the canonical Hecke algebra generators always have eigenvalues -1 and q (this follows directly from the last relation in (2.99)). Thus, if we denote the eigenvalues of the braiding for π_1^1 by α_1 and α_2, then we have, using (2.101) and (2.115):

$$\begin{aligned} \alpha_1 &= (q_1)^{-3/4} q_1 q_2 = e^{\frac{-i\pi}{k(k+2)}} \\ \alpha_2 &= (q_1)^{-3/4}(-1) q_2 = -e^{\frac{-i(2k+1)\pi}{k(k+2)}} \end{aligned} \tag{2.118}$$

[10]For modular properties of cosets, one can consult for example [65],[15]. The relationship between modular and braiding properties of CFTs and quantum groups is clarified in [68].

On the other hand, the eigenvalues β_1 and β_2 for the braiding associated with π_{e_1} can be found using (2.101) and this yields

$$\begin{aligned}\beta_1 &= (q_4)^{-\frac{k+1}{2k}}(-1)(q_2)^{-\frac{k+1}{2k}}q_2 = -e^{\frac{-i(2k+1)\pi}{k(k+2)}} = \alpha_2 \\ \beta_2 &= (q_4)^{-\frac{k+1}{2k}}(-1)(q_2)^{-\frac{k+1}{2k}}(-1) = e^{\frac{-i\pi}{k(k+2)}} = \alpha_1 \end{aligned} \tag{2.119}$$

so that the eigenvalues of the braidings are equal, as they should be.

2.6 Quantum group picture and braiding for the RR-states

In this section, we will describe the Read-Rezayi states as systems of point particles with a hidden quantum group symmetry. We also give an explicit description of braiding representations that are associated with these states. All of this will be done in subsection 2.6.1. In section 2.6.2, we will give an alternative description of the resulting braid group representations, which does not make any explicit use of quantum groups. In this description, it is also somewhat easier to change the number of quasiholes in the system. Finally, in section 2.6.3, we show how the results a of Nayak and Wilczek [17] for the Pfaffian state arise as a special case.

2.6.1 RR-states and hidden quantum group symmetry

In section 2.3.2, we described the RR-states as conformal blocks in a CFT which was the tensor product of the parafermion CFT and a CFT for a chiral boson. In section 2.5.5, we derived a relation between the parafermion CFT and the quantum groups $\mathcal{A}_{q_1,q_2} = U_{q_1}(sl(2)) \otimes \mathbb{CZ}_{4k,q_2}$ and $\mathcal{B}_{q_3,q_4} = U_{q_3}(sl(k)) \otimes U_{q_4}(sl(k))$. In section 2.5.4, we gave a quantum group for the chiral boson. Clearly, we can thus make a quantum group which will describe fusion and braiding for the Read-Rezayi states by tensoring the parafermion and boson quantum groups. However, since the extra boson factor does not affect the fusion of the relevant representations and only adds some scalar factors to the braiding matrices, we will choose to work with $\mathcal{A}_{q_1,q_2}$ and $\mathcal{B}_{q_3,q_4}$ and to add the scalar factors by hand. Thus, we see the following picture of the RR-states emerge. The RR-system of electrons and quasiholes can be seen as a system of point particles with hidden quantum group symmetry (cf. section 2.4.9). The electrons, which were represented by the operator $\psi = \Phi_2^0 = \Phi_{2e_1}$ in the CFT-picture, are now point particles which carry the representation π_2^0 of $\mathcal{A}_{q_1,q_2}$ or the representation π_{2e_1} of $\mathcal{B}_{q_3,q_4}$. Similarly, the quasiholes, which used to be represented by the field $\sigma = \Phi_1^1 = \Phi_{e_1}$, now carry the representation π_1^1 of $\mathcal{A}_{q_1,q_2}$ or the representation π_{e_1} of $\mathcal{B}_{q_3,q_4}$. The state of an RR-system with N electrons and n quasiholes may then be described as a vector in the tensor product of N π_2^0 (or π_{2e_1}) modules and n π_1^1 (or π_{e_1}) modules. However, not all of the vectors in this tensor product correspond to physical states. First of all, we have to restrict to the states in a truncated tensor product with a given bracketing, as explained in sections 2.4.3 and 2.4.6. Second, there is a restriction that comes from the fact that the conformal block in (2.37) vanishes unless all the fields that appear in it fuse into the vacuum sector. This now means that the system as a whole is in one of the $\mathcal{A}_{q_1,q_2}$-representations π_0^0, π_k^k or in the $\mathcal{B}_{q_3,q_4}$-representation π_0. Thus, the physical states in the tensor product are those that lie in a truncated tensor product and are in a global quantum group representation that corresponds to the CFT's vacuum sector. The second condition has the same consequence as the corresponding condition for the CFT; one has to have $N + n$ equal to zero modulo k, because otherwise there are no states that fulfill this condition. The condition can be interpreted as saying that the incorporation of more electrons in an RR-system and the creation of quasiholes in such a system are $\mathcal{A}$-charge preserving processes. It follows as in the CFT-picture that quasiholes may only be created in multiples of k at a time (if the number of electrons is kept fixed).

Now let us look at the braiding of electrons and quasiholes. For convenience, we will do this in terms of $\mathcal{A}_{q_1,q_2}$, but the treatment in terms of $\mathcal{B}_{q_3,q_4}$ will give equivalent results (see the discussion at the end of section 2.5.5). Since the representation π_2^0 is one dimensional, the braiding between electrons is Abelian. This means any exchange of electrons will just give a phase factor. To find this factor, one may use the formulae (1.39) and (1.28) for the universal R-matrix and for the representations of $U_{q_1}(sl(2))$, the analogous formulae (2.115) and (2.111) for the universal R-matrix of $\mathbb{CZ}_{4k,q_2}$, and the explicit factors from the boson vertex operators which appear in the expression (2.37) for the wave function. All this together just gives a factor of -1, as is appropriate for fermions. Similarly, one may show that there is no non-Abelian braiding between electrons and quasiholes and that the braiding factor for electron-quasihole exchanges is equal to one. Hence, as far as the braiding is concerned, the electrons and quasiholes can be treated separately. Since only the quasiholes have non-Abelian braiding, we will from now on focus on these.

The braiding associated to a system of identical particles with hidden quantum group symmetry can be elegantly described in terms of a basis that is labeled by the paths on the fusion diagram of the quantum group representation carried by the particles (we described this in detail in sections 2.4.8 to 2.4.10). The quasiholes of the RR-states carry the $\mathcal{A}_{q_1,q_2}$-representation π_1^1 and the fusion diagram for this representation is the same as that for the parafermionic field Φ_1^1, which is in turn the same as the fusion diagram for the spin-$\frac{1}{2}$-representation of $U_{q_1}(sl(2))$. The braiding representations associated to this $U_q(sl(2))$-representation were described in detail in section 2.4.10 (and they were also included in the material of section 2.5.3). The only difference between the braid representations described there and the braiding for the RR-quasiholes lie in a scalar factor for every exchange, which comes from the $\mathbb{CZ}_{4k,q_2}$ part of $\mathcal{A}_{q_1,q_2}$ and from the explicit factors in the wave function (2.37). Thus, a basis for the space of states with n quasiholes in fixed positions is labeled by the paths on the fusion diagram of figure 2.1 which start at the point $(0,0)$ and which end at the point (n,Λ), where $\Lambda = -N \bmod k$, so that the $\mathcal{A}_{q_1,q_2}$-charge of the whole system corresponds to the vacuum sector of the CFT. We will call the braid group representation on this space ρ_n^Λ. In this representation, the braid group generator τ_m will only mix paths which differ from each other only at the m^{th} node. Given any path p, there will be at most one path p' which differs from p only at the m^{th} node and we will call this the partner path of p at node m. If a path does not have a partner path at node m, then the action of τ_m on this path will be multiplication by a scalar factor. To give this factor, let us first take $q = e^{2\pi i/(k+2)}$, so that we have

$$q_1 = q, \quad q_2 = q^{-\frac{1}{2}-\frac{1}{k}}. \tag{2.120}$$

The path will then get a factor of $-q^{-1+\frac{1-M}{2(kM+2)}}$ if it changes direction at the m^{th} node and a factor of $q^{\frac{1-M}{2(kM+2)}}$ otherwise. When M takes its lowest physical value of 1, these factors reduce to $-q^{-1}$ and 1 respectively. If a path does have a partner path at its m^{th} node, then the path and its partner path will have the same representations at nodes $m-1$ and $m+1$ and these representations will have the same dimension. Let us call this dimension d. The exchange τ_m will then act on the vector space generated by the path and its partner path through the matrix

$$\rho_n^\Lambda(\tau_m) \equiv \frac{-q^{-\frac{1}{2}+\frac{1-M}{2(kM+2)}}}{\lfloor d \rfloor_q} \begin{pmatrix} q^{-d/2} & \sqrt{\lfloor d+1 \rfloor_q \lfloor d-1 \rfloor_q} \\ \sqrt{\lfloor d+1 \rfloor_q \lfloor d-1 \rfloor_q} & -q^{d/2} \end{pmatrix}. \tag{2.121}$$

Here we have ordered the basis so that the first of the basis vectors corresponds to the path with the highest representation at node m of the diagram. The matrix above is symmetric and unitary and should be compared with (2.95). The braid group representations ρ_n^Λ are irreducible by the same arguments as those we used for the braid group representations associated with $U_q(sl(2))$. Information on their dimensions has been gathered in section 2.3.4.

2.6.2 Tensor product description of the braiding

We will now set up an alternative description for the braid group representations ρ_n^Λ of the previous section. In this description, the representation spaces are seen as subspaces of n-fold tensor product spaces. This makes it somewhat closer in spirit to the description Nayak and Wilczek have given of the braiding for the Pfaffian state [17], a fact we will utilize in section 2.6.3. We also feel that the description of this section is useful in itself, because it shows very clearly how braidings in systems with an arbitrary number of quasiholes can be performed by a recipe that depends very little on this number.

Let us start by defining $V_{k,l}$ to be the k-dimensional vector space spanned by orthonormal vectors which represent the possible l^{th} steps in a path on the fusion diagram of figure 2.1. We write:

$$V_{k,l} = \begin{cases} \text{Span}\{v_{0,1}, v_{2,3}, \ldots, v_{k-2,k-1}, v_{2,1}, v_{4,3}, \ldots, v_{k,k-1}\} & (k \text{ even }, l \text{ odd}) \\ \text{Span}\{v_{1,2}, v_{3,4}, \ldots, v_{k-1,k}, v_{1,0}, v_{3,2}, \ldots, v_{k-1,k-2}\} & (k \text{ even }, l \text{ even}) \\ \text{Span}\{v_{0,1}, v_{2,3}, \ldots, v_{k-1,k}, v_{2,1}, v_{4,3}, \ldots, v_{k-1,k-2}\} & (k \text{ odd }, l \text{ odd}) \\ \text{Span}\{v_{1,2}, v_{3,4}, \ldots, v_{k-2,k-1}, v_{1,0}, v_{3,2}, \ldots, v_{k,k-1}\} & (k \text{ odd }, l \text{ even}) \end{cases} \tag{2.122}$$

Here the indices on each basis vector represent the weights at the starting points and end points of the piece of path represented by the vector. In order to simplify the description, we have also, for $l \leq k$, included some vectors which do not actually correspond to bits of path in the fusion diagram of figure 2.1 (for example the vector $v_{2,1}$ at $l = 1$). Clearly, any continuous path of length n through the fusion diagram may be represented by a canonical basis vector of the "domino" form $v_{\Lambda_1,\Lambda_2} \otimes v_{\Lambda_2,\Lambda_3} \otimes v_{\Lambda_3,\Lambda_4} \otimes \ldots \otimes v_{\Lambda_{n-1},\Lambda_n}$ in the tensor product space $V_{k,1} \otimes V_{k,2} \otimes \ldots \otimes V_{k,n}$. The paths in the representation space of ρ_n^Λ can be isolated by requiring $\Lambda_0 = 0$ and $\Lambda_n = \Lambda$.

We can now define a matrix representation $\Gamma_{k,n}$ of the relations (1.11) on the given tensor product space which has a very simple form and which reduces to the braid group representation ρ_n^Λ when one restricts to the subspace of the tensor product which corresponds to the paths in ρ_n^Λ. The action of the τ_i is defined as follows. $\Gamma_{k,n}(\tau_i)$ is always of the form $1 \otimes \ldots \otimes 1 \otimes R_{k,i} \otimes 1 \otimes \ldots \otimes 1$, where the matrix $R_{k,i}$ acts only on $V_{k,i} \otimes V_{k,i+1}$. On the basis vectors which are of domino form in the i^{th} and $(i+1)^{\text{th}}$ factor, we take

$$\begin{aligned} R_{k,i} v_{\Lambda_i,\Lambda_i+1} \otimes v_{\Lambda_i+1,\Lambda_i+2} &= \alpha v_{\Lambda_i,\Lambda_i+1} \otimes v_{\Lambda_i+1,\Lambda_i+2} \\ R_{k,i} v_{\Lambda_i,\Lambda_i-1} \otimes v_{\Lambda_i-1,\Lambda_i-2} &= \alpha v_{\Lambda_i,\Lambda_i-1} \otimes v_{\Lambda_i-1,\Lambda_i-2} \\ R_{k,i} v_{\Lambda_i,\Lambda_i+1} \otimes v_{\Lambda_i+1,\Lambda_i} &= \frac{-\alpha q^{-\frac{\Lambda_i}{2}-1}}{\lfloor \Lambda_i + 1 \rfloor_q} v_{\Lambda_i,\Lambda_i+1} \otimes v_{\Lambda_i+1,\Lambda_i} - \\ & \quad \frac{\alpha q^{-\frac{1}{2}} \sqrt{\lfloor \Lambda_i + 2 \rfloor_q \lfloor \Lambda_i \rfloor_q}}{\lfloor \Lambda_i + 1 \rfloor_q} v_{\Lambda_i,\Lambda_i+1} \otimes v_{\Lambda_i+1,\Lambda_i} \quad (1 \leq \Lambda_i \leq k-1) \end{aligned}$$

$$\begin{aligned}
R_{k,i}v_{\Lambda_i,\Lambda_i-1}\otimes v_{\Lambda_i-1,\Lambda_i} &= \frac{\alpha q^{\frac{\Lambda_i}{2}}}{\lfloor \Lambda_i+1\rfloor_q} v_{\Lambda_i,\Lambda_i-1}\otimes v_{\Lambda_i-1,\Lambda_i} - \\
&\quad \frac{\alpha q^{-\frac{1}{2}}\sqrt{\lfloor \Lambda_i+2\rfloor_q \lfloor \Lambda_i\rfloor_q}}{\lfloor \Lambda_i+1\rfloor_q} v_{\Lambda_i,\Lambda_i+1}\otimes v_{\Lambda_i+1,\Lambda_i} \quad (1\le \Lambda_i\le k-1)\\
R_{k,i}v_{\Lambda_i,\Lambda_i+1}\otimes v_{\Lambda_i+1,\Lambda_i} &= -\alpha q^{-1} v_{\Lambda_i,\Lambda_i+1}\otimes v_{\Lambda_i+1,\Lambda_i} \quad (\Lambda_i=0)\\
R_{k,i}v_{\Lambda_i,\Lambda_i-1}\otimes v_{\Lambda_i-1,\Lambda_i} &= -\alpha q^{-1} v_{\Lambda_i,\Lambda_i-1}\otimes v_{\Lambda_i-1,\Lambda_i} \quad (\Lambda_i=k)
\end{aligned} \tag{2.123}$$

where we have defined

$$\alpha = q^{\frac{1-M}{2(kM+2)}} \tag{2.124}$$

This factor of course reduces to 1 when $M=1$. For basis vectors v which are not of the domino form in $V_{k,i}\otimes V_{k,i+1}$, we define $R_{k,i}v=0$. With this definition, the matrices in $\Gamma_{k,n}$ satisfy the relations (1.11), but they are not invertible (of course they are invertible if we restrict to the space generated by vectors of the domino form). Alternatively, one may set $R_{k,i}v=v$. In that case, the matrices in $\Gamma_{k,n}$ are invertible, but they no longer satisfy the second relation in (1.11) (of course, they still do satisfy this relation on the "domino state space"). It is not difficult to see that, on the set of vectors which corresponds to the paths of ρ_n^Λ, the representation defined here indeed reduces to ρ_n^Λ. The matrices $R_{k,i}$, for given k, depend only on the parity of i, which means that knowledge of $R_{k,1}$ and $R_{k,2}$ is enough to determine the representation $\Gamma_{k,n}$ and hence also all the ρ_n^Λ completely. This is quite useful, because it gives us an easy way to go from a description of n particles to a description of $n+1$ particles; we just add another tensor factor and use the same matrices $R_{k,1}$ and $R_{k,2}$ as before to implement particle exchanges. We hope that this explicit recipe can be a small first step towards a second quantized description of particles with non-Abelian statistics.

2.6.3 Reproducing the results for the Pfaffian state

Now let us check that our results reproduce those of Nayak and Wilczek [17] for $k=2$, $n=2m$ even, N even and $M=1$. In this case, the relevant paths on the fusion diagram have to end at the coordinates $(0,2m)$ in case m is even and at the coordinates $(2,2m)$ in case m is odd. The fusion diagram for $k=2$ is given in figure 2.7 Each of these paths can be uniquely characterized

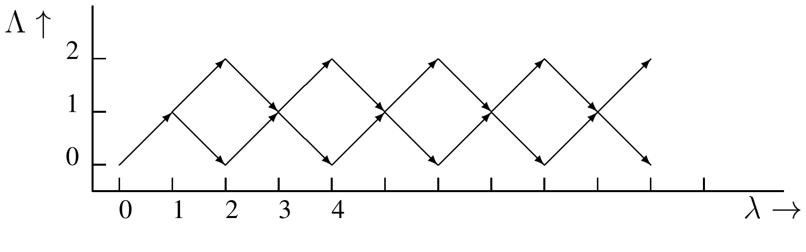

Figure 2.7: fusion diagram for the quasiholes at $k=2$. The diagram must be thought extended indefinitely in the λ-direction

by stating whether or not it changes direction at each of its odd numbered vertices. If m is even, then the paths have to change direction an even number of times in order to end up at the point $(0,2m)$. If m is odd then the number of changes of direction also has to be odd in order for the path to end up at the point $(2,2m)$. Thus, for m even, we may represent any path of length $2m$ by a ket $|\, s_1, s_2, \dots s_m\,\rangle$, where each of the s_i is a sign, a plus sign denoting a change of direction and a minus sign no change. The physically relevant paths are then the paths for which the product of all these signs is a plus sign. For m odd, we may do the same, but now with a minus sign denoting a change of direction and a plus sign denoting no change. The relevant states are then once more the ones whose overall sign is positive. Both for m odd and for m even, we thus describe a 2^{m-1} dimensional space whose basis vectors are labeled in the same way as those of Nayak and Wilczek. Just as Nayak and Wilczek have done, we will interpret this space as a subspace of an m-fold tensor product of two dimensional spaces, each of which has basis $\{|+\rangle, |-\rangle\}$. Now let us check that the action of the braiding matrices on these states is also the same as in [17].

For $k=2$, the tensor product $V_{k,1} \otimes V_{k,2}$ contains four states with the domino property: the states $v_{0,1} \otimes v_{1,0}$, $v_{0,1} \otimes v_{1,2}$, $v_{2,1} \otimes v_{1,2}$ and $v_{2,1} \otimes v_{1,0}$. Using these as an ordered basis of relevant states, the matrix $R_{k,1}$ can now be found by filling in (2.123). It is given by

$$R_{k,1} = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.125}$$

From this, we may read off that the action of the braid group generator τ_{2l+1} on the sign states $|\, s_1, \ldots, s_m \,\rangle$ is given by

$$\tau_{2l+1} |\, s_1, \ldots, s_m \,\rangle = \begin{cases} 1 |\, s_1, \ldots, s_m \,\rangle & (s_{2l+1} = m+1 \bmod 2) \\ i |\, s_1, \ldots, s_m \,\rangle & (s_{2l+1} = m \bmod 2) \end{cases} \tag{2.126}$$

In this equation, we let the value $+$ of the symbol s_{2l+1} correspond to $0 \bmod 2$ and we let the value $-$ correspond to $1 \bmod 2$. We see that τ_{2l+1} acts only on the $(2l+1)^{\text{th}}$ factor of the tensor product of sign spaces and on this factor it is given by the following 2×2 matrix:

$$\tau_{2l+1} \equiv \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} & (m = 1 \bmod 2) \\ \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} & (m = 0 \bmod 2) \end{cases} \tag{2.127}$$

In Nayak and Wilcek's work, the action of τ_{l+1} also corresponds to the action of a diagonal matrix in the $(l+1)^{\text{th}}$ tensor product factor. In this case, the matrix does not depend on m and it is given by (cf. (2.51))

$$\tau_{2l+1}^{NW} \equiv e^{i\frac{\pi}{4}} e^{\frac{i\pi}{4}\sigma_3} = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} \tag{2.128}$$

Here, σ_3 denotes the third Pauli matrix. We see that the Nayak-Wilczek matrix is the same as ours, up to a change in the order of the basis when m is odd.

The tensor product $V_{k,2} \otimes V_{k,3}$ contains only two states with the domino property: the states $v_{1,0} \otimes v_{0,1}$ and $v_{0,1} \otimes v_{1,0}$. Using this as an ordered basis for the relevant states, the matrix $R_{k,2}$ can again be found from (2.123) and is given by

$$R_{k,2} = \begin{pmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{-1+i}{2} & \frac{1+i}{2} \end{pmatrix} \tag{2.129}$$

From this, we may read off the action of the braid group generator τ_{2l} on the states $|\, s_1, \ldots, s_m \,\rangle$. This generator acts only on the $(2l)^{\text{th}}$ and $(2l+1)^{\text{th}}$ tensor factors of the sign space and on those it is given by the matrix:

$$\tau_{2l} \equiv \begin{pmatrix} \frac{1+i}{2} & 0 & 0 & \frac{-1+i}{2} \\ 0 & \frac{1+i}{2} & \frac{-1+i}{2} & 0 \\ 0 & \frac{-1+i}{2} & \frac{1+i}{2} & 0 \\ \frac{-1+i}{2} & 0 & 0 & \frac{1+i}{2} \end{pmatrix} \tag{2.130}$$

where the basis on which this matrix acts, is $\{|++\rangle, |+-\rangle, |-+\rangle, |--\rangle\}$. The matrix above is identical to Nayak and Wilczek's matrix for τ_{2l}, which we gave in (2.51). Hence, we have reproduced Nayak and Wilczek's result. Note that the change of order in the basis which was needed for τ_{2l+1} when m is odd has no effect on the matrix for τ_{2l}, which is why the result does not depend on the parity of m this time.

2.7 Discussion and outlook

We have shown how quantum groups may be used to give an algebraic description of the braiding and fusion properties of the excitations of non-Abelian quantum Hall systems. Due to the relationship between conformal field theory and quantum groups, it is in principle possible to find such a description for any quantum Hall state that has a CFT-description. As an application, we obtained the explicit braiding matrices for the quasihole excitations over the Read-Rezayi series of states. In a special case, these reduce to the matrices given by Nayak and Wilczek in [17], as they should.

The obvious question to ask is now whether one can somehow make predictions about physical quantities from the results we have derived. The answer to this depends very much on what quantities one considers as physical. For example, one may fairly easily calculate amplitudes for Aharonov-Bohm scattering of quasiholes from the braiding matrices we have given, but it seems unlikely that the control over quasiholes that one needs to test these will soon be reached in experiments.

One would probably have better chances of making contact with experiment if one could find effects of the non-Abelian braiding in some transport properties of the quantum Hall state. To be able to make predictions about such quantities, one would most probably need to have a better understanding of the relation between the overcomplete set of states with localized quasiholes which we deal with here and a basis of the Hilbert space of the quantum Hall state. Suitable bases of the spaces of zero modes for the Read-Rezayi-states are constructed in [31, 32] and a logical next step in the program of understanding the consequences of non-Abelian braiding in quantum Hall states would thus be to express the states with localized quasiholes in terms of these bases and vice versa.

The degeneracy associated with many quasihole states also gives a contribution to the entropy of a non-Abelian quantum Hall state. If one could measure the entropy sufficiently well (which is not the case at present) and separate this contribution from the (many) other contributions, then one could in principle determine the quantum dimension of the quantum group representation carried by the quasiholes.

An important theoretical question is whether there is some intuitive way of understanding which features of the underlying theory cause the quantum symmetry exhibited by the effective theories at the plateaus. If such an intuitive picture could be found it would probably be very helpful in extracting physics from the effective theories. A good place to start looking would seem to be the paper [92] of Ivanov, which provides an understanding of the degeneracy of the many quasihole states of the Pfaffian starting from the theory of p-wave superconductors. Another road towards a better understanding of the quantum group symmetry could start from the approach of Ho and Capelli, Georgiev and Todorov [33, 34], who construct non-Abelian Hall states from Abelian ones. It would be interesting to have a description of the projections onto non-Abelian theories performed in these papers in the quantum group theoretical framework.

There are also some questions of a more mathematical nature which arise naturally from our work. For example, one would like to generalize the way we associated quantum groups to coset CFTs to more general cosets than the ones we considered. Such a generalization would also have applications to physics, since it would enable us to describe more of the trial Hall states that have been proposed by means of quantum groups. A first step in this program would be to look at the generalizations of the parafermions that were defined by Gepner [93] or at the cosets of [60, 61], which are described by W-algebras. We expect that most arguments we gave for the parafermions will go through unchanged for these theories. In connection with this, there should be identities like (2.77) for the "$6j$-symbols" of quantum universal enveloping algebras more general than $U_q(sl(2))$; one identity for each external automorphism of the corresponding Affine Lie algebra. A generalization of the identities (2.77) in a different direction has recently been obtained in [94]. One may also ask whether the groups generated by the braiding matrices we have found are finite and/or can be characterized in a nice way. Some light has recently been shed on such matters by Read [95] (see also [96]).

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Chapter 3

Hopf symmetry breaking and confinement in planar gauge theory

Many two-dimensional physical systems have symmetries which are mathematically described by quantum groups. In this chapter we study the phases that appear when such a symmetry is broken spontaneously. As our model systems, we take gauge theories in 2+1 dimensions whose gauge symmetry is spontaneously broken to a finite group. These enjoy a quantum group symmetry which includes the residual gauge symmetry. This Hopf symmetry provides a framework in which fundamental excitations (electric charges) and topological excitations (magnetic fluxes) can be treated on equal footing. Using our general formalism, we can thus study symmetry breaking by condensates with both electric and magnetic quantum numbers and we can investigate the Higgs and confinement phenomena which accompany the formation of these condensates. As usual, confinement of particles is linked to the formation of string-like defects and these defects are also classified. We find that symmetry breaking by an electric condensate leads to magnetic confinement and vice-versa. The general formalism is elucidated by many examples which involve electric, magnetic and even dyonic condensates.

3.1 Introduction

One of the roads towards an understanding of confinement starts with the proposal of 't Hooft and Mandelstam [97, 98] to think of it in terms of the breaking of a dual or magnetic symmetry by a condensate of magnetic monopoles. While this idea has been very fruitful, it has not yet led to a rigorous proof of confinement. One reason for this is that the supposed magnetic symmetry is not manifest in the usual formulation of gauge theory. It is therefore difficult to study its breaking in detail. Other approaches also try to link the phenomenon of electric confinement to condensate physics in the magnetic sector, but there seems to be no general consensus as to which magnetic excitations should be the ones to condense.

We will study the Higgs and confinement transitions in a class of theories where both the electric and the magnetic symmetry are manifest and where we have a clear picture of the possible magnetic excitations. The theories in question are (2+1)-dimensional gauge theories in which the gauge group G is broken down to a discrete group H. The full electric-magnetic symmetry in such "discrete gauge theories" is described by a quantum group or quasitriangular Hopf algebra: the quantum double $D(H)$ of the discrete unbroken gauge group. If H is Abelian, then $D(H)$ is just the group algebra of the group $\tilde{H} \times H$, so that we have the electric group H

and a dual magnetic group $\tilde{H} \cong H$. However, if H is non-Abelian, then $D(H)$ is not a group algebra and so the total symmetry is not described by a group. As a consequence, the discussion of $D(H)$-symmetry breaking cannot proceed in the usual way when H is non-Abelian; we need to generalize the concepts involved in the discussion of symmetry breaking so that we can study symmetry breaking not only for symmetries described by groups, but also for symmetries described by quantum groups.

The construction of a formalism for the description of spontaneous quantum group symmetry breaking is not just a hurdle in the study of (2+1)-dimensional gauge theories, but should rather be seen as a problem of great independent interest. Many models of low-dimensional systems in high energy and in condensed matter physics are known to exhibit quantum group symmetries and a general formalism for the breaking of such symmetries could be applied to classify and study the phases of all these models. Therefore, a large part of this chapter (sections 3.5 and 3.6) is devoted to setting up a formalism for the description of symmetry breaking and confinement in theories whose symmetry is described by a finite dimensional semisimple quasitriangular Hopf algebra. This class of Hopf algebras includes for example all group algebras of finite groups and their quantum doubles. However, there are also many physically interesting quantum groups which are not included in this class; one may for example think of the q-deformed enveloping algebras at roots of unity that appeared in the quantum Hall systems of chapter 2. Nevertheless, a symmetry breaking formalism for finite dimensional quantum groups is a good first step and we expect that some features of the scheme we present here will turn out to be generic.

