Quantisation of 2+1 dimensional Gravity as a Chern-Simons theory

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Abstract

In this thesis we consider the quantisation of Chern Simons theories.
The interpretation of Chern Simons theory as a theory of gravity, if we consider gauge groups $ISO(3-p,p), ISO(4-p,p)$, provides an extra motivation for studying quantisations, so we investigate the theory of gravity in 2+1 dimensions as a topological Chern Simons theory.
The nature of phase space is that of functions on multiple copies of the group, which implies the need of a description in terms of Hopf algebra's.
Through the dual picture of the Hopf algebra of functions on the group, the Hopf algebra called the universal enveloping algebra, a mathematical structure is described which allows for a deformation quantisation scheme. Crucial, already on a classical level on which it directly describes a Poisson bracket, is the existence of an element of the tensor product of two copies of the gauge algebra, called the r-matrix. This Lie bialgebra structure defining object is the main object of study in a formalised discussion of different deformation quantisations of a theory.
A deformation quantisation of the originally commuting algebra of functions on a group (or multiple copies of it), turning it into a noncommuting algebra, is characterised by the universal $\mathcal{R}$-element. This $\mathcal{R}$-element can be expanded in powers of a quantum parameter h, which is 0 in the classical limit, where the first order part in h is the classical r-matrix.
The quantisation of Chern Simons theory with gauge group $ISO(3)$, equivalent to Euclidean gravity without cosmological constant, as argued from different reasoning than the deformation quantisation point of view, leads to the quantum group symmetry of $D(SU(2))$, the quantum double.
At the end of this thesis, we show that this result agrees with a quantisation from the deformation quantisation point of view. In particular, it is shown that the universal $\mathcal{R}$-element can be expanded in a quantum parameter, where the first order part is a 'canonical' r-element for the algebra $iso(3)$.

Contents

1 Introduction

How can a trivial theory be interesting? 2+1-dimensional gravity seems to be a trivial theory if we look at it (not too) carefully. Einstein equations in this low-dimensional space imply curvature is zero, and regarding the theory of general relativity in 2+1 dimensions as a gauge theory means the ingredients of the theory are flat gauge connections (i.e. gauge fields for which the curvature is zero). Different configurations of such gauge fields are the only distinguishable properties of the theory. That is, since we are dealing with gauge theory, we must distinguish between different gauge field configurations that are not related by a gauge transformation, since physical aspects of the theory may not depend on the choice of gauge. Since all flat gauge connections can be transformed into each other by a gauge transformation, there is only one equivalence class of gauge connections, an observation which trivialises the theory. One would think.

But of course we would not be physicists if we would think a bit further and find a way to detrivialise the theory. A flat gauge field configuration is not uniquely defined up to gauge transformations if the manifold to which this configuration is associated has holes in it. The gauge field along a closed loop around such a hole can take different values along the loop, as long it returns to the same value at the starting point. This single-valuedness allows for *twistings* of the gauge field along this loop, and the associated configuration can not just be gauge transformed into each different twisted configuration of the gauge field.

This actually rather simple thought gives an opening to a whole subject of physics, the topological field theory. The topology of our base manifold determines the observable physical part of the theory, the phase space.
Canonical procedures in physics give properties of phase spaces that follow directly from the action one starts off with. However, the observation that we are dealing with flat connections gives an enormous reduction of the infinite dimensional phase space one would think of from a field theoretical point of view.

In this thesis I have described phase space reductions and quantisation aspects of the theory of 2+1 dimensional gravity as a Chern Simons theory.
In particular the quantisation of this theory is described from a certain point of view, that of deformation quantisation.
This is of course interesting, because the words gravity and quantisation in ordinary 3 + 1 dimensional physics always appear to be in conflict with each other. Deleting one dimension is, in contrast to what one might think, not as meaningless as just a reduction of the degrees of freedom. In fact it turns out that this changes the nature of this theory into a topological one, as we explained above.
This implies that the concept of distance is of no significance, whereas the only feature that does have influence on observables of the theory is the topology of the manifold we are considering. We will regard the 2+1 dimensional manifold

as a 2-surface with time throughout this thesis.
The topological nature of the theory of gravity in 2+1 dimensions is supported by the observation made by Witten in 1988 [1], that it can be regarded as a Chern Simons gauge theory of the gauge group $ISO(2,1)$ in the Minkowskian case.
The equations of motion imply that the gauge connection is flat everywhere on the manifold, so that the theory is trivial if the surface Σ we are considering is simply connected.
Nontrivial solutions arise when we allow for twisting of the gauge field, which is only possible if the fundamental group of the surface, $\pi_1(\Sigma)$, is nontrivial, thus if we allow handles or punctures in the surface.
Particles can be added to this theory by inserting them in the punctures, where they act as delta-type sources of curvature.
For the Euclidean theory of gravity, which is shown to be equivalent to Chern Simons theory with gauge group $ISO(3)$, a quantum theory has been developed in [2], mainly from arguments based on the topological nature of the theory. In particular this topological nature implies particles have to interact in a topological way, described by braiding and fusion relations.
The result of this procedure was that the quantum theory for $ISO(3)$ gauge group is described by a *quantum double* symmetry, of the group $D(SU(2))$. This identification of a quantum group symmetry is a natural result for a quantised Chern Simons theory as we shall see in this thesis. This illustrates the importance of Chern Simons theory as a physical model to which one can indeed apply the mathematical structures of quantum group theory.

At the starting point of my research the goal was to clarify the relations between the different Chern Simons theories of gravity and their quantisations. As pointed out by Witten [1], one can add a cosmological constant to the theory, for which case in stead of the Euclidean group and the Poincare group in 3 dimensions one observes the relevant gauge group is $SO(4), SO(3,1)$ or $SO(2,2)$, depending on the sign of the cosmological constant and on the space-time signature.

In the end I have given an overview of quantisation of Chern Simons theory from the deformation quantisation point of view, describing an important part of the mathematical framework needed, and which gives insight in the structure of the theory, also on a classical level. While I was doing my research, Schroers [3] showed that the quantum double quantisation, which was claimed to be the correct result for the zero cosmological constant case from reasoning based on topological arguments as mentioned above, can be interpreted as a deformation quantisation. This result is given as a beautiful example at the end of this thesis.

The outline of the thesis will be the following:
The second section (following the introduction) is in fact an introduction to Chern Simons theory. The action is given, the phase space and observables are identified.

In the third section we show the equivalence of 2+1 dimensional gravity and Chern Simons theories, where we include the description of the Euclidean and Minkowskian cases with nonzero cosmological constant.
The fourth section is a description of classical phases space, its reductions, and Poisson brackets. The mathematical framework of Hopf algebras and bialgebra structures is introduced from a general point of view as well as the application to Chern Simons theories.
In the fifth section the matter of quantisation is discussed, where we focus on deformation quantisation and quantum groups.
The sixth section describes the above mentioned deformation of $ISO(3)$ theory resulting in $D(SU(2))$ identified by Schroers [3].
The seventh section is a concluding section, in which we make some final remarks, and formulate some interesting questions left open.

2 Chern Simons Theory

2.1 The action and gauge invariance

2.1.1 The Chern Simons action

Chern Simons theory is a so called *topological gauge theory* on a 3-manifold, described by the Chern-Simons action, defined by

$$S_{CS} = \frac{k}{4\pi} \oint_M tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \tag{1}$$

Here $A = A^a T_a$ is a Lie algebra valued connection corresponding to a gauge group G, in fact it is a connection on a principal G-bundle over M. M is the 3-manifold and by the trace we actually mean a nondegenerate invariant inner product for the representation T_a of the Lie algebra. For $G = SU(n)$ one can for instance take the Casimir invariant $T_a T_a$, which indeed yields taking the trace over algebra elements in the fundamental representation for which $\frac{1}{2} Tr(T_a T_b) = \delta_{ab}$. As we shall see, in stead of the trace, in some cases we will be able to define different inner products, corresponding to different invariants on algebras such as the algebra of $SO(4)$. The bilinear form must be nondegenerate if we require that the action contains kinetic terms for all components of the gauge field, relative to the algebra basis T_a. The quantity k is a coupling constant. As we shall see, the relevance of the value of k depends on the topology of the group G.
The Chern-Simons theory is called a topological theory because of the absence of space time metric dependence. The action M does not depend on a specific space time metric, which means the notion of distance is not required. The only property of the manifold that can be significant is therefore its topology.

The Chern-Simons action is invariant under infinitesimal gauge transformations

$$\delta A_\mu(x) = D_\mu \varepsilon(x) = \partial_\mu \varepsilon(x) + [A_\mu, \varepsilon](x)$$

where ϵ is an infinitesimal Lie algebra valued parameter.

2.1.2 Large gauge transformations

Finite gauge transformations, not connected to the identity, do not leave the action totally invariant, which we show by gauge-transforming the different components of the Lagrangian:

The connection A transforms in the adjoint representation:

$$A \ \rightarrow \ g(A + d)g^{-1},$$

which yields

$$dA \ \rightarrow \ gdAg^{-1} \ + \ (dg) \wedge Ag^{-1} \ - \ gA \wedge dg^{-1} \ + \ dg \wedge dg^{-1}$$

and

$$A \wedge A \;\rightarrow\; gA \wedge Ag^{-1} \;-\; dg \wedge Ag^{-1} \;+\; gA \wedge dg^{-1} \;-\; dg \wedge dg^{-1}.$$

The invariant inner product will allow cyclic permutations of group and algebra valued elements. Together with the property

$$gdg^{-1} = -(dg)g^{-1}$$

we arrive at:

$$\begin{aligned} Tr\{A \wedge dA + \frac{2}{3} A \wedge A \wedge A\} \quad \rightarrow \quad & Tr\{A \wedge dA \; + \frac{2}{3} A \wedge A \wedge A \\ & - d(A \wedge g^{-1} dg) \\ & - \frac{1}{3} gdg^{-1} \wedge gdg^{-1} \wedge gdg^{-1}\} \end{aligned} \tag{2}$$

Thus after a gauge-transformation we are left with two extra terms in the action, one which is a total derivative and vanishes if we state that the gauge transformation is the identity at the boundary of the three-manifold. The second term however is only a priori zero if the topology of the manifold is trivial, which would mean all gauge transformations are continuously connected to the identity.
To see this, consider the gauge transformation as a C^∞ mapping from the manifold M to the group manifold. Requiring it to be C^∞ means that in a trivial topology of the manifold M the mapping can always be retrieved by iterated infinitesimal gauge transformations if the gauge group is connected.
We will be interested in cases for which the topology of the manifold M is not trivial. The extra term in (2) is a topological invariant when integrated over. It depends on the topology of the group manifold, in fact, it must be an element of the group $\pi_3(G)$, which in most cases if the group is compact, is equal to $\mathbb{Z}$. The integer we are thus left with is called the winding number, see Birmingham et al. [4].
Requiring physical invariance under *twisted* gauge transformations restricts the coupling constant k to be an integer if we choose the right normalisation of the inner product on the algebra. The addition of the winding number term to the action will then not be manifest in physical sectors of the theory if we consider the role of the action in the Feynman path integral formulation. The expression e^{iS} will be invariant if the extra term is an integer multiple of 2π, which is exactly what the restriction implies.

2.1.3 The action written in components

Using the explicit algebra basis, the commutation relations

$$[T_a, T_b] = f_{ab}{}^c T_c$$

and the inner product $\langle T_a, T_b \rangle$, we can give a more explicit version of (1):

$$\begin{aligned} S_{CS} &= k \int_M dx^3 \Big(\langle T_a, T_b \rangle \epsilon^{ijk} A_i{}^a (\partial_j A_k{}^b - \partial_k A_j{}^b) \\ &\quad + \frac{2}{3} \langle \frac{1}{2} [T_a, T_b], T_c \rangle \epsilon^{ijk} A_i{}^a A_j{}^b A_k{}^c \Big) \\ &= \langle T_c, T_d \rangle k \int_M dx^3 \epsilon^{ijk} \Big(A_i{}^c (\partial_j A_k{}^d - \partial_k A_j{}^d) + \frac{1}{3} f_{ab}{}^c A_i{}^a A_j{}^b A_k{}^d \Big). \quad (3) \end{aligned}$$

2.2 Hamiltonian formulation and phase space

2.2.1 The Hamilton formalism

If we specify our manifold M to be of the form $\Sigma \times \mathbb{R}$, we can identify it with a space-time manifold in which $\mathbb{R}$ represents time and at each time t, Σ is a *time slice*, a space 2-manifold. The theory now admits a Hamiltonian formulation, in which one can derive equations of motion through variation of the action and identify Poisson-brackets, which describe the 'classical mechanics' of the system.
We identify the 0-coordinate with time and are left with two space dimensions i and j. Our Chern-Simons action now reads

$$S_{CS} = -\frac{k}{2} \int dt \int_\Sigma dz^2 \epsilon^{ij} Tr \Big(A_i \partial_t A_j - A_t F_{ij} \Big) \quad (4)$$

From this action we can derive equations of motion by demanding the variation of the action induced by variation of the fields to be zero. The variation of A_t in particular leads to the equation:

$$F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j] = 0 \quad (5)$$

In fact A_t is nothing but a Lagrange multiplier implementing the zero curvature (or fieldstrength) constraint on the phase space.
The above ingredients tell us a lot about the phase space of the theory. It consists of all flat connections on the Riemann surface Σ, which for the physical sectors should be divided out by the gauge symmetry. What phase space will look like, again depends on the topology of the surface Σ as we will argue next.

2.2.2 Wilson loops

Similar to the case of the gauge transformations, the gauge field configuration can be viewed as a map from the space of all gauge fields to the manifold. The topological nature of the theory is reflected in the classification of the different configurations of the gauge field. The way to distinguish between physically different configurations is by considering a path ordered exponential of the line integral over the gauge field along a path γ on the surface Σ :

$$W(\gamma_1) = P e^{\int_{\gamma_1} A} \quad (6)$$

This expression gives a representing group element for the configuration of the gauge field along the path γ_1 and is called a Wilson line.
As the connection gauge field A transforms in the adjoint representation, gauge transformations act through conjugation with the value of the gauge group element at the endpoints of the path. If we consider closed loops we find that the expression

$$W(C) = Tr(Pe^{\oint_C A}) \tag{7}$$

is an invariant of the theory and therefore a physical observable. The observables W(C) are called Wilson loops, see again [4]
Expression (6) is certainly not a gauge invariant quantity. A gauge transformation can take it to another group element, which corresponds to the result of evaluating the expression (6) along a path γ_2, found by continuously deforming γ_1.
In the case of (7) one can identify the gauge transformation with deforming the loop C, which in this case does not change the value of $W(C)$ as long as (5) is imposed.
Thus we can conclude that the only way in which two loops C_1 and C_2 can lead to physically different gauge field configurations is when the surface has a nontrivial topology and at least one of the loops is noncontractible. Different inequivalent gauge field configurations can be represented by different embeddings of closed loops into the surface Σ, which corresponds to the definition of the fundamental group $\pi_1(\Sigma)$.

If we do not yet take gauge equivalence into account and thus do not yet take the trace in expression (7), every inequivalent noncontractible loop can be mapped into the group with help of the path ordered exponential in (6). Different noncontractible loops can be combined to give one new loop. This defines a multiplication of loops, which must be consistent with the group multiplication. Gauge transformations act consistently, in the sense that they respect this multiplication rule, as we can see from

$$\begin{aligned} W(\gamma_1) = h_1, \quad W(\gamma_2) &= h_2 \\ W(\gamma_1) * W(\gamma_2) &= h_1 h_2 \\ g : h_i &\to g h_i g^{-1} \\ g : h_1 h_2 \to g h_1 g^{-1} g h_2 g^{-1} &= g h_1 h_2 g^{-1}. \end{aligned} \tag{8}$$

The physical phase space of the theory is now the space of all homomorphisms from the fundamental group of the surface Σ to the gauge group G, which has to be divided out by conjugation with the gauge group:

$$\Gamma = \mathrm{Hom}(\pi_1(\Sigma); G)/\sim, \tag{9}$$

where $\sim$ stands for the conjugation equivalence. This space will obviously not be very interesting if the surface Σ is simply connected, because it then consist of only one point. Noncontractible loops arise when we assign punctures to

the surface (points that do not allow loops to be pulled through) or when its genus is not zero. A Wilson loop observable (7 will physically correspond to a conjugacy class rather than to a group element, due to the gauge equivalence.

2.2.3 Sources of curvature

The observable can be restricted to a certain conjugacy class if the loop encloses a source of curvature. This can be shown by using a nonabelian version of Stokes' theorem:

$$\oint_C A = \int_S DA = \int_S F \tag{10}$$

Here S is the surface area with boundary C. Thus for the Wilson loop we get

$$W(C) = \mathrm{Tr}(\exp(\int_S F)). \tag{11}$$

For contractible loops we have that the total interior of C belongs to Σ, for which we have the constraint equation $F = 0$, and so the Wilson loop will correspond to the trivial conjugacy class, the identity. For noncontractible loops this no longer holds necessarily. A way of inserting particles in the theory is by putting them in points which do not belong to Σ, i.e. loops cannot be pulled through these so called *punctures*. These particles then carry charges of the gauge group and act as delta-sources for curvature:

$$F^a = \sum_i Q^a{}_i \delta^{(2)}(z - z_i). \tag{12}$$

Here $Q^a{}_i$, the charges of the i'th particle, are again components relative to the Lie algebra. They restrict the representative group element corresponding to the Wilson loop which encloses them to lie in a certain conjugacy class (due to the gauge transformations which correspond to conjugations).

2.2.4 Poisson-brackets

The identification of the time-coordinates and space-coordinates of the manifold as in (2.2.1) also admits the identification of poisson brackets for the gauge fields, which will become important in particular at the point of quantising the theory. We will not discuss quantisation yet, but we will just give the poisson brackets, which can actually be identified before discussing the phase space as we did above. They are usually read off by distinguishing between fields which do appear in the action with their time derivatives and fields which don't. One defines them only for the first category. The result is:

$$\{A_i{}^a(z_1), A_j{}^b(z_2)\} = \frac{2}{k}\langle T_a, T_b\rangle \epsilon_{ij} \delta^{(2)}(z_1 - z_2). \tag{13}$$

The reader who is familiar with gauge theories will be surprised to find that the i and j components of the same gauge field are each others conjugates, whereas, for instance in Maxwell theory, the electric field is conjugate to the A-field. With these brackets one identifies the constraint algebra which is another way

to finally arriving at the physical sector of the theory. The constraint algebra is simply found by evaluating the brackets of the different components of the field strength:

$$\{F^a(z_1), F^b(z_2)\} = f^{ab}{}_c F^c \delta^{(2)}(z_1 - z_2) \tag{14}$$

One finds that this reproduces the algebra of the gauge group. Another consequence of this observation is that the inclusion of particles in the theory as suggested in (12) yields these particles carry charges obeying the same poisson-algebra, which should follow from the matter term added to the action describing these particles. Again these are matters of particular interest when discussing quantisation, and we will give a more explicit description of the construction of the particle phase space, which is a *coadjoint orbit*, for the case of $ISO(3)$ gauge group in section 4.

An important final remark on this matter is that one can distinguish between two different approaches towards discussing the physically interesting parts of the theory and finally quantisation.

The first approach is to use the canonically defined poisson brackets (13) to define an operator algebra acting on a Hilbert space, which is obviously far too large, because of the gauge equivalence and remaining constraints. The constraint algebra will be interpreted as an operator algebra selecting out the physical sector of the theory.

The second approach is to first divide out the gauge equivalence and constraints, identifying what is left of phase space and after this quantising the theory. It might then be difficult to find Poisson brackets on this reduced phase space that are consistent with the constraints and the gauge equivalence. The advantage however is that one can perform a dramatical reduction of the degrees of freedom by this procedure. We will in fact see that in our case the phase space reduces from an infinite dimensional one to a finite dimensional space (the description in terms of Wilson loop observables actually illustrates this).

3 2+1 Gravity as a Chern Simons theory

In this section we will give the relation between general relativity in 2+1 dimensions and Chern Simons theory, as pointed out by Witten in a quite famous article in 1988 [1]. Most of the important results will be repeated here..