Let us preview some of these features. Before symmetry breaking, we have a theory whose fields or particles carry representations of a Hopf algebra $\mathcal{A}$. This Hopf symmetry is broken by the formation of a condensate and the residual symmetry is described by the maximal Hopf subalgebra $\mathcal{T}$ of $\mathcal{A}$ which leaves the condensate invariant. However, not all the irreps of $\mathcal{T}$ correspond to free particles in the broken theory; some are confined. The unconfined excitations can be classified by a third Hopf algebra $\mathcal{U}$, which is a quotient of $\mathcal{T}$. The strings that confined excitations pull in the condensate are labeled by means of the so called Hopf kernel of the projection of $\mathcal{T}$ onto $\mathcal{U}$. In section 3.12, we give a much more elaborate summary of the whole symmetry breaking formalism, including a diagram of the most important algebras and maps involved (figure 3.2). To keep one's orientation, it may be useful to look forward to this figure regularly while making one's way toward it.

Now let us mention some of the results on discrete gauge theories that we have obtained using our theory of symmetry breaking. These include descriptions of

- electric gauge symmetry breaking and the corresponding magnetic confinement,
- symmetry breaking by manifestly gauge invariant magnetic condensates and the ensuing electric confinement,
- symmetry breaking and confinement by various other types of condensates, such as non-gauge invariant magnetic condensates and dyonic condensates.

All these results will be illustrated with explicitly worked examples. These will include a complete treatment of the discrete gauge theories whose gauge group is an odd dihedral group.

The detailed setup of the chapter is as follows. In section 3.2, we give a quick introduction to discrete gauge theories. We describe the fundamental and topological excitations and the topological interactions between these excitations, which play an important role in the confinement discussion. At the end of this section, we also give a non-technical preview of the effects of symmetry breaking by a condensate of fundamental, electrically charged particles. In section 3.3, we describe the quantum groups which reproduce the spectrum and interactions of discrete gauge theories: the quantum doubles of finite groups. Some examples of quantum doubles which we use throughout the chapter are introduced in section 3.4. In section 3.5, we develop a general method for the study of spontaneous symmetry breaking in systems with a quantum group symmetry and in section 3.6, we give a general discussion of the confinement phenomena that accompany this symmetry breaking. Section 3.7 gives some motivation for the choice of the specific condensates we study in the rest of the chapter. In section 3.8, we discuss the phases of discrete gauge theories that occur when the electric symmetry is broken by condensation of an electrically charged particle that does not carry magnetic flux. In section 3.9, we discuss the phases that occur when the magnetic symmetry is broken by a gauge invariant magnetic condensate. In section 3.10, we discuss the simultaneous breaking of the electric and magnetic symmetries as a consequence of the condensation of a pure, non gauge invariant, magnetic flux. In section 3.11, we present some results on dyonic condensates. Finally, in section 3.12, we give a summary and a brief outlook.

3.2 Discrete gauge theories: physical setting

We use the term discrete gauge theory for a (2+1)-dimensional Yang-Mills-Higgs theory in which the Higgs field has broken the (continuous) gauge group G down to a finite group H. Such theories are discussed in detail in [99, 100, 101, 102, 103, 104]. As a consequence of the symmetry breaking these theories contain topological defects which are labeled by elements of $\pi_1(G/H)$. By the exact homotopy sequence, this corresponds to $\pi_0(H) = H$ when $\pi_1(G)$ is trivial. It follows that, when G is simply connected, the defects are characterized by elements of the unbroken group H[1]. The element of H that characterizes a defect may be identified as the value of a Wilson loop integral around the defect. In analogy with electromagnetism, we will call the value of this loop integral the "magnetic flux" through the loop and we will call the defects magnetic fluxes. In this setting, fluxes are thus group valued. It is clear that the values of the Wilson loop integrals are indeed elements of the unbroken group H, since, if they were not, parallel transport around the closed loop would not leave the Higgs field's expectation value invariant.

The action of the unbroken group H on fluxes is given by conjugation: a flux $g \in H$ is sent to hgh^{-1} by the element h of H. This transformation rule is just the transformation rule for the Wilson loop integral. As a consequence, the fluxes are organized into gauge multiplets, one for each conjugacy class of H. Thus, the distinct types of flux-carrying particle are labeled by the conjugacy classes $A, B, \ldots$ of H, while a particle of, say, type A, has an internal Hilbert space of dimension equal to the number of elements of the conjugacy class A.

Apart from the topological fluxes, we also allow fundamental charged particles which are labeled by the irreducible representations of the unbroken group H. The internal Hilbert space of particle that carries the irrep α of H is just the module V_α of α and the action of the gauge group on this space is just the action given by the matrices of the irrep α. In addition to charges and fluxes, one has dyons: particles which carry both flux and charge. The charges of dyons with flux $g \in H$ are characterised by the irreducible representations of the centraliser associated

[1] One may argue that even in a theory where G is not simply connected, the set of *stable* fluxes will still be labeled by elements of H, due to the presence of (Dirac) magnetic monopoles in the three-dimensional theory that underlies the two-dimensional theory we are considering here [104]

with the conjugacy class of g in H. In other words, the charge of a dyon is characterised by a representation of the subgroup of the gauge group that leaves the flux of the dyon invariant. The set of electric charges available to a dyon thus depends on the flux of the dyon, indicating that there must be a non-trivial interplay between the electric and magnetic symmetries in this theory.

Now that we have given the natural set of quantum numbers labeling the different sectors (or particle charges) in this theory, let us turn to the interactions. Because the unbroken group is discrete, the gauge fields of the theory are massive, with mass proportional to the length v of the vacuum expectation value of the Higgs field. As a consequence, the electric and magnetic gauge interactions are screened with a screening length inversely proportional to v. We are interested in the low energy or long distance limit of these theories, or equivalently in the limit in which the expectation value of the Higgs field becomes large. In this limit, the theory becomes topological; the only interacions between the particles that survive are ultra-short range interactions, that may be described by fusion rules, and non-local Aharonov-Bohm interactions (the Aharonov-Bohm effect is not screened by the Higgs effect [105, 106, 102, 104]). These Aharonov-Bohm interactions may be described by the action of a (coloured) braid group on the multi-particle states involved and hence we will refer to it as "braiding"

The fusion rules for charges are given by the tensor product decomposition of H-irreps. The fusion product of two fluxes is found by concatenation of the associated Wilson loops, which leads to the conclusion that the fusion product of fluxes $g_1, g_2 \in H$ is the flux labeled by $g_1 g_2 \in H$. The effect of braiding a charge α around a flux $g \in H$ is given by the action of g on V_α. If α is a one-dimensional irrep of H, then this is the usual Aharonov-Bohm phase factor, but if α is higher-dimensional, then the action of g on V_α will be described by the matrix $\alpha(g)$, which will not necessarily reduce to a phase factor. It follows that, if the unbroken group is non-Abelian, discrete gauge theories allow for non-Abelian braiding between charges and fluxes; if a charge is first taken around a flux g_1 and then around a flux g_2, then the effect on the wave function of the charge may be different than if it is taken first around g_2 and then around g_1, simply because one may have $\alpha(g_1)\alpha(g_2) \neq \alpha(g_2)\alpha(g_1)$. The braiding between fluxes may be found by contour manipulation. If a particle with flux g_1 is taken around a particle with flux g_2, then its flux will change from g_1 to $g_2 g_1 g_2^{-1}$, i.e. braiding between fluxes is given by conjugation. Braiding and fusion of dyons will be described in section 3.3.

Much of the rest of this chapter is devoted to the development of a mathematical formalism which will help us describe the phases that result when the symmetry of a discrete gauge theory is broken by the formation of a condensate. However, in the special case where the condensate is purely electric, one may already get a fairly accurate picture of what happens using only the information in this section. The reason for this is that the formation of an electric condensate can be viewed as a simple modification of the Higgs condensate which broke G down to H in the first place. After this modification, the new residual gauge group will be the subgroup N of H which leaves the new condensate invariant. The spectrum of free excitations should thus consist of fluxes labeled by conjugacy classes of N, charges labeled by irreps of N and dyons with flux and centralizer charge. In short, everything should be as it was before, except that the role of H has been taken over by N. Any fluxes $h \in H \backslash N$ that were present when the new condensate formed, will pull a string. That is, their presence makes it impossible for the new Higgs condensate to be single valued and hence causes the expectation value of the Higgs field to develop a line-like discontinuity. The energy associated with this discontinuity will grow linearly with its length and as a consequence, the fluxes h outside N will be confined in "hadrons" whose overall flux does lie in N. The line discontinuities themselves can be viewed as domain walls between regions with different values v_1, v_2 of the Higgs expectation value. They may thus be characterized by an element $h \in H$ such that $hv_1 = v_2$, but this characterization is not unique, since hnv_1 will also equal v_2 for any $n \in N$. Therefore, the strings (or walls) should be characterised by a coset hN in H/N, or more precisely by a gauge orbit of such cosets. This picture of what happens when a purely electric condensate is formed is an important part of the intuition that will be used in the rest of this chapter and it is a non-trivial test of the formalism we will develop that it must repoduce this picture.

3.3 The quantum double of a finite group

3.3.1 The double and its dual

The ribbon Hopf algebra that describes the fusion and braiding of the discrete gauge theory with unbroken group H is the quantum double $D(H)$ of H. As a vector space, $D(H)$ is $F(H) \otimes \mathbb{C}H$, the tensor product of the group algebra $\mathbb{C}H$ of H and its dual, the space $F(H)$ of functions on H. Since H is finite, we may identify this vector space with $F(H \times H)$, the space of functions on $H \times H$, and we may write elements of the double as such functions. On the double, we have the usual structures of a Hopf algebra: a multiplication $\bullet$, identity 1, comultiplication Δ, counit ϵ and antipode S:

$$\begin{array}{rcl} 1(x,y) & := & \delta_e(y) \\ (f_1 \bullet f_2)(x,y) & := & \int_H f_1(x,z)\, f_2(z^{-1}xz, z^{-1}y)\, dz \\ \epsilon(f) & := & \int_H f(e,y)\, dy \\ (\Delta f)(x_1,y_1;x_2,y_2) & := & f(x_1x_2,y_1)\, \delta_{y_1}(y_2) \\ (Sf)(x,y) & := & f(y^{-1}x^{-1}y, y^{-1}). \end{array} \tag{3.1}$$

Here, the integrals over H are a convenient notation for the sum over all elements of H. We see that $D(H)$ is generated as an algebra by the elements $1 \otimes g$ ($g \in H$) and $\delta_g \otimes e$ ($g \in H$). The elements $1 \otimes g$ together form the gauge group H, while the elements $\delta_g \otimes e$ are a basis of $F(H)$ and can be interpreted as projections on the set of states with flux g in the theory. Both multiplication and comultiplication of the double are consistent with this interpretation. The universal R-matrix of $D(H)$ is given by the formula

$$R(x_1,y_1;x_2,y_2) = \delta_e(y_1)\delta_e(x_1y_2^{-1}) \tag{3.2}$$

and the ribbon element c is given by

$$c(x,y) = \bullet \circ (S \otimes \mathrm{id})(R_{21}) = \delta_e(xy). \tag{3.3}$$

The dual $D(H)^*$ of $D(H)$ is $\mathbb{C}H \otimes F(H)$ as a vector space. This space may again be identified with $F(H \times H)$, so that we may realize both the structures of $D(H)$ and those of $D(H)^*$ on this space. The multiplication $\star$, unit 1^*, comultiplication Δ^*, counit ϵ^* and antipode S^* of $D(H)^*$ are given by

$$\begin{array}{rcl} 1^*(x,y) & := & \delta_e(x) \\ (f_1 \star f_2)(x,y) & := & \int_H f_1(z,y)\, f_2(z^{-1}x,y)\, dz \\ \epsilon^*(f) & := & \int_H f(x,e)\, dx \\ (\Delta^* f)(x_1,y_1;x_2,y_2) & := & f(x_1,y_1y_2)\, \delta_{x_2}(y_1^{-1}x_1y_1) \\ (S^*f)(x,y) & := & f(y^{-1}x^{-1}y, y^{-1}). \end{array} \tag{3.4}$$

$D(H)$ and $D(H)^*$ have a canonical Hermitian inner product, given by the same formula for both $D(H)$ and $D(H)^*$:

$$(f_1, f_2) = \int_H \int_H f_1(x, y)\, \overline{f_2(x, y)}\, dx\, dy. \tag{3.5}$$

The matrix elements of the irreps of both $D(H)$ and $D(H)^*$ are orthogonal with respect to this inner product. This follows from the theory of Woronowicz [107] for compact matrix quantum groups, which holds both for $D(H)$ and $D(H)^*$ [108], but it may also be proved directly.

3.3.2 Irreducible representations

The irreducible representations π^A_α of $D(H)$ have been classified in [109], using the fact that the double is a based ring in the sense of Lusztig [110]. An alternative way to classify the irreps of quantum doubles of groups makes use of the fact that $D(H)$ is a transformation group algebra [111]. Since we will be making rather extensive use of transformation group algebras in the sequel, we will follow this path. We follow the notation and conventions of [111]. First, we give a simplified definition of a transformation group algebra, adjusted to our needs, which involve only finite groups:

Definition 6 *Let H be a finite group acting on a finite set X. Then $F(X \times H)$ is called a transformation group algebra if it is equipped with the multiplication $\bullet$ given by*

$$F_1 \bullet F_2(x, y) = \int_H F_1(x, z) F_2(z^{-1}x, z^{-1}y) dz. \tag{3.6}$$

For a more general definition and references, see [112]. When we take $X = H$ and the action of H given by conjugation, then we regain the algebra structure of quantum double $D(H)$. There is a general theorem which classifies the irreducible representations of all transformation group algebras as defined above, but before we give this, we must first define the Hilbert spaces that the representations will act upon. Let N be a subgroup of H, let α be a unitary representation of N, and let V_α be its module, then we define

$$F_\alpha(H, V_\alpha) := \{\phi : H \to V_\alpha | \phi(xn) = \pi_\alpha(n^{-1})\phi(x), \forall x \in H, \forall n \in N\}. \tag{3.7}$$

The irreps of our transformation group algebras are then described by the following theorem, which is a simple consequence of theorem 3.9 in [113]

Theorem 1 *Let $F(X \times H)$ be a transformation group algebra and let $\{\mathcal{O}_A\}$ be the collection of H-orbits in X (A takes values in some index set). For each A, choose some $\xi_A \in \mathcal{O}_A$ and let N_A be the stabilizer of ξ_A in H. Then, for each pair $(\mathcal{O}_A, \alpha)$ of an orbit $\mathcal{O}_A$ and an irrep α of the stabilizer N_A of this orbit, we have an irreducible unitary representation τ^A_α of $F(X \times H)$ on $F_\alpha(H, V_\alpha)$ given by*

$$(\tau^A_\alpha(F)\phi)(x) := \int_H F(x\xi_A, z)\phi(z^{-1}x) dz. \tag{3.8}$$

Moreover, all unitary irreducible representations of $F(X \times H)$ are of this form and irreps τ^A_α and τ^B_β are equivalent only if $\mathcal{O}_A = \mathcal{O}_B$ and $\alpha \cong \beta$.

In the case of $D(H)$, the orbits $\mathcal{O}_A$ are just the conjugacy classes of H. The irreps Π^A_α of $D(H)$ are thus labeled by pairs (A, α) of a conjugacy class label A and an irrep α of the centralizer N_A of a specified element $g_A \in C_A$. We see that the spectrum of irreps of $D(H)$ is in one to one correspondence with the spectrum of excitations of the discrete gauge theory that we described in section 3.2. In particular, the pure (uncharged) magnetic fluxes correspond to the Π^A_1, where 1 denotes the trivial representation of H, and the pure charges correspond to the Π^e_α. We will call the element $g_A \in A$ the *preferred element* of A. Any choice of preferred element yields the same isomorphism class of representations of $D(H)$. The carrier space of Π^A_α is just the space $F_\alpha(H, V_\alpha)$ defined above and for brevity we will denote it by V^A_α The dimension d^A_α of V^A_α is the product of the number of elements of the conjugacy class A and the dimension d_α of α. To see this, note that the functions in V^A_α are completely determined once their value on one element of each N_A-coset is chosen. Now the number of N_A-cosets is just $|A|$, the number of elements of A, which shows that $d^A_\alpha = |A| d_\alpha$. In fact, there is a canonical correspondence between cosets of N_A and elements of A: the coset hN_A corresponds to the element hg_Ah^{-1}. Thus, a state with pure flux hg_Ah^{-1} will be represented by a wave function with support on hN_A.

The action of an element $F \in D(H)$ on V^A_α is given by the formula in the theorem above, which in the case of $D(H)$ becomes

$$\left(\pi^A_\alpha(F)\phi\right)(x) := \int_H dz\, F(xg_Ax^{-1}, z)\, \phi(z^{-1}x). \tag{3.9}$$

From this formula, it is easy to see that the action of the gauge group elements $1 \otimes g$ in the purely electric representation Π^e_α is indeed isomorphic to the action of the gauge group in the representation α. Also, the action of the gauge group on magnetic fluxes is given by conjugation, which can be seen as follows: the state with flux hg_Ah^{-1} is represented by the function $\delta_{hN_A} \in V^A_1$, i.e. by the characteristic function of the coset hN_A. The action of the element $1 \otimes g \in D(H)$ sends this function to the function ghN_A, which in turn corresponds to the flux $g(hg_Ah^{-1})g^{-1}$.

The *spin* of a particle that transforms in the irrep Π^A_α is given by the action of the ribbon element c. We have

$$\left(\pi^A_\alpha(c)\phi\right)(x) := \alpha(g_A^{-1})\phi(x). \tag{3.10}$$

Since the element g_A^{-1} is central in N_A, the matrix $\alpha(g_A^{-1})$ is a constant multiple of the unit matrix: we have $\alpha(g_A^{-1}) = s^A_\alpha I$, where $s^A_\alpha \in \mathbb{C}$ is a root of unity which we call the *spin factor* of Π^A_α. Clearly, we have $s^A_\alpha = \frac{1}{\mathrm{d}_\alpha}\mathrm{Tr}(\alpha(\mathrm{g}_\mathrm{A}^{-1}))$, where d_α is the dimension of the representation α of N_A. A consistent description of the *braiding* for arbitrary representations of $D(H)$ is given by the R-matrix (3.2).

3.3.3 Matrix elements and Characters

The fusion rules for representations of $D(H)$ and of more general transformation group algebras may be calculated by means of a character formalism. The character of a representation π of a transformation group algebra $\mathcal{T} = C_c(X \times H)$ is a linear functional $\chi : \mathcal{T} \to \mathbb{C}$ defined as follows: $\chi(t)$ is the trace of the matrix $\pi(t)$. Clearly, the character of the representation π depends only on the isomorphism class of π. Let us calculate the character of an irrep τ^A_α of $\mathcal{T}$ from the formula in theorem 1. The first thing we need is a basis for the vector space $F_\alpha(G, V_\alpha)$. To get this, we choose a basis e^α_i for V_α and a representative for each left coset of N_A. For a given coset of N_A, all the elements of this coset send the preferred element ξ_A of the orbit $\mathcal{O}_A$ to the same element ζ of this orbit. Moreover, each element $\zeta \in \mathcal{O}_A$ uniquely determines a coset. Therefore, we call the representative elements of the cosets x_ζ and these x_ζ are just arbitrarily chosen elements of H with the property that $x_\zeta \xi_A = \zeta$. A basis for $F_\alpha(G, V_\alpha)$ is now given by the functions ϕ^i_ζ defined by

$$\phi^i_\zeta(y) = 1_{x_\zeta N_A}(y)\alpha(y^{-1}x_\zeta)e^\alpha_i. \tag{3.11}$$

ϕ^i_ζ is the unique element of $F_\alpha(G, V_\alpha)$ which takes the value e^α_i at x_ζ and one may easily verify that these elements do indeed form a basis for $F_\alpha(G, V_\alpha)$. The matrix elements of τ^A_α in this basis are given by

$$\tau^A_\alpha(F)^{i,j}_{\zeta,\eta} = \int_{N_A} F(x_\eta \xi_A, x_\eta n x_\zeta^{-1})\alpha_{i,j}(n)\, dn, \tag{3.12}$$

where the $\alpha_{i,j}$ are the matrix elements of α with respect to the basis of e^α_i. As a consequence, the character χ^A_α of τ^A_α is given by

$$\chi^A_\alpha(F) = \int_{\mathcal{O}_A} d\zeta \int_{N_A} dn\, F(x_\zeta \xi_A, x_\zeta n x_\zeta^{-1})\chi_\alpha(n), \tag{3.13}$$

where χ_α denotes the character of α. We can remove the arbitrarily chosen elements x_ζ from this formula by adding an integration over N_A, thus changing the integration over $\mathcal{O}_A$ into an integration over H:

$$\chi^A_\alpha(F) = \int_H dz \int_{N_A} dn\, F(z\xi_A, znz^{-1})\chi_\alpha(n). \tag{3.14}$$

When $\mathcal{T} = D(H)$, this reduces to the formula given in [111]. Clearly, the characters are fully determined by their values on a basis for $\mathcal{T}$. When X is finite, we can take the basis of delta functions $\delta_\eta \delta_h$ ($\eta \in X, h \in H$) and we can take the characters to be elements of $F(X \times H)$ by writing $\chi^A_\alpha(\eta, h) := \chi^A_\alpha(\delta_\eta \delta_h)$. We have

$$\chi^A_\alpha(\eta, h) = 1_{N_\eta}(h) 1_{\mathcal{O}_A}(\eta)\chi_\alpha(x_\eta^{-1} h x_\eta). \tag{3.15}$$

When $\mathcal{T} = D(H)$, this gives the formula for the characters in [109].

We may define an inner product $\langle \cdot, \cdot \rangle$ on the space of functions $X \times H$ by the formula

$$\langle \chi_1, \chi_2 \rangle = \int_X d\eta \int_H dh\, \chi_1(\eta, h)\overline{\chi_2(\eta, h)}. \tag{3.16}$$

One may check that the characters are orthogonal with respect to this inner product:

$$\left\langle \chi^A_\alpha, \chi^B_\beta \right\rangle = |H|\delta_{A,B}\delta_{\alpha,\beta}. \tag{3.17}$$

When $\mathcal{T} = D(H)$, the inner product defined here is just the invariant inner product (3.5) on $D(H)^*$ and the orthogonality of the characters with respect to this inner product follows from Woronowicz's general theory. The decomposition of a tensor product of irreps of $D(H)$ may be found by calculating the inner products of the characters of the irreps with the character of the tensor product. In this way, the fusion properties of pure fluxes and charges that we have described in section 3.2 are reproduced and we may also calculate the fusion rules for dyons.

3.4 Examples of quantum doubles

We briefly present examples of quantum doubles of finite groups. These will be the standard examples in our discussion of symmetry breaking in the remainder of this chapter.

3.4.1 $D(H)$ for Abelian H

The quantum double of an Abelian group H is isomorphic to the group algebra of $H \times H$ as a Hopf algebra. One way to see this is the following: First recall that any finite Abelian group is isomorphic to some $\mathbb{Z}_{k_1} \times \ldots \times \mathbb{Z}_{k_n}$, where $k_i | k_j$ for $i < j$. Then recall that, by Pontryagin duality, the character group of an Abelian group H is isomorphic to H. We may thus denote the elements of H by n-tuples $(m_1, \ldots, m_n)$, with $0 \leq m_i \leq k_i$ and we may also label the characters $\chi_{m_1,\ldots,m_n}$ of H with such n-tuples in such a way that the map $(m_1, \ldots, m_n) \to \chi_{m_1,\ldots,m_n}$ is an isomorphism of groups. The canonical way to do this labeling is such that $\chi_{m_1,\ldots,m_n}$ is the character given by

$$\chi_{m_1,\ldots,m_n}(l_1, \ldots, l_n) = \exp(2\pi i(\frac{m_1 l_1}{k_1} + \ldots + \frac{m_n l_n}{k_n})). \tag{3.18}$$

The characters are linearly independent functions on H and thus $D(H)$ is spanned by the elements $\chi \otimes \delta_h$, where χ is a character of H and h is an element of H. But one calculates easily that

$$\begin{aligned}
(\chi_1 \otimes \delta_{h_1}) \bullet (\chi_2 \otimes \delta_{h_2}) &= (\chi_1\chi_2 \otimes \delta_{h_1 h_2}) \\
1_{D(H)} &= 1 \otimes \delta_e \equiv e_{H\times H} \\
\Delta(\chi \otimes \delta_h) &= (\chi \otimes \delta_h) \otimes (\chi \otimes \delta_h) \\
S(\chi \otimes \delta_h) &= (\bar{\chi} \otimes \delta_{h^{-1}}) = (\chi \otimes \delta_h)^{-1} \\
\epsilon(\chi \otimes \delta_h) &= 1,
\end{aligned} \tag{3.19}$$

so that, comparing to (1.19), we see that we indeed have $D(H) \cong \mathbb{C}(H \times H)$ as a Hopf algebra.

As a consequence, the irreducible representations of $D(H)$ are just the tensor products $\chi_1 \otimes \chi_2$ that may be formed from two irreps χ_1, χ_2 of H. These correspond to the irreps Π^A_α that we described in section 3.3.2 in the following way. When H is Abelian, all conjugacy classes of A consist of just one element, so that the label A may be identified with the element g_A of H. This element may in turn be identified with a character χ_A using the isomorphism between H and its character group that we indicated above. Also, we have $N_A = H$ and so α is already a character of H. One may now check easily that the irrep Π^A_α of $D(H)$ corresponds to the irrep $\chi_A \otimes \alpha$ of $H \otimes H$. The tensor product of two $D(H)$-irreps is just the usual tensor product of $H \times H$ representations. The only difference with the usual representation theory of $H \times H$ is that the representations now have non-trivial spin-factors and non-trivial braiding, given by the ribbon element and the R-matrix of $D(H)$ respectively. We have

$$\begin{aligned}
s^A_\alpha &= \alpha(g_A^{-1}) \\
\Pi^A_\alpha \otimes \Pi^B_\beta(R) &= \beta(g_A).
\end{aligned} \tag{3.20}$$

As one may read off using (3.18), these are just the usual phase factors one expects for Abelian dyons, involving products of charge and flux quantum numbers.

3.4.2 $D(D_{2m+1})$

Perhaps the simplest non-Abelian groups are the dihedral groups D_n which describe the symmetries of the regular n-gons. Among these is the smallest non-Abelian group: the dihedral group D_3, which is isomorphic to the symmetric group S_3. D_n has $2n$ elements: the unit, $n-1$ non-trivial rotations and n reflections. It can be presented on two generators as follows:

$$D_n = \{s, r | \, s^2 = r^n = 1, sr = r^{n-1}s\}. \tag{3.21}$$

The $2n$ elements may all be written in the form r^k or sr^k (with $e = r^0$). The powers of r are the rotations, the elements that involve s are the reflections.

We will deal exclusively with the odd dihedral groups D_{2m+1}. D_{2m+1} has $m+2$ conjugacy classes, which we will label by their preferred elements and which we will denote by their preferred elements in square brackets (e.g.$[r]$) if confusion between class and element might arise. The classes are

$$\begin{array}{rcl} [e] & = & \{e\} \\ [r^k] & = & \{r^k, r^{-k}\} \quad (0 < k \leq m) \\ [s] & = & \{sr^k | \, 0 \leq k \leq 2m+1\}. \end{array} \tag{3.22}$$

The centralizers of these classes are given by

$$N_e = D_{2m+1}, \quad N_{r^k} = \langle \, r \, \rangle \cong \mathbb{Z}_{2m+1}, \quad N_s =< s >\cong \mathbb{Z}_2. \tag{3.23}$$

D_{2m+1} has two one dimensional representations: the trivial representation, which we will denote J_0 and a representation J_1 given by $J_1(r) = 1, \; J_1(s) = -1$. The remaining irreps of D_{2m+1} are all two dimensional and faithful. We will call them $\alpha_1, \ldots, \alpha_m$ and they may be given by

$$\alpha_k(r) = \begin{pmatrix} \cos(\frac{2k\pi}{2m+1}) & -\sin(\frac{2k\pi}{2m+1}) \\ \sin(\frac{2k\pi}{2m+1}) & \cos(\frac{2k\pi}{2m+1}) \end{pmatrix} \qquad \alpha(s) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.24}$$

The character table for D_{2m+1} can now be read off; it is given in table 3.1 The representations

	$[e]$	$[r^k]$	$[s]$
J_0	1	1	1
J_1	1	1	-1
α_j	2	$q^{jk} + q^{-jk}$	0

Table 3.1: character table for D_{2m+1}. We have defined $q = e^{2\pi i/(2m+1)}$.

of the $\mathbb{Z}_{2m+1}$ and $\mathbb{Z}_2$ centralizers will be denoted $\beta_0, \beta_1, \ldots, \beta_{2m}$ and $\gamma_0, \gamma_1 \equiv \gamma$ respectively. They are as given in the previous section. The representations of $D(D_{2m+1})$ will thus be labeled $\Pi^e_{J_0}, \Pi^e_{J_1}, \Pi^e_{\alpha_k}, \Pi^{r^k}_{\beta_l}, \Pi^s_1$ and Π^s_γ. All in all this yields $2(m^2+m+2)$ representations. The dimensions d^A_α and spin factors s^A_α of these irreps are given in table 3.2: The fusion rules of the irreps of $D(D_{2m+1})$ may be determined by means of the characters (3.15) and the orthogonality relations (3.17). They have been given explicitly in [114]. One may also show that tensor products of multiple $D(D_{2m+1})$-irreps can carry non-Abelian representations of the braid group.