3.1 General Relativity

Einsteins theory of general relativity starts with the equivalence principle, which states that in any point x in space-time one can choose a locally orthonormal basis of coordinates with Minkowskian metric $\eta_{ab} = (-1, 1, 1)$. If we denote the orthonormal coordinates by $\xi^a(x)$ this means we can write down the metric relative to arbitrary coordinates x^μ

$$dx^\mu dx_\mu = g_{\mu\nu} dx^\mu dx^\nu = g_{\mu\nu} \frac{\partial x^\mu}{\partial \xi^a} \frac{\partial x^\nu}{\partial \xi^b} d\xi^a d\xi^b = \eta_{ab} d\xi^a d\xi^b$$

and so

$$g_{\mu\nu} = \frac{\partial \xi^a}{\partial x^\mu} \frac{\partial \xi^b}{\partial x^\nu} \eta_{ab}. \tag{15}$$

We define the triad field $e_a{}^\mu(x) = \frac{\partial \xi^a}{\partial x^\mu}$. This is the field which transforms vector fields written in arbitrary coordinates into vector fields relative to the locally orthonormal basis:

$$X^a(x) = e_\mu{}^a X^\mu(x).$$

In fact $g_{\mu\nu}$ is a tensor that also transforms according to

$$g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab}.$$

We require the physical world to be independent of the choice of coordinate system, which means it should be invariant under the local change of frame, which can be identified whith local Lorentz transformations. For a local Lorentz invariance we need to introduce a covariant derivative, and with it a gauge field called the spin connection. Thus we can compare derivatives of vectors in different coordinate systems. The covariant derivative reads:

$$D_\mu X^a = \partial_\mu X^a + \omega_\mu{}^a{}_b X^b.$$

Where $\omega_\mu{}^a{}_b$ is the spin connection. With respect to general coordinate transformations (where we now use indices μ, ν in stead of a, b.) the connection $\omega_\mu{}^a{}_b$ is replaced by the Christoffel symbols $\Gamma^\nu{}_{\mu\rho}$.

$$D_\mu X^\nu = \partial_\mu X^\nu + \Gamma^\nu{}_{\mu\rho} X^\rho. \tag{16}$$

We can explicitly relate $\Gamma^\nu{}_{\mu\rho}$ to $\omega_\mu{}^b{}_a$ and $e_\mu{}^a$ by demanding

$$D_\mu X^a = e_\nu{}^a D_\mu X^\nu \tag{17}$$

which leads to

$$\Gamma^{\nu}{}_{\mu\rho} = e_{\nu}{}^{a}(\partial_{\mu} e_{\rho}{}^{a} + \omega_{\mu}{}^{b}{}_{a} e_{\rho}{}^{b}). \tag{18}$$

In general relativity the connection $\Gamma^{\nu}{}_{\mu\rho}$ is assumed to be metric compatible and therefore symmetric. The connection we obtain in this way is referred to as the Levi-Civita connection and the fact that it should be symmetric gives the zero torsion condition:

$$T^{a}{}_{\mu\nu} = \partial_{\mu} e_{\nu}{}^{a} - \partial_{\nu} e_{\mu}{}^{a} + \omega_{\mu}{}^{a}{}_{b} e_{\nu}{}^{b} - \omega_{\nu}{}^{a}{}_{b} e_{\mu}{}^{b} = 0. \tag{19}$$

This is in fact a constraint equation, implying that the ω-connections depend on the e's. From now on we switch to the vector notation of the Lorentz group $SO(2,1)$, where, in stead of the generators J^{ab}, we use $J_a = \frac{1}{2}\epsilon_{abc} J^{bc}$. The spin connection now reads

$$\omega_{\mu}{}^{a}{}_{b} = \epsilon^{a}{}_{bc} \omega_{\mu}{}^{c}. \tag{20}$$

The Riemann curvature tensor for the $SO(2,1)$ gauge field is given by:

$$R^{a}{}_{\mu\nu} = \partial_{\mu} \omega_{\nu}{}^{a} - \partial_{\nu} \omega_{\mu}{}^{a} + \epsilon^{a}{}_{bc} \omega_{\mu}{}^{b} \omega_{\nu}{}^{c} \tag{21}$$

With help of the Riemann curvature tensor we construct the Einstein-Hilbert action:

$$S_{EH} = \int d^3x \epsilon^{\mu\nu\rho} e_{\rho a} R^{a}{}_{\mu\nu} \tag{22}$$

Variation of this action with respect to the gauge fields should yield the Einstein equations.

3.2 The Chern-Simons action for gauge group $ISO(2,1)$

3.2.1 $ISO(2,1)$ and its algebra

In this subsection we will be comparing the previous discussion to the Chern-Simons theory with gauge group $ISO(2,1)$. To write down the Chern-Simons action with $ISO(2,1)$ gauge group we need to know the algebra with its commutation relations and an invariant bilinear form as we discussed in section 1. The group $ISO(2,1)$, the poincare group in 2+1 dimensions, is the semidirect product of the group of Lorentz transformations and the group of translations in 3 dimensions:

$$ISO(2,1) = SO(2,1) \ltimes T(3) \tag{23}$$

The generators of the group form an algebra with the following commutation relations:

$$\begin{aligned} [J_a, J_b] &= \epsilon_{ab}{}^{c} J_c \\ [J_a, P_b] &= \epsilon_{ab}{}^{c} P_c \\ [P_a, P_b] &= 0 \end{aligned} \tag{24}$$

Here the J_a are generators of the 3-dimensional Lorentz group in vector representation. The elements P_a are the generators of translations. The algebra

indices are raised and lowered with the Minkowski metric $\eta_{ab} = (-1, 1, 1)$ to make sure we are dealing with $ISO(2,1)$ in stead of $ISO(3)$.

Fortunately the group does have a nondegenerate invariant bilinear form, a form that commutes with all algebra elements. The expression $J^a P_a$ satisfies these conditions as we can check:

$$\begin{aligned}
[J^a P_a, J_b] &= J^a [P_a, J_b] + [J^a, J_b] P_a = \epsilon_{ab}{}^c (J^a P_c + \eta^{ad} J_c P_d) = 0 \\
[J^a P_a, P_b] &= J^a [P_a, P_b] + [J^a, P_b] P_a = \epsilon_{ab}{}^c \eta^{ad} P_c P_d = \epsilon^{ca}{}_b P_c P_a = 0
\end{aligned} \tag{25}$$

The inner product on the algebra can now be identified:

$$\begin{aligned}
\langle J_a, P_b \rangle &= \eta_{ab} \\
\langle J_a, J_b \rangle &= 0 \\
\langle P_a, P_b \rangle &= 0
\end{aligned} \tag{26}$$

We can now write down the gauge field A in this algebra basis:

$$A_\mu(x) = \omega_\mu{}^a(x) J_a + e_\mu{}^a(x) P_a \tag{27}$$

The choice of notation is not arbitrary as we will see when we explicitly calculate what the action looks like. We substitute this expression for the gauge field in the general action as given in (3):

$$\begin{aligned}
S_{CS} = \quad & \frac{k}{2} \int_M \Big[\langle J_a, P_b \rangle \omega^a \wedge d e^b + \langle P_a, J_b \rangle e^a \wedge d\omega^b \\
+ \quad & \frac{2}{3} \left(\frac{1}{2} \langle [P_a, J_b], J_c \rangle e^a \wedge \omega^b \wedge \omega^c \right. \\
+ \quad & \left. \frac{1}{2} \langle [J_a, P_b], J_c \rangle \omega^a \wedge e^b \wedge \omega^c + \frac{1}{2} \langle [J_a, J_b], P_c \rangle \omega^a \wedge \omega^b \wedge e^c \right) \Big] \\
= \quad & \frac{k}{2} \int_M \left(\omega^a \wedge d e_a + e^a \wedge d\omega_a + \epsilon_{abc} e^a \wedge \omega^b \wedge \omega^c \right) \\
= \quad & \frac{k}{2} \int_M \epsilon^{\mu\nu\rho} \left(e_{\mu a} (\partial_\nu \omega_\rho{}^a - \partial_\rho \omega_\nu{}^a + \epsilon^a{}_{bc} \omega_\nu{}^b \omega_\rho{}^c) \right) \\
= \quad & \int d^3x \epsilon^{\mu\nu\rho} e_{\rho a} R^a{}_{\mu\nu}
\end{aligned} \tag{28}$$

Here we have made use of partial integration in the first term, and assumed that the gauge field is zero at the boundary of the manifold M.
What we thus arrive at is nothing but the Einstein-Hilbert action on a 3-manifold. Making this identification, we find that the gauge field of local Lorentz transformations is indeed the spin-connection $\omega_\mu{}^a$ and the triad $e_\mu{}^a$ must be interpreted as the gauge field of translations!
This observation is of crucial importance and turns the theory of 2+1 dimensional gravity into a topological one. In gauge theory the field $e_\mu{}^a$ can take the value zero, whereas on the other hand, we required the triad to be invertible

in equation (15). This implies that in the gauge theory formulation, the base manifold M for general relativity cannot be a metric space. See Witten [1] for a more thorough discussion.

3.2.2 Further identification with 2+1 gravity

Let us point out what the equations of motion look like for the case of $ISO(2,1)$ gauge group.
Varying the action (28) with respect to the gauge fields ${e_\mu}^a$ and ${\omega_\mu}^a$ gives us the equations of motion:

$$\begin{aligned} {R_{\mu\nu}}^a &= \partial_\mu {\omega_\nu}^a - \partial_\nu {\omega_\mu}^a + {\epsilon^a}_{bc}{\omega_\mu}^b{\omega_\nu}^c = 0, \\ {T_{\mu\nu}}^a &= \partial_\mu {e_\nu}^a - \partial_\nu {e_\mu}^a + {\epsilon^a}_{bc}({\omega_\mu}^b{e_\nu}^c + {\omega_\nu}^b{e_\mu}^c) = 0. \end{aligned} \tag{29}$$

These equations exactly correspond to Einsteins' equations in vacuum, $R_{\mu\nu} = 0$ and the zero torsion condition $T_{\mu\nu} = 0$, which turns ${\Gamma^\mu}_{\nu\rho}$ into the levi-civita connection and thus gives a relation between ${e_\mu}^a$ and ${\omega_\mu}^a$.
Finally we show that the role of infinitesimal gauge transformations in the Chern-Simons gauge formulation is related to that of the diffeomorphisms together with local Lorentz transformations in the geometrical Einstein formulation: An infinitesimal gauge transformation is generated by a Lie-algebra valued parameter $\varepsilon = \rho^a P_a + \tau^a J_a$, where ρ^a and τ^a are infinitesimally small. We compute:

$$\begin{aligned} \delta A_\mu &= -D_\mu\varepsilon = -\partial_\mu\varepsilon - [A_\mu, \varepsilon] \\ &= -(\partial_\mu\rho^a + {\epsilon^a}_{bc}{\omega_\mu}^b\rho^c + {\epsilon^a}_{bc}{e_\mu}^b\tau^c)P_a - (\partial_\mu\tau^a + {\epsilon^a}_{bc}{\omega_\mu}^b\tau^c)J_a \end{aligned}$$

which yields

$$\delta {\omega_\mu}^a = -(\partial_\mu\tau^a + {\epsilon^a}_{bc}{\omega_\mu}^b\tau^c) \tag{30}$$

$$\delta {e_\mu}^a = -(\partial_\mu\rho^a + {\epsilon^a}_{bc}{\omega_\mu}^b\rho^c + {\epsilon^a}_{bc}{e_\mu}^b\tau^c) \tag{31}$$

The action is indeed invariant under these transformations as one is invited to verify.
The question is if these transformations coincide with the local Lorentz transformations and diffeomorphisms allowed in general relativity.
Let us compare the transformation laws to those of the diffeomorphism generated by a vector field $-v^\mu$:

$$\tilde{\delta} {\omega_\mu}^a = -v^\nu(\partial_\nu {\omega_\mu}^a - \partial_\mu {\omega_\nu}^a) - \partial_\mu(v^\nu {\omega_\nu}^a) \tag{32}$$

$$\tilde{\delta} {e_\mu}^a = -v^\nu(\partial_\nu {e_\mu}^a - \partial_\mu {e_\nu}^a) - \partial_\mu(v^\nu {e_\nu}^a) \tag{33}$$

If we take the vector field to agree with $\rho^a = v^\mu e_\mu{}^a$, and compare the terms in (31) which are proportional to ρ^a, the diffeomorphism, there is still a difference between the transformations (31) and (33):

$$\tilde{\delta} e_\mu{}^a - \delta e_\mu{}^a = -v^\nu (D_\nu e_\mu{}^a - D_\mu e_\nu{}^a) + \epsilon^{abc} v^\nu \omega_{\nu b} e_{\mu c}. \tag{34}$$

The expression

$$D_\nu e_\mu{}^a - D_\mu e_\nu{}^a$$

is nothing but the torsion, which should vanish as the equations of motion tell us. What remains then is the term $\epsilon^{abc} v^\nu \omega_{\nu b} e_{\mu c}$, which corresponds to a local Lorentz transformation with parameter $\tau^a = v^\mu \omega_\mu{}^a$.
Similarly, the difference between the transformations of $\omega_\mu{}^a$ is

$$\tilde{\delta} \omega_\mu{}^a - \delta \omega_\mu{}^a = -v^\nu (D_\nu \omega_\mu{}^a - D_\mu \omega_\nu{}^a) + \epsilon^{abc} v^\nu \omega_{\nu b} \omega_{\mu c}. \tag{35}$$

Here the we have again chosen $\tau^a = v^\mu \omega_\mu{}^a$. Now part of the difference is given by the curvature $R^a = D_\nu \omega_\mu{}^a - D_\mu \omega_\nu{}^a$, becoming zero after imposing the equations of motion, and the other part again a local Lorentz transformation.

Our conclusion is that the Lorentz transformations and diffeomorphisms agree with the gauge transformations of $ISO(2,1)$ as long as we impose the equations of motion following from the Chern-Simons action.

3.3 The case of a nonzero cosmological constant

We can generalize the description of 2+1 gravity to a picture in which a cosmological constant is added to the theory. The cosmological constant appears in the Einstein Hilbert action as an extra term in the Riemann tensor, quadratic in the triad. The action should therefore contain a third power of this translational part of the gauge field. This would mean that the expression

$$\frac{1}{2} \langle [P_a, P_b], P_c \rangle (e^a \wedge e^b \wedge e^c) \tag{36}$$

should be nonzero. If we do not change the inner product on the algebra, this can be accomplished by changing the commutation relations (24) to

$$\begin{aligned} [J_a, J_b] &= \epsilon_{ab}{}^c J_c \\ [J_a, P_b] &= \epsilon_{ab}{}^c P_c \\ [P_a, P_b] &= -\lambda \epsilon_{ab}{}^c J_c \end{aligned} \tag{37}$$

Here $-\lambda$ is the cosmological constant, as we recognize after writing down the new Chern-Simons action:

$$S_{CS} = \frac{1}{2} \int_M \epsilon^{\mu\nu\rho} \left(e_{\mu a} (\partial_\nu \omega_\rho{}^a - \partial_\rho \omega_\nu{}^a) + \epsilon_{abc} e_\mu{}^a (\omega_\nu{}^b \omega_\rho{}^c - \frac{\lambda}{3} e_\nu{}^b e_\rho{}^c) \right) \tag{38}$$

The curvature tensor has changed to:

$$R_{\mu\nu}{}^{a} = \partial_\mu \omega_\nu{}^{a} - \partial_\nu \omega_\mu{}^{a} + \epsilon^{a}{}_{bc}\omega_\mu{}^{b}\omega_\nu{}^{c} - \frac{\lambda}{3} e_\nu{}^{b} e_\rho{}^{c} \tag{39}$$

Whereas, the torsion $T_{\mu\nu}{}^{a}$ does not change. Changing the commutation relations like we did above obviously changes the gauge group. In fact, starting off with $ISO(2,1)$, the commutation relations (37) can be identified to essentially being the generating algebra of the group $SO(3,1)$ or $SO(2,2)$, depending on whether the cosmological constant $-\lambda$ is positive or negative respectively. See the appendix for a description of the algebras of $SO(4-p,p)$

Thus we can conclude that adding a cosmological constant to the $ISO(2,1)$ gauge theory of gravity with Minkowskian signature changes the gauge group to $SO(3,1)$ in case the cosmological constant is positive, and $SO(2,2)$ if the cosmological constant is negative.
Besides these different groups for 2+1 dimensional gravity with Minkowskian signature, we can consider the case of Euclidean gravity, where we replace the gauge group $ISO(2,1)$ by the Euclidean group $ISO(3)$. Until now, the only thing that changes is the metric used to raise and lower algebra-indices. The commutation relations (24) with all indices down will be different in the two cases.
If in the Euclidean case the cosmological constant is allowed to take nonzero values, we are dealing with $SO(4)$ gauge theory ($-\lambda > 0$) or $SO(3,1)$ ($-\lambda < 0$).

3.4 Wigner contraction of the gauge algebra and the role of the coupling constant

3.4.1 The Wigner contraction

We can smoothen the relation between the different gauge-groups for the different values of the cosmological constant by redefining the generators of boosts in the algebra $SO(3,1)$: The conventional normalisation of these algebra-elements is such that:

$$[P_a, P_b] = \epsilon_{ab}{}^{c} J_c$$

where the Lorentz metric is used to raise and lower indices. To arrive at (37) we define:

$$\tilde{P}_a = \sqrt{-\lambda} P_a. \tag{40}$$

The commutation relations are now correct for both the $\lambda < 0$ and the $\lambda > 0$ case and one can interpret the $ISO(2,1)$ theory as the $\lambda \to 0$ limit. This contraction is called a Wigner contraction, see Gilmore [5].

3.4.2 A new basis for the algebra

An important observation is that the algebras of $SO(4-p,p)$ admit another nondegenerate invariant bilinear form than the one chosen in (26). As we shall

see, this leads to a Chern Simons action which is no longer equivalent to the Einstein-Hilbert action, but does yield the same equations of motion. The invariant is the operator $J^a J_a + P^a P_a$, leading to an inner product:

$$\begin{aligned} \langle J_a, J_b \rangle &= \delta_{ab} \\ \langle P_a, P_b \rangle &= \delta_{ab} \\ \langle J_a, P_b \rangle &= 0. \end{aligned} \tag{41}$$

One can calculate that the action becomes:

$$\begin{aligned} S = & \int_M d^3 z \epsilon^{\mu\nu\rho} \Big(\omega_{\mu a} (\partial_\nu \omega_\rho{}^a - \partial_\rho \omega_\nu{}^a + \frac{2}{3} \epsilon_{abc} \omega_\nu{}^b \omega_\rho{}^c) \\ & + \lambda e_{\mu a} (\partial_\nu e_\rho{}^a - \partial_\rho e_\nu{}^a) + 2\lambda \epsilon_{abc} \omega_\mu{}^a e_\nu{}^b e_\rho{}^c \Big) \\ = & \int_M d^3 z \epsilon^{\mu\nu\rho} (\omega_{\mu a} R^a{}_{\nu\rho} + \lambda e_{\mu a} T^a{}_{\nu\rho}) \end{aligned} \tag{42}$$

Surprisingly, variation of this different action leads again to the familiar equations of motion:

$$\begin{aligned} R^a &= 0 \\ T^a &= 0 \end{aligned} \tag{43}$$

We can clarify the role of the second bilinear form by changing to another algebra basis:

$$J^+_a = \frac{1}{2}(J_a + \frac{\tilde{P}_a}{\sqrt{-\lambda}}), \qquad J^-_a = \frac{1}{2}(J_a - \frac{\tilde{P}_a}{\sqrt{-\lambda}}) \tag{44}$$

satisfying commutation relations:

$$\begin{aligned} [J^+_a, J^+_b] &= \epsilon_{ab}{}^c J^+_c \\ [J^-_a, J^-_b] &= \epsilon_{ab}{}^c J^-_c \\ [J^+_a, J^-_b] &= 0. \end{aligned} \tag{45}$$

One can recognize two mutually commuting $SO(3)$ algebras when starting off with $SO(4)$, two $SO(2,1)$ algebras when starting off with $SO(2,2)$ and a the complexification of $SO(3)$ in the $SO(3,1)$ case. The two different inner products on the algebra correspond to the bilinear forms

$$J^{+a} J^+_a - J^{-a} J^-_a \tag{46}$$

for the inner product that leads to the Einstein-Hilbert action, and

$$J^{+a} J^+_a + J^{-a} J^-_a \tag{47}$$

for the second action. Thus the inner products are:

$$\begin{aligned} \langle J^+_a, J^+_b \rangle &= \frac{1}{2} \delta_{ab} \\ \langle J^-_a, J^-_b \rangle &= -\frac{1}{2} \delta_{ab} \\ \langle J^+_a, J^-_b \rangle &= 0 \end{aligned} \tag{48}$$

and

$$\begin{aligned}\langle J^+_a, J^+_b\rangle &= \frac{1}{2}\delta_{ab}\\ \langle J^-_a, J^-_b\rangle &= \frac{1}{2}\delta_{ab}\\ \langle J^+_a, J^-_b\rangle &= 0\end{aligned} \tag{49}$$

respectively.
One can write down the action in this new basis, where we introduce the new components of the gauge field:

$$\begin{aligned}A^{+\ a}_{\mu} &= \omega_{\mu}^{\ a} + \sqrt{-\lambda}e_{\mu}^{\ a}\\ A^{-\ a}_{\mu} &= \omega_{\mu}^{\ a} - \sqrt{-\lambda}e_{\mu}^{\ a}\end{aligned} \tag{50}$$