	$\Pi^e_{J_0}$	$\Pi^e_{J_1}$	$\Pi^e_{\alpha_j}$	$\Pi^{r^k}_{\beta_l}$	Π^s_1	Π^s_γ
d^A_α	1	1	2	2	$2m+1$	$2m+1$
s^A_α	1	1	1	q^{-kl}	1	-1

Table 3.2: dimensions and spin factors for the irreps of $D(D_{2m+1})$

3.5 Symmetry Breaking

3.5.1 Hopf symmetry breaking

Consider the situation where a condensate has formed, carrying the representation Π^A_α of $D(H)$. The ground state or "vacuum" of the theory is then a background of identical particles, all in the same state $\phi \in V^A_\alpha$. We model this situation with a tensor product state $\phi \otimes \phi \otimes \ldots \otimes \phi$. This state breaks the $D(H)$-symmetry of the theory and we want to find out what the residual symmetry algebra of the system after this breaking is. Now if the original symmetry were described by a group, then finding the residual symmetry would in principle be straightforward; we would find out which of the group elements leave the condensate state ϕ invariant, i.e. we would find the stabilizer of ϕ, and this stabilizer would be the residual symmetry. If the original symmetry is described by a Hopf algebra, then we cannot use this recipe, for several reasons. First of all, we cannot expect to find a subalgebra of the Hopf algebra which leaves ϕ invariant in the usual sense of the word; any such subalgebra would have to contain the element 0 which would obviously send ϕ to 0. Hence, we need a new definition of invariance. Fortunately, there is a natural definition, namely the following (cf. [7])

Definition 7 *Let $\mathcal{A}$ be a Hopf algebra with counit ϵ, let $a \in \mathcal{A}$ and let ϕ be a vector in some $\mathcal{A}$-module. Then we say that ϕ is left invariant by a if the action of a on ϕ is given by $a \cdot \phi = \epsilon(a)\phi$.*

This definition is natural, since it just says that the vector ϕ transforms under $a \in \mathcal{A}$ in the same way that the vacuum would. Also, if the Hopf algebra $\mathcal{A}$ is a group algebra, then we see that this definition of invariance reduces to the usual one on the group elements. Nevertheless, when we apply the above definition of invariance to a group algebra, then we see that the subalgebra which leaves a vector ϕ invariant is *not* the group algebra of the stabilizer of ϕ. In fact, it is a much larger algebra, which is not a Hopf algebra. On the other hand, the maximal Hopf subalgebra of the group algebra which leaves ϕ invariant (with the above definition of invariance), is exactly the group algebra of the stabilizer of ϕ. This follows easily from the fact that the Hopf subalgebras of a group algebra $\mathbb{C}G$ are exactly the group algebras of the subgroups of G (this is well known to Hopf algebra theorists, but we also give an explanation of why it is so in section 3.5.3). This suggests that we should define the residual symmetry algebra after breaking as follows:

Definition 8 *Suppose we have a theory with Hopf symmetry $\mathcal{A}$. If a condensate of particles in the state ϕ forms in this theory, then the residual symmetry algebra is the maximal Hopf subalgebra of $\mathcal{A}$ that leaves ϕ invariant. We will call this algebra the* Hopf stabilizer *of ϕ*

A maximal Hopf subalgebra with a certain property is defined as a Hopf subalgebra with this property which is not a subalgebra of a larger Hopf subalgebra with his property. The maximal Hopf subalgebra in the above definition is unique, since, if we have two different Hopf subalgebras which leave the same vector invariant, then the subalgebra generated by these two is itself a Hopf subalgebra which leaves this vector invariant and which contains the original two Hopf subalgebras. The above definition reduces to the usual definition in the case of group algebras and it has the further advantage that the residual symmetry algebra will always be a Hopf algebra[2]. The spectrum of the residual algebra will thus have the desirable properties that we discussed in chapter 1; there will be a natural description of many-particle states, there will be a trivial or vacuum representation and given an irrep of the algebra that labels a "particle" (an excitation over the condensate), there will also be an irrep (possibly the same) that labels the "antiparticle". The fact that the residual symmetry algebra is a Hopf algebra also makes sure that the invariance of ϕ implies the invariance of all the states $\phi \otimes \phi \otimes \ldots \otimes \phi$. This follows easily from the fact that $(\epsilon \otimes \epsilon) \circ \Delta = \epsilon$. Thus, we might have taken the condensate to be a superposition of states with different numbers of particles (still all in the state ϕ) and this would have yielded exactly the same residual algebra.

3.5.2 Hopf subalgebras and Hopf quotients

In view of the definition 8 of the residual symmetry algebra after the formation of a condensate, it is useful to find out all we can about Hopf subalgebras of the quantum double $D(H)$, or more generally, about Hopf subalgebras of finite dimensional semisimple Hopf algebras. In the present section, we give a characterization of the Hopf subalgebras of such Hopf algebras, which will provide us with a way of finding all these Hopf subalgebras and in particular the residual symmetry algebras of definition 8 in a systematic way. Along with the results on Hopf subalgebras, we also prove some results on Hopf quotients or quotient Hopf algebras which will be useful in our discussion of confinement further on. The main theorems in this section are closely related to results in [116] and [117]. We include elementary proofs here in order to make our treatment more self-contained. We write the results in a form which is useful for our needs, rather than maximally general or compact. A lot of background for this section can be found in [4].

We define a Hopf quotient as follows

Definition 9 *Let $\mathcal{A}$ and $\mathcal{B}$ be Hopf algebras. If we have a surjective Hopf map $\Gamma : \mathcal{A} \to \mathcal{B}$, then we call $\mathcal{B}$ a* Hopf quotient *of $\mathcal{A}$.*

The Hopf algebra $\mathcal{B}$ is in fact isomorphic to the quotient of $\mathcal{A}$ by the kernel of the map Γ, explaining the terminology. Our first step in characterizing Hopf subalgebras and Hopf quotients is to relate them to each other, using the following proposition

Proposition 1 *Let $\mathcal{A}$ and $\mathcal{B}$ be finite dimensional Hopf algebras and let $\Gamma : \mathcal{A} \to \mathcal{B}$ be a Hopf map. Then the dual map $\Gamma^* : \mathcal{B}^* \to \mathcal{A}^*$ is also a Hopf map. Moreover, if Γ is injective then Γ^* is surjective and if Γ is surjective then Γ^* is injective. Finally, if we identify $\mathcal{A}$ and $\mathcal{A}^{**}$ and $\mathcal{B}$ and $\mathcal{B}^{**}$ in the canonical way, then we have $\Gamma^{**} = \Gamma$.*

Proof. The proof that Γ^* is a Hopf map is straightforward calculation. As an example, we show that $\Gamma^* \otimes \Gamma^* \circ \Delta_{\mathcal{B}^*} = \Delta_{\mathcal{A}^*} \circ \Gamma^*$. For any $f \in \mathcal{B}^*$, we have

$$\Gamma^* \otimes \Gamma^* \circ \Delta_{\mathcal{B}^*}(f) = f \circ \mu_{\mathcal{B}} \circ \Gamma \otimes \Gamma = f \circ \Gamma \circ \mu_{\mathcal{A}} = \Delta_{\mathcal{A}^*} \circ \Gamma^*(f). \tag{3.25}$$

[2]In fact, it is also semisimple since any Hopf subalgebra of a finite dimensional semisimple Hopf algebra is itself semisimple. This is proved in [4], using the Nichols-Zoeller theorem [115].

We used the fact that Γ is a Hopf map in the second equality. The other properties that make Γ^* into a Hopf map can be verified analogously. The proof that $\Gamma^{**} = \Gamma$ is also easy. We identify the element $a \in \mathcal{A}$ with the functional $E_a \in \mathcal{A}^{**}$ that is evaluation in A, i.e. $E_a : f \to f(a)$. Similarly, we identify $b \in \mathcal{B}$ with $E_b \in \mathcal{B}^{**}$. One may check that these identifications are Hopf isomorphisms. The action of Γ^{**} on a is calculated as follows:

$$\Gamma^{**}(a)(f) = \Gamma^{**}(E_a)(f) = E_a \circ (\Gamma^*)(f) = E_a(f \circ \Gamma) = E_{\Gamma(a)}(f) = \Gamma(a)(f) \tag{3.26}$$

Hence, we see that indeed $\Gamma^{**} = \Gamma$. The statements about injectivity or surjectivity of Γ^* are basic properties of the dual map. □
From this proposition, we have the following corollaries

Corollary 1 *Let $\mathcal{A}$ be a finite dimensional Hopf algebra and let $\mathcal{B}$ be a Hopf subalgebra of $\mathcal{A}$. Then $\mathcal{B}^*$ is a Hopf quotient of $\mathcal{A}^*$. The corresponding surjective Hopf map is restriction to $\mathcal{B}$, which is the dual map of the embedding of $\mathcal{B}$ in $\mathcal{A}$*

Corollary 2 *Let $\mathcal{A}$ be a finite dimensional Hopf algebra and let $\mathcal{B}$ be a Hopf quotient of $\mathcal{A}$, with corresponding surjective Hopf map Γ. Then $\mathcal{A}^*$ has a Hopf subalgebra isomorphic to $\mathcal{B}^*$, namely the image of the embedding Γ^*.*

The next proposition shows that if $\mathcal{B}$ is a Hopf quotient of $\mathcal{A}$, then the set of representations of $\mathcal{B}$ naturally corresponds to a subring of the representation ring of $\mathcal{A}$.

Proposition 2 *Let $\mathcal{A}$ be a Hopf algebra and let $\mathcal{B}$ be a Hopf quotient of $\mathcal{A}$. Denote the associated surjective Hopf map from $\mathcal{A}$ to $\mathcal{B}$ by Γ. Then the representations of $\mathcal{B}$ are in one-to-one correspondence with the representations of $\mathcal{A}$ that factor over Γ. Also, this correspondence preserves irreducibility. It follows that, if $\mathcal{A}$ is semisimple, then so is $\mathcal{B}$. The correspondence map between representations also commutes with taking conjugates and tensor products of representations. As a consequence, the tensor product of irreps of $\mathcal{A}$ that factor over Γ will decompose in the same way as the tensor product of the corresponding irreps of $\mathcal{B}$.*

Proof. Let ρ be a representation of $\mathcal{B}$. Then $\rho \circ \Gamma$ is a representation of $\mathcal{A}$, since Γ is a Hopf map. Moreover, if ρ is irreducible then so is $\rho \circ \Gamma$, since Γ is surjective. On the other hand, let $\tau : \mathcal{A} \to M_{n \times n}$ be a representation of $\mathcal{A}$ which factors over Γ, that is, $\tau = \rho \circ \Gamma$ for some map $\rho : \mathcal{B} \to M_{n \times n}$. Then ρ is a representation of $\mathcal{B}$, since Γ is surjective and irreducibility of τ implies that ρ is irreducible. Also, τ uniquely determines ρ and vice versa. Hence, the representations of $\mathcal{B}$ are in one-to-one correspondence with the representations of $\mathcal{A}$ which factor over Γ and irreducibility is preserved in this correspondence. Semisimplicity of $\mathcal{A}$ is equivalent to the property that all $\mathcal{A}$-modules decompose into irreducibles. This holds in particular for all $\mathcal{A}$-modules in which the action of $\mathcal{A}$ factors over Γ, and hence also for all $\mathcal{B}$-modules, implying that $\mathcal{B}$ is semisimple. The remaining statements follow easily from the fact that Γ is a Hopf map. If $\tau = \rho \circ \Gamma$ then $\overline{\tau} = \overline{\rho} \circ \Gamma$. This can be seen by looking at the matrix elements $\overline{\tau}_{i,j}$ of $\overline{\tau}$:

$$\overline{\tau}_{i,j} = (\tau_{j,i} \circ S_{\mathcal{A}}) = \rho_{j,i} \circ \Gamma \circ S_{\mathcal{A}} = \rho_{j,i} \circ S_{\mathcal{B}} \circ \Gamma = \overline{\rho}_{i,j} \otimes \Gamma. \tag{3.27}$$

Here, we have used $\Gamma \circ S_{\mathcal{A}} = S_{\mathcal{B}} \circ \Gamma$. For the tensor product of irreps $\tau^a = \rho \circ \Gamma$ and $\tau^b = \rho^b \circ \Gamma$, we have

$$\tau^a \otimes \tau^b \circ \Delta_{\mathcal{A}} = \rho^a \otimes \rho^b \circ \Gamma \otimes \Gamma \circ \Delta_{\mathcal{A}} = \rho^a \otimes \rho^b \circ \Delta_{\mathcal{B}} \circ \Gamma, \tag{3.28}$$

where we used that $\Gamma \otimes \Gamma \circ \Delta_{\mathcal{A}} = \Delta_{\mathcal{B}} \circ \Gamma$. We see that $\tau^a \otimes \tau^b \circ \Delta_{\mathcal{A}}$ and $\rho^a \otimes \rho^b \circ \Delta_{\mathcal{B}}$ act on the same module by the same matrices (since Γ is surjective). Hence the decomposition of

tensor product representations of $\mathcal{A}$ will be the same as the decomposition of tensor product representations of $\mathcal{B}$. $\square$
Before the next proposition, we need another definition

Definition 10 *We call a set X of irreps of a Hopf algebra $\mathcal{A}$* closed under tensor products and conjugation *if $\tau \in X \Rightarrow \overline{\tau} \in X$ and if $\tau^a, \tau^b \in X$ implies that all the irreps in the decomposition of the tensor product of τ^a and τ^b are contained in X.*

Note that, in the previous proposition, the set of irreps of $\mathcal{A}$ that factor over Γ is an example of a set of irreps of $\mathcal{A}$ that close under tensor products and conjugation. Also note that a set of irreps that closes under tensor products and conjugation will always contain the counit. We now prove a basic fact about sets of irreducibles of $\mathcal{A}$ that close under conjugation and tensor products:

Proposition 3 *Let $\mathcal{A}$ be a semisimple Hopf algebra and let X be a set of irreps of $\mathcal{A}$ that closes under conjugation and tensor products. Then the linear space V_X spanned by the matrix elements of the representations in X is a Hopf subalgebra of $\mathcal{A}^*$*

Proof. First, let us take the product of two matrix elements. We have $\mu_{\mathcal{A}^*}(\tau^a_{i,j}\tau^b_{k,l}) = (\tau^a \otimes \tau^b \circ \Delta_{\mathcal{A}})_{(i,j),(k,l)}$. In other words, the product of matrix elements of τ^a and τ^b in $\mathcal{A}^*$ is a matrix element of the tensor product representation $\tau^a \otimes \tau^b \circ \Delta_{\mathcal{A}}$. Since $\mathcal{A}$ is semisimple, this tensor product may be decomposed into irreps and the matrix elements of the tensor product representation are linear combinations of the matrix elements of the irreps this decomposition. But since these irreps are contained in X, it follows that V_X is closed under multiplication. Clearly, V_X also contains $1_{\mathcal{A}^*} = \epsilon_A$, so V_X is a unital subalgebra of $\mathcal{A}^*$. We also have $S(V_X) \subset V_X$, since

$$S_{\mathcal{A}^*}(\tau_{i,j}) = \tau_{i,j} \circ S_{\mathcal{A}} = \overline{\tau}_{j,i} \tag{3.29}$$

and $\tau \in X \Rightarrow \overline{\tau} \in X$. For the comultiplication of a matrix element of any representation of $\mathcal{A}$, we have

$$\Delta_{\mathcal{A}^*}(\tau_{i,j}) = \sum_k \tau_{i,k} \otimes \tau_{k,j} \tag{3.30}$$

and hence we have $\Delta_{\mathcal{A}^*}(V_X) \subset V_X \otimes V_X$. $\square$
Now we arrive at one of our main goals, which is a partial converse of the previous proposition:

Theorem 2 *Let $\mathcal{A}$ be a finite dimensional semisimple Hopf algebra over the complex numbers. Let $\mathcal{B}$ be a Hopf subalgebra of $\mathcal{A} = \mathcal{A}^{**}$. Then $\mathcal{B}$ is spanned by the matrix elements of a set of irreps of $\mathcal{A}^*$ which closes under conjugation and tensor products.*

Proof. Let ι be the inclusion of $\mathcal{B}$ into $\mathcal{A}$. Then $\mathcal{B}^*$ is a Hopf quotient of $\mathcal{A}^*$ with the associated Hopf map given by ι^* (cf. corollary 1). Because $\mathcal{A}$ is semisimple and defined over the complex numbers, its dual $\mathcal{A}^*$ is also semisimple (see [118] and also [4]). Hence, using proposition 2, $\mathcal{B}^*$ is also semisimple. But then it follows that the matrix elements of the irreps of $\mathcal{B}^*$ span $\mathcal{B}^{**} = \mathcal{B}$. On the other hand, we know from proposition 2 that the irreps of $\mathcal{B}^*$ are identified (through $\iota^{**} = \iota$) with a set of irreps of $\mathcal{A}^*$ which closes under conjugation and tensor products.
$\square$
This theorem is the characterization of Hopf subalgebras that we will be using in our discussion of $D(H)$-symmetry breaking in section 3.5.3. More generally, it can simplify the problem of finding all the Hopf subalgebras of a finite dimensional semisimple Hopf algebra $\mathcal{A}$ enormously. If the irreps of $\mathcal{A}^*$ and the decompositions of their tensor products are known, then finding all sets of irreps of $\mathcal{A}^*$ that close under tensor products is a process that can be carried out easily on a computer. Finally, we prove a similar statement about Hopf quotients:

Theorem 3 *Let $\mathcal{A}$ be a finite dimensional semisimple Hopf algebra over the complex numbers. Then any Hopf quotient of $\mathcal{A} = \mathcal{A}^{**}$ is isomorphic to a quotient obtained by restriction to a Hopf subalgebra of $\mathcal{A}^*$ generated by matrix elements of a set of irreps of $\mathcal{A}$ which closes under conjugation and tensor products.*

Proof. Let $\mathcal{B}$ be a Hopf quotient of $\mathcal{A}$ and let Γ be the associated projection. Then $\Gamma^*(\mathcal{B}^*) \cong \mathcal{B}^*$ is a Hopf subalgebra of $\mathcal{A}^*$ (cf. corollary 2) and $\mathcal{B} = \mathcal{B}^{**}$ is isomorphic to the quotient of $\mathcal{A} = \mathcal{A}^{**}$ obtained by restriction to $\Gamma^*(\mathcal{B}^*)$. Since $\mathcal{A}$ is semisimple, $\mathcal{A}^*$ is also semisimple. Thus, we can now apply the previous theorem to the pair $(\mathcal{A}^*, \Gamma^*(\mathcal{B}^*))$ and it follows that $\Gamma^*(\mathcal{B}^*)$ is the desired Hopf subalgebra. $\square$

3.5.3 Hopf subalgebras of quantum doubles

In section 3.5.2, we showed that the Hopf subalgebras of a finite dimensional semisimple Hopf algebra (such as $D(H)$) are in one-to-one correspondence with sets of irreps of the dual Hopf algebra that close under tensor products and conjugation. Therefore, we now construct the representations of the dual algebra $D(H)^*$. From (3.4), we see that, as an algebra (but not as a Hopf algebra), $D(H)^*$ is isomorphic to $\mathbb{C}H \otimes F(H)$. As a consequence, the irreducible representations of $D(H)^*$ are tensor products of irreps of $\mathbb{C}H$ and irreps of $F(H)$. The irreps of $\mathbb{C}H$ just correspond to the irreps of H and we will denote them ρ_i. The irreps of $F(H)$ are all one dimensional and are labeled by the elements of H. We have an irrep E_g for each $g \in H$, given by

$$E_g(f) = f(g). \tag{3.31}$$

We can thus label each representation of $D(H)^*$ by a pair (ρ_i, g). Tensor products of the irreps of $D(H)^*$ may be formed by means of Δ^*. Although this coproduct is not the same as the usual coproduct for $\mathbb{C}H \otimes F(H)$, the decomposition of tensor products into irreps is not affected by this (the Clebsch-Gordan coefficients for the decomposition are affected). Thus we have

$$\rho_i \otimes \rho_j = \bigoplus_k N_{ij}^k \rho_k \implies (\rho_i, g) \otimes (\rho_j, h) = \bigoplus_k N_{ij}^k (\rho_k, gh) \tag{3.32}$$

where the N_k^{ij} are the usual multiplicities in the decomposition of tensor products of H-irreps. Also, we have

$$\overline{(\rho_i, g)} = (\bar{\rho}_i, g^{-1}). \tag{3.33}$$

From these formulae, we see that any set X of irreps of $D(H)^*$ that closes under tensor products and conjugation is associated to a set of irreps of H and a set of irreps of $F(H)$ with the same property. These sets just consist of the irreps that may occur as a factor of one of the irreps in X. Consequently, for any Hopf subalgebra $\mathcal{B}$ of $D(H)$, there are minimal Hopf subalgebras $\mathcal{C}$ of $F(H)$ and $\mathcal{D}$ of $\mathbb{C}H$ such that $\mathcal{B} \subset \mathcal{C} \otimes \mathcal{D} \subset D(H)$. In the other direction, we see that, for any pair of Hopf subalgebras $\mathcal{C} \subset F(H)$ and $\mathcal{D} \subset \mathbb{C}H$, the vector space $\mathcal{C} \otimes \mathcal{D}$ is a Hopf subalgebra of $D(H)$. Note that this Hopf subalgebra is usually not isomorphic to $\mathcal{C} \otimes \mathcal{D}$ as a Hopf algebra. Also, not all Hopf subalgebras of $D(H)$ are of this form. However, the ones that are will be quite important in the sequel. Therefore, we now find all the Hopf subalgebras of the group algebra $\mathbb{C}H$ and of the function algebra $F(H)$. This also gives us two simple examples of the use of theorem 2.

Proposition 4 *The Hopf subalgebras of a group algebra $\mathbb{C}H$ are the group algebras of the subgroups of H.*

Proof. The Hopf algebra dual to $\mathbb{C}H$ is the vector space $F(H)$ of functions on H, with Hopf algebra structure given by

$$\begin{aligned} 1^* : g \mapsto 1, \qquad \mu^*(f_1, f_2) : g \mapsto f_1(g) f_2(g), \qquad \Delta^*(f) : (g_1, g_2) \mapsto f(g_1 g_2), \\ \epsilon^* : f \mapsto f(e), \qquad S^*(f) : g \mapsto f(g^{-1}), \end{aligned} \tag{3.34}$$

where g, g_1, g_2 are arbitrary elements of the group H. Note that, in the formula for the comultiplication, we have identified $F(H) \otimes F(H)$ with $F(H \times H)$ in the usual way. The irreducible representations E_g of $F(H)$ were given in (3.31). One checks easily that the tensor product of two if these irreps, as defined using Δ^*, is given by

$$E_g \otimes E_h = E_{gh}. \tag{3.35}$$

Also, we have $\bar{\pi}_g = E_{g^{-1}}$. Hence, the sets of irreps of $(\mathbb{C}H)^*$ that close under conjugation and tensor products correspond exactly to the subgroups of H. The proposition follows. □

Proposition 5 *A Hopf subalgebra of an algebra $F(H)$ of functions on a group H is isomorphic to the algebra $F(H/K)$ of functions on the quotient of H by some normal subgroup K.*

Proof. Let $\mathcal{C}$ be a Hopf subalgebra of $F(H)$ and let us denote the irreps of $\mathbb{C}H$ whose matrix elements span $\mathcal{C}$ by ρ_i. Then the intersection of the kernels of the ρ_i is a normal subgroup K of H and any function in $\mathcal{C}$ will be constant on the cosets of K. We can also say that $\mathcal{C}$ really consists of functions on the quotient group H/K. Now let us show the opposite inclusion. If we form the direct sum $\oplus_i \rho_i$ of all the representations ρ_i, then this representation of H will have exactly K as its kernel and hence it can be identified with a faithful representation of H/K. Now it is a theorem in the theory of finite groups that the tensor powers of any faithful representation of a group contain all irreducible representations of this group (see for instance [119]). Hence all irreps of H/K are contained in the tensor powers of $\oplus_i \rho_i$ and hence the matrix elements of these irreps are contained in $\mathcal{C}$. But since the matrix elements of the irreps of H/K span $F(H/K)$, it follows that $\mathcal{C} \cong F(H/K)$. □

Thus we see that for any Hopf subalgebra $\mathcal{B}$ of $D(H)$, there is a maximal normal subgroup K of H and a minimal subgroup N of H such that $\mathcal{B}$ is in fact a Hopf subalgebra of $F(H/K) \otimes \mathbb{C}N$. Also, every subalgebra of $D(H)$ of the form $F(H/K) \otimes \mathbb{C}N$ is automatically a Hopf subalgebra. These particular Hopf subalgebras are in fact also transformation group algebras, with the group N acting on H/K by conjugation. This will be very useful later on, since it will allow us to apply the representation theory of transformation group algebras that we described in section 3.3.

Now let us turn to the problem of finding the Hopf subalgebra of $D(H)$ which leaves a given condensate vector ϕ invariant.

Proposition 6 *The Hopf stabilizer $\mathcal{T} \subset D(H)$ of a given vector $\phi \in V_\alpha^A$ is spanned by the matrix elements of those irreps (ρ, g) of $D(H)^*$ for which*

$$\forall x : \phi(gx) = \frac{\chi_\rho(g_A)}{d_\rho} \phi(x). \tag{3.36}$$

Here, χ_ρ denotes the character of the irrep ρ of H and d_ρ denotes its dimension. This equation for (ρ, g) can only be satisfied if $\frac{\chi_\rho(g_A)}{d_\rho}$ is a root of unity. $\mathcal{T}$ is a transformation group algebra of the form $F(H/K) \otimes \mathbb{C}N$ if and only if this root of unity equals 1 for all (ρ, g) which satisfy (3.36)

Proof. $\mathcal{T}$ is by definition the maximal Hopf subalgebra of $D(H)$ which leaves ϕ invariant. Therefore it is spanned by the matrix elements of a set of irreps (ρ_i, g) of $D(H)^*$ which closes under conjugation and tensor products (cf. theorem 2). The requirement that the matrix elements of the irrep (ρ_i, g) leave ϕ invariant is just

$$(\rho_i)_{ab}(xg_Ax^{-1})\phi(g^{-1}x) = \delta_{ab}\phi(x). \tag{3.37}$$

If we take the trace of the left and right hand side of this equation, we obtain

$$\chi_{\rho_i}(g_A)\phi(g^{-1}x) = d_{\rho_i}\phi(x). \tag{3.38}$$

Here, we have used the invariance of χ_{ρ_i} under conjugation to remove the conjugation with x. This equation is equivalent to (3.36), so any solution to (3.37) satisfies (3.36). The converse is also true. From (3.36), we see that ϕ has to be an eigenvector of the action of g^{-1} with eigenvalue $\frac{\chi_{\rho_i}(g_A)}{d_{\rho_i}}$. Since g has finite order, this implies that $\frac{\chi_{\rho_i}(g_A)}{d_{\rho_i}}$ is a root of unity. This means that $\rho_i(g_A)$ must be $\frac{\chi_{\rho_i}(g_A)}{d_{\rho_i}}$ times the unit matrix, since $\chi_{\rho_i}(g_A)$ is the sum of the eigenvalues of $\rho_i(g_A)$, which are all roots of unity (ρ_i is unitary). But if this holds, then $\rho_i(xg_Ax^{-1})$ is also $\frac{\chi_{\rho_i}(g_A)}{d_{\rho_i}}$ times the unit matrix and hence (3.37) is satisfied.

Using that the coproduct of $D(H)^*$ corresponds to the product of $D(H)$ and also that Π^A_α and ϵ are algebra homomorphisms, one may easily show that the set of irreps (ρ_i, g) whose matrix elements solve (3.37) (or 3.36) closes under tensor products. It also clearly closes under conjugation. Therefore, $\mathcal{T}$ is spanned by the matrix elements of those irreps.

If, for all irreps (ρ_i, g) whose matrix elements span $\mathcal{T}$, we have $\frac{\chi_\rho(g_A)}{d_\rho} = 1$, then all the ρ_i are paired up with the same set of elements g of H, namely those elements whose action leaves ϕ invariant. These elements form a subgroup N_ϕ of H. IN this situation, $\mathcal{T}$ is the transformation group algebra $F(H/K) \otimes N_\phi \subset D(H)$, where K is the intersection of the kernels of the ρ_i. If one of the roots of unity $\frac{\chi_{\rho_i}(g_A)}{d_\rho}$ does not equal 1, then the representation (ρ_i, e) of $D(H)^*$ does not occur in $\mathcal{T}$, but (ρ_i, g) does, for some $g \neq e$. Hence $\mathcal{T}$ cannot be a transformation group algebra of the form $F(H/K) \otimes N$ in this case. $\square$

3.6 Confinement

3.6.1 Confinement and Hopf quotients

As we have seen, the formation of a condensate of particles in the state ϕ breaks the Hopf symmetry $\mathcal{A}$ of a theory down to the Hopf stabilizer $\mathcal{T} \subset \mathcal{A}$ of ϕ. The particles in the effective theory which has the condensate as its ground state will thus carry irreducible representations of $\mathcal{T}$. However, not all the particles in the effective theory will occur as free particles; some will be confined. The intuition behind this is simple: if a particle in the effective theory has non-trivial monodromy with the condensate particles, then it will "draw a string" in the condensate. That is, the condensate's order parameter has to have a (half)line discontinuity as a consequence of the non-trivial parallel transport around the location of the particle. This line discontinuity corresponds physically to a domain wall and will cost a fixed amount of energy per unit of length[3] and hence it may not extend to infinity. As a consequence, single particles that have

[3]Note that we have not specified the Hamiltonian in our model, but we assume here that, behind the scenes, there is a "Higgs potential" which causes the symmetry breaking condensation. Such a potential will make strings cost an amount of energy that increases linearly with their length.

non-trivial braiding with the condensate cannot occur. On the other hand, configurations such as a particle and its antiparticle connected by a (short) finite length string may occur and we may compare these to the mesons of QCD. Similarly, one may have baryon-like excitations, which are bound states of three or more elementary excitations which do not match in pairs. Thus, we expect all the irreps of the Hopf stabilizer of the condensate to occur as particles in the broken theory, but some of them will occur as free particles, while others will occur only as constituents of mesonic or baryonic excitations.