Writing down the action in this algebra basis leads us to:

$$\begin{aligned}S_{EH} = \frac{k}{4\sqrt{-\lambda}}\int_M \epsilon^{\mu\nu\rho}\Big[&\Big(2A^+_{\mu a}\partial_\nu A^{+\ a}_{\rho} + \frac{2}{3}\epsilon_{abc}A^{+\ a}_{\mu}A^{+\ b}_{\nu}A^{+\ c}_{\rho}\Big)\\ &-\Big(2A^-_{\mu a}\partial_\nu A^{-\ a}_{\rho} + \frac{2}{3}\epsilon_{abc}A^{-\ a}_{\mu}A^{-\ b}_{\nu}A^{-\ c}_{\rho}\Big)\Big]\end{aligned} \tag{51}$$

where one can see that the action has to be divided by a factor $\sqrt{-\lambda}$ to get the right correspondence with the Einstein-Hilbert action.
The second inner product gives:

$$\begin{aligned}S = \frac{k}{4}\int_M \epsilon^{\mu\nu\rho}\Big[&\Big(2A^+_{\mu a}\partial_\nu A^{+\ a}_{\rho} + \frac{2}{3}\epsilon_{abc}A^{+\ a}_{\mu}A^{+\ b}_{\nu}A^{+\ c}_{\rho}\Big)\\ &+\Big(2A^-_{\mu a}\partial_\nu A^{-\ a}_{\rho} + \frac{2}{3}\epsilon_{abc}A^{-\ a}_{\mu}A^{-\ b}_{\nu}A^{-\ c}_{\rho}\Big)\Big].\end{aligned} \tag{52}$$

Both actions are thus invariant under the $SO(4-p,p)$ gauge group, which we can now rewrite:

$$\begin{aligned}SO(4) &= SO(3)\times SO(3)\\ SO(3,1) &= SO(3,\mathbb{C})\\ SO(2,2) &= SO(2,1)\times SO(2,1)\end{aligned} \tag{53}$$

Notice however that the physically interesting limit $-\lambda \to 0$ is not yet very clear in this picture.

3.4.3 The coupling constant

The coupling constant k plays different roles for these different gauge groups. As we stated in section 2, this constant is in some cases restricted to certain values if we demand invariance under large gauge transformations. This depended on the homotopy group $\pi_3(G)$. For most compact semisimple groups,

$\pi_3(G) = \mathbb{Z}$ and k was restricted to integer multiples of $\frac{1}{4\pi}$. For the gauge groups, we are considering, we have different results. For instance, the gauge group $SO(2,2)$ is not compact and $\pi_3(SO(2,2)) = 0$. Thus k can take any value.

An interesting case is the case of $SO(4)$. We now find that

$$\pi_3(SO(3) \times SO(3)) = \mathbb{Z} \times \mathbb{Z} \tag{54}$$

We can conclude from this that the two terms in the action (51), which transform independently, can have different coupling constant coefficients, whereas the two independent $SO(3)$ gauge transformations can have independent winding numbers. However, to recover the Euclidean Einstein-Hilbert action with positive cosmological constant, the two couplings must coincide. Distinct couplings would lead to a mix of the two actions (51) and (52). One could argue that the theory admits this mix, because the two actions do lead to the same equations of motion and should therefore give equivalent theories. This question will be shortly referred to in section 7.
We did not yet until now discuss the role of the coupling constant in the case of zero cosmological constant. In the $ISO(3)$ and the $ISO(2,1)$ case we are again dealing with noncompact groups. The Minkowskian case is similar to $SO(2,2)$, because

$$\pi_3(SO(2,1)) = 0 \tag{55}$$

so that the winding number will be 0 too and the coupling constant can take any value.
For $ISO(3)$ we can always scale away any coupling constant by absorbing it in the triad $e_\mu{}^a$, because it appears in the action as a first power in every term. Thus for $ISO(3)$, it can again take any value. A same argument holds for $SO(3,1)$. Although it has a compact part $SO(3)$, the coupling constant can be scaled away by a normalisation of the 'boost' part of the gauge field, $e_\mu{}^a$. In the remaining part of this thesis, we will be focussing our attention mainly on the gauge group $ISO(3)$, for which a beautiful quantisation theory has been developed.

4 Phase space and poisson structures

As we already emphasized at the end of section 2, there are two ways of proceeding the investigation of the Chern Simons theories and developing quantum theories. The first was to directly quantise the canonical Poisson brackets following from the action and the second was to first impose constraints and divide out gauge symmetry, reducing the phase space to a relatively simple one.
In this section we will first describe the canonical Poisson structures of $ISO(3)$ without imposing constraints and dividing out gauge symmetries. After that we will discuss phase space identification and reduction from different points of view. A general description of the concepts of phase space as a symplectic manifold and the relation with Poisson structures is given first.
We introduce a single particle in the case of $ISO(3)$ gauge group, and discuss its phase space, the *coadjoint orbit*.
We will discuss Poisson structures and finally give a general discription of the Poisson structure for Chern Simons theory on a manifold $\Sigma \times \mathbb{R}$, where Σ has arbitrary genus and curvature sources are inserted in N punctures of this manifold. The Poisson structure will be determined by a *classical r-matrix*, related to a *bialgebra structure* of the gauge group algebra. The meaning of these words will become clear later on.
One should bear in mind that the descriptions given here have their own advantages when it comes to quantisation of the theory, that they are in fact preparations for different quantisation schemes, which we will describe in the section following the present section. In many books on quantisation only part of the theory concerns the actual quantisation. Many considerations are valid on a classical level, which one is often bound to forget when focussing on quantisation theory. To make a clear distinction between the classical part of the theory and the quantum-description, we stick to describing phase space in a classical way as long as possible, where now and then we might refer to arguments that come from theories of quantisation.
We will already mention that the description in terms of symplectic manifolds is a 'preparation' to the method of geometric quantisation, whereas the formalism in terms of classical r-matrices and bialgebra structures is useful when performing deformation quantisation, both methods will be explained in section 5.

4.1 Poisson brackets

For the purpose of describing phase space we first turn to the Hamiltonian formulation as we did in section 2. The manifold M is again specified to be of the form $\Sigma \times \mathbb{R}$ and we will restrict our attention to the case of $ISO(3)$ gauge group. The action (2.2.1) now becomes

$$S = \frac{k}{2} \int dt \int_\Sigma d^2z \epsilon^{ij} (-2e_i{}^a \partial_t \omega_j{}^a + e_0{}^a R_{ij}{}^a + \omega_0{}^a T_{ij}{}^a) \tag{56}$$

where we have used the notations:

$$
\begin{aligned}
R_{ij}{}^{a} &= (\partial_i \omega_j{}^{a} - \partial_j \omega_i{}^{a} + \epsilon^{a}{}_{bc}\omega_i{}^{b}\omega_j{}^{c}), \\
T_{ij}{}^{a} &= (\partial_i e_j{}^{a} - \partial_j e_i{}^{a} + \epsilon^{a}{}_{bc}(\omega_i{}^{b} e_j{}^{c} + \omega_j{}^{b} e_i{}^{c})).
\end{aligned} \tag{57}
$$

The spatial curvature can now be written as

$$
F_{ij} = R_{ij}{}^{a} J_a + T_{ij}{}^{a} P_a \tag{58}
$$

The constraints arising from the variation of the fields $e_0{}^{a}$ and $\omega_0{}^{a}$ are that all components of the curvature should vanish,

$$
\epsilon^{ij} R_{ij}{}^{a} = 0, \quad \epsilon^{ij} T_{ij}{}^{a} = 0. \tag{59}
$$

Besides this, we can investigate the constraint algebra after deriving the Poisson brackets.
The Poisson brackets turn out to be

$$
\begin{aligned}
\{\omega_i{}^{a}(z_1), e_j{}^{b}(z_2)\} &= \frac{2}{k}\epsilon^{ij}\eta^{ab}\delta^{(2)}(z_1 - z_2) \\
\{\omega_i{}^{a}(z_1), \omega_j{}^{b}(z_2)\} &= 0 \\
\{e_i{}^{a}(z_1), e_j{}^{b}(z_2)\} &= 0
\end{aligned} \tag{60}
$$

in analogy with the general case (13) And calculating the constraint algebra we arrive at:

$$
\begin{aligned}
\{T_a(z_1), T_b(z_2)\} &= \epsilon_{ab}{}^{c} T_c(z_1)\delta^{(2)}(z_1 - z_2) \\
\{T_a(z_1), R_b(z_2)\} &= \epsilon_{ab}{}^{c} R_c(z_1)\delta^{(2)}(z_1 - z_2) \\
\{R_a(z_1), R_b(z_2)\} &= 0
\end{aligned} \tag{61}
$$

in which we recognize the algebra of the gauge group, which is no surprise [1]. Notice that the role of the rotation generators is taken over by the Torsion components, and the Riemann tensor components behave as the translation generators in this isomorphism. Again this is in analogy with the section 2 case (14), where we also mentioned that when coupling particles to the gauge field, we find that the charges should equivalently obey the same Poisson algebra. We thus already know that the charges acting as sources of torsion should obey an algebra isomorphic to that of the rotation generators, whereas the sources of curvature act as translation generators.

4.2 Phase space and symplectic manifolds

In many cases, where one can develop a suitable Hamiltonian formulation, a useful, more geometric way of describing phase space is in terms of symplectic spaces. The ingredients needed to develop the description in this manner are:

1) A phase space, found by identifying the degrees of freedom in a theory, possibly subject to constraints.

$$\Gamma(x,p) = \mathcal{M}$$

In general, on phase space we will not yet have identified coordinates and momenta, this is why we will denote it as a manifold $\mathcal{M}$ rather than $\Gamma(x,p)$, although thinking of it as the space of coordinates and momenta is what we finally want to do. On this phase space one has functions which represent observables of the theory:

$$f \in F(\mathcal{M})$$

2) Another ingredient that can be very useful to give a Hamiltonian description of phase space is a symplectic form: A closed, nondegenerate, antisymmetric 2-form ω [1] that is defined everywhere on the phase space and provides a linear isomorphism

$$T_x\mathcal{M} \to T_x^*\mathcal{M} : X \mapsto X \lrcorner \omega \tag{62}$$

where $\lrcorner$ denotes the contraction of the vector field with the first entry of the 2-form. With help of the symplectic form, one can assign vector fields to every observable f. The Hamiltonian vector field X_f associated to f is defined by the equation:

$$X_f \lrcorner \omega - df = 0. \tag{63}$$

One can derive from this equation that a Hamiltonian vector field generates a symplectic action, i.e. an action that leaves the symplectic form invariant. This definition allows one to find the poisson brackets, which should satisfy:

$$\{g, f\} = X_f(g). \tag{64}$$

3) An important step in the procedure is the choice of a polarisation. This is essentially a choice of coordinates versus momenta. There is a canonical choice if the total phase space can be identified to be a cotangent bundle $T^*\mathcal{M}$. The base space $\mathcal{M}$ of this manifold can then be parametrised by the coordinates, whereas the basis coordinates of the cotangent vectors are identified with momenta. There is then a natural symplectic form, which will lead to commuting coordinates.
In many cases one will be able to find this local coordinate system (p_i, x^j), relative to which the symplectic form ω looks like:

$$\omega = dp_i \wedge dx^i. \tag{65}$$

Using this we can make the above equations more explicit and more recognisable:

$$X_f = X_f{}^{p_i} \frac{\partial}{\partial p_i} + X_f{}^{x^j} \frac{\partial}{\partial x^j} \tag{66}$$

$$X_f \lrcorner \omega - df = X_f{}^{p_i} dx^i - X_f{}^{x^j} dp_j - \frac{\partial f}{\partial p_i} dp_i - \frac{\partial f}{\partial x^j} dx^j = 0 \tag{67}$$

[1] This introduction of another ω variable should not lead to confusion

Thus we identify:

$$\begin{aligned} X_f{}^{p_i} &= \frac{\partial f}{\partial x^i} \\ X_f{}^{x^j} &= -\frac{\partial f}{\partial p_j} \end{aligned} \tag{68}$$

And thus (64) leads to the familiar Poisson brackets

$$\{f, g\} = \frac{\partial f}{\partial x^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial x^i} \tag{69}$$

The advantage of this formalism is that one immediately has a mapping from the Poisson algebra to the Lie algebra of Hamiltonian vector fields on which one can perform a canonical quantisation, by making use of the symplectic form. In the end one would like to map the Hamiltonian vector fields to operators which satisfy the quantum condition, which we will describe later on (133). We refer to [6] for a detailed discussion.

4.3 Adding a particle: Coadjoint orbits

We will now describe how a particle can be added to the theory. Particles will, as described at the end of section 2, be introduced by inserting them at the punctures of the Riemann surface Σ, where they act as delta sources for curvature. Their internal phase space is coupled to the phase space of the gauge field through minimal coupling, which in the particle Lagrangian can be achieved by letting

$$d_t \to D_t = d_t + e_t + \omega_t. \tag{70}$$

An internal phase space of this particle must be described by gauge charges that under Poisson brackets generate the algebra isomorphic to the Lie algebra. This is a consequence of the equation (12) and the above Poisson relations (61), as already remarked at the end of section 2.2.4.
An elegant way of constructing such a particle Lagrangian, which results in an explicit parametrisation of internal phase space is the coadjoint orbit method of Kirrilov and Kostant. It is a rather geometrical procedure, that in general works for any semi-simple group, but it can also be applied to some semi-direct product groups, such as $ISO(3)$, see [7] and [8].
A great advantage of the procedure is the fact that it identifies a symplectic structure on the internal particle phase space. Here we again use an argument from geometric quantisation theory. As we mentioned before, the method of geometric quantisation, which we will describe further on, is actually applicable in particular to problems in which phase space turns out to have a symplectic structure.

4.3.1 Poisson algebra of functions on $\mathfrak{g}^*$

In general, a coalgebra $\mathfrak{g}^*$ being a vector space, its tangent space $T(\mathfrak{g}^*)$ can be identified with the base space itself, $\mathfrak{g}^*$, so that the cotangent space $T^*(\mathfrak{g}^*)$ can be identified with the dual of $\mathfrak{g}^*$, the algebra $\mathfrak{g}$. We will make this more explicit:

For $\xi, \eta \in \mathfrak{g}^*$ and $f \in F(\mathfrak{g}^*)$ we define the linear bijection $\xi : \mathfrak{g}^* \to T_\xi(\mathfrak{g}^*)$ denoted by $\eta \mapsto \eta_\xi$ where:

$$\eta_\xi(f) = \frac{d}{dt} f(\xi + t\eta)|_{t=0} \tag{71}$$

A cotangent element df_ξ is now identified with an algebra element by demanding:

$$\langle \eta, df_\xi \rangle = \eta_\xi(f) = \frac{d}{dt} f(\xi + t\eta)|_{t=0} \tag{72}$$

Conversely, $x \in \mathfrak{g}$ can be viewed as an element of $F(\mathfrak{g}^*)$ by the identification

$$x(\xi) = \langle \xi, x \rangle. \tag{73}$$

This means for the element $dx \in T^*(\mathfrak{g}^*)$, it is again an algebra element:

$$\langle (dx)_\xi, \eta \rangle = \frac{d}{dt} \langle x, \xi + t\eta \rangle|_{t=0} = \langle x, \eta \rangle. \tag{74}$$

One can show that the function $\{f, g\}$ given by

$$\{f, g\}(\xi) = \langle [(df)_\xi, (dg)_\xi], \xi \rangle \tag{75}$$

defines a poisson structure on $\mathfrak{g}^*$. For elements x, y, regarded as functions on $\mathfrak{g}^*$, we thus find

$$\{x, y\}(\xi) = \langle [x, y], \xi \rangle \tag{76}$$

so

$$\{x, y\} = [x, y]. \tag{77}$$

This bracket is referred to as the Kirillov-Kostant bracket. We can regard basis elements x_i of $\mathfrak{g}$ as coordinate functions on $\mathfrak{g}^*$ by writing ξ relative to the cobasis t^j, where $\langle t^j, x_i \rangle = \delta^j{}_i$:

$$\xi = \xi_j t^j, \quad \langle x_i, \xi \rangle = \xi_i \tag{78}$$

With this identification the Poisson algebra of the coordinate functions on the coalgebra is isomorphic to the Lie algebra with its Lie bracket.

4.3.2 The coadjoint orbit

Now that we have discovered that the vector space of the coalgebra is equipped with a poisson bracket, we can ask ourselves the question what symplectic manifold can be associated with it, which could be suitable for quantisation. A way in which we can construct a symplectic manifold is by picking an element of $\mathfrak{g}^*$ and acting on it with a symplectic transformation. The submanifold of $\mathfrak{g}^*$ thus found is consequently a symplectic manifold. Symplectic acions are generated by Hamiltonian vector fields, as described by (63). Specifying this for the vector space $\mathfrak{g}^*$ as described above, we find the action of Hamiltonian vector fields is:

$$(X_f)_\xi(g)_\xi = \{g, f\}(\xi) \tag{79}$$

and for functions $x, y \in \mathfrak{g}$ we find:

$$(X_x)_\xi(y)_\xi = \{y, x\}(\xi) = \langle [y, x], \xi \rangle \tag{80}$$

This corresponds to the definition of the coadjoint action of algebra elements on ξ:

$$-\langle [x, y], \xi \rangle = -\langle ad_x(y), \xi \rangle = -\langle y, ad_x^*(\xi) \rangle \tag{81}$$

Thus the generating action is the coadjoint action of elements in the algebra $\mathfrak{g}$. This means the subspace of $\mathfrak{g}^*$ which is a symplectic space is found by the coadjoint action of the group element g^{-1} if g is a group element generated by $x \in \mathfrak{g}$. Of course all algebra elements generate symplectic actions, so that we can identify the *symplectic leaf* corresponding to a chosen element $\xi \in \mathfrak{g}^*$ to be its *coadjoint orbit* of ξ under the group action.

4.3.3 The particle Lagrangian

The construction of the corresponding Lagrangian which yields the correct Poisson brackets requires the existence of a locally defined *canonical one-form* θ_ξ on phase space (which is the coadjoint orbit), for which we have that $-d\theta_\xi = \omega_\xi$. This relation will in general not be valid globally. In fact a condition for a symplectic space to be quantizable is the *integrability condition*, which means that the symplectic form should be an integral element of the second cohomology group. We will not go into this matter here, but refer to Woodhouse [6]. If a one-form θ_Y is found, one can define a first order Lagrangian by

$$Ldt = \theta(\zeta) - H(\zeta)dt \tag{82}$$

where ζ are coordinates parametrising phase space, so that the action becomes

$$S = \int \Big(\theta_\xi, \frac{d}{dt}\Big) dt \tag{83}$$

where we have ignored the Hamiltonian $H(\zeta)$ which does not contribute to the definition of poisson brackets. The large brackets in the action denote the pairing between the vector fields (in this case $\frac{d}{dt}$) and one-forms. We can clarify the picture by giving the example of the canonical one-form for a cotangent space with coordinates x^i and momenta p_i. In this case one can check that the canonical one-form

$$\theta = p_i dx^i \tag{84}$$

defines the familiar kinetical part of the Lagrangian.

A well studied manifold related to this problem is the phase space of a particle moving on the group G. It has a canonical symplectic structure ω, because it can be identified with the cotangent bundle T^*G, and has a canonical one-form θ. The reduction of this manifold by dividing out by symmetries of the group action turns out to be isomorphic to a coadjoint orbit of the covector one started off with, which is still a symplectic manifold. On the orbit the canonical one-form is given by:

$$\theta_\xi(g, \xi) = \langle \xi, (g^{-1}dg) \rangle \tag{85}$$

where the element $g^{-1}dg$ is a one-form with values in $\mathfrak{g}$.