There are some requirements which should hold for the set of representations of $\mathcal{T}$ that do not get confined. Clearly, this set should contain the vacuum representation or counit of T. Also, it should be closed under tensor products and charge conjugation; we would not want two non-confined particles to fuse to a confined particle, and, given that a particle is not confined, we would like the same to hold for its charge-conjugate. From these conditions on the non-confined representations it follows, using the results of section 3.5.2, that the matrix elements of the representations of the non-confined irreps of $\mathcal{T}$ span a Hopf-subalgebra of $\mathcal{T}^*$. We will call this subalgebra $\mathcal{U}^*$. Again using the results in section 3.5.2, it follows that the dual $\mathcal{U}$ of $\mathcal{U}^*$ will be a Hopf algebra whose irreducible representations are exactly the representations of $\mathcal{T}$ which are not confined (and whose matrix elements span $\mathcal{U}^*$). The dual map of the embedding of $\mathcal{U}^*$ into $\mathcal{T}^*$ is a Hopf map from $\mathcal{T}$ onto $\mathcal{U}$ and therefore $\mathcal{U}$ is a Hopf quotient of $\mathcal{T}$. This Hopf quotient $\mathcal{U}$ may be seen as the symmetry which classifies the non-confined excitations of the system. A schematic picture of the main symmetry algebras defined in this chapter and of their relations may be found on page 126.

To determine which irreps of $\mathcal{T}$ correspond to free particles and which are confined, we need to have a notion of braiding between a representation π of the original Hopf algebra and a representation ρ of $\mathcal{T}$. Clearly, the braiding should be derived from the R-matrix of $\mathcal{A}$. Let us write this as $R = \sum_k R^1_k \otimes R^2_k$. Unfortunately, we cannot just define the matrix for an exchange of a ρ and a π as $\sigma \circ (\rho \otimes \pi)(R)$, since the R^1_k are not usually elements of $\mathcal{T}$. However, we can take the exchange matrix to be $\sigma \circ ((\rho \circ P) \otimes \pi)(R)$, where P is the orthogonal projection of $\mathcal{A}$ onto $\mathcal{T}$. We also define the braid matrix for the product $\pi \otimes \rho$ as $\sigma \circ (\pi \otimes (\rho \circ P))(R)$. A representation ρ of $\mathcal{T}$ should now correspond to a free particle excitation if these braiding matrices have trivial action on the product of the condensate vector with an arbitrary vector in the module of ρ. That is, for a non-confined representation ρ, we would like to demand

$$\begin{aligned} \sum_k \rho(P(R^1_k)) \otimes \pi(R^2_k)\phi &= \rho(1) \otimes \phi \\ \sum_k \pi(R^1_k)\phi \otimes \rho(P(R^2_k)) &= \phi \otimes \rho(1). \end{aligned} \tag{3.39}$$

This gives us a requirement on every matrix element of each of the non-confined representations ρ. These matrix elements are of course elements of $\mathcal{T}^*$ and we may in fact write down a corresponding requirement for arbitrary elements of $\mathcal{T}^*$. To do this, we first define a left and a right action of $\mathcal{T}^*$ on the module V_π of the representation π of $\mathcal{A}$. We take

$$\begin{aligned} f \cdot v &:= \sum_k f(P(R^1_k))\pi(R^2_k)v \\ v \cdot f &:= \sum_k f(P(R^2_k))\pi(R^1_k)v, \end{aligned} \tag{3.40}$$

where $f \in \mathcal{T}^*$ and $v \in V_\pi$.

Proposition 7 *Let $\star$ denote the multiplication on $\mathcal{T}^*$. Then*

$$\begin{aligned} (f_1 \star f_2) \cdot v &= f_1 \cdot (f_2 \cdot v) \\ v \cdot (f_1 \star f_2) &= (v \cdot f_1) \cdot f_2. \end{aligned} \tag{3.41}$$

Proof. We have

$$\begin{aligned} (f_1 \star f_2) \cdot v &= (f_1 \otimes f_2 \otimes \pi) \circ (\Delta \otimes \mathrm{id}) \circ (P \otimes \mathrm{id})(R)v \\ &= (f_1 \otimes f_2 \otimes \pi) \circ (P \otimes P \otimes \mathrm{id}) \circ (\Delta \otimes \mathrm{id})(R)v \\ &= (f_1 \otimes f_2 \otimes \pi) \circ (P \otimes P \otimes \mathrm{id})(R_{13}R_{23})v \\ &= f_1(P(R_l^1))f_2(P(R_k^1))\pi(R_l^2 R_k^2) \\ &= f_1(P(R_l^1))\pi(R_l^2)f_2(P(R_k^1))\pi(R_k^2)v \\ &= f_1 \cdot (f_2 \cdot v). \end{aligned} \tag{3.42}$$

In the fourth equality, we used that π is a representation of $\mathcal{A}$. In the third equality, we used $(\Delta \otimes \mathrm{id})(R) = R_{13}R_{23}$. In the second equality, we used the fact that the orthogonal projection P commutes with the comultiplication, that is

$$(P \otimes P) \circ \Delta = \Delta \circ P. \tag{3.43}$$

One may see that this holds by evaluating both sides on the basis of $\mathcal{A}$ that is given by the matrix elements of the irreps of $\mathcal{A}^*$. Using this property of P and $(\mathrm{id} \otimes \Delta)(R) = R_{13}R_{12}$, one may similarly prove that $v \cdot (f_1 \star f_2) = (v \cdot f_1) \cdot f_2$. □
The requirements (3.39) on the matrix elements of ρ may now be generalized to

$$f \cdot \phi = \phi \cdot f = f(1)\phi = \epsilon^*(f)\phi. \tag{3.44}$$

Hence, the requirement that the representation ρ has trivial braiding with the condensate becomes the requirement that the left and right action of the matrix elements of ρ, as defined above, leave the condensate invariant (in the sense of definition 7). Thus, we may say that passing from the unbroken symmetry $\mathcal{T}$ to the unconfined symmetry $\mathcal{U}$ is equivalent to breaking the dual $\mathcal{T}^*$ down to $\mathcal{U}^*$ by the condensate ϕ. If we take this point of view, then the fact that we are talking about braiding is hidden in the definition of the actions above.

Unfortunately, it turns out that (3.44) does not always have solutions. In particular, the counit $\epsilon_{\mathcal{T}}$ of $\mathcal{T}$ does not always solve (3.44) (or (3.39)). This is linked to the fact that the "action" we defined above preserves the multiplication, but not necessarily the unit $\epsilon_{\mathcal{T}}$ of $\mathcal{T}^*$.

Therefore, to ensure that $\mathcal{U}^*$ contains at least the "vacuum representation" $\epsilon_{\mathcal{T}}$ of $\mathcal{T}^*$, we change the condition (3.44) to

$$\begin{aligned} f \cdot \phi &= f(1)\epsilon \cdot \phi \\ \phi \cdot f &= f(1)\phi \cdot \epsilon. \end{aligned} \tag{3.45}$$

In other words, we no longer demand that the elements of $\mathcal{U}^*$ leave the condensate invariant, but instead, we ask that they act on the condensate in the same way as ϵ. To put it yet another way, we say that the representations of $\mathcal{T}$ that are not confined are those representations that have the same braiding with the condensate as the vacuum representation. Note that, since $\epsilon_{\mathcal{T}} = 1_{\mathcal{T}^*}$, we may also write the above condition as

$$\begin{aligned} f \cdot (\epsilon \cdot \phi) &= \epsilon^*(f)(\epsilon \cdot \phi) \\ (\phi \cdot \epsilon) \cdot f &= \epsilon^*(f)(\phi \cdot \epsilon). \end{aligned} \tag{3.46}$$

We may thus still see confinement as a dual symmetry breaking, but now $\mathcal{T}^*$ is not broken to $\mathcal{U}^*$ by the original condensate ϕ, but by the vectors $\epsilon \cdot \phi$ and $\phi \cdot \epsilon$. Clearly, when $\epsilon \cdot \phi = \phi$ and $\phi \cdot \epsilon = \phi$, the new condition on elements of $\mathcal{U}^*$ reduces to (3.44).

We have now defined an algebra $\mathcal{U}$ (through its dual $\mathcal{U}^*$) whose representations should classify the non-confined excitations over the condensate. There should be an action of the braid group on the Hilbert space for a number of such excitations. Therefore, we should like $\mathcal{U}$ to be quasitriangular with an R-matrix related to the R-matrix of the original Hopf algebra $\mathcal{A}$ (for example $(P \otimes P)(R_{\mathcal{A}})$). As we will see in the examples, the conditions (3.45) are often enough to ensure that $\mathcal{U}$ has such a quasitriangular structure. Nevertheless, it does not always seem to be the case (this will be made somewhat clearer in section 3.10). Therefore we expect that the requirements (3.45) will in general have to be supplemented by some extra condition and the non-confined algebra could then be smaller than the algebra $\mathcal{U}$ defined here.

3.6.2 Domain walls and Hopf kernels

In the previous section, we talked about confined particles pulling strings in the condensate. These were line discontinuities in the condensate's wave function, induced by the non-trivial parallel transport around the confined particle. Evidently, the internal state of the condensate particles on one side of such a string will differ from that on the other side. Therefore, we may also view these strings as domain walls between regions with different condensates which exhibit the same symmetry breaking pattern[4]. We would like to classify such walls. Clearly, a wall is uniquely determined by the confined particles on which it may end, or in other words, by a representation of the residual algebra $\mathcal{T}$ that does not correspond to a representation of its non-confined quotient $\mathcal{U}$. However, there may be several irreps of $\mathcal{T}$ that cause the same parallel transport in the condensate and these will all pull the same string (or wall). In fact, let ρ be an irrep of $\mathcal{T}$ and let τ be a non-confined irrep of $\mathcal{T}$, then any irrep of $\mathcal{T}$ in the tensor product representation $(\rho \otimes \tau) \circ \Delta$ will pull the same string as ρ, since the non-confined irrep τ has trivial braiding with the condensate. In short, we may say that walls are unaffected by fusion with non-confined particles.

In view of the above, we expect that the wall that corresponds to a $\mathcal{T}$-representation ρ is already determined by the restriction of ρ to a subalgebra $\mathcal{W}$ of $\mathcal{T}$. This subalgebra should be such that, if τ is a non-confined irrep of $\mathcal{T}$, then the restriction to $\mathcal{W}$ of the tensor product representation $(\rho \otimes \tau) \circ \Delta$ should be isomorphic to a direct sum of copies of the restriction of ρ to $\mathcal{W}$ (the number of copies being the dimension of τ). Now it turns out that such a $\mathcal{W} \subset \mathcal{T}$ exists, and in fact, there are two logical options. Denote the Hopf map from the residual symmetry algebra $\mathcal{T}$ onto the non-confined algebra $\mathcal{U}$ by Γ. The *left Hopf kernel* $\mathrm{LKer}(\gamma)$ of Γ is then the subset of $\mathcal{T}$ defined as

$$\mathrm{LKer}(\Gamma) := \{t \in \mathcal{T} \,|\, (\Gamma \otimes \mathrm{id}) \circ \Delta(t) = 1_{\mathcal{U}} \otimes t\} \tag{3.47}$$

and similarly, the *right Hopf kernel* of Γ is defined as

$$\mathrm{RKer}(\Gamma) := \{t \in \mathcal{T} \,|\, (\mathrm{id} \otimes \Gamma) \circ \Delta(t) = t \otimes 1_{\mathcal{U}}\}. \tag{3.48}$$

These are our two candidates for $\mathcal{W}$. One may check that the left Hopf kernel is a right coideal subalgebra (that is, $\mathrm{LKer}(\Gamma)$ is a subalgebra and $\Delta(\mathrm{LKer}(\Gamma)) \subset \mathcal{T} \otimes \mathrm{LKer}(\Gamma)$) and similarly

[4] One may in principle also have domain walls between condensates with different symmetry breaking patterns, but this requires the parameters which govern the system (the symmetry breaking potential) to vary as one crosses the walls, breaking translational symmetry at the level of the Lagrangian or Hamiltonian

that the right Hopf kernel is a left coideal subalgebra. Moreover, one has $S(\mathrm{RKer}(\Gamma)) = \mathrm{LKer}(\Gamma)$ and $S(\mathrm{LKer}(\Gamma)) = \mathrm{RKer}(\Gamma)$. Thus, $\mathrm{LKer}(\Gamma)$ is a Hopf subalgebra of $\mathcal{T}$ exactly if $\mathrm{LKer}(\Gamma) = \mathrm{RKer}(\Gamma)$. In the examples we will meet, this will not usually be the case. Note that, even if $\mathrm{LKer}(\Gamma) \neq \mathrm{RKer}(\Gamma)$, the representations of $\mathrm{LKer}(\Gamma)$ and $\mathrm{RKer}(\Gamma)$ are in one to one correspondence: the representation ρ of $\mathrm{LKer}(\Gamma)$ corresponds to the representation $\bar{\rho} := \rho^t \circ S$ of $\mathrm{RKer}(\Gamma)$. If $\mathrm{LKer}(\Gamma)$ and $\mathrm{RKer}(\Gamma)$ are semisimple algebras (which we will assume), then this isomorphism of representations induces an isomorphism of algebras and so $\mathrm{LKer}(\Gamma) \cong \mathrm{RKer}(\Gamma)$. In other words: our candidates for $\mathcal{W}$ are isomorphic and it does not really matter which one we take.

Conjecture 1 *The wall corresponding to the $\mathcal{T}$-irrep Ω^B_β is characterized by the restriction of Ω^B_β to either* $\mathrm{LKer}(\Gamma)$ *or* $\mathrm{RKer}(\Gamma)$.

We provide the following evidence for this conjecture

1. If ρ is a representation of $\mathrm{LKer}(\Gamma)$ and τ is a representation of $\mathcal{T}$, then we define the tensor product as $(\tau \otimes \rho) \circ \Delta$. This is well-defined, since $\mathrm{LKer}(\Gamma)$ is a right coideal of $\mathcal{T}$. It is also clearly a representation of $\mathrm{LKer}(\Gamma)$. Now suppose that τ corresponds to a representation $\tilde{\tau}$ of $\mathcal{U}$, that is, $\tau = \tilde{\tau} \circ \Gamma$. Then, the defining property of $\mathrm{LKer}(\Gamma)$ guarantees that we have

$$(\tau \otimes \rho) \circ (\Gamma \otimes \mathrm{id}) \circ \Delta(t) = \tau(1) \otimes \rho(t) \tag{3.49}$$

for all $t \in \mathrm{LKer}(\Gamma)$. In other words, $\mathrm{LKer}(\Gamma)$ is indeed defined in such a way that its representations are not affected by fusion with representations of the non-confined algebra $\mathcal{U}$, just as walls are not affected by fusion with non-confined particles. Clearly, we may also define the tensor product $(\bar{\rho} \otimes \tau) \circ \Delta$ of an $\mathrm{RKer}(\Gamma)$ representation $\bar{\rho}$ with a representation τ of $\mathcal{T}$ and again, the fusion will be trivial if τ corresponds to a representation of $\mathcal{U}$.

2. Every representation of $\mathcal{T}$ corresponds to a representation of $\mathrm{LKer}(\Gamma)$ by restriction. In particular, if ρ is an irrep of $\mathcal{T}$ which factors over Γ, that is $\rho = \tau \circ \Gamma$, with τ a representation of $\mathcal{U}$, then we have, for all $t \in \mathrm{LKer}(\Gamma)$:

$$\begin{aligned} \rho(t) &= \tau \circ \Gamma(t) \\ &= \tau \circ \Gamma \circ (\mathrm{id} \otimes \epsilon) \circ \Delta(t) \\ &= \tau \circ (\mathrm{id} \otimes \epsilon) \circ (\Gamma \otimes \mathrm{id}) \circ \Delta(t) = \epsilon(t)\tau(1). \end{aligned} \tag{3.50}$$

Thus, we see that the non-confined irreps of $\mathcal{T}$ all correspond to the trivial representation of $\mathrm{LKer}(\Gamma)$. This result is consistent with the fact that the non-confined representations of $\mathcal{T}$ do not pull strings. Again, a similar result holds for $\mathrm{RKer}(\Gamma)$.

3. If B is a finite dimensional Hopf algebra, C a Hopf quotient of B and A the corresponding left Hopf kernel, then it is known by a theorem of Schneider (theorem 2.2 in [120], see also [4] for background) that B is isomorphic to a crossed product of A and C as an algebra and also as a left A-module and as a right C-comodule. Such crossed products are defined as follows:

Definition 11 *Let C be a Hopf algebra, let A be an algebra and let $\sigma : C \otimes C \to A$ be a convolution-invertible linear map. Also, suppose we have a linear map from $B \otimes A$ to A, which we write as $b \otimes a \mapsto b \cdot a$. We require that A is a twisted B-module, that is, $1 \cdot a = a$ for all a and*

$$c \cdot (d \cdot a) = \sum \sigma(c_{(1)}, d_{(1)})(c_{(2)}d_{(2)} \cdot a)\sigma^{-1}(c_{(3)}, d_{(3)}) \tag{3.51}$$

Here we use Sweedler notation for the coproduct. We also require that σ is a cocycle, that is

$$\sigma(c,1) = \sigma(1,c) = \epsilon(c)1$$
$$\sum \left(c_{(1)}\cdot\sigma(d_{(1)},e_{(1)})\right)\sigma(c_{(2)},d_{(2)}e_{(2)}) = \sum \sigma(c_{(1)},d_{(1)})\sigma(c_{(2)}d_{(2)},e) \tag{3.52}$$

and that C measures A:

$$c\cdot 1 = \epsilon(c)1, \quad c\cdot(ab) = \sum(c_{(1)}\cdot a)(c_{(2)}\cdot b). \tag{3.53}$$

Now the crossed product algebra $A\#_\sigma B$ *is the vector space $A\otimes B$ with the product given by*

$$(a\otimes c)(b\otimes d) = \sum a(c_{(1)}\cdot b)\sigma(c_{(2)},d_{(1)})\otimes c_{(3)}d_{(2)}. \tag{3.54}$$

These crossed products were introduced in [121, 122]. An accessible treatment may be found in [4] or [8]. When it is given that C measures A, one may show that the conditions that involve σ are equivalent to the associativity of the product of $A\#_\sigma C$. Some elementary properties of the crossed product are

- A is embedded into $A\#_\sigma C$ through $a\mapsto a\otimes 1$, that is, we have $(a\otimes 1)(b\otimes 1) = (ab\otimes 1)$.
- The map $j: C\to A\#_\sigma C$ given by $c\mapsto 1\otimes c$ is clearly a C-comodule morphism when $A\#_\sigma C$ and C are given the comodule structures $\mathrm{id}_A\otimes\Delta_C$ and Δ_C respectively, but it is usually not an algebra morphism; we have $(1\otimes c)(1\otimes d) = \sum(\sigma(c_{(1)},d_{(1)})\otimes c_{(2)}d_{(2)})$.
- When σ is trivial, that is $\sigma(c,d) = \epsilon(c)\epsilon(d)$, the cross product is just the ordinary smash product; we have $(a\otimes b)(c\otimes d) = \sum a(c_{(1)}\cdot b)\otimes c_{(2)}d_{(2)}$. In this case, j is an algebra morphism.

We see that our residual algebra $\mathcal{T}$ is isomorphic to the cross product $\mathrm{LKer}(\Gamma)\#_\sigma\mathcal{U}$ for some cocycle σ. This lends support to the idea that $\mathcal{T}$-excitations are characterized by a wall, corresponding to a representation of $\mathrm{LKer}(\Gamma)$ and by further quantum numbers, which can be associated to the non-confined algebra $\mathcal{U}$. If the cross product was just the tensor product $\mathrm{LKer}(\Gamma)\otimes\mathcal{U}$, then these "non confined quantum numbers" would be labels of $\mathcal{U}$-representations, but here, we cannot expect this, because the actions of $\mathrm{LKer}(\Gamma)$ and $\mathcal{U}$ on a $\mathcal{T}$-module will not commute. In fact, $\mathcal{U}$ is typically not even a subalgebra of $\mathcal{T}$. Therefore, finding the quantum numbers associated to $\mathcal{U}$ for the general case is a non-trivial task, which we postpone to future work[5].

3.6.3 Confinement for transformation group algebras

Suppose the $D(H)$ symmetry of a discrete gauge theory has been broken by a condensate $\phi\in V_\alpha^A$ and the residual symmetry algebra $\mathcal{T}$ is a transformation group algebra of the kind referred to in section 3.5.3. The explicit definition is

$$\mathcal{T} = \{F\in D(H)|F(xk,y) = F(x,y)1_N(y)\ (\forall k\in K)\}, \tag{3.55}$$

where N is a subgroup of H and K is a normal subgroup of H. Such algebras will arise frequently in our examples. Here, we investigate which representations of such a $\mathcal{T}$ are confined

[5] Note that much more than we have written here is known when $\mathrm{LKer}(\Gamma)$ is a Hopf algebra. For such results, see for instance [123, 124, 125]

and which are not. In particular, we will find that there is a set of non-confined representations of $\mathcal{T}$ such that the irreps in this set are in one to one correspondence with those of $D(N/(N\cap K))$.

First, we find some properties of the condensate vector ϕ. If $\mathcal{T}$ is of the form given above then the invariance of ϕ under elements of the form $1\otimes n$ (with $n\in N$) and $f\otimes e$ implies that we have

$$\begin{array}{ll}(\forall n\in N): & \phi(nx)=\phi(x)\\ (\forall (f\otimes e)\in\mathcal{T}): & f(xg_Ax^{-1})\phi(x)=f(e)\phi(x).\end{array}\tag{3.56}$$

The second equation and $\phi\neq 0$ imply that there is an $x\in H$ such that $f(xg_Ax^{-1})=f(e)$ for all f that are constant on K-cosets. As a consequence, we have $xg_Ax^{-1}\in K$ and hence, since K is normal in H, we have $A\subset K$ and in particular $g_A\in K$.

Now let us write down an explicit formula for the orthogonal projection P of $D(H)$ onto $\mathcal{T}$:

$$P(F)(x,y)=\frac{1}{|K|}\int_K \mathrm{d}k\, F(xk,y)1_N(y).\tag{3.57}$$

In the following, we will sometimes omit the characteristic function of N and just keep in mind that the projected function has support in N. With the above formula, we can find $(P\otimes\mathrm{id})(R)$ and $(\mathrm{id}\otimes P)(R)$ from the formula (3.2) for the R-matrix of $D(H)$:

$$\begin{aligned}(P\otimes\mathrm{id})(R)(x_1,y_1,x_2,y_2) &= \frac{1}{|K|}\int_K \mathrm{d}k\,\delta_e(x_1k(y_2)^{-1})\delta_e(y_1)\\ (\mathrm{id}\otimes P)(R)(x_1,y_1,x_2,y_2) &= \delta_e(x_1(y_2)^{-1})\delta_e(y_1)1_N(y_2).\end{aligned}\tag{3.58}$$

Using these formulae, we can write down the left and right actions of $\mathcal{T}^*$ on the condensate vector, as defined in equation (3.40). They are given by

$$\begin{aligned}(\tau\cdot\phi)(x) &= \int_H \mathrm{d}z\,(\tau(F_L))(z)\phi(z^{-1}x)\\ (\phi\cdot\tau)(x) &= (\tau(F_R))\phi(x),\end{aligned}\tag{3.59}$$

where τ is an arbitrary element of $\mathcal{T}^*$ and we have defined

$$\begin{aligned}F_L(a,b\,;z) &= \frac{1}{|K|}\int_K \mathrm{d}k\,\delta_e(akz^{-1})\delta_e(b)\\ F_R(a,b\,;x,z) &= \delta_e(xg_Ax^{-1}b^{-1})1_N(b).\end{aligned}\tag{3.60}$$

F_R and F_L should be read as functions of a and b with parameters x and z. We want to find the maximal Hopf subalgebra of $\mathcal{T}^*$ for which the condition (3.45) holds. This will be spanned (as a linear space) by the matrix elements of a set of representations of $\mathcal{T}$. Since $\mathcal{T}$ is isomorphic to a transformation group algebra, we know its representations (see section 3.3.2). They are labeled by an orbit B of the action of N on H/K and by an irreducible representation β of the stabilizer $N_B\subset N$ of this orbit. The matrix elements of the representation labeled by B and β in the basis of formula (3.11) can be read off from formula (3.12), which in this case becomes

$$\tau^B_\beta(F)^{i,j}_{\zeta,\eta}=\int_{N_B} F(x_\eta g_B x_\eta^{-1},x_\eta n x_\zeta^{-1})\beta_{i,j}(n)\,\mathrm{d}n.\tag{3.61}$$

In this formula, we have $x_\eta,x_\zeta\in N$ as in (3.12), while g_B is an arbitrary element of the K-coset ξ_B that features in (3.12). Note that it does not matter which element of this coset we take, since the function F in the integrand is constant on K-cosets in its left argument.

Proposition 8 *The requirements (3.45) which determine which of the irreps τ_β^B of $\mathcal{T}$ are not confined, reduce to*

$$(\forall \eta \in \mathcal{O}_B): \ \frac{1}{|K|}\int_K \mathrm{d}k\, \phi(k x_\eta g_B^{-1} x_\eta^{-1} x) = \frac{1}{|K|}\int_K \mathrm{d}k\, \phi(kx) \tag{3.62}$$

$$(g_A \notin N) \vee \big(\forall x \in \mathrm{supp}(\phi),\, \forall \eta \in \mathcal{O}_B: \ \beta(x_\eta^{-1} x g_A x^{-1} x_\eta) = I\big). \tag{3.63}$$

Here, $\mathcal{O}_B$ is the orbit of ξ_B in H/K and I is the unit matrix in the module of β.

Proof. Substituting (3.61) into to the formulae for the actions above, we find for the left action

$$\begin{aligned}((\tau_\beta^B)_{\zeta,\eta}^{i,j} \cdot \phi)(x) &= \frac{1}{|K|}\int_H \mathrm{d}z \int_{N_B} \mathrm{d}n \int_K \mathrm{d}k\, \delta_e(x_\eta g_B x_\eta^{-1} k z^{-1})\delta_e(x_\eta n x_\zeta^{-1})\beta_{i,j}(n)\phi(z^{-1}x) \\ &= \frac{1}{|K|}\int_{N_B} \mathrm{d}n \int_K \mathrm{d}k\, \delta_e(x_\eta n x_\zeta^{-1})\beta_{i,j}(n)\phi(k^{-1} x_\eta g_B^{-1} x_\eta^{-1} x) \\ &= 1_{N_B}(x_\eta^{-1}x_\zeta)\beta_{i,j}(x_\eta^{-1}x_\zeta)\frac{1}{|K|}\int_K \mathrm{d}k\, \phi(k x_\eta g_B^{-1} x_\eta^{-1} x) \end{aligned} \tag{3.64}$$

and similarly, for the right action

$$\begin{aligned}(\phi \cdot (\tau_\beta^B)_{\zeta,\eta}^{i,j})(x) &= \int_{N_B} \mathrm{d}n\, \delta_e(x g_A x^{-1} x_\zeta n^{-1} x_\eta^{-1})\beta_{i,j}(n)\phi(x) \\ &= 1_{N_B}(x_\eta^{-1} x g_A x^{-1} x_\zeta)\beta_{i,j}(x_\eta^{-1} x g_A x^{-1} x_\zeta)\phi(x). \end{aligned} \tag{3.65}$$

As a special case, we can find the left and right action of the counit $\epsilon \in \mathcal{T}^*$, which corresponds to the one-dimensional representation $\tau_1^{[e]}$. We have

$$\begin{aligned}(\epsilon \cdot \phi)(x) &= \frac{1}{|K|}\int_K \mathrm{d}k\, \phi(kx) \\ (\phi \cdot \epsilon)(x) &= 1_N(g_A)\phi(x). \end{aligned} \tag{3.66}$$

The final ingredient we need in order to write down the requirements (3.45) for the matrix elements, is the value of $(\tau_\beta^B)_{\zeta,\eta}^{i,j}$ in $1_{\mathcal{T}} = 1_{D(H)}$. This is given by

$$(\tau_\beta^B)_{\zeta,\eta}^{i,j}(1_{D(H)}) = 1_{N_B}(x_\eta^{-1}x_\zeta)\beta_{ij}(x_\eta^{-1}x_\zeta). \tag{3.67}$$

Thus, the conditions (3.45) that the matrix elements of τ_β^B have to fulfill, in order for τ_β^B not to be confined, become

$$1_{N_B}(x_\eta^{-1}x_\zeta)\beta_{i,j}(x_\eta^{-1}x_\zeta)\frac{1}{|K|}\int_K \mathrm{d}k\, \phi(k x_\eta g_B^{-1} x_\eta^{-1} x) = 1_{N_B}(x_\eta^{-1}x_\zeta)\beta_{ij}(x_\eta^{-1}x_\zeta)\frac{1}{|K|}\int_K \mathrm{d}k\, \phi(kx) \tag{3.68}$$

and

$$1_{N_B}(x_\eta^{-1} x g_A x^{-1} x_\zeta)\beta_{i,j}(x_\eta^{-1} x g_A x^{-1} x_\zeta)\phi(x) = 1_N(g_A)1_{N_B}(x_\eta^{-1}x_\zeta)\beta_{ij}(x_\eta^{-1}x_\zeta)\phi(x). \tag{3.69}$$

In the special case where $\eta = \zeta$ and $i = j$, the condition (3.68) reduces to (3.62). On the other hand, if (3.62) holds, then (3.68) will also hold for general (η, ζ, i, j) and hence (3.62) is equivalent to (3.68). The condition (3.69) is trivially satisfied when g_A is not contained in N (this is the first alternative in (3.63)). When g_A is an element of N (and hence of $N \cap K$, using (3.56)), it may also be simplified; in the special case where $\eta = \zeta$, (3.69) reduces to

$$1_{N_B}(x_\eta^{-1} x g_A x^{-1} x_\eta)\beta_{i,j}(x_\eta^{-1} x g_A x^{-1} x_\eta)\phi(x) = \delta_{ij}\phi(x). \tag{3.70}$$

Now since $g_A \in N \cap K$, it follows that $x_\eta^{-1} x g_A x^{-1} x_\eta \in N \cap K$. But $N \cap K$ acts trivially on H/K and hence $N \cap K \subset N_B$ for any B. Hence the condition above reduces to the second alternative in (3.63). In the other direction, it is not difficult to see that (3.69) will be satisfied for general (η, ζ, i, j) if (3.63) is satisfied. Thus, we see that (3.69) is equivalent to (3.63). □
Proposition 8 indicates how far we can go towards the general solution of (3.45) without specifying the condensate vector ϕ. The following proposition describes a set of solutions that is present for any ϕ, but that is not always the full set of solutions.