4.3.4 The $ISO(3)$ case

The internal phase space of the particle is now parametrised by coordinates ξ^a, components of the coalgebra vector. In the $ISO(3)$ case we specify an internal state in this way by $\xi^* = j^a J_a^* + p^a P_a^*$, where the j^a and p^a are now the coordinates on the coadjoint orbit. This corresponds to the algebra element

$$\xi = p^a J_a + j^a P_a, \tag{86}$$

which identifies the curvature sources. Invariants of the coadjoint orbits are the length of the vector $p^a p_a = \mu^2$ and the inner product $p^a j_a = \mu s$. We can identify the first relation with a mass-shell condition, which means μ corresponds to a Euclidean mass. We identify s with spin.
For $\mu > 0$, on the coadjoint orbit, which we shall denote by $\mathcal{O}_{\mu s}$, the p^a describe a 2-sphere and in each point the relation $p^a j_a = \mu s$ leaves 2 degrees of freedom for the j^a, so geometrically the orbit looks like a cotangent bundle of a 2-sphere with radius μ.
On the $\mathcal{O}_{\mu s}$ the Kirillov-Kostant bracket in fact by construction reproduces the algebra isomorphic to the Lie algebra.

$$\begin{aligned} \{j_a, j_b\} &= \epsilon_{ab}{}^c j_c \\ \{j_a, p_b\} &= \epsilon_{ab}{}^c p_c \\ \{p_a, p_b\} &= 0 \end{aligned} \tag{87}$$

Indeed there is a correct correspondence between the poisson algebra of the j_a, p_a and the poisson algebra with the curvature components T^a, R^a, as we see by reconsidering equations (12), (58) and (61).
The total phase space of the theory is normally given by the tensor product of the phase space of the particles and the phase space of the gauge fields. With the above construction we can make an identification.
The phase space of the gauge fields in the absence of handles of the surface is a conjugacy class determined by the holonomy around a puncture in which the curvature is nonzero, as described in section 2. The phase space of the particle is the coadjoint orbit $\mathcal{O}_{\mu s}$, which is closely related to a conjugacy class. The orbit of j^a and p^a is in fact determined by the action:

$$\begin{aligned} g : \exp(j^a P_a + p^a J_a) &\mapsto \exp(Co_{g^{-1}}(j^a P_a + p^a J_a)) \\ &= \exp(Ad_g(j^a P_a + p^a J_a)) \\ &= g \exp(j^a P_a + p^a J_a) g^{-1} \end{aligned} \tag{88}$$

which is the same as conjugation of the element $\exp(\int_\sigma F d\sigma)$, if we identify orbits μ modulo 2π.

On a classical level, we thus have a description for the case of a single particle. However, phase space is not well-described by the coadjoint orbits when we consider multiparticle systems.
The largest problem arises when taking into account the multiplication rule (8) for combining different noncontractible loops into one loop. We remind the

reader that this multiplication rule was grouplike, i.e. the group structure of phase space is a natural feature.
On a two-particles level this would for instance mean that combining two noncontractible loops around the two different particles into a single noncontractible loop would be an operation, usually referred to as *fusion* that maps the phase space of functions on two copies of the group to a space of functions on one copy of the group, this single group element being obtained by the usual group multiplication of the two one started off with. A crucial point here is the compatibility with this action one demands from the Poisson bracket. Unfortunately the Kirillov-Kostant construction of the Poisson bracket for a coadjoint orbit does not obey this compatibility relation, in which one recognises the difference between a coadjoint orbit and the conjugacy class structure of the phase space, the latter being the more correct point of view for considering multi-particle sectors.

4.4 Poisson-Lie groups, Hopf algebra's and bialgebra structures

We have seen that the underlying group structure of the reduced phase space is a natural feature of Chern Simons theory. To tackle this problem, in this subsection we introduce a few concepts of a formalism in which one is finally able to define Poisson brackets on the reduced phase space. The beginning of this subsection to some extent will be a sequence of definitions and introductions of concepts, necessary to formalise our discussion. For more details, we refer to Chari and Pressley [9].

4.4.1 Poisson-Lie groups

As we argued above, we must focus our attention on the properties of phase spaces that are grouplike, that is, the phase space is a Lie group G and so the space of functions on phase space is:

$$F(\mathcal{M}) = F(G) \tag{89}$$

If this algebra is equipped with a Poisson bracket, the main difference with respect to a more trivial phase space is the restriction that this Poisson bracket must be compatible with the group action. This is the case if the group action is a poisson map, of which we will give the definition now:
A smooth map $F : M \to N$ between the two manifolds M and N, both equipped with a Poisson structure, is called a Poisson map if:

$$\{f_1, f_2\}_N \circ F = \{f_1 \circ F, f_2 \circ F\}_N \tag{90}$$

If we translate this to the case in which the map is the usual group action which we denote by,

$$m \ : \ G \otimes G \to G \ : \ (g_1, g_2) \mapsto g_1 g_2 \tag{91}$$

we need the definition of the tensor product space Poisson bracket: If M and N are Poisson manifolds (i.e. they are equipped with a Poisson structure), the natural Poisson bracket on $M \times N$ is defined as

$$\{f_1, f_2\}_{M\times N}(x, y) = \{f_1(., y), f_2(., y)\}_M(x) + \{f_1(x, .), f_2(x, .)\}_N(y) \tag{92}$$

for $x \in M$ and $y \in N$.
Applying this to the case where $M \times N = G \times G$ we can now explicitly write down the condition following from demanding the group action to be a Poisson map:

$$\{f_1, f_2\}(g_1 g_2) = \{f_1 \circ L_{g_1}, f_2 \circ L_{g_1}\}(g_2) + \{f_1 \circ R_{g_2}, f_2 \circ R_{g_2}\}(g_1) \tag{93}$$

By L_g and R_g we mean the left and right actions of the group element g respectively. If this condition is satisfied for a Poisson bracket on the algebra of functions on the group $F(G)$, G is called a Poisson-Lie group.

4.4.2 Hopf algebras, the general definition

The algebra of functions on a group, which must satisfy certain relations due to the underlying group structure is a natural example of an object called a *Hopf algebra*, if it is equipped with some additional features. We will first give the general definition of a Hopf algebra and then show how the algebra of functions on a group G can be identified to be one:

A Hopf algebra over a commutative ring k is a k-module $\mathcal{A}$ such that:

i) $\mathcal{A}$ is a bialgebra, i.e. it is both an algebra and a coalgebra over k.

Here we will give a short explanation on the terms algebra and coalgebra:
By an algebra we mean that the k-module $\mathcal{A}$ is equipped with a multiplication map, that is a map $\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, together with a unit map, $\eta : k \to \mathcal{A}$, such that the order in the mappings expressed by the following diagrams is of no significance, i.e. these diagrams 'commute':

$$\begin{array}{ccc} \mathcal{A} \otimes k & \xrightarrow{\mathbb{1}\otimes\eta} & \mathcal{A} \otimes \mathcal{A} \\ \simeq \downarrow & & \downarrow \mu \\ \mathcal{A} & \xrightarrow[\mathbb{1}]{} & \mathcal{A} \end{array} \qquad \begin{array}{ccc} k \otimes \mathcal{A} & \xrightarrow{\eta\otimes\mathbb{1}} & \mathcal{A} \otimes \mathcal{A} \\ \simeq \downarrow & & \downarrow \mu \\ \mathcal{A} & \xrightarrow[\mathbb{1}]{} & \mathcal{A} \end{array} \tag{94}$$

$$\begin{array}{ccc} \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} & \xrightarrow{\mu\otimes\mathbb{1}} & \mathcal{A} \otimes \mathcal{A} \\ \mathbb{1}\otimes\mu \downarrow & & \downarrow \mu \\ \mathcal{A} \otimes \mathcal{A} & \xrightarrow[\mu]{} & \mathcal{A} \end{array} \tag{95}$$

For the k-module to be a coalgebra, it must be equipped with the comultiplication map $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ and the counit $\epsilon : \mathcal{A} \to k$, for which the following diagrams commute:

$$\begin{array}{ccc} \mathcal{A} \otimes k & \xleftarrow{\mathbb{1}\otimes\epsilon} & \mathcal{A} \otimes \mathcal{A} \\ \simeq \uparrow & & \uparrow \Delta \\ \mathcal{A} & \xleftarrow[\mathbb{1}]{} & \mathcal{A} \end{array} \qquad \begin{array}{ccc} k \otimes \mathcal{A} & \xleftarrow{\epsilon\otimes\mathbb{1}} & \mathcal{A} \otimes \mathcal{A} \\ \simeq \uparrow & & \uparrow \Delta \\ \mathcal{A} & \xleftarrow[\mathbb{1}]{} & \mathcal{A} \end{array} \tag{96}$$

$$\begin{array}{ccc} \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A} & \xleftarrow{\Delta\otimes\mathbb{1}} & \mathcal{A}\otimes\mathcal{A} \\ {\scriptstyle \mathbb{1}\otimes\Delta}\uparrow & & \uparrow{\scriptstyle \Delta} \\ \mathcal{A}\otimes\mathcal{A} & \xleftarrow[\Delta]{} & \mathcal{A} \end{array} \tag{97}$$

A bialgebra is thus equipped with all these maps $(\mu, \Delta, \eta, \epsilon)$.
ii) The comultiplication $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ and the counit $\epsilon : \mathcal{A} \to k$ are homomorphisms of algebras

iii) The multiplication $\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ and the unit $\eta : k \to \mathcal{A}$ are homomorphisms of coalgebras.

iv) There is a bijective k-module map $S^{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ such that the now following diagrams commute:

$$\begin{array}{ccc} \mathcal{A}\otimes\mathcal{A} & \xrightarrow{S\otimes\mathbb{1}} & \mathcal{A}\otimes\mathcal{A} \\ {\scriptstyle \Delta}\uparrow & & \downarrow{\scriptstyle \mu} \\ \mathcal{A} & \xrightarrow[\eta\circ\epsilon]{} & \mathcal{A} \end{array} \qquad \begin{array}{ccc} \mathcal{A}\otimes\mathcal{A} & \xrightarrow{\mathbb{1}\otimes S} & \mathcal{A}\otimes\mathcal{A} \\ {\scriptstyle \Delta}\uparrow & & \downarrow{\scriptstyle \mu} \\ \mathcal{A} & \xrightarrow[\eta\circ\epsilon]{} & \mathcal{A} \end{array} \tag{98}$$

S is called the antipode.
With all these structures, a Hopf algebra can be characterised by a set $(\mathcal{A}, \mu, \Delta, \eta, \epsilon, S)$.