Proposition 9 *Independently of the condensate vector ϕ, there is a set of unconfined irreps of $\mathcal{T}$ which closes under conjugation and tensor products. The corresponding Hopf quotient of $\mathcal{T}$ is isomorphic to the quantum double of the group $N/(N \cap K)$. This quantum double may be realized naturally on the space of functions on $N \times N$ which are constant on $(N \cap K)$-cosets in both arguments. The Hopf surjection $\Gamma : \mathcal{T} \to D(N/(N \cap K))$ is then given by*

$$\Gamma(f)(x, y) = \int_{N \cap K} f(x, yk) dk. \tag{3.71}$$

Proof. First, we find our set of irreps. Note that the left hand side of (3.62) is $\frac{1}{|K|}$ times the sum of the values of ϕ over the K-coset $x_\eta g_B^{-1} x_\eta^{-1} x K$. Similarly, the right hand side involves a sum over xK. Using the fact that $\phi(nx) = \phi(x)$ for any $n \in N$ (cf. (3.56)), we see that (3.62) will be satisfied if $x_\eta g_B^{-1} x_\eta^{-1} x K = nxK$ for some $n \in N$, or equivalently if there is an $n \in N$ such that $g_B K = nK$. Furthermore, (3.63) is clearly satisfied for all β that are trivial on $N \cap K \subset N_B$. Thus, the irreps τ_β^B of $\mathcal{T}$ for which $g_B K = nK$ and $\beta|_{N \cap K} = 1$ are never confined.

Second, we show that these irreps are in one-to one correspondence with the irreps of $D(N/(N \cap K))$. To see this, first note that, for $n_1, n_2 \in N$, we have

$$n_1 K = n_2 K \Leftrightarrow n_1(K \cap N) = n_2(K \cap N). \tag{3.72}$$

In fact, let $\bar{N}$ denote the subgroup of H/K which consists of the classes nK with $n \in N$, then this correspondence is an isomorphism between $\bar{N}$ and $N/(N \cap K)$. It follows that the N-orbits in H/K whose elements lie in $\bar{N}$ are in one to one correspondence with the conjugacy classes of $N/(K \cap N)$. Now fix an arbitrary such orbit $B \subset \bar{N}$. The irreps β of the stabilizer $N_B \subset N$ of this orbit which are trivial on $K \cap N$ are in one to one correspondence with the irreps of $N_B/(K \cap N)$. But $N_B/(K \cap N)$ is exactly the centralizer of the conjugacy class of $N/(K \cap N)$ that corresponds to B. Hence the non-confined irreps of T_A are labeled by a conjugacy class of $N/(K \cap N)$ and an irrep of the centralizer of this class in $N/(K \cap N)$. But this means that they are in one to one correspondence with the irreps of $D(N/(K \cap N))$.

Now let us have a closer look at the map $\Gamma : \mathcal{T} \to D(N/(N \cap K))$. For convenience, we will realize $D(N/(N \cap K))$ on the space of functions on $N \times N$ which are constant on $(N \cap K)$-cosets in both arguments. The isomorphism with the usual formulation in terms of functions on $N/(N \cap K)$ is taken as follows. Let $\bar{f} \in F(N/(N \cap K) \times N/(N \cap K))$ then $\bar{f}$ corresponds to the function $f \in F(N \times N)$ given by $f(x, y) = \bar{f}(x(N \cap K), y(N \cap K))$. The demand that this identification is an isomorphism fixes the Hopf algebra structure on $F(N \times N)$. For example, the product of two functions on $N \times N$ may now be written as

$$f_1 \bullet f_2(x, y) = \frac{1}{|N \cap K|} \int_N f_1(x, z) f_2(z^{-1} x z, z^{-1} y) dz. \tag{3.73}$$

It is straightforward to prove that Γ, as defined above, is indeed a Hopf homomorphism. For example, to see that Γ preserves the product, we write

$$\begin{aligned}\Gamma(f_1) \bullet \Gamma(f_2)(x,y) &= \frac{1}{|N\cap K|}\int_N dz \int_{N\cap K} dk_1 \int_{N\cap K} dk_2\, f_1(x, zk_1) f_2(z^{-1}xz, z^{-1}yk_2)\\ &= \frac{1}{|N\cap K|}\int_N dz \int_{N\cap K} dk_1 \int_{N\cap K} dk_2\, f_1(x,z) f_2(z^{-1}xz, k_1 z^{-1}yk_2)\\ &= \int_N dz \int_{N\cap K} dk_2\, f_1(x,z) f_2(z^{-1}xz, z^{-1}yk_2) = \Gamma(f_1 \bullet f_2)(x,y). \end{aligned} \tag{3.74}$$

In going from the first to the second line, we used the invariance of the integral over N and the fact that f_2 is constant on K-cosets in its left argument. In going from the second to the third line, we used the invariance of the k_2-integral to remove the k_1-dependence from the integrand and subsequently removed the k_1-integral. The rest of the proof that Γ is a Hopf algebra morphism is similar. It is also easy to see that Γ is surjective.

To complete the proof of the proposition, we need to show that the set of unconfined irreps we have found consists precisely of those irreps of $\mathcal{T}$ that factor over Γ. Thus, let $\Pi^{\tilde{B}}_{\tilde{\beta}}$ be an irrep of $D(N/N\cap K)$ and let $\chi^{\tilde{B}}_{\tilde{\beta}}$ be its character, as given in (3.13). Then we have

$$\chi^{\tilde{B}}_{\tilde{\beta}}(\Gamma(f)) = \int_{\tilde{B}} d\zeta \int_{N_{\tilde{B}}} dn \int_{N\cap K} dk\, f(x_\zeta g_{\tilde{B}} x_\zeta^{-1}, x_\zeta n x_\zeta^{-1} k)\chi_{\tilde{\beta}}(n). \tag{3.75}$$

Here, we have abused notation slightly: in stead of elements of $N/(N\cap K)$ one should read representatives of these elements in N where appropriate. It should be clear that the choice of representatives does not affect the result. We may now change the sum over the conjugacy class $\tilde{B} \subset N/(N\cap K)$ into a sum over the corresponding N-orbit $B \subset \bar{N} \subset H/K$. Similarly, we may change the sums over $N_{\tilde{B}}$ and $N\cap K$ into one sum over $N_B \subset N$. This yields

$$\chi^{\tilde{B}}_{\tilde{\beta}}(\Gamma(f)) = \int_B d\zeta \int_{N_B} dn\, f(x_\zeta g_B x_\zeta^{-1}, x_\zeta n x_\zeta^{-1})\chi_\beta(n), \tag{3.76}$$

where β is the irrep of N_B that corresponds to the irrep $\tilde{\beta}$ of $N_{\tilde{B}}$ (of course, β is trivial on $N\cap K$). The expression above is just the value on f of the character of the irrep τ^B_β of $\mathcal{T}$. τ^B_β indeed belongs to our set of unconfined irreps and from the one to one correspondence between irreps of $D(N/(K\cap N))$ and irreps in our unconfined set, we see that we must get all irreps in the unconfined set in this way. Thus, the set of unconfined irreps of $\mathcal{T}$ that we have found corresponds precisely to the set of irreps of $\mathcal{T}$ that factor over Γ and the proposition follows. □
Note that the R-matrix and ribbon-element of $D(N/(K\cap N))$ provide the set of non-confined irreps that we have found above with a well defined braiding and spin. It is not clear that we will have such properties for the full set of solutions to (3.62) and (3.63). Therefore, we expect that the physically relevant set of solutions to these equations is the one given in the proposition above.This issue will not be very important in the sequel, since the set of solutions in the proposition is actually complete in all our examples.

Proposition 10 *The left and right Hopf kernels of* Γ *are given by*

$$\mathrm{LKer}(\Gamma) = \{f \in \mathcal{T} | \forall x_1 \in N : f(x_1x_2, y) = f(x_2, y) \wedge \forall y \notin N\cap K : f(x,y) = 0\} \tag{3.77}$$
$$\mathrm{RKer}(\Gamma) = \{f \in \mathcal{T} | \forall x_2 \in N : f(x_1x_2, y) = f(x_1, y) \wedge \forall y \notin N\cap K : f(x,y) = 0\} \tag{3.78}$$

Proof. We have

$$\begin{aligned}\Gamma \otimes \mathrm{id}(\Delta(f))(x_1, y_1, x_2, y_2) &= \int_{N\cap K} f(x_1x_2, y_1k)\delta_e(y_1ky_2^{-1}) \\ (1_{D(N/(N\cap K))} \otimes f)(x_1, y_1, x_2, y_2) &= 1_{N\cap K}(y_1)f(x_2, y_2) \qquad (3.79)\end{aligned}$$

and the left Hopf kernel of Γ consists of those functions f for which the right hand sides of these equations are equal:

$$\int_{N\cap K} f(x_1x_2, y_1k)\delta_e(y_1ky_2^{-1}) = 1_{N\cap K}(y_1)f(x_2, y_2). \qquad (3.80)$$

Now if we take $y_1 = y_2$, then this requirement reduces to

$$f(x_1x_2, y_1) = 1_{N\cap K}(y_1)f(x_2, y_1), \qquad (3.81)$$

from which we see that $f(x, y)$ equals zero for all $x \in H$ when y is not an element of $N \cap K$, while for $y \in N \cap K$, we have $f(x_1x_2, y) = f(x_2, y)$ for all $x_1 \in N$. On the other hand, all f which satisfy these requirements automatically satisfy (3.80). One may see this by noting that both the left hand side and the right hand side of (3.80) can be non-zero only if both y_1 and y_2 are elements of $K \cap N$, in which case left hand side and right hand side are equal. The formula for $\mathrm{LKer}(\Gamma)$ now follows. The proof of the expression for $\mathrm{RKer}(\Gamma)$ is similar and we leave it to the reader. □

If we once again let $\bar{N}$ be the subgroup of H/K that consists of the cosets nK of the elements of N, then we see that we have the following

Corollary 3 *As algebras:*

$$\begin{aligned}\mathrm{LKer}(\Gamma) &\cong F(\bar{N}\backslash(H/K)) \otimes \mathbb{C}(N \cap K) \\ \mathrm{RKer}(\Gamma) &\cong F((H/K)/\bar{N}) \otimes \mathbb{C}(N \cap K) \qquad (3.82)\end{aligned}$$

Proof. To see that the isomorphisms are algebra isomorphisms, note that the elements of $1 \otimes \mathbb{C}(K \cap N)$ commute with those of $F(\bar{N}\backslash(H/K)) \otimes 1$ and $F((H/K)/\bar{N}) \otimes 1$. This is because the elements of $F(H/K) \otimes 1$ already commuted with those of $1 \otimes \mathbb{C}(K \cap N)$ in $\mathcal{T}$. □

As a consequence of this corollary, each irreducible representation of the left kernel is a product of an irrep of $F(\bar{N}\backslash(H/K))$ and an irrep of $N \cap K$. The irreps of $F(\bar{N}\backslash(H/K))$ are of course labeled by the elements of $\bar{N}\backslash(H/K)$ and hence each irrep of $\mathrm{LKer}(\Gamma)$ is labeled by an element of $\bar{N}\backslash(H/K)$ and an irrep of $N \cap K$. Similarly, each irrep of $\mathrm{RKer}(\Gamma)$ is labeled by an element of $(H/K)/\bar{N}$ and an irrep of $N \cap K$.

3.7 Requirements on condensates

Before we turn to the study of explicit examples of symmetry breaking and confinement, let us first motivate the choices of condensate vectors that we will use in our examples.

Up to now we have assumed that one may form a condensate of any kind of particle in the theory, in any internal state ϕ. However, if we want to have true Bose condensates, then we should demand that the state ϕ has trivial self-braiding and also trivial spin factor[6]. In other words:

[6] In some applications, it could be more useful to think of our condensate as a background of particles in the same *internal* state, but not necessarily with the same external quantum numbers. Then the restrictions we give here are not necessary. Examples of "condensates" of particles with a non-trivial spin factor would be the fractional quantum Hall ground states proposed in [49, 50]

- The condensate must have trivial spin factor, i.e $\alpha(g_A) = I$.
- The condensate must have trivial self-braiding, i.e.

$$\sigma \circ \Pi^A_\alpha \otimes \Pi^A_\alpha(R)\phi \otimes \phi = \phi \otimes \phi. \tag{3.83}$$

The examples that we will treat in the rest of this chapter will all have trivial spin and self-braiding. The rest of this section is devoted to finding out which kinds of electric, magnetic and dyonic condensates will satisfy these requirements.

For any purely *electric* condensates $\phi \in V^e_\alpha$ (see section 3.8), the requirements are both trivially satisfied.

A vector ϕ in a purely *magnetic* $D(H)$-module V^A_1 will automatically have trivial spin, but may have non-trivial self-braiding. Nevertheless, there will always be at least two gauge orbits of magnetic states with trivial self-braiding for every class A which has more than a single element. The first of these orbits contains all the states with pure fluxes hg_Ah^{-1}, which have wave functions 1_{hN_A}. We will study the corresponding condensates in section 3.10). The second orbit, which will be studied in section 3.9, consists of the single gauge invariant state which is the superposition of all these pure fluxes. Its wave function is the function that sends all elements of H to 1. Of course if A has only a single element, then these orbits coincide. Note that, when the orbits are different, they will also have different symmetry breaking patterns. In particular, the gauge singlet will leave the electric group unbroken, while the states in the other orbit will not. To see that the states in the two orbits we have mentioned do indeed have trivial self-braiding and to see if there are more states with this property, we write down the expression for the self-braiding of an arbitrary $f \in V^A_1$. We have

$$\begin{aligned} f \otimes f \quad &: \quad (g,h) \mapsto f(g)f(h) \\ \sigma \circ R(f \otimes f) \quad &: \quad (g,h) \mapsto f(g)f(gg_A^{-1}g^{-1}h) \end{aligned} \tag{3.84}$$

and hence f has trivial self-braiding precisely when

$$f(g)f(h) = f(g)f(gg_A^{-1}g^{-1}h) \quad (\forall g,h \in H). \tag{3.85}$$

One may readily check that the states we have already mentioned are always solutions to this equation. Depending on H and A, there may also be extra solutions. For example, if $A \subset N_A$, then all f are allowed, since in that case $hN_A = gg_A^{-1}g^{-1}hN_A$.

Dyons (see section 3.11) can have non-trivial spin, but dyons with trivial spin also exist for many groups H. In fact, given a magnetic flux A, there will be dyons with flux $A \neq [e]$ and trivial spin factor precisely when g_A is contained in a proper normal subgroup of its centralizer N_A. For Abelian H, this just means that the cyclic group generated by g_A must be a proper subgroup of H. For non-Abelian H, one may note that g_A is contained in the center of N_A, which is a proper normal subgroup if N_A is still non-Abelian. If N_A is Abelian, then we have the requirement that the cyclic group generated by g_A must be a proper subgroup of N_A. When H is Abelian, the requirement of trivial self-braiding is equivalent to that of trivial spin and hence all the spinless dyons we have found may be condensed. When H is non-Abelian, this is not the case and the requirement of trivial self-braiding then restricts the possibilities further. In particular, using the ribbon property of $D(H)$, it gives the necessary condition that two of the condensed dyons should be able to fuse into a particle with trivial spin. In spite of this restriction, there are still many non-Abelian groups H which allow for dyonic states with trivial self-braiding. One may for example show that they occur for any non-Abelian H with a non trivial center.

3.8 Electric condensates

3.8.1 Symmetry breaking

In this section, we study symmetry breaking by an electric condensate $\phi \in \Pi^e_\alpha$. The first thing to do is to find the residual symmetry algebra, which is the Hopf stabilizer of ϕ. This means finding all representations (ρ, g) of $D(H)^*$ which solve equation (3.36) in the special case where the flux A is trivial. In this case, we see immediately that all ρ are allowed. The requirement on g is just that $\phi(gx) = \phi(x)$, or equivalently, $\phi(g^{-1}x) = \phi(x)$, for all $x \in H$. Using the invariance property of ϕ, this reduces to $\alpha(x^{-1}gx)\phi(x) = \phi(x)$ and using the invariance property once again, we see that this reduces to the single requirement

$$\alpha(g)\phi(e) = \phi(e). \tag{3.86}$$

Thus, if we define $v := \phi(e) \in V_\alpha$, then g has to be an element of the stabilizer N_v of v. Since ρ was unrestricted, it follows that the residual symmetry algebra is the Hopf subalgebra $\mathcal{T}_v(H)$ of the double which is $F(H) \otimes \mathbb{C}N_v$ as a vector space, or in terms of functions on $H \times H$:

$$\mathcal{T}_v(H) := \{F \in D(H) | \mathrm{supp}(F) \subset H \times N_v\}. \tag{3.87}$$

$\mathcal{T}_v(H)$ is a transformation group algebra, with N_v acting on H by conjugation. Hence we may immediately write down all its irreducible representations, using theorem 1. They are labeled by an N_v-orbit $\mathcal{O}$ in H and by a representation τ of the stabilizer $N_\mathcal{O}$ of of a chosen element $g_\mathcal{O} \in \mathcal{O}$ in N_A. We will denote them $\Omega^\mathcal{O}_\tau$. The Hilbert space on which $\Omega^\mathcal{O}_\tau$ acts is the space $F_\tau(N_v, V_\tau)$ defined in (3.7). We will call it $V^\mathcal{O}_\tau$ for short. The action of $\Omega^\mathcal{O}_\tau$ on this space is given by the formula in theorem 1, which in this case becomes

$$\left(\Omega^\mathcal{O}_\tau(F)\phi\right)(x) := \int_{N_v} dz\, F(xg_\mathcal{O}x^{-1}, z)\, \phi(z^{-1}x). \tag{3.88}$$

The characters $\psi^\mathcal{O}_\tau$ of these representations are given by formula 3.14 or equivalently by formula (3.15). We have

$$\psi^\mathcal{O}_\tau(\eta, h) = 1_{N_\eta}(h) 1_\mathcal{O}(\eta) \psi_\tau(x_\eta^{-1} h x_\eta). \tag{3.89}$$

Using these characters and the inner product (3.16), one may calculate the fusion rules for $\mathcal{T}_v(H)$-irreps.

Clearly, any representation of $D(H)$ also gives a representation of $\mathcal{T}_v(H)$ by restriction. When we consider the irreps of $D(H)$ as $\mathcal{T}_v(H)$-representations in this way, they will usually no longer be irreducible. Their decomposition into $\mathcal{T}_v(H)$-irreps may be calculated by taking the inner product (3.16) of their character with the characters $\psi^\mathcal{O}_\tau$. The character of the $D(H)$-irrep Π^B_β, seen as a $\mathcal{T}_v(H)$-irrep, is just the restriction of the original character χ^B_β; we have

$$\chi^B_\beta(\eta, h) = 1_{N_B}(h) 1_B(\eta) \chi_\alpha(x_\eta^{-1} h x_\eta). \tag{3.90}$$

From this formula and the formula for $\psi^\mathcal{O}_\tau$, we see immediately that the irreps $\Omega^\mathcal{O}_\tau$ of $\mathcal{T}_v$ which constitute Π^B_β will all have $\mathcal{O} \subset B$. Also we see that a purely magnetic $D(H)$-irrep Π^B_1 will decompose into the purely magnetic $\mathcal{T}_v(H)$-irreps $\Omega^\mathcal{O}_1$ with $\mathcal{O} \subset B$. A purely electric irrep Π^e_β of $D(H)$ will decompose into the purely electric irreps Ω^e_τ of $\mathcal{T}_v(H)$ which are such, that the irrep τ of $N_A \subset H$ is contained in the decomposition of the irrep β of H.

3.8.2 Confinement

Let us now determine which representations of the residual algebra $\mathcal{T}_v(H)$ of the previous section will be confined and which will not. The non-confined representations have to satisfy the conditions (3.45). Since $\mathcal{T}_v(H)$ is isomorphic to a transformation group algebra, we may apply the results of section 3.6.3 (with $N = N_v$ and $K = \{e\}$) to simplify these to the conditions (3.62) and (3.63). From section 3.6.3, proposition 9, we know that these equations are solved at least by those $\Omega^{\mathcal{O}}_\tau$ for which $g_{\mathcal{O}}K = nK$ for some $n \in N_v$ and τ is trivial on K. Since we have $K = \{e\}$ here, this reduces to just the requirement that $g_{\mathcal{O}} \in N_v$. We have also shown that this set of irreps closes under conjugation and tensor products and that they are in fact the irreps of a quotient $\mathcal{U}_v(H)$ of $\mathcal{T}_v(H)$ that is isomorphic to $D(N_v/(N_v \cap K))$, which is here just $D(N_v)$.

It turns out that the irreps $\Omega^{\mathcal{O}}_\tau$ with $g_{\mathcal{O}} \in N_v$ are actually all the irreps that meet the requirements (3.62) and (3.63). Let us check this. In the case at hand, (3.63) is always satisfied, since g_A is the unit element of H. Thus, we are left with condition (3.62). Since the $K = \{e\}$, this reduces to

$$\phi(x_\eta g_{\mathcal{O}}^{-1} x_\eta^{-1} x) = \phi(x). \tag{3.91}$$

Using the invariance property of ϕ, this becomes $\alpha(x^{-1}x_\eta g_{\mathcal{O}} x_\eta^{-1}x)\phi(x) = \phi(x)$. Multiplying with $\alpha(x)$ from the left and using the invariance of ϕ once more, we see $\alpha(x_\eta g_{\mathcal{O}} x_\eta^{-1})\phi(e)$ must equal $\phi(e)$. If we now recall that $v = \phi(e)$ and that the x_η are elements of N_v, then we see that we are left with the requirement that $g_{\mathcal{O}}$ should be an element of N_v. Thus, the class of solutions that we had already is indeed complete and the non-confined algebra is just the quantum double of N_v.

The fact that the non-confined algebra $\mathcal{U}$ is the quantum double $D(N_v)$ of the stabilizer N_v of the condensate vector comes as no surprise; the original $D(H)$-theory was obtained from a gauge theory with a continuous gauge group G by breaking this group down to H through condensation of an electric excitation. All we have done by condensing one of the electric particles of the $D(H)$-theory is to modify the electric condensate of the G-theory in such a way that the residual gauge group is now N_v rather than H. We referred to this replacement of H with N_v already at the end of section 3.2 and it is encouraging to see that our formalism for symmetry breaking and confinement in quantum groups produces the result we anticipated there.

The result that the non confined irreps of $\Omega^{\mathcal{O}}_\tau$ of $\mathcal{T}_v$ are exactly those for which $g_{\mathcal{O}} \in N_v$ is also in accordance with our intuitive treatment in section 3.2; the $\Omega^{\mathcal{O}}_\tau$ whose "flux" $g_{\mathcal{O}}$ acts trivially on the condensate are not confined, because they will have trivial braiding with the condensate. The remaining $\Omega^{\mathcal{O}}_\tau$ will be confined, because they pull strings in the condensate. In fact, all the expectations we voiced in section 3.2 come true and are now under precise mathematical control. "Hadronic" excitations with overall flux in N_v can be classified by means of the fusion rules of $\mathcal{T}_v$, which can be obtained using the inner product (3.16) on the space of characters. Also, the theory of section 3.6.2 implies that the classification of strings or domain walls does indeed involve the elements of H/N_v, as we now show.

In section 3.6.2 we asserted that the string associated with an irrep $\Omega^{\mathcal{O}}_\tau$ may be characterized by the restriction of $\Omega^{\mathcal{O}}_\tau$ to the left or right Hopf kernel of the projection Γ of $\mathcal{T}_v(H)$ onto $D(N_v)$. Let us take the right kernel. From (3.78) we see that the elements of the right kernel are all of the form $f \otimes \delta_e$, where f is constant on left cosets of N_v in H; the right kernel is isomorphic to the algebra of functions on the left N_v-cosets in H The irreps E_{hN} of $\mathrm{RKer}(\Gamma)$ are labeled by these cosets and given by

$$E_{hN}(f \otimes \delta_e) = f(h). \tag{3.92}$$

It is easy to find the restriction of $\Omega_\tau^{\mathcal{O}}$ to $\mathrm{RKer}(\Gamma)$. Let ϕ_ζ^i be the basis elements for $V_\tau^{\mathcal{O}}$ as defined through formula (3.11), that is $\phi_\zeta^i(y) = 1_{x_\zeta N_{\mathcal{O}}}(y)\tau(y^{-1}x_\zeta)e_i^\tau$. Note that the ζ is are in this case just elements of H and that we have $x_\zeta g_{\mathcal{O}} x_\zeta^{-1} = \zeta$. Also, $N_{\mathcal{O}}$ is just the stabilizer of $g_{\mathcal{O}}$ in N. Using this, we have

$$\begin{aligned}(\Omega_\tau^{\mathcal{O}}(f\otimes\delta_e)\phi_\zeta^i)(y) &= f(yg_{\mathcal{O}}y^{-1})\phi(y)\\ &= f(yg_{\mathcal{O}}y^{-1})1_{x_\zeta N_{\mathcal{O}}}(y)\tau(y^{-1}x_\zeta)e_i^\tau\\ &= f(\zeta)\phi(y) = E_{\zeta N}(f)\phi(y).\end{aligned} \tag{3.93}$$

So we see that each of the ϕ_ζ^i spans a one dimensional $\mathrm{RKer}(\Gamma)$-submodule of $\Omega_\tau^{\mathcal{O}}$ isomorphic to the module of $E_{\zeta N}$. This gives the decomposition of $\Omega_\tau^{\mathcal{O}}$ into $\mathrm{RKer}(\Gamma)$ modules: for each ζ in the orbit $\mathcal{O}$, we have d_τ copies of $E_{\zeta N}$. Of course, some of the cosets ζN may coincide and then $E_{\zeta N}$ will occur a multiple of d_τ times in the decomposition. In particular, if $\Omega_\tau^{\mathcal{O}}$ is not confined, then the orbit $\mathcal{O}$ is just a conjugacy class of N and we see that $\Omega_\tau^{\mathcal{O}}$ corresponds to $|\mathcal{O}|d_\tau$ copies of the trivial $\mathrm{RKer}(\Gamma)$-representation E_N, a result which we showed in general already in section 3.6.2. Here, it is also easy to see that none of the confined irreps of $\mathcal{T}_v$ has this property. In other words, none of the non-confined irreps pull strings, while all the confined ones do. The result we have got for the labeling of the walls is what we should have expected; a string is created by inserting a flux $g \notin N_v$ into the condensate. This string may be characterized by the fact that, if the condensate state on one side of the string is given by $\phi(0) = v$, then it must be given by $\alpha(g)v$ on the other side. But this means that the fluxes gn, with $n \in N_v$, will all pull the same string as the flux g, since $\alpha(gn)v = \alpha(g)v$. Hence, the string may already be characterized by the coset gN_v. However, the flux g which pulls the string may be transformed into the fluxes ngn^{-1} by gauge transformations with elements $n \in N_v$. Hence the walls should indeed be labeled by the set of cosets $ngn^{-1}N_v$ which is just the set of cosets ζN of the elements ζ in the N_v-orbit of g.

3.8.3 Examples of electric condensates

Abelian H

Suppose a particle in the irrep Π_α^e of $D(H)$ has condensed in the state $v \in V_\alpha^e$. We have seen that the residual symmetry algebra $\mathcal{T}_v(H)$ is the Hopf subalgebra of $D(H) \cong \mathbb{C}(H\times H) \cong F(H\times H)$ which consists of the functions supported by $H \times N_v$ (cf. 3.87). Because H is Abelian, the irrep α is one-dimensional and hence N_v is just the kernel N_α of α. Thus, we have $\mathcal{T}_v(H) \cong F(H\times N_\alpha) \cong \mathbb{C}(H\times N_\alpha)$. Here, the action of N_α on H is trivial, since H is Abelian and hence the irreps Ω_β^h of $\mathcal{T}_v$ are labeled by an element h of H and an irrep β of N_α. The decomposition of $D(H)$-irreps into $\mathcal{T}_v(H)$-irreps is straightforward: we have $\Pi_\beta^h \equiv \Omega_{\tilde\beta}^h$, where $\tilde\beta$ is the restriction of β to N_α.

The irreps Ω_β^h of $\mathcal{T}_v$ which are not confined are those for which $h \in N_v$ and they are of course in one to one correspondence with the irreps of $D(N_v)$. The corresponding Hopf projection $\Gamma : \mathcal{T}_v(H) \to D(N_v)$ is just restriction of the functions in $\mathcal{T}_v(H)$ to N_v in the left argument. The left and right Hopf kernels of Γ coincide and they are both isomorphic to the space of functions on the quotient H/N_v. the representations of this function space are just the evaluation functionals on the classes hN_v and as before, we denote them E_{hN_v}. The restriction of an irrep Ω_β^h of $\mathcal{T}$ to $\mathrm{LKer}(\Gamma)$ is simply given by $\Omega_\beta^h \equiv E_{hN_v}$.

To illustrate what happens a bit more explicitly, we will work out the case of $H = \mathbb{Z}_n$. This group is generated by a single element which we will call r and it has n irreps $\alpha_0, \ldots, \alpha_{n-1}$, given by

$$\alpha_k : \; r^a \mapsto e^{2\pi i k a/n}. \tag{3.94}$$

The kernel of α_k consists of those r^a for which $ak = 0 \bmod n$. The minimal non-zero a for which this holds is the quotient $n/\gcd(n,k) =: x$. Hence, we have $N_\alpha = \langle\, r^x \,\rangle \cong \mathbb{Z}_{\gcd(n,k)}$. The corresponding residual symmetry algebra is

$$\mathcal{T}_v(\mathbb{Z}_n) \cong F(\mathbb{Z}_n \times \mathbb{Z}_{\gcd(n,k)}) \cong \mathbb{C}\mathbb{Z}_n \times \mathbb{Z}_{\gcd(n,k)}. \tag{3.95}$$

Thus we see that the electric symmetry can be broken in as many different ways as n has divisors (the magnetic symmetry is never broken). The irreps $\Omega^{r^a}_{\alpha_l}$ of the residual algebra are labeled by an element $r^a \in H$ and an irrep α_l (with $0 \leq l < \gcd(n,k)$) of $\mathbb{Z}_{\gcd(n,k)}$. The decomposition of the irreps $\Pi^{r^a}_{\alpha_l}$ (with $0 \leq l < n$) of $\mathbb{Z}_n$ is then given by

$$\Pi^{r^a}_{\alpha_l} \equiv \Omega^{r^a}_{\alpha_{l \bmod \gcd(n,k)}}. \tag{3.96}$$

This follows from the fact that $\alpha_l(r^{px}) = e^{2\pi i l p x/n} = e^{2\pi i l p/\gcd(n,k)}$.

The irreps $\Omega^{r^a}_{\alpha_l}$ which are not confined are those for which $r^a \in N_\alpha$, i.e. $a = px$, with $0 \leq p < \gcd(n,k)$. These correspond to the irreps $\Pi^{p}_{\alpha_l}$ of the non-confined algebra $\mathcal{U}$, which is given by

$$\mathcal{U}_v(\mathbb{Z}_n) \cong D(N_\alpha) \cong D(\mathbb{Z}_{\gcd(n,k)}). \tag{3.97}$$

The Hopf map Γ from $F(\mathbb{Z}_n \times \mathbb{Z}_{\gcd(n,k)})$ to $\mathcal{U}$ is just restriction to $\mathbb{Z}_{\gcd(n,k)}$ in the left argument. The Hopf kernel of this map consist of the functions $f \otimes \delta_e$ in $F(\mathbb{Z}_n \times \mathbb{Z}_{\gcd(n,k)})$ for which f is constant on the cosets of $N_\alpha = \mathbb{Z}_{\gcd(n,k)}$. Hence, we have

$$\mathrm{LKer}(\Gamma) \cong F(\mathbb{Z}_x). \tag{3.98}$$

The representations of $\mathrm{LKer}(\Gamma)$ may be denoted E_a with $(0 \leq a < x)$ and are given by $E_a(f \otimes \delta_e) = f(r^a)$. The restriction of $\Omega^{r^a}_{\alpha_l}$ to $\mathrm{LKer}(\Gamma)$ is given by

$$\Omega^{r^a}_{\alpha_l} \equiv E_{a \bmod x}. \tag{3.99}$$

$H = D_{2m+1}$

We will treat all the possible types of electric condensate in order.

1. First, we take a condensate $v \in V^e_{J_1}$. We then have $N_v = \langle\, r \,\rangle \cong \mathbb{Z}_{2m+1}$ and hence

$$\mathcal{T}_v(D_{2m+1}) \cong F(D_{2m+1}) \,\tilde{\otimes}\, \mathbb{C}\mathbb{Z}_{2m+1}. \tag{3.100}$$

Here and in the sequel, the tilde on the tensor product sign indicates that the factor on the right acts on the factor on the left through conjugation. To find the irreps of $\mathcal{T}_v$, we first need to find the orbits of the adjoint $\mathbb{Z}_{2m+1}$-action on D_{2m+1} and their stabilizers. One easily finds that the orbits are $\{e\}, \{r\}, \{r^2\}, \ldots, \{r^{2m}\}$ and $\{s, sr, sr^2, \ldots, sr^{2m}\}$ Of these, all the orbits that contain a single element have stabilizer $\langle\, r \,\rangle \cong \mathbb{Z}_{2m+1}$, while the remaining orbit has the trivial stabilizer $\{e\}$. Thus, the irreps of $\mathcal{T}_v$ may be denoted $\Omega^{r^k}_{\beta_l}$ (with $0 \leq k, l < 2m+1$) and Ω^s. Here, we let r^k and s denote the orbits of r^k and s, in order not to overload the notation. We see that $\mathcal{T}_v$ has $(2m+1)^2 + 1$ irreps, which are all one-dimensional, except for Ω^s, which is

$2m+1$-dimensional. It follows that the squares of the dimensions add up to $2(2m+1)^2$, which equals the dimension of $\mathcal{T}_v$, as it should. The decomposition of $D(D_{2m+1})$-irreps into $\mathcal{T}_v$-irreps may be found directly or by means of the orthogonality relations for the characters of $\mathcal{T}_v$. We have

$$\begin{array}{ll} \Pi^e_{J_0} \equiv \Omega^e_{\beta_0} & \Pi^{r^k}_{\beta_l} \equiv \Omega^{r^k}_{\beta_l} \oplus \Omega^{r^{-k}}_{\beta_{-l}} \\ \Pi^e_{J_1} \equiv \Omega^e_{\beta_0} & \Pi^s_{\gamma_0} \equiv \Omega^s \\ \Pi^e_{\alpha_k} \equiv \Omega^e_{\beta_k} \oplus \Omega^e_{\beta_{-k}} & \Pi^s_{\gamma_1} \equiv \Omega^s. \end{array} \tag{3.101}$$

Of the representations of $\mathcal{T}_v$, Ω^s is confined, since $s \notin N_v$. The others are not confined and are in one to one correspondence with the irreps of the non-confined algebra

$$\mathcal{U}_v(D_{2m+1}) \cong D(N_v) = D(\mathbb{Z}_{2m+1}). \tag{3.102}$$

The right Hopf kernel of the projection Γ of $\mathcal{T}_v(H)$ onto $D(N_v)$ is isomorphic to the algebra of functions on the set of left $\langle\, r\,\rangle$-cosets. There are only two such cosets, namely $R := \langle\, r\,\rangle$ and $S := s\langle\, r\,\rangle$ and hence two corresponding one-dimensional representations E_R and E_S of the right kernel. The decomposition of $\mathcal{T}_v$-irreps into $\mathrm{RKer}\Gamma$-irreps is given by

$$\Omega^{r^k}_{\beta_l} \equiv E_R \quad \Omega^s \equiv (2m+1)E_S. \tag{3.103}$$

2. Now we take a condensate v in the module $V^e_{\alpha_j}$. The stabilizer N_v of v consists by definition of all the elements g of $D(D_{2m+1})$ for which v is an eigenvector of $\alpha_j(g)$ with eigenvalue 1. This includes in particular all the elements of the kernel of α_j. From the character table of $D(D_{2m+1})$ (table 3.1), one may read off that this kernel consists of those elements r^a for which $q^{ja}+q^{-ja}=2$, where $q=e^{2\pi i/(2m+1)}$, or in other words, for which $\cos(2\pi ja/(2m+1))=1$. It follows that one has to have $ja = 0 \bmod 2m+1$. The smallest non-zero a for which this holds is $(2m+1)/\mathrm{gcd}(2m+1,j) =: x$. Thus, one has $N_{\alpha_j} = \langle\, r^x\,\rangle \cong \mathbb{Z}_{\mathrm{gcd}(2m+1,j)}$.

Of course, the stabilizer N_v of v may be larger than N_{α_j}, if v is an eigenvector of $\alpha_j(g)$ for some $g \notin N_{\alpha_j}$. Thus, in order to find out what kinds of stabilizers are possible, it is a good idea to have a look at the eigenvalues of the matrices $\alpha_j(g)$. From the explicit matrices in (3.24), we see that the eigenvalues of $\alpha_j(r^p)$ are q^{jp} and q^{-jp}, with $q=e^{2\pi i/(2m+1)}$. It follows that, if one of the eigenvalues of r^p equals 1, so does the other. Hence, the only elements of $\langle\, r\,\rangle$ whose matrices have eigenvalues equal to one are those that are already contained in the kernel of α_j. The eigenvalues of each of the matrices $\alpha_j(sr^p)$ are 1 and -1. Thus, we have two possibilities: either v is not left invariant by any of the matrices $\alpha_j(sr^p)$, in which case $N_v = N_{\alpha_j} = \langle\, r^x\,\rangle \cong \mathbb{Z}_{\mathrm{gcd}(2m+1,j)}$, or v is left invariant by some of the $\alpha_j(sr^p)$. In this case, we may without loss of generality choose v to be the invariant vector of $\alpha_j(s)$, since each of the sr^p is a conjugate of s in D_{2m+1} and hence the invariant vectors of the sr^p are in the same gauge orbit as the invariant vector of s. With this choice, one sees easily that $N_v = \langle\, r^x\,\rangle \cup s\langle\, r^x\,\rangle \cong D_{\mathrm{gcd}(2m+1,j)}$. We will now treat the two possibilities for N_v in order.

2.a When $N_v = \langle\, r^x\,\rangle \cong \mathbb{Z}_{\mathrm{gcd}(2m+1,j)}$, we have

$$\mathcal{T}_v(D_{2m+1}) \cong F(D_{2m+1})\,\tilde{\otimes}\,\mathbb{C}\mathbb{Z}_{\mathrm{gcd}(2m+1,j)}. \tag{3.104}$$

The orbits of the $\langle\, r^x\,\rangle$-action on D_{2m+1} are $\{e\},\{r\},\ldots,\{r^{2m}\}$, with stabilizer $\langle\, r^x\,\rangle$, and $s\langle\, r^x\,\rangle$, $sr\langle\, r^x\,\rangle,\ldots,sr^{x-1}\langle\, r^x\,\rangle$, with stabilizer $\{e\}$. This means that the irreps of $\mathcal{T}$ may be denoted as $\Omega^{r^k}_{\beta_l}$ (with $0 \le k < 2m+1$, $0 \le l < \mathrm{gcd}(2m+1,j)$) and Ω^{sr^k} (with $0 \le k < x$). Here, we have once again denoted orbits by representative elements. We see that there are $(2m+1)\mathrm{gcd}(2m+1,j) + (2m+1)/\mathrm{gcd}(2m+1,j)$ irreps. Of these, $(2m+1)\mathrm{gcd}(2m+1,j)$

are one-dimensional and the remaining $(2m+1)/\mathrm{gcd}(2m+1,j)$ (the Ω^{sr^p}) are $\mathrm{gcd}(2m+1,j)$-dimensional, so that the squares of the dimensions again add up to the dimension of $\mathcal{T}$, which is $2(2m+1)\mathrm{gcd}(2m+1,j)$. The decomposition of $D(D_{2m+1})$-irreps reads

$$\begin{array}{ll} \Pi^e_{J_0} \equiv \Omega^e_{\beta_0} & \Pi^{r^k}_{\beta_l} \equiv \Omega^{r^k}_{\beta_l} \oplus \Omega^{r^{-k}}_{\beta_{-l}} \\ \Pi^e_{J_1} \equiv \Omega^e_{\beta_0} & \Pi^s_{\gamma_0} \equiv \bigoplus_{0\leq p<x} \Omega^{sr^p} \\ \Pi^e_{\alpha_l} \equiv \Omega^e_{\beta_l} \oplus \Omega^e_{\beta_{-l}} & \Pi^s_{\gamma_1} \equiv \bigoplus_{0\leq p<x} \Omega^{sr^p} \end{array} \tag{3.105}$$

where the labels l and $-l$ should be read modulo $2m+1$ on the left hand side and modulo $\mathrm{gcd}(2m+1,j)$ on the right hand side. The non-confined irreps are those $\Omega^{r^k}_{\beta_l}$ for which $r^k \in \langle\, r^x \,\rangle$ and they are in one correspondence with the irreps of the non-confined algebra

$$\mathcal{U}_v(D_{2m+1}) \cong D(\mathbb{Z}_{\mathrm{gcd}(j,2m+1)}). \tag{3.106}$$

The right and the left kernel of the Hopf map $\Gamma : \mathcal{T} \to \mathcal{U}$ are equal and isomorphic to the algebra of functions on the quotient group $D_{2m+1}/\langle\, r^x \,\rangle$. Since this quotient group is isomorphic to D_x, we have

$$\mathrm{RKer}(\Gamma) \cong F(D_x). \tag{3.107}$$

The representations of $\mathrm{RKer}(\Gamma)$ are labeled by the elements R^k, SR^k of D_x and we denote them E_{R^k}, E_{SR^k}. The decomposition of $\mathcal{T}$-irreps into $\mathrm{RKer}(\Gamma)$-irreps is given by

$$\Omega^{r^k}_{\beta_l} \equiv E_{R^k} \quad \Omega^{sr^k} \equiv xE_{SR^k} \tag{3.108}$$

where, on the right hand side, k should be taken modulo x.
2.b When $N_v \cong D_{\mathrm{gcd}(2m+1,j)}$, we have

$$\mathcal{T}_v(D_{2m+1}) \cong F(D_{2m+1})\,\tilde{\otimes}\,\mathbb{C}D_{\mathrm{gcd}(2m+1,j)}. \tag{3.109}$$

The orbits of the $D_{\mathrm{gcd}(2m+1,j)}$ on D_{2m+1} are $\{e\}$, $\{r,r^{-1}\}$, $\{r^2,r^{-2}\},\ldots,\{r^m,r^{-m}\}$, $s\langle\, x \,\rangle$ and $sr\langle\, x \,\rangle \cup sr^{x-1}\langle\, x \,\rangle$, $sr^2\langle\, x \,\rangle \cup sr^{x-2}\langle\, x \,\rangle \ldots$. The stabilizer of e is of course all of $D_{\mathrm{gcd}(2m+1,j)}$, the stabilizer of r^k is $\langle\, r^x \,\rangle \cong \mathbb{Z}_{\mathrm{gcd}(2m+1,j)}$ and the stabilizer of s is $\langle\, s \,\rangle \cong \mathbb{Z}_2$. The stabilizer of the orbits $sr^p\langle\, x \,\rangle \cup sr^{x-p}\langle\, x \,\rangle$ is just $\{e\}$ Hence, the irreps of $\mathcal{T}_v$ may be denoted $\Omega^e_{J_0}$, $\Omega^e_{J_1}$, $\Omega^e_{\alpha_k}$ (with $1 \leq k \leq \frac{1}{2}(\mathrm{gcd}(2m+1,j)-1)$), $\Omega^{r^k}_{\beta_l}$ (with $0 < k \leq m$, $0 \leq l < \mathrm{gcd}(2m+1,j)$), $\Omega^s_{\gamma_0}, \Omega^s_{\gamma_1}$ and finally Ω^{sr^p} (with $1 \leq p < \frac{1}{2}(x-1)$). This yields $3+\frac{1}{2}(2m+1)(\mathrm{gcd}(2m+1,j)+1/\mathrm{gcd}(2m+1,j))$ irreps in total and one may check that the squares of their dimensions sum correctly to the dimension of $\mathcal{T}_v$, which is $4(2m+1)\mathrm{gcd}(2m+1,j)$. The decomposition of the $\Pi^e_{\alpha_k}$ into $\mathcal{T}_v$-irreps is now

$$\Pi^e_{\alpha_k} \equiv \begin{cases} \Omega^e_{J_0} \oplus \Omega^e_{J_1} & ([k]=0) \\ \Omega^e_{\alpha_{[k]}} & ([k] \leq \frac{1}{2}(\mathrm{gcd}(2m+1,j)-1) \\ \Omega^e_{\alpha_{\mathrm{gcd}(2m+1,j)-[k]}} & ([k] > \frac{1}{2}(\mathrm{gcd}(2m+1,j)-1) \end{cases} \tag{3.110}$$

Here, $[k]$ denotes $k \bmod \mathrm{gcd}(2m+1,j)$. The decomposition of the other D_{2m+1}-irreps into $\mathcal{T}_v$-irreps is given by

$$\begin{array}{ll} \Pi^e_{J_0} \equiv \Omega^e_{J_0} & \Pi^s_{\gamma_0} \equiv \Omega^{s_{\gamma_0}} \oplus \bigoplus_{1\leq p<\frac{1}{2}(x-1)} \Omega^{sr^p} \\ \Pi^e_{J_1} \equiv \Omega^e_{J_1} & \Pi^s_{\gamma_1} \equiv \Omega^{s_{\gamma_1}} \oplus \bigoplus_{1\leq p<\frac{1}{2}(x-1)} \Omega^{sr^p} \\ \Pi^{r^k}_{\beta_l} \equiv \Omega^{r^k}_{\beta_l} & \end{array} \tag{3.111}$$

The labels l on the left should be read modulo $2m+1$, while those on the right hand side should be read modulo $\gcd(2m+1,j)$. The non-confined irreps of $\mathcal{T}_v$ are $\Omega^e_{J_0}, \Omega^e_{J_1}$, the $\Omega^e_{\alpha_k}$, $\Omega^s_{\gamma_0}, \Omega^s_{\gamma_1}$ and those $\Omega^{r^k}_{\beta_l}$ for which $r^k \in \langle\, r^x \,\rangle$. These irreps correspond to the irreps of the non-confined algebra

$$\mathcal{U}_v(D_{2m+1}) \cong D(D_{\gcd(2m+1,j)}). \tag{3.112}$$

The right kernel of the Hopf map $\Gamma : \mathcal{T} \to \mathcal{U}$ is isomorphic to the algebra of functions on the space of left cosets of $N_v \cong D_{\gcd(2m+1,j)}$ in D_{2m+1}. There are x distinct cosets, namely the cosets of $e, r, \ldots, r^{x-1}$. We will denote these $E, R, \ldots, R^{x-1}$. The corresponding irreps of $\mathrm{RKer}(\Gamma)$ will again be denoted E_{R^k}. The restriction of the irreps of $\mathcal{T}_v$ to $\mathrm{LKer}(\Gamma)$ is given by

$$\begin{array}{lll} \Omega^e_{J_0} \equiv E_E & \Omega^{r^k}_{\beta_l} \equiv E_{R^k} & \Omega^{sr^p} \equiv \gcd(2m+1,j)(E_{R^p} \oplus E_{R^{x-p}}) \\ \Omega^e_{J_1} \equiv E_E & \Omega^{s_{\gamma_0}} \equiv \gcd(2m+1,j)E_E & \\ \Omega^e_{\alpha_l} \equiv 2E_E & \Omega^{s_{\gamma_1}} \equiv \gcd(2m+1,j)E_E & \end{array} \tag{3.113}$$

where the index k should be read modulo $2m+1$ on the left hand side and modulo x on the right. In the restriction of Ω^{sr^p}, we see our first example of a situation where the wall created by a $\mathcal{T}$-particle carries a representation of $\mathrm{RKer}(\Gamma)$ that contains two distinct irreps of $\mathrm{RKer}(\Gamma)$, namely E_{R^p} and $E_{R^{x-p}}$. The isotypical components of these irreps are sent onto each other by gauge transformation with $s \in N_v$, since $sR^ps^{-1} = R^{x-p}$.

3.9 Gauge invariant magnetic condensates

3.9.1 Symmetry breaking

There is precisely one gauge invariant state in every magnetic representation Π^A_1. This state is represented by the constant function

$$\phi : h \mapsto 1 \tag{3.114}$$

on H. To find the Hopf stabilizer of ϕ, we need to find the irreps (ρ, g) of $D(H)^*$ which solve equation (3.36). Since ϕ is constant equal to one, this reduces to

$$\rho(g_A) = I. \tag{3.115}$$

Hence, the unbroken symmetry algebra is the algebra generated by the matrix elements of the representations (ρ, g) for which g_A is contained in the kernel of ρ. Now define K_A as the minimal normal subgroup of H that contains g_A (and hence all of A). Since the kernel of a representation is a normal subgroup, the irreducible representations ρ which have g_A in their kernel will be precisely the ones which contain all of K_A in their kernel. Such irreps are in one-to-one correspondence with the irreps of H/K_A [119] and since the matrix elements of the irreps of H/K_A generate $F(H/K_A)$, the algebra generated by the matrix elements of the irreps of H which contain g_A in their kernel is precisely the algebra $\mathcal{T}_A$ of functions on G which are constant on the cosets of K_A. Hence, the unbroken symmetry algebra in this case is the Hopf subalgebra $\mathcal{T}_A(H)$ of $D(H)$ defined by

$$\mathcal{T}_A(H) := \{F \in D(H) | \forall k \in K_A : F(xk, y) = F(x, y)\}. \tag{3.116}$$

Clearly, $\mathcal{T}_A \cong F(H/K_A \times H)$ as a linear space and we see that $\mathcal{T}_A$ is a transformation group algebra, with H acting on K_A by conjugation. This means we can once again make use of theorem 1 to write down the irreps of $\mathcal{T}_A$. They are labeled by an H-orbit $\mathcal{O} \subset H/K$ and an irrep τ of the stabilizer $N_{\mathcal{O}}$ of a chosen element $g_{\mathcal{O}} \in \mathcal{O}$. The irrep labeled by $\mathcal{O}$ and τ will be denoted $\Omega_\tau^{\mathcal{O}}$. It acts on the Hilbert space $F_\tau(H, V_\tau)$ in the usual way:

$$\left(\Omega_\tau^{\mathcal{O}}(F)\phi\right)(x) := \int_H dz\, F(xg_{\mathcal{O}}x^{-1}, z)\,\phi(z^{-1}x). \tag{3.117}$$

The character $\psi_\tau^{\mathcal{O}}$ of $\Omega_\tau^{\mathcal{O}}$ is given as a function on $H/K_A \times H$ by (cf. (3.15))

$$\psi_\tau^{\mathcal{O}}(\eta, h) = 1_{N_\eta}(h) 1_{\mathcal{O}}(\eta) \psi_\tau(x_\eta^{-1} h x_\eta). \tag{3.118}$$

The decomposition of any $\mathcal{T}_A(H)$-module into irreps may be found by calculating the inner products (defined in (3.16)) between the character of the module and the above characters of the irreps. Of course, we can view any $D(H)$-module as a $\mathcal{T}_A(H)$-module by restriction. The characters χ_β^B of the irreps Π_β^B of $D(H)$, viewed as $\mathcal{T}_A(H)$-modules are given by

$$\chi_\beta^B(gK_A, h) = \sum_{k \in K_A} 1_{N_{gk}}(h) 1_B(gk) \chi_\beta(x_{gk}^{-1} h x_{gk}). \tag{3.119}$$

We see that all the irreps $\Omega_\tau^{\mathcal{O}}$ in the decomposition of Π_β^B must be such that B is a subset of the set of elements of H that constitute the K_A-classes in $\mathcal{O}$. Clearly, there is only a single orbit $\mathcal{O}$ for which this holds. The decomposition of a purely electric representation Π_β^e is very simple: such a representation is irreducible and isomorphic to the purely electric irrep $\Omega_\beta^{\mathcal{O}=K_A}$ (Note that $N_{K_A} = H$). On the other hand, the decomposition of a purely magnetic representation Π_1^B may contain irreps $\Omega_\tau^{\mathcal{O}}$ which are not purely magnetic (i.e. τ may be non-trivial).

3.9.2 Confinement

We will now find out which of the irreps $\Omega_\tau^{\mathcal{O}}$ of $\mathcal{T}_A$ are confined and which are not. The non-confined irreps have to satisfy the requirements (3.45). Since $\mathcal{T}_A$ is isomorphic to a transformation group algebra, these reduce to the conditions (3.62) and (3.63), with $K = K_A$ and $N = H$. We have seen in section 3.6.3, proposition 9, that these requirements will be satisfied by the set of irreps $\Omega_\tau^{\mathcal{O}}$ for which $g_{\mathcal{O}}K = nK$ for some $n \in N$ and for which τ is trivial on K. The first of these requirements is trivial here, since $N = H$ and so this set consist of all $\Omega_\tau^{\mathcal{O}}$ for which τ is trivial on K_A. These irreps correspond to the irreps of the quotient $D(H/K_A)$ of $\mathcal{T}_A$. In the case at hand, it turns out that this set of of solutions is actually complete and hence the non-confined symmetry algebra $\mathcal{U}_A$ is just the quantum double of the quotient group H/K_A. Let us demonstrate this.

Equation (3.62) is trivially satisfied by the matrix elements of all the irreps of $\mathcal{T}_A$, since ϕ is the constant function 1. Thus, we are left with the requirement (3.63). Since the support of ϕ is all of H, this becomes

$$(\forall x \in H) \quad \tau(xg_A x^{-1}) = I \tag{3.120}$$

or in other words

$$A \subset \mathrm{Ker}(\tau). \tag{3.121}$$

The requirement that $A \subset \mathrm{Ker}(\tau)$ is equivalent to the requirement that $K_A \subset \mathrm{Ker}(\tau)$, since K_A is just the subgroup of H generated by the elements of A. Hence, the non-confined irreps $\Omega_\tau^{\mathcal{O}}$ of $\mathcal{T}_A$ are exactly those for which τ is trivial on K_A, as we claimed.

The results we have obtained are quite satisfying when one thinks back of the intuition that went into our method of finding the non-confined irreps. We wanted the non-confined irreps to have trivial braiding with the condensate. For a purely magnetic condensate, this means roughly that the flux of the condensate should commute with the flux of the non-confined irreps and should act trivially on the charges of the non-confined irreps. The first of these conditions is automatically met: the flux state of the condensate commutes with any other flux state (the class sum is a central element of the group algebra of H). Therefore, there is no requirement on $\mathcal{O}$. The second condition is implemented by the demand that τ is trivial on K_A, the group which is generated by the fluxes in the class A. We also wanted to have well-defined fusion, spin and braiding among the non-confined particles and these are now provided by the Hopf structure, R-matrix and ribbon-element of $D(H/K_A)$.

Finally, let us say something about the characterization of strings (or walls). From proposition 10, we see that the Hopf kernel of the projection $\Gamma : \mathcal{T}_A(H) \to D(H/K_A)$ is just the set of elements $1 \otimes f \in \mathcal{T}_A(H)$ for which f has support in K_A. In this case, the left and right Hopf kernels coincide and hence the kernel is itself a Hopf algebra. This Hopf algebra is clearly isomorphic to the group algebra $\mathbb{C}K_A$ (cf. corollary 3) and hence the irreps of $\mathrm{LKer}(\Gamma)$ correspond to the irreps of K_A. If ρ is an irrep of K_A then we also write ρ for the corresponding irrep of $\mathrm{LKer}(\Gamma)$ and with this slight abuse of notation, we may write

$$\rho(1 \otimes \delta_k) = \rho(k) \tag{3.122}$$

for all $k \in K_A$. We will now calculate the decomposition of a representation $\Omega^{\mathcal{O}}_\tau$ of $\mathcal{T}_A(H)$ into representations of $\mathrm{LKer}(\Gamma)$ by means of the formula (3.13) for the character $\psi^{\mathcal{O}}_\tau$ of $\Omega^{\mathcal{O}}_\tau$. For $g \in K_A$, we have

$$\begin{aligned}(\psi^{\mathcal{O}}_\tau(1 \otimes \delta_g) &= \int_{\mathcal{O}} d\zeta \int_{N_{\mathcal{O}}} dn\, \delta_g(x_\zeta n x_\zeta^{-1})\chi_\tau(n) \\ &= \int_{\mathcal{O}} d\zeta\, \chi_\tau(x_\zeta^{-1} g x_\zeta). \end{aligned} \tag{3.123}$$

From this, we see that the restriction of $\Omega^{\mathcal{O}}_\tau$ to $\mathrm{LKer}(\Gamma) \cong \mathbb{C}K_A$ contains exactly the irreps of K_A that are contained in the restriction of β to K_A, together with the irreps obtained from these by composition with the automorphisms of K_A that are given by conjugation with the x_ζ^{-1}. As in the case of electric condensates, we see that the non-confined irreps are exactly all those that have trivial restriction to the Hopf kernel of Γ.

3.9.3 Examples of gauge invariant condensates

Abelian H

For Abelian H, every state in a purely magnetic representation Π^A is gauge invariant, so this section covers all purely magnetic condensates for Abelian groups. Suppose we condense a state in the purely magnetic representation labeled by the element g_A of H. Then we know that the residual symmetry algebra $\mathcal{T}_A(H)$ is the Hopf subalgebra of $D(H)$ which consists of the functions that are constant on cosets of K_A in their left argument. Here K_A is the minimal normal subgroup of H that contains g_A, which, when H is Abelian, is just the cyclic group generated by g_A. As an algebra, $\mathcal{T}_A(H)$ is isomorphic to the transformation group algebra $F(H/K_A \times H)$, where H acts on H/K_A by conjugation. When H is Abelian, the action of H

on H/K_A is thus trivial. The orbits are then just the elements of $H/K_A = H/\langle g_A \rangle$ and the stabilizer of each orbit is all of H. Thus, the irreps of $\mathcal{T}_A(H)$ may be denoted $\Omega_\alpha^{hK_A}$, where hK_A is an element of H/K_A and α is an irrep of H. The action of $\mathcal{T}_A$ in the irrep $\Omega_\alpha^{hK_A}$ is given in formula (3.117). The irreps of $D(H)$ may be easily decomposed into irreps of $\mathcal{T}_A(H)$; we have $\Pi_\alpha^h \equiv \Omega_\alpha^{hK_A}$. The non-confined irreps of $\mathcal{T}_A(H)$ are those $\Omega_\alpha^{hK_A}$ for which α is trivial on K_A These correspond to the irreps of the non-confined algebra $\mathcal{U} \cong D(H/K_A) \cong D(H/\langle g_A \rangle)$. The kernel of the Hopf map $\Gamma : \mathcal{T} \to \mathcal{U}$ is isomorphic to $\mathbb{C}K_A$ and hence its irreps are just the irreps of K_A. Since $K_A = \langle g_A \rangle$, it follows that the number of these irreps equals the order of the element g_A. We may indeed give the irreps explicitly; denoting them as ρ_k (with $0 \leq k < \mathrm{ord}(g_A)$), we have $\rho_k((g_A)^p) = \exp(2\pi i k p/\mathrm{ord}(g_A))$ The restriction of the irreps of $\mathcal{T}_A$ to $\mathrm{LKer}(\Gamma)$ is also easily found. We have $\Omega_\alpha^{hK_A} \equiv \alpha|_{K_A}$. In other words, the wall corresponding to $\Omega_\alpha^{hK_A}$ can be labeled by the phase factor $\alpha(g_A)$.

We once again explicitly work out the case of $H = \mathbb{Z}_n$. As in section 3.8.3, we will denote our preferred generator for $\mathbb{Z}_n$ by r and we write $\alpha_0, \ldots, \alpha_{n-1}$ for the irreps of $\mathbb{Z}_n$. Now suppose we condense the magnetic flux $g_A = r^k$. Then we have $K_A = \langle r^k \rangle = \langle r^{\gcd(k,n)} \rangle \cong \mathbb{Z}_x$, where $x = n/\gcd(k,n)$. As a consequence, we have $H/K_A \cong \mathbb{Z}_n/\mathbb{Z}_x \cong \mathbb{Z}_{\gcd(k,n)}$. Thus,

$$\mathcal{T}_{[r^k]}(\mathbb{Z}_n) \cong F(\mathbb{Z}_{\gcd(k,n)} \times \mathbb{Z}_n) \cong \mathbb{C}(\mathbb{Z}_{\gcd(k,n)} \times \mathbb{Z}_n) \tag{3.124}$$

and we see that there is one type of broken symmetry for each divisor of n. The irreps $\Omega_{\alpha_l}^{r^a K_A}$ of $\mathcal{T}$ may be labeled by an element r^a of $\mathbb{Z}_{\mathrm{gcd}}(k,n)$ and an irrep α_l of $\mathbb{Z}_n$. The decomposition of the irreps $\Pi_{\alpha_l}^{r^a}$ (with $0 \leq a < n$) of $\mathbb{Z}_n$ is then given by

$$\Pi_{\alpha_l}^{qr^a} \equiv \Omega_{\alpha_l}^{r^a \bmod \gcd(n,k) K_A} \tag{3.125}$$

The unconfined irreps of $\mathcal{T}$ are those $\Omega_{\alpha_l}^{r^a K_A}$ for which $\alpha_l(r^k) = \exp(2\pi i k l/n) = 1$, or equivalently $kl = 0 \bmod n$. These are exactly the ones for which l is a multiple of x and we see that they correspond to the irreps of

$$\mathcal{U}_{[r^k]}(\mathbb{Z}_n) \cong D(H/\langle r^k \rangle) \cong D(\mathbb{Z}_{\gcd(k,n)}). \tag{3.126}$$

The kernel of the Hopf map $\Gamma : \mathcal{T} \to \mathcal{U}$ is isomorphic to $\mathbb{C}\mathbb{Z}_x$ and has representations $\tilde{\alpha}_j$ (with $(0 \leq j < x)$) defined in the usual way, with $r^{\gcd(k,n)}$ as the preferred generator. That is, we take $\tilde{\alpha}_j(r^{\gcd(k,n)}) = \exp(2\pi i j/x)$. The restriction of $\mathcal{T}$-irreps to $\mathrm{RKer}\Gamma$ is given by

$$\Omega_{\alpha_l}^{r^a K_A} \equiv \tilde{\alpha}_{l \bmod x}. \tag{3.127}$$

One should notice the duality between the situation described here and that for symmetry breaking by electric condensates described in section 3.8.3.

$H = D_{2m+1}$

1. First, we take the condensate state in the module $V_{\beta_0}^{r^k}$. To find the residual symmetry algebra $\mathcal{T}_{[r^k]}$, we first need to find the minimal normal subgroup $K_{[r^k]}$ of D_{2m+1} that contains r^k. This is just the subgroup generated by the elements of the conjugacy class of r^k, which are r^k and r^{-k}. In other words, we have $K_{[r^k]} = \langle r^k \rangle = \langle r^{\gcd(k,2m+1)} \rangle \cong \mathbb{Z}_x$, where $x = (2m+1)/\gcd(k,2m+1)$. One checks easily that $D_{2m+1}/\langle r^k \rangle \cong D_{\gcd(k,2m+1)}$, where this $D_{\gcd(k,2m+1)}$ is generated in the usual way by the rotation $R = r\langle r^k \rangle$ and the reflection $S = s\langle r^k \rangle$. We will also use the notation E for the class $e\langle r^k \rangle$. The residual algebra is now

$$\mathcal{T}_{[r^k]}(D_{2m+1}) \cong F(D_{\gcd(k,2m+1)}) \tilde{\otimes} \mathbb{C}D_{2m+1}. \tag{3.128}$$

The orbits of the D_{2m+1} action are exactly the conjugacy classes of $D_{\gcd(k,2m+1)}$. These are $\{E\}$, $\{R^p, R^{-p}\}$ (with $0 \leq p < (\gcd(k,2m+1)-1)/2$) and finally $\{S, SR, \ldots, SR^{(\gcd(k,2m+1)-1)}\}$. The stabilizers of these orbits are $N_E = D_{2m+1}$, $N_{R^p} = \langle\, r \,\rangle \cong \mathbb{Z}_{2m+1}$ and $N_S = E \cup sE \cong D_x$. Thus the irreps of $\mathcal{T}_{[r^k]}$ may be written as $\Omega^E_{J_0}$, $\Omega^E_{J_1}$, $\Omega^E_{\alpha_j}$ (with $1 \leq j \leq m$), $\Omega^{R^p}_{\beta_l}$ (with $1 \leq p \leq \frac{1}{2}(\gcd(k,2m+1)-1)$, $0 \leq l < 2m+1$), $\Pi^S_{J_0}$, $\Pi^S_{J_1}$ and $\Pi^S_{\alpha_j}$ (with $1 \leq j \leq \frac{1}{2}(x-1)$). This yields $3 + \frac{1}{2}(2m+1)(\gcd(k,2m+1) + \frac{1}{\gcd(k,2m+1)})$ irreps in total and one may check that the squares of their dimensions add up to the dimension of $\mathcal{T}_{[r^k]}$, which is $4\gcd(k,2m+1)(2m+1)$. The decomposition of $D(D_{2m+1})$-irreps is as follows:

$$\begin{array}{ll} \Pi^e_{J_0} \equiv \Omega^E_{J_0} & \Pi^{r^p}_{\beta_l} \equiv \Omega^{R^p}_{\beta_l} \\ \Pi^e_{J_1} \equiv \Omega^E_{J_1} & \Pi^s_{\gamma_0} \equiv \Omega^S_{J_0} \oplus \bigoplus_j \Omega^S_{\alpha_j} \\ \Pi^e_{\alpha_l} \equiv \Omega^E_{\alpha_l} & \Pi^s_{\gamma_1} \equiv \Omega^S_{J_1} \oplus \bigoplus_j \Omega^S_{\alpha_j} \end{array} \tag{3.129}$$

In the decomposition of $\Pi^s_{\gamma_0}$, we see that a chargeless flux may be turned into a charged flux upon formation of a magnetic condensate.

The irreps $\Omega^{\mathcal{O}}_\tau$ of $\mathcal{T}_{[r^k]}$ that are not confined are exactly those for which τ is trivial on $K_{[r^k]} = \langle\, r^k \,\rangle$. These are $\Omega^E_{J_0}$, $\Omega^E_{J_1}$, the $\Omega^E_{\alpha_l}$ and $\Omega^{R^p}_{\alpha_l}$ for which l is a multiple of x, $\Omega^S_{J_0}$ and $\Omega^S_{J_1}$. These correspond to the irreps of the non-confined algebra

$$\mathcal{U}_{[r^k]}(D_{2m+1}) \cong D(D_{2m+1}/\langle\, r^k \,\rangle) \cong D(D_{\gcd(k,2m+1)}). \tag{3.130}$$

The kernel of the Hopf map $\Gamma : \mathcal{T} \to \mathcal{U}$ is isomorphic to $\mathbb{C}\langle\, r^k \,\rangle = \mathbb{C}\mathbb{Z}_x$. We will denote its representations by ρ_l (with $0 \leq l < x$). They are defined in the usual way, with $r^{\gcd(k,2m+1)}$ taken as the preferred generator. The restriction of the irreps of $\mathcal{T}_{[r^k]}$ to $\mathrm{LKer}(\Gamma)$ is given by

$$\begin{array}{lll} \Omega^E_{J_0} \equiv \rho_0 & \Omega^E_{\alpha_l} \equiv \rho_l \oplus \rho_{-l} & \Omega^S_{J_0} \equiv \gcd(k,2m+1)\rho_0 \\ \Omega^E_{J_1} \equiv \rho_0 & \Omega^{r^p}_{\beta_l} \equiv \rho_l \oplus \rho_{-l} & \Omega^S_{J_1} \equiv \gcd(k,2m+1)\rho_0 \\ & & \Omega^S_{\alpha_j} \equiv \gcd(k,2m+1)(\rho_j \oplus \rho_{-j}) \end{array} \tag{3.131}$$

Here, the indices on the ρ's should be read modulo x.

2. Now we take the condensate state in the module $V^s_{\gamma_0}$. Since the minimal normal subgroup of D_{2m+1} that contains s is D_{2m+1} itself, this condensate breaks the magnetic part of $D(D_{2m+1})$ completely and we are left with just the electric group D_{2m+1}, that is, we have

$$\mathcal{T}_{[s]}(D_{2m+1}) \cong \mathbb{C}D_{2m+1}. \tag{3.132}$$

Thus, the irreps of $\mathcal{T}_{[s]}$ are just the irreps J_0, J_1 and $\alpha_1, \ldots, \alpha_m$ of D_{2m+1}. The decomposition of D_{2m+1}-irreps into these gauge group irreps is

$$\begin{array}{lll} \Pi^e_{J_0} \equiv J_0 & & \Pi^s_{\gamma_0} \equiv J_0 \oplus \bigoplus_j \alpha_j \\ \Pi^e_{J_1} \equiv J_1 & \Pi^{r^k}_{\beta_l} \equiv \begin{cases} J_0 \oplus J_1 & (l=0) \\ \alpha_l & (1 \leq l \leq m) \\ \alpha_{2m+1-l} & (m+1 \leq l < 2m+1) \end{cases} & \Pi^s_{\gamma_1} \equiv J_1 \oplus \bigoplus_j \alpha_j \end{array} \tag{3.133}$$

Again, we see that pure fluxes may be turned into particles which carry a charge with respect to the residual symmetry.

Of the irreps of $\mathcal{T}_{[s]}$, only the trivial representation J_0 is not confined. In other words, all non-confined excitations over this condensate are “color” singlets. This means that

$$\mathcal{U}_{[s]}(D_{2m+1}) \cong \mathbb{C}\{e\}. \tag{3.134}$$

The Hopf kernel of the associated map $\Gamma : \mathcal{T}_{[s]} \to \mathcal{U}_{[s]}$ is all of $\mathcal{T}_{[s]}$. Hence the “restriction” to $\mathrm{LKer}(\Gamma)$ is trivial; the walls are just labeled by the irreps of $\mathcal{T}_{[s]}$.

3.10 Condensates of pure magnetic flux

3.10.1 Symmetry breaking

We will now study symmetry breaking by a state with pure flux yg_Ay^{-1} in the conjugacy class $A \subset H$. The vector $\phi \in \Pi_1^A$ that corresponds to this state is given by $\phi(x) = 1_{yN_A}(x)$. According to proposition 6, the residual symmetry algebra $\mathcal{T}_{yg_Ay^{-1}}(H)$ is spanned by the matrix elements of the set of irreps (ρ, g) of $D(H)^*$ which have the property that ϕ is an eigenstate of the action of g^{-1} with eigenvalue equal to $\frac{\chi_\rho(g_A)}{d_\rho}$. In the case at hand, it is clear that the only eigenvalue of the action of any element of H that may occur is the value 1. It follows that ρ must be such that $\chi_\rho(g_A) = d_\rho$ and hence such that g_A lies in the kernel of ρ. Given such a ρ, we can find the corresponding elements g by solving the equation $\phi(gx) = \phi(x)$. In this case,we have

$$1_{yN_A}(x) = 1_{yN_A}(gx) = 1_{g^{-1}yN_A}(x). \tag{3.135}$$

Now the functions 1_{yN_A} and $1_{g^{-1}yN_A}$ are equal exactly if $g^{-1} \in yN_Ay^{-1}$, or equivalently, $g \in yN_Ay^{-1} = N_{yg_Ay^{-1}}$. Thus, the admissible irreps (ρ, g) are those for which g_A lies in the kernel of ρ and g commutes with the condensed flux yg_Ay^{-1}. Following the same arguments as in section 3.9.1, we see that the allowed ρ span exactly the space of functions on H that are constant on the cosets of the minimal normal subgroup K_A of H that contains the class A. Thus the residual symmetry algebra is the Hopf subalgebra $\mathcal{T}_{yg_Ay^{-1}}(H)$ of $D(H)$ defined by

$$\mathcal{T}_{yg_Ay^{-1}}(H) := \left\{ F \in D(H) | \forall k \in K_A : F(xk, y) = F(x, y) 1_{N_{yg_Ay^{-1}}}(y) \right\} \tag{3.136}$$

Clearly, $\mathcal{T}_{yg_Ay^{-1}} \cong F(H/K_A \times N_{yg_Ay^{-1}})$ as a vector space and we see that $\mathcal{T}_{yg_Ay^{-1}}$ is a transformation group algebra, with $N_{yg_Ay^{-1}}$ acting on K_A through conjugation. Thus we may again use theorem 1 to write down the irreps of $\mathcal{T}_{yg_Ay^{-1}}$. They are labeled by an $N_{yg_Ay^{-1}}$-orbit $\mathcal{O} \subset H/K_A$ and an irrep τ of the stabilizer $N_\mathcal{O}$ of a chosen element $g_\mathcal{O} \in \mathcal{O}$. The irrep labeled by $\mathcal{O}$ and τ will be denoted $\Omega_\tau^\mathcal{O}$. It acts on the Hilbert space $F_\tau(N_{yg_Ay^{-1}}, V_\tau)$ in the usual way:

$$\left(\Omega_\tau^\mathcal{O}(F)\phi\right)(x) := \int_{N_{yg_Ay^{-1}}} dz\, F(xg_\mathcal{O}x^{-1}, z)\, \phi(z^{-1}x). \tag{3.137}$$

The character $\psi_\tau^\mathcal{O}$ of $\Omega_\tau^\mathcal{O}$ is given as a function on $H/K_A \times N_{yg_Ay^{-1}}$ by (cf. (3.15))

$$\psi_\tau^\mathcal{O}(\eta, n) = 1_{N_\eta}(h) 1_\mathcal{O}(\eta) \psi_\tau(x_\eta^{-1} h x_\eta). \tag{3.138}$$

The characters χ_β^B of the irreps Π_β^B of $D(H)$, viewed as $\mathcal{T}_{yg_Ay^{-1}}(H)$-modules are given by

$$\chi_\beta^B(gK_A, n) = \sum_{k \in K_A} 1_{N_{gk}}(n) 1_A(gk) \chi_\beta(x_{gk}^{-1} n x_{gk}), \tag{3.139}$$

where $n \in N_{yg_Ay^{-1}}$.

3.10.2 Confinement

We want to find out which of the irreps Ω_β^B of $\mathcal{T}_{yg_Ay^{-1}}$ are confined and which are not. To keep things simple, we take the condensed flux yg_Ay^{-1} to be just g_A. This can be done without any real loss of generality, since g_A was chosen arbitrarily in A. Again, the non-confined irreps have to satisfy the requirements (3.45) and again, these reduce to (3.62) and (3.63) (with $K = K_A$ and $N = N_{g_A} = N_A$), since $\mathcal{T}_{g_A}$ is isomorphic to a transformation group algebra. In the case at hand, where $\phi = 1_{N_A}$, (3.62) reduces to the requirement that

$$(\forall x \in N_A, \forall \eta \in B) \;\; |x_\eta g_{\mathcal{O}}^{-1} x_\eta^{-1} x K_A \cap N_A| = |x K_A \cap N_A|. \tag{3.140}$$

By definition of x_η and $g_{\mathcal{O}}$, we have $x_\eta g_{\mathcal{O}}^{-1} x_\eta K_A = \eta$. using this and multiplying the sets in the above equation with x^{-1} from the right, we see that it reduces to

$$(\forall \eta \in B) \;\; |\eta K_A \cap N_A| = |K_A \cap N_A|. \tag{3.141}$$

We know that $g_A \in K_A \cap N_A$ and thus that, if the above requirement is to hold, $\eta K_A \cap N_A$ must be non-empty. But this implies that $\eta = n K_A$ for some $n \in N_A$. On the other hand, if this is the case, then the above equation is always satisfied. Hence, the orbits B which are not confined are those whose elements can be written in the form nK_A for some $n \in N_A$.

The condition (3.63) becomes

$$(\forall x \in N_A, \forall \eta \in B) \;\; \beta(x_\eta^{-1} x g_A x^{-1} x_\eta) = I. \tag{3.142}$$

Since $x \in N_A$ and $x_\eta \in N_A$ for all η, this reduces further to yield the condition

$$\beta(g_A) = I \tag{3.143}$$

on β. Basically, this says that β must be trivial on the minimal normal subgroup of N_B that contains g_A.

Let us compare the solutions that we have found to the set of solutions that we had found already in proposition 9 in section 3.6.3. The latter set consists of all $\Omega_\tau^{\mathcal{O}}$ for which the orbit $\mathcal{O}$ is made up of cosets of the form nK_A (with $n \in N_A$) and for which τ is trivial on $K_A \cap N_A$. Thus, we see that we have not found any extra orbits $\mathcal{O}$, but, depending on H, A and $N_{\mathcal{O}}$, we may have found extra irreps τ of $N_{\mathcal{O}}$, since the minimal normal subgroup of $N_{\mathcal{O}}$ that contains g_A can be smaller than $K_A \cap N_A$.

Thus we come back to a point that we touched upon already in section 3.6.1, namely the fact that we are in doubt whether it is always possible to give the full set of solutions to (3.45) a well-defined spin and and a well-defined braiding. We do know that braiding and spin are well-defined for the set of solutions that we had already found in section 3.6.3, since these are in one to one correspondence with the irreps of the quantum double of $N_A/(K_A \cap N_A)$. Therefore, we expect that the non-confined symmetry algebra for the condensates treated in this section should be $D(N_A/(K_A \cap N_A))$.

If the unconfined algebra is $D(N_A/(K_A \cap N_A))$, then the walls that are created by $\mathcal{T}_{g_A}$-excitations can be classified by the left or right Hopf kernel of the map $\Gamma : \mathcal{T}_{g_A} \to D(N_A/(K_A \cap N_A))$. We will take the right kernel, as given in proposition 10. Corollary 3 tells us that this Hopf kernel is isomorphic as an algebra to the tensor product $F((H/K_A)/\overline{N_A}) \otimes \mathbb{C}(N_A \cap K_A)$, where $\overline{N_A}$ is the subgroup of H/K_A which consists of elements of the form nK_A, with $n \in N$. In fact, $\mathrm{RKer}(\Gamma)$ is spanned by the elements of $\mathcal{T}$ which are of the form $1_{hK_AN_A} \otimes \delta_g$ with $g \in N_A \cap K_A$. The irreps of $\mathrm{RKer}(\Gamma)$ are tensor products of an irrep $E_{[\zeta]}$ of $F((H/K_A)/\overline{N_A})$ and an irrep ρ_l of $N_A \cap K_A$. We will denote them $E_{[\zeta]} \otimes \rho_l$. Here, $[\zeta]$ is notation for the $\overline{N_A}$-coset of ζ in H/K_A. We have

$$E_{[\zeta]} \otimes \rho_l(1_{hK_AN_A} \otimes g) = \delta_{[\zeta],[hK_A]} \rho_l(g). \tag{3.144}$$

The decomposition of the $\mathcal{T}_{g_A}$-irrep $\Omega_\tau^{\mathcal{O}}$ into $\mathrm{RKer}(\Gamma)$ irreps may be found using the formula (3.13) for the character $\psi_\tau^{\mathcal{O}}$. We have

$$\begin{aligned}
\psi_\tau^{\mathcal{O}}(1_{hK_AN_A}\otimes\delta_g) &= \int_{\mathcal{O}} d\zeta \int_{K_A\cap N_A} dn\, 1_{hK_AN_A}(x_\zeta \xi_{\mathcal{O}} x_\zeta^{-1})\delta_g(x_\zeta n x_\zeta^{-1})\chi_\alpha(n) \\
&= \int_{\mathcal{O}} d\zeta\, 1_{hK_AN_A}(x_\zeta \xi_{\mathcal{O}} x_\zeta^{-1})\chi_\alpha(x_\zeta^{-1} g x_\zeta) \\
&= \int_{\mathcal{O}} d\zeta\, \delta_{[\zeta],[hK_A]}\chi_\alpha(x_\zeta^{-1} g x_\zeta). \qquad (3.145)
\end{aligned}$$

From this, we read off that $\Omega_\tau^{\mathcal{O}}$ is the sum over $\zeta\in\mathcal{O}$ of those $E_{[\zeta]}\otimes\rho_l$ for which ρ_l is related to one of the $N_A\cap K_A$-irreps contained in τ by conjugation with x_ζ^{-1}. Of course, there may be multiplicities in the decomposition, for example if the coset $[\zeta]$ is the same for several $\zeta\in\mathcal{O}$. Also, note that the non-confined irreps of $\mathcal{T}_{g_A}$ all correspond to the trivial irrep $E_{[K_A]}\otimes 1$, as they should.

3.10.3 Examples of pure flux condensates

Pure flux condensates whose flux is central in H are gauge invariant and examples may be found in section 3.9.3. Here we treat the case where the flux of the condensate is non-central, so that not only the magnetic part of the double, but also the electric group is broken.

$H = D_{2m+1}$

1. Suppose the condensed flux is $r^k\in D_{2m+1}$. In that case, the residual symmetry algebra is the transformation group algebra $F(D_{2m+1}/K_{r^k})\tilde{\otimes}\mathbb{C}N_{r^k}$, where K_{r^k} is the minimal normal subgroup that contains r^k and where $N_{r^k}=\langle r\rangle\cong\mathbb{Z}_{2m+1}$ is the centralizer of r^k in D_{2m+1}. From section 3.9.3, we know that $K_{r^k}=\langle r^k\rangle=\langle r^{\gcd(k,2m+1)}\rangle\cong\mathbb{Z}_x$, where $x=(2m+1)/\gcd(k,2m+1)$. We also recall that $D_{2m+1}/\mathbb{Z}_x\cong D_{\gcd(k,2m+1)}$. Hence,

$$\mathcal{T}_{r^k}(D_{2m+1})\cong F(D_{\gcd(k,2m+1)})\tilde{\otimes}\mathbb{C}\mathbb{Z}_{2m+1}. \qquad (3.146)$$

The $D_{\gcd(k,2m+1)}$ is generated $R=r\mathbb{Z}_x$ and $S=s\mathbb{Z}_x$ and we will write E for its unit element $e\mathbb{Z}_x$. The orbits of the action of N_{r^k} on this $D_{\gcd(k,2m+1)}$ are $\{E\}$, $\{R\},\ldots,\{R^{\gcd(k,2m+1)-1}\}$ and $\{S,SR,\ldots,SR^{\gcd(k,2m+1)-1}\}$. The stabilizer of the orbits with one element is of course $N_{r^k}\cong\mathbb{Z}_{2m+1}$, while the orbit of S has stabilizer $K_{r^k}\cong\mathbb{Z}_x$. It follows that the representations of $\mathcal{T}_{r^k}$ may be written $\Omega_{\beta_l}^{R^p}$ (with $0\le p<\gcd(k,2m+1)$, $0\le l<2m+1$) and $\Omega_{\beta_l}^{S}$ (with $0\le l<x$). We see that $\mathcal{T}_{r^k}$ has $(2m+1)\gcd(k,2m+1)$ irreps of dimension 1 and x irreps of dimension $\gcd(k,2m+1)$. The squares of the dimensions add to the dimension of $\mathcal{T}_{r^k}$, which is $2(2m+1)^2/\gcd(k,2m+1)$. The decomposition of $D(D_{2m+1})$-irreps into $\mathcal{T}_{r^k}$-irreps is as follows:

$$\begin{array}{ll}
\Pi^e_{J_0}\equiv\Omega^E_{\beta_0} & \Pi^{r^k}_{\beta_l}\equiv\Omega^{R^k}_{\beta_l}\oplus\Omega^{R^{-k}}_{\beta_{-l}} \\
\Pi^e_{J_1}\equiv\Omega^E_{\beta_0} & \Pi^s_{\gamma_0}\equiv\bigoplus_l\Omega^S_{\beta_l} \\
\Pi^e_{\alpha_l}\equiv\Omega^E_{\beta_l}\oplus\Omega^E_{\beta_{-l}} & \Pi^s_{\gamma_1}\equiv\bigoplus_l\Omega^S_{\beta_l}
\end{array} \qquad (3.147)$$

The irreps of $\mathcal{T}_{r^k}$ which are not confined are the $\Omega^{R^k}_{\beta_l}$ for which $\beta_l(r^k)=1$, or in other words, those for which l is a multiple of x. It follows that the unconfined representations are automatically in one to one correspondence with the irreps of

$$\mathcal{U}_{r^k}(D_{2m+1})\cong D(N_{r^k}/(K_{r^k}\cap N_{r^k}))\cong D(\mathbb{Z}_{\gcd(k,2m+1)}). \qquad (3.148)$$

The right Hopf kernel of the Hopf map $\Gamma : \mathcal{T}_{r^k} \to D(\mathbb{Z}_{\gcd(k,2m+1)})$ is isomorphic to $F(\mathbb{Z}_2) \otimes \mathbb{C}\mathbb{Z}_x$ and we may denote its representations as $E_{[E]} \otimes \rho_l$ and $E_{[S]} \otimes \rho_l$ (with $0 \le l < x$). Here, $[E]$ and $[S]$ denote the $\overline{N_{r^k}}$-cosets of the K_{r^k}-cosets E and S and ρ_l denotes the l^{th} representation of $\mathbb{Z}_x$, defined in the usual way, with $r^{\gcd(k,2m+1)}$ taken as the preferred generator of $\mathbb{Z}_x$. The restriction of the irreps of $\mathcal{T}_{r^k}$ to $\mathrm{RKer}\Gamma$ is given by

$$\Omega^{R^k}_{\beta_l} \equiv E_{[E]} \otimes \rho_{l \bmod x} \quad \Omega^{S}_{\beta_l} \equiv \gcd(k, 2m+1) E_{[S]} \otimes \rho_{l \bmod x} \tag{3.149}$$

2. Now suppose the condensate has flux $s \in D_{2m+1}$. The minimal normal subgroup of D_{2m+1} that contains s is D_{2m+1} itself and the normalizer N_s of s is just $\{e, s\} \cong \mathbb{Z}_2$. Hence, this condensate leaves us with the symmetry algebra

$$\mathcal{T}_s(D_{2m+1}) \cong F(\mathbb{Z}_2) \cong \mathbb{C}\mathbb{Z}_2. \tag{3.150}$$

The irreps of this $\mathbb{Z}_2$ may be labeled Ω_{J_0} and Ω_{J_1} and the decomposition of the irreps of $D(D_{2m+1})$ is then given by

$$\begin{array}{ll} \Pi^e_{J_0} \equiv \Omega_{J_0} & \Pi^{r^k}_{\beta_l} \equiv \Omega_{J_0} \oplus \Omega_{J_1} \\ \Pi^e_{J_1} \equiv \Omega_{J_1} & \Pi^s_{\gamma_0} \equiv (m+1)\Omega_{J_0} \oplus m\Omega_{J_1} \\ \Pi^e_{\alpha_l} \equiv \Omega_{J_0} \oplus \Omega_{J_1} & \Pi^s_{\gamma_1} \equiv m\Omega_{J_0} \oplus (m+1)\Omega_{J_1} \end{array} \tag{3.151}$$

Since $J_1(s) = -1 \neq 1$, it follows that Ω_{J_1} is confined, so that the only non-confined irrep of $\mathcal{T}_s$ is the "color singlet" Ω_{J_0}. Hence

$$\mathcal{U}_s(D_{2m+1}) \cong \mathbb{C}\{e\} \tag{3.152}$$

and the corresponding Hopf kernel equals $\mathcal{T}_s$.

3.11 Dyonic condensates

Attempts to study dyonic condensates in the same generality as electric or magnetic condensates meet with some problems of a technical nature. For example, the residual algebra after symmetry breaking does not have to be a transformation group algebra of the kind we discussed in section 3.6.3 (see the second part of section 3.11.1 for an example). Therefore, we will only treat some examples with specific groups and condensate vectors here, in order to give an idea of what one may expect. In the process, we also complete our treatment of condensates in theories where the gauge group is an odd dihedral group.

3.11.1 $H = \mathbb{Z}_n$

First of all, let us check which condensates satisfy the requirements of trivial spin and self-braiding that we gave in section 3.7. As before, we denote our favorite generator of $\mathbb{Z}_n$ as r and we denote the representations of this group, defined in the usual way, as α_l (with $0 \le l < n$). The representations of $D(\mathbb{Z}_n)$ may then be written $\Pi^{r^k}_{\alpha_l}$. The spin factor $s^{r^k}_{\alpha_l}$ of $\Pi^{r^k}_{\alpha_l}$ is just $\exp(-2\pi i k l/n)$ and so the requirement of trivial spin selects those $\Pi^{r^k}_{\alpha_l}$ for which we have

$$kl = 0 \bmod n. \tag{3.153}$$

Thus, given k, the allowed l are those which are 0 modulo $n/\gcd(k,n)$ and given l, the allowed k are those which are 0 modulo $n/\gcd(l,n)$. From this, we see immediately that, if n is a prime, there will be no allowed dyonic condensates (either l or k has to be zero). We will thus assume from now on that n is composite. For Abelian groups, the requirement of trivial self-braiding is automatically satisfied for states with trivial spin, so the $\Pi^{r^k}_{\alpha_l}$ with $kl = 0 \bmod n$ all give good condensates.

To find the residual algebra $\mathcal{T}^k_l$ for a $\Pi^{r^k}_{\alpha_l}$-condensate, we have to find the representations of $D(\mathbb{Z}_n)^*$ that satisfy equation (3.37). Since $D(\mathbb{Z}_n)^* \cong \mathbb{C}\mathbb{Z}_n \otimes F(\mathbb{Z}_n)$, its representations may be labeled by an irrep α_q of $\mathbb{Z}_n$ and an element r^p of $\mathbb{Z}_n$. The equation (3.37) then selects those (α_p, r^q) for which $\alpha_p(r^k)\alpha_l(r^q) = 1$, or more explicitly, those (α_p, r^q) for which $\exp(2\pi i(kp + lq)/n) = 1$. This means that, to find the allowed p and q, we have to solve the equation

$$kp + lq = 0 \bmod n. \tag{3.154}$$

Rather than looking at the general solution of this equation for all k and l, we will examine two illustrative special cases:

1. First, let us take (n,k,l) such that $n = kl$ and $\gcd(k,l) = 1$. In this situation, we can easily find the solution to equation (3.154). Since $kp + lq$ is a multiple of n, say mn, we may solve for p to get

$$p = \frac{1}{k}(mn - lq) = lm - \frac{l}{k}q, \tag{3.155}$$

using $n = kl$ in the second equality. Since p has to be integer, it follows that $\frac{l}{k}q$ must be an integer. Since $\gcd(k,l) = 1$, the fraction $\frac{l}{k}$ is irreducible and hence $\frac{l}{k}q$ can only be integer if q is a multiple of k. But then it follows from the equation above that p is a multiple of l. On the other hand, it is clear from $n = kl$ that any (p,q) for which p is an l-fold and q is a k-fold will solve (3.154). Thus the residual algebra $\mathcal{T}^k_l$ is spanned by the (matrix elements of) the representations (α_p, r^q) for which $p = 0 \bmod l$ and $q = 0 \bmod k$. Now since $n = kl$, the irreps α_p of $\mathbb{Z}_n$ with $p = 0 \bmod l$ correspond exactly to the irreps of the quotient group $\mathbb{Z}_n/\langle\, r^k \,\rangle \cong \mathbb{Z}_k$. Hence,

$$\mathcal{T}^k_l \cong F(\mathbb{Z}_n/\langle\, r^k \,\rangle) \,\tilde{\otimes}\, \mathbb{C}\langle\, r^k \,\rangle \cong F(\mathbb{Z}_k) \,\tilde{\otimes}\, \mathbb{C}\mathbb{Z}_l. \tag{3.156}$$

We see that $\mathcal{T}^k_l$ is a transformation group algebra of the kind treated in section 3.6.3, where both the normal subgroup K and the subgroup N of these sections equal $\langle\, r^k \,\rangle \cong \mathbb{Z}_l$ in this case. The representations of $\mathcal{T}^k_l$ may thus be denoted Ω^r_s, with $0 \le r < k$ and $0 \le s < l$ and the restriction of the irreps of $D(\mathbb{Z}_n)$ to $\mathcal{T}^k_l$ is given by

$$\Pi^{r^a}_{\alpha_b} \equiv \Omega^{a \bmod k}_{b \bmod l}. \tag{3.157}$$

Using the theory of section 3.6.3, one may see that all the Ω^r_s with $(r,s) \neq (0,0)$ confined. The unconfined algebra $\mathcal{U}^k_l$ is thus the group algebra of the trivial group and the Hopf kernel of the Hopf map $\Gamma : \mathcal{T}^k_l \to \mathcal{U}^k_l$ is all of $\mathcal{T}^k_l$, implying that walls and $\mathcal{T}^k_l$-particles are in one-to-one correspondence.

2. Now consider the case where $l = -k \bmod n$. Equation (3.154) then becomes

$$k(p - q) = 0 \bmod n \tag{3.158}$$

so that the allowed (p,q) are those for which $p = q \bmod n/\gcd(k,n)$. It follows that

$$\mathcal{T}^k_l \cong \mathbb{C}(\mathbb{Z}_n \times \mathbb{Z}_{\gcd(k,n)}), \tag{3.159}$$

where (α_1, r) generates the $\mathbb{Z}_n$ and where either $(\alpha_{n/\mathrm{gcd}(k,n)}, e)$ or $(\alpha_0, r^{n/\mathrm{gcd}(k,n)})$ can be taken as the generator for the $\mathbb{Z}_{\mathrm{gcd}(k,n)}$. We will take the latter possibility. One should notice that, in contrast to everything we have seen up to now, the full residual algebra is not generated by the residual magnetic and the residual electric symmetry algebra. The residual electric and magnetic algebra are generated by $(\alpha_0, r^{n/\mathrm{gcd}(k,n)})$ and $(\alpha_{n/\mathrm{gcd}(k,n)}, e)$ respectively and are both isomorphic to $\mathbb{C}\mathbb{Z}_{\mathrm{gcd}(k,n)} \cong F(\mathbb{Z}_{\mathrm{gcd}(k,n)})$. The total residual algebra is $\mathbb{C}(\mathbb{Z}_n \times \mathbb{Z}_{\mathrm{gcd}(k,n)})$ and contains for example the element (α_1, r), which cannot be generated from the elements of the residual electric and magnetic algebras. Clearly, the residual algebra is not a transformation group algebra of the kind treated in section 3.6.3. This phenomenon is not limited to Abelian H, but can also occur for non-Abelian H. In fact, one may check that it does so already for some condensates in a $D(D_4)$-theory.

The representations of $\mathcal{T}_l^k$ may be written $\chi_{a,b}$, with $0 \leq a < n$, $0 \leq b < \mathrm{gcd}(k,n)$. They are defined in the usual way, through

$$\begin{aligned} \chi_{a,b}(\alpha_1, r) &= e^{2\pi i a/n} \\ \chi_{a,b}(\alpha_0, r^{n/\mathrm{gcd}(k,n)}) &= e^{2\pi i b/\mathrm{gcd}(n,k)}. \end{aligned} \tag{3.160}$$

On the magnetic part of $\mathcal{T}_l^k$, $\chi_{a,b}$ is given by $\chi_{a,b}(\alpha_{n/\mathrm{gcd}(k,n)}, e) = e^{2\pi i(a-b)/\mathrm{gcd}(n,k)}$, as follows from the definition above. The restriction of the irreps of $D(\mathbb{Z}_n)$ to T is given by

$$\Pi_{\alpha_q}^{r^p} \equiv \chi_{p+q,\,q}, \tag{3.161}$$

where the second q on the right hand side should be read modulo $\mathrm{gcd}(k,n)$.

Since we cannot apply the theory of section 3.6.3 here, we have to refer back to the requirements (3.45) in section 3.6.1 in order to determine which of the representations of $\mathcal{T}_l^k$ are confined and which are not. After some algebra, the first of these requirements, applied to $f = \chi_{a,b}$, reduces to

$$e^{-2\pi i(a-b)k/n} = 1, \tag{3.162}$$

from which it follows that

$$a = b \bmod n/\mathrm{gcd}(n,k). \tag{3.163}$$

Note that $n/\mathrm{gcd}(n,k)$ is a divisor of both n and $\mathrm{gcd}(n,k)$, since $k^2 = 0 \bmod n$. As a consequence, the above equation retains its usual meaning, despite the fact that a is only defined modulo n and b is only defined modulo $\mathrm{gcd}(n,k)$.

The second requirement in (3.45), applied to $f = \chi_{a,b}$, becomes

$$e^{2\pi i bk/n} = 1, \tag{3.164}$$

so that we have

$$b = 0 \bmod n/\mathrm{gcd}(n,k). \tag{3.165}$$

Hence, the unconfined representations of $\mathcal{T}_l^k$ are just those $\chi_{a,b}$ for which both a and b are multiples of $n/\mathrm{gcd}(n,k)$. All in all, this leaves $\mathrm{gcd}(n,k)$ possibilities for a and $(\mathrm{gcd}(n,k))^2/n$ possibilities for b, so that the non-confined algebra $\mathcal{U}_l^k$ is given by

$$\mathcal{U}_l^k \cong \mathbb{C}(\mathbb{Z}_{\mathrm{gcd}(n,k)} \times \mathbb{Z}_{(\mathrm{gcd}(n,k))^2/n}). \tag{3.166}$$

As an example, consider the case of $D(\mathbb{Z}_9)$, with a condensate given by $k = -l = 3$. The only non-confined irreps of $\mathcal{T}_{-3}^3$ are then $\chi_{0,0}$, $\chi_{3,0}$ and $\chi_{6,0}$ and the unconfined algebra is $\mathbb{C}\mathbb{Z}_3$. We give a graphical representation of our results for this case in figure 3.1.

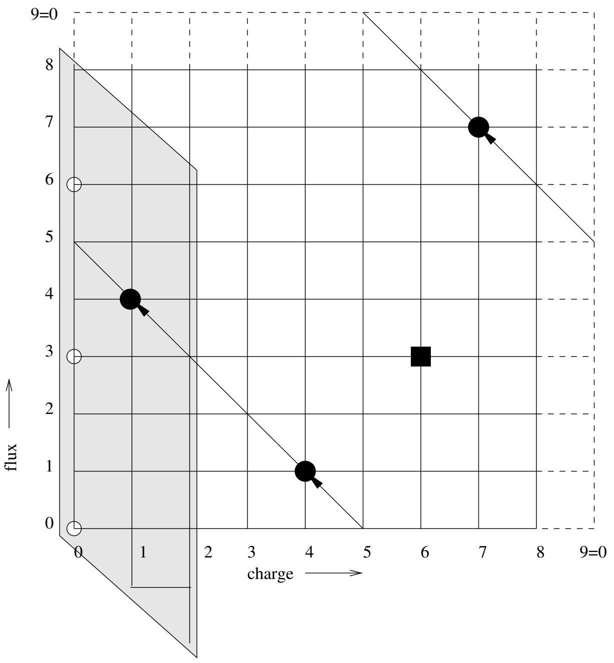

Figure 3.1: Flux-charge lattice for a $D(\mathbb{Z}_9)$ theory. We assume that a condensate of particles with flux $r^3 \equiv 3$ and charge $\alpha_{-3} \equiv -3 \equiv 6$ forms. The condensed irrep is indicated as a square dot. The residual symmetry algebra $\mathcal{T}_{-3}^{3}$ is a group algebra $\mathbb{C}(\mathbb{Z}_9 \times \mathbb{Z}_3)$. Two $D(H)$-irreps in the lattice are equivalent as $\mathcal{T}_{-3}^{3}$-irreps if one can be reached from the other through translations by the "condensate vector" $(-3, 3)$. This way, $D(H)$-irreps are identified in trios. The trio in the picture corresponds to the $\mathcal{T}$-irrep $\chi_{5,1}$. The shaded region contains one representative from each trio and is thus a diagram of all $\mathcal{T}$-irreps. The small white circles indicate the three unconfined irreps of $\mathcal{T}$, which correspond to the irreps of $\mathcal{U} \cong \mathbb{C}\mathbb{Z}_3$.

3.11.2 $H = D_{2m+1}$

In this section, we complete our treatment of condensates in the odd dihedral gauge theories. First, we find out which states in dyonic representations of $D(D_{2m+1})$ satisfy the conditions of trivial spin and trivial self-braiding. From table 3.2, we read off that the only dyonic irreps of $D(D_{2m+1})$ which have trivial spin are those $\Pi_{\beta_l}^{r^k}$ for which $\exp(2\pi ikl/(2m+1)) = 1$, or in other words, for which $kl = 0 \bmod 2\mathrm{m}+1$. It follows that there are no admissible dyonic condensates when $2m+1$ is prime. If $2m+1$ is not a prime, then there will be dyons with trivial spin and one may check easily that any state in the module of one of the $\Pi_{\beta_l}^{r^k}$ with $kl = 0 \bmod 2\mathrm{m}+1$ also has trivial self-braiding. Therefore, all states in the modules of these dyonic irreps may in principle be condensed.

Now suppose that we have condensed a state ϕ in the module of $\Pi_{\beta_l}^{r^k}$. To find the residual symmetry algebra, we have to solve equation (3.36). The representations (ρ, g) of $D(D_{2m+1})^*$ which satisfy this equation are those for which $\frac{\chi_\rho(r^k)}{d_\rho}$ is a root of unity and ϕ is an eigenvector of g^{-1} with eigenvalue equal to this root of unity. Thus, let us first find all irreps ρ of D_{2m+1} for which $\frac{\chi_\rho(r^k)}{d_\rho}$ is a root of unity. From table 3.1 one may read off that these are J_0, J_1 and those α_j for which $2\cos(2\pi jk/(2m+1)) = 2$, or equivalently $jk = 0 \bmod 2m+1$. This leaves exactly those j which are multiples of $x_k := (2m+1)/\gcd(k, 2m+1)$. In all these cases, $\frac{\chi_\rho(r^k)}{d_\rho}$ actually equals 1, or equivalently, ρ is trivial on r^k. It follows that the allowed ρ correspond to the irreps of the quotient group $D_{2m+1}/\langle\, r^k \,\rangle$, which is isomorphic to $D_{\gcd(k,2m+1)}$. The residual symmetry algebra will now be spanned by the (ρ, g) with ρ in the set we have just found and g

an element of the subgroup of D_{2m+1} that leaves ϕ invariant. This subgroup will depend on ϕ. Therefore, let us write ϕ more explicitly as $a\phi_{r^k} + b\phi_{r^{-k}}$. Here ϕ_{r^k} and $\phi_{r^{-k}}$ are just the two basis functions for $V_{\beta_l}^{r^k}$ as defined through (3.11). We have dropped the index i in (3.11), since the module of β_l is one-dimensional. Using the formula (3.12) for the matrix elements of $\Pi_{\beta_l}^{r^k}$ with respect to this basis, we can now write the action of the elements of D_{2m+1} on ϕ explicitly as

$$\begin{aligned} \Pi_{\beta_l}^{r^k}(1 \otimes \delta_{r^p})(a\phi_{r^k} + b\phi_{r^{-k}}) &= e^{\frac{2\pi ilp}{2m+1}} a\phi_{r^k} + e^{\frac{-2\pi ilp}{2m+1}} b\phi_{r^{-k}} \\ \Pi_{\beta_l}^{r^k}(1 \otimes sr^p)(a\phi_{r^k} + b\phi_{r^{-k}}) &= e^{\frac{-2\pi ilp}{2m+1}} b\phi_{r^k} + e^{\frac{2\pi ilp}{2m+1}} a\phi_{r^{-k}}. \end{aligned} \tag{3.167}$$

From the first of these equations, we see that, independently of the choice of (a, b), r^p will leave ϕ invariant only if $\exp(2\pi ilp/(2m+1)) = 1$. In other words, p has to be a multiple of $x_l := (2m+1)/\gcd(2m+1, l)$. From the second equation above, we see that sr^p will leave ϕ invariant only if $b = \exp(2\pi ilp/(2m+1))a$. If no such relation between a and b exists, then none of the elements $sr^p \in D_{2m+1}$ will leave ϕ invariant and the subgroup of D_{2m+1} that does leave ϕ invariant is just the $\mathbb{Z}_{\gcd(2m+1,l)}$ generated by r^{x_l}. If we do have $b = \exp(2\pi ilp/(2m+1))a$ for some p, then the required subgroup of D_{2m+1} is the $D_{\gcd(2m+1,l)}$ generated by r^{x_l} and sr^p. All these $D_{\gcd(2m+1,l)}$ subgroups actually represent the same physics, since they are conjugates in D_{2m+1} (or equivalently, the corresponding condensates are all related by gauge transformations). We have thus found two distinct possibilities for the residual symmetry algebra $\mathcal{T}$, which we will call $\mathcal{T}_l^k$ and $\bar{\mathcal{T}}_l^k$. These algebras are given by

$$\begin{aligned} \mathcal{T}_l^k &\cong F(D_{2m+1}/\langle r^k \rangle) \tilde{\otimes} \mathbb{C}\langle r^{x_l} \rangle \cong F(D_{\gcd(2m+1,k)}) \tilde{\otimes} \mathbb{C}\mathbb{Z}_{\gcd(2m+1,l)} \\ \bar{\mathcal{T}}_l^k &\cong F(D_{2m+1}/\langle r^k \rangle) \tilde{\otimes} \mathbb{C}\langle r^{x_l}, s \rangle \cong F(D_{\gcd(2m+1,k)}) \tilde{\otimes} \mathbb{C}D_{\gcd(2m+1,l)}. \end{aligned} \tag{3.168}$$

Both $\mathcal{T}_l^k$ and $\bar{\mathcal{T}}_l^k$ are thus transformation group algebras of the for $F(H/K) \tilde{\otimes} \mathbb{C}N$ (for $\mathcal{T}_l^k$, we have $K = \langle r^k \rangle$ and $N = \langle r^{x_l} \rangle$, whereas for $\bar{\mathcal{T}}_l^k$, we have $K = \langle r^k \rangle$ and $N = \langle r^{x_l}, s \rangle$). Because of this, the decomposition of $D(D_{2m+1})$-irreps into $\mathcal{T}$-irreps proceeds in the same way as for the electric and magnetic cases. Also, the theory of section 3.6.3 may be applied to treat confinement. One finds that the unconfined algebras $\mathcal{U}_l^k$ and $\bar{\mathcal{U}}_l^k$ are isomorphic to the quantum doubles of the groups $N/(K \cap N)$. Now it turns out that we have $K \subset \langle r^{x_l} \rangle$ and hence $K \subset N$ in both cases. To see this, remember that we have $kl = 0 \bmod 2\mathrm{m}+1$ and thus $\gcd(k, 2m+1)\gcd(l, 2m+1) = 0 \bmod 2\mathrm{m}+1$. Hence,

$$\gcd(k, 2m+1)\gcd(l, 2m+1) = q\,(2m+1) \tag{3.169}$$

for some integer q and it follows that $\gcd(k, 2m+1) = qx_l$ and $\langle r^k \rangle \subset \langle r^{x_l} \rangle$ (note that $\langle r^k \rangle = \langle r^{\gcd(k,2m+1)} \rangle$). The integer q has the property that it divides both $\gcd(k, 2m+1)$ and $\gcd(l, 2m+1)$. Using this, one now sees easily that

$$\begin{aligned} \mathcal{U}_l^k &\cong D(\mathbb{Z}_q) \\ \bar{\mathcal{U}}_l^k &\cong D(D_q). \end{aligned} \tag{3.170}$$

When $q = 1$, this means that there is full confinement of $\mathcal{T}_l^k$-irreps, while on the other hand, there are still four unconfined $\bar{\mathcal{T}}_l^k$-irreps, since $\bar{\mathcal{U}}_l^k \cong D(\mathbb{Z}_2)$.

The Hopf kernels of the maps $\Gamma : \mathcal{T}_l^k \to \mathcal{U}_l^k$ and $\bar{\Gamma} : \bar{\mathcal{T}}_l^k \to \bar{\mathcal{U}}_l^k$ can also be determined, following the treatment in section 3.6.3. We find that

$$\begin{aligned} \mathrm{RKer}(\Gamma) &\cong F(D_{x_l}) \otimes \mathbb{C}\mathbb{Z}_{x_k} \\ \mathrm{RKer}(\bar{\Gamma}) &\cong F(D_{\gcd(k,2m+1)}/D_{\gcd(k,2m+1)/x_l}) \otimes \mathbb{C}\mathbb{Z}_{x_k}. \end{aligned} \tag{3.171}$$

3.12 Summary and Outlook

The general picture that emerges from our investigation of symmetry breaking and confinement in discrete gauge theories can be seen in figure 3.2.

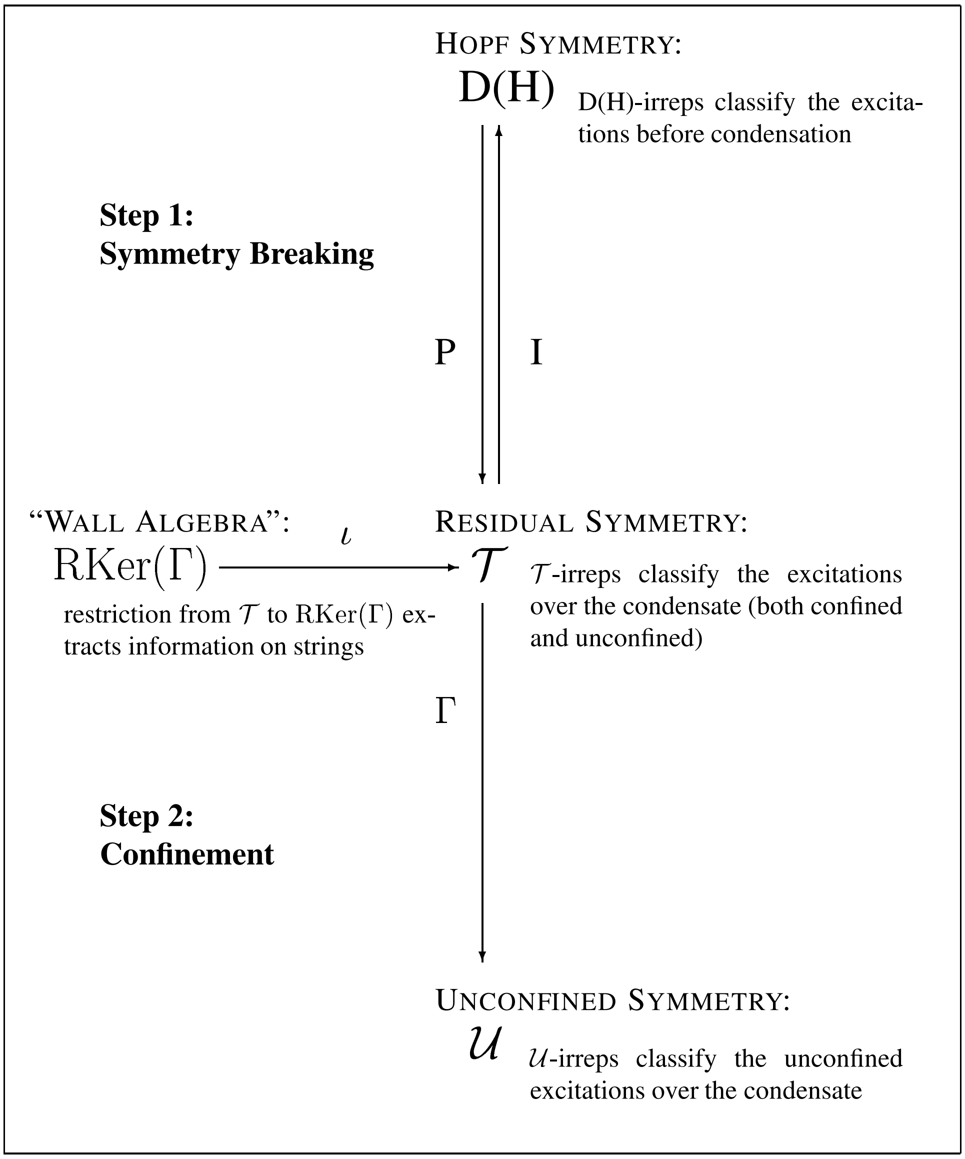

Figure 3.2: Schematic picture of the structures that play a role in this chapter

In words, it is as follows. The formation of a condensate induces symmetry breaking from $D(H)$ to the Hopf subalgebra $\mathcal{T} \subset D(H)$ which is the Hopf stabilizer of the condensate state. The ensuing confinement is described by a Hopf projection Γ of $\mathcal{T}$ onto an "unconfined" symmetry algebra $\mathcal{U}$, whose irreps label the free charges over the condensate. Walls or strings in the condensate are labeled by the restrictions of $\mathcal{T}$-irreps to the right Hopf kernel of Γ. In the diagram, I denotes the (Hopf) inclusion of $\mathcal{T}$ into $D(H)$, ι denotes the inclusion of $\mathrm{LKer}(\Gamma)$ into $\mathcal{T}$ and P denotes the orthogonal projection of $D(H)$ onto $\mathcal{T}$, which we use in our definition of $\mathcal{U}$. To the information in the diagram, we should add that all "baryonic" excitations on the condensate can be constructed by fusing together a number of confined particles, labeled by $\mathcal{T}$-irreps, in such a way that the overall fusion product has a non-confined charge, labeled by a

$\mathcal{U}$-irrep.

Note that the role that the unconfined algebra $\mathcal{U}$ plays in the $D(H)$-theory is quite comparable to the role that $D(H)$ plays in the gauge theory with continuous gauge group G of which our discrete gauge theory is a Higgsed version. Just like $D(H)$ classifies the free excitations over the Higgs condensate in the continuous gauge theory, $\mathcal{U}$ classifies the free excitations over the condensate in the $D(H)$-theory. In fact, the different unconfined algebras we have found for specific condensates are typically themselves quantum doubles of a group related to H. For example:

- For purely electric condensates, we have found that $\mathcal{U}$ is the quantum double of the stabilizer N of the condensate in H. This is just what we expected, since the only effect of condensing one of the electric particles of the $D(H)$-theory is to modify the electric condensate of the G-theory in such a way that the residual gauge group is now N rather than H.

- For gauge invariant magnetic condensates, we have found that $\mathcal{U}$ is the quantum double of the quotient group H/K, where K is the group generated by the elements of conjugacy class that labels the condensate. This is also in accordance with the intuition, since the division by K can be seen as a consequence of the fact that, after condensation, the flux of any particle can only be determined up to the condensed flux.

In a sense, we can describe the condensed phases of the $D(H)$-theory even better than the $D(H)$-theory itself describes the Higgs phase of the G-theory, since the algebra $\mathcal{T}$ that we obtain after symmetry breaking gives us information on the possible substructures of the free excitations over the condensate.

Nevertheless, there is still much work to be done. First of all, from the requirements (3.45) that we found in section 3.6, it is not clear that the set of irreps of $\mathcal{U}$ will always have a well-defined braiding. Although this does happen in the examples with electric and gauge invariant magnetic condensates (where $\mathcal{U}$ is quasitriangular), we do not expect that the equations (3.45) will guarantee this in general. Therefore, we expect that supplementary conditions will be necessary for a completely satisfactory definition of $\mathcal{U}$. Secondly, it would be good to have some "independent" theoretical confirmation of the results we have presented. One could for example try to find the phases that we are predicting in numerical calculations on a lattice.

It is also important to generalize the techniques for the breaking of Hopf algebra symmetries that we have developed, both to the case where the symmetry algebra is infinite dimensional and to the case where it is no longer a Hopf algebra, but only a quasi-Hopf algebra or even a weak quasi-Hopf algebra or Hopf algebroid. This would extend the applicability of our symmetry breaking scheme enormously. A theory of symmetry breaking for infinite dimensional Hopf algebras could for example have interesting applications in the study of (2+1)-dimensional gravity using the quantum group theoretical framework of [13, 14]. A generalization to weak quasi-Hopf algebras would bring any physical system which has a description in terms of Chern-Simons theory or two dimensional conformal field theory within the reach of our methods. One application could be the construction of a hierarchy of fractional quantum Hall states much like the one proposed by Haldane and Halperin [49, 50], but using non-Abelian quantum Hall states such as those studied in chapter 2 as the starting point. One could form a condensate of quasihole excitations over such a state to obtain a new Hall state at a different filling factor.

Finally, it would of course be extremely interesting if the treatment of symmetry breaking and confinement that we give here could be extended to gauge theories in 3+1 (or higher) dimensions. One might begin to think of such an extension starting from the ideas presented in [126].

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Samenvatting

Hier komt de samenvatting in het Nederlands.

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