4.4.3 The Hopf algebra of functions on a group $F(G)$ and the group Hopf algebra $\mathbb{C}(G)$

Now that we know the general definition of a Hopf algebra, we can proceed by showing how $F(G)$ can be regarded as a Hopf algebra. The algebra of functions on a finite group inherits a natural Hopf algebra structure $(F(G), m, \Delta, \eta, \epsilon, S)$ from the group structure, given by:
i) The pointwise multiplication:

$$m(f_1 \otimes f_2)(g) = f_1(g) f_2(g)$$

ii) The comultiplication induced by group multiplication:

$$\Delta f(g_1 \otimes g_2) = f(g_1 g_2)$$

iii) The unit:

$$\eta(\lambda)(g) = \lambda$$

iv) The co-unit:

$$\epsilon f = f(e)$$

v) The antipode:

$$S(f(g)) = f(g^{-1})$$

An important property of a Hopf algebra which we will use further on, is that its dual $\mathcal{A}^*$ is again a Hopf algebra. There is a natural dual for the algebra of

functions on the group, consisting of the *group Hopf algebra* $\mathbb{C}(G)$, the algebra of polynomials of group elements. The pairing $\langle , \rangle : \mathcal{A} \otimes \mathcal{A}^* \to k$ is obviously given by

$$\langle f, g \rangle = f(g) \tag{99}$$

where $f \in F(G), g \in \mathbb{C}(G)$. For a dual algebra $\mathcal{A}^*$, one demands that the dual of the multiplication map on $\mathcal{A}$ is the same as comultiplication on $\mathcal{A}^*$ with respect to this pairing and vice-versa. Bearing this in mind, it is now not difficult to determine the Hopf algebra structure on $F(G)^* \simeq \mathbb{C}(G)$:
i) Usual group multiplication:

$$m(g_1 \otimes g_2) = g_1 g_2$$

ii) The comultiplication:

$$\Delta g = g \otimes g$$

iii) The unit:

$$\eta(\lambda) = \lambda e$$

iv) The co-unit:

$$\epsilon g = 1$$

v) The antipode:

$$S(g) = g^{-1}$$

4.4.4 Another relevant Hopf algebra: The universal enveloping algebra

We will now define a Hopf algebra that will be important in the rest of our story about dualities and Poisson structures, the universal enveloping algebra (UEA): The universal enveloping algebra is denoted by $U(\mathfrak{g})$ and its definition is that it consists of all tensor product spaces of the algebra $\oplus_n \mathfrak{g}^{\otimes n}$ divided out by the equivalence relation $XY - YX = [X, Y]$, identifying the commutator. The Universal enveloping algebra is again a Hopf algebra, which means it is, besides the above properties, equipped with a unit $\mathbb{1}$ and with the following defining maps:
i) The multiplication:

$$m(X \otimes Y) = XY$$

ii) The comultiplication:

$$\Delta(X) = X \otimes \mathbb{1} + \mathbb{1} \otimes X$$

iii) The unit:

$$\eta(\lambda) = \lambda \mathbb{1}$$

iv) The counit:

$$\epsilon(X) = 0$$

v) The antipode:

$$S(X) = -X$$

In addition to this definition we prove that the coproduct satisfies the condition $\Delta(XY - YX) = \Delta([X, Y])$:

$$\begin{aligned}\Delta(XY - YX) &= \Delta(X)\Delta(Y) - \Delta(Y)\Delta(X) \\ &= (X \otimes \mathbb{1} + \mathbb{1} \otimes X)(Y \otimes \mathbb{1} + \mathbb{1} \otimes Y) \\ &\quad -(Y \otimes \mathbb{1} + \mathbb{1} \otimes Y)(X \otimes \mathbb{1} + \mathbb{1} \otimes X) \\ &= [X, Y] \otimes \mathbb{1} + \mathbb{1} \otimes [X, Y] \\ &= \Delta([X, Y]) \end{aligned} \tag{100}$$

4.4.5 General picture of Poisson-Hopf algebras and co-Poisson-Hopf algebras

The above definition of Hopf algebra structures can be extended to the case which is of interest for us: The case in which in addition to the usual structure, a Poisson structure is added to the Hopf algebra. We will again first give a general definition of a Poisson-Hopf algebra and its dual, the co-Poisson-Hopf algebra, without specifying to the case of functions on a group.
The general definition for a Poisson-Hopf algebra is:
$\mathcal{A}$ is a Hopf algebra $(\mathcal{A}, \eta, \mu, \epsilon, \Delta, S)$ equipped with a skew-symmetric map $\{\ ,\ \}_{\mathcal{A}} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ satisfying two identities:
The Jacobi identity:

$$\{a_1, \{a_2, a_3\}_{\mathcal{A}}\}_{\mathcal{A}} + \{a_3, \{a_1, a_2\}_{\mathcal{A}}\}_{\mathcal{A}} + \{a_2, \{a_3, a_1\}_{\mathcal{A}}\}_{\mathcal{A}} = 0 \tag{101}$$

and the Leibniz identity:

$$\{a_1 a_2, a_3\}_{\mathcal{A}} = \{a_1, a_3\}_{\mathcal{A}} a_2 + a_1 \{a_2, a_3\}_{\mathcal{A}} \tag{102}$$

for all $a_1, a_2, a_3 \in \mathcal{A}$, together with the compatibility condition:

$$\{\Delta(a_1), \Delta(a_2)\}_{\mathcal{A}\otimes\mathcal{A}} = \Delta(\{a_1, a_2\}_{\mathcal{A}}) \tag{103}$$

where

$$\{a_1 \otimes a_1', a_2 \otimes a_2'\}_{\mathcal{A}\otimes\mathcal{A}} = \{a_1, a_2\}_{\mathcal{A}} \otimes \mu(a_1', a_2') + \mu(a_1, a_2) \otimes \{a_1', a_2'\}_{\mathcal{A}} \tag{104}$$

We can again turn to the dual picture now. Where we regarded the comultiplication in the dual Hopf algebra $\mathcal{A}^*$ as the mapping dual to the multiplication on the Hopf algebra $\mathcal{A}$, we can now introduce a mapping dual to the Poisson bracket, which is called the *Poisson co-bracket*. One can indeed think of it as a structure on the dual Hopf algebra $\mathcal{A}^*$, but to emphasize that one can apply a co-Poisson structure to an arbitrary Hopf algebra, we will define it on an arbitrary Hopf algebra $\mathcal{B}$.
The dual picture of a Poisson bracket on a Hopf algebra is a Poisson co-bracket on a Hopf algebra $\mathcal{B}$, which is a map $\delta : \mathcal{B} \to \mathcal{B} \otimes \mathcal{B}$ satisfying the co-Jacobi identity:

$$\sum_{cycl} (\delta \otimes \mathbb{1})(\delta b) = 0 \quad \forall b \in \mathcal{B}. \tag{105}$$

Here by the cyclic sum we mean the sum over the three cyclic permutations of the factors in $\mathcal{B}\otimes\mathcal{B}\otimes\mathcal{B}$ resulting from the operator $(\delta\otimes\mathbb{1})\delta$. The second identity that must be satisfied is the co-Leibniz identity:

$$(\Delta\otimes\mathbb{1})\delta = (\mathbb{1}\otimes\delta)\Delta + \sigma_{23}(\delta\otimes\mathbb{1})\Delta \tag{106}$$

Where $\sigma_{23}(b_1\otimes b_2\otimes b_3) = b_1\otimes b_3\otimes b_2$ for $b_1, b_2, b_3 \in \mathcal{B}$. The compatibility condition in the dual picture becomes:

$$\delta(b_1b_2) = \delta(b_1)\Delta(b_2) + \Delta(b_1)\delta(b_2) \qquad \forall b_1, b_2 \in \mathcal{B}. \tag{107}$$

4.4.6 Hopf algebra dualities and $F(G)$ versus $U(\mathfrak{g})$

All properties of dual Hopf algebra's $\mathcal{A}^*$ are in the finite dimensional case defined by the Hopf algebra $\mathcal{A}$ itself as one can show. As we mentioned before, the coproduct on the dual Hopf algebra $\mathcal{A}^*$ is in that case determined by the action on the Hopf algebra $\mathcal{A}$ through:

$$\langle\Delta(b), a_1\otimes a_2\rangle = \langle b, m(a_1\otimes a_2)\rangle \quad \forall b\in\mathcal{A}^*, \text{and} a_1, a_2\in\mathcal{A} \tag{108}$$

where $\langle,\rangle$ denotes the pairing between the Hopf algebra $\mathcal{A}$ and its dual $\mathcal{A}^*$ so essentially, Δ on $\mathcal{A}^*$ could be denoted by m^*. The dual of the Hopf algebra of functions on a Lie group G is often identified to be the universal enveloping algebra associated with the Lie algebra $\mathfrak{g}$ corresponding to the group G (see Chari and Pressley [9]), in contrast with the previous discussion in which we used the duality between $F(G)$ and $\mathbb{C}(G)$. The difference of course lies in a different choice of pairing $\langle,\rangle$. We can make this duality more explicit by giving the pairing between two dual elements, found by assigning left invariant differential operators to every element of $U(\mathfrak{g})$: Let $X\in U(\mathfrak{g}), f\in F(G)$ and $g\in G$.

$$\langle X, f\rangle(g) = \frac{d}{dt} f(e^{Xt}g)|_{t=0} \tag{109}$$

This pairing induces injective maps from $F(G)$ and $U(\mathfrak{g})$ to the duals $U(\mathfrak{g})^*$ and $F(G)^*$ respectively. The images of these maps are at least subspaces of the complete duals. The pointwise multiplication on $F(G)$ and the coproduct on $U(\mathfrak{g})$ are consistently dual to each other, as shown below:

$$\begin{aligned}\langle\Delta(X), f\otimes h\rangle(\Delta(g)) &= \langle(X\otimes\mathbb{1}+\mathbb{1}\otimes X), f\otimes h\rangle(g\otimes g)\\ &= \langle\mathbb{1}, f\rangle(g)\langle X, h\rangle(g) + \langle X, f\rangle(g)\langle\mathbb{1}, h\rangle(g)\\ &= f(g)\Big(\frac{d}{dt}h(e^{tX}g)|_{t=0}\Big) + \Big(\frac{d}{dt}f(e^{tX}g)|_{t=0}\Big)h(g)\end{aligned} \tag{110}$$

whereas

$$\begin{aligned}\langle X, m(f\otimes h)\rangle(g) &= \frac{d}{dt}\Big(f(e^{Xt}g)h(e^{Xt}g)\Big)|_{t=0}\\ &= f(g)\Big(\frac{d}{dt}h(e^{tX}g)|_{t=0}\Big) + \Big(\frac{d}{dt}f(e^{tX}g)|_{t=0}\Big)h(g)\end{aligned} \tag{111}$$

4.4.7 Lie bialgebra structures and r-matrices

Untill now we have mainly introduced new concepts, without actually making any direct progress in our problem of finding a Poisson structure for the phase space of functions on a group. However, our new concepts do provide a promising way of approaching this problem in an abstract manner.
The key point is the observation that the universal enveloping algebra can be equipped with an object that can be interpreted as a co-Poisson structure. On the space of functions on a group, this would, with the above duality, give a natural Poisson structure.
For this observation, we need the additional concepts of a *Lie bialgebra structure*, which can be described in terms of a *classical r-matrix*.
A Lie bialgebra structure is a skew symmetric linear map

$$\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$$

called the cocommutator, which is a 1-cocycle. On the dual algebra $\mathfrak{g}^*$ it defines the map

$$\delta^* : \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*$$

which must be the commutator in $\mathfrak{g}^*$. The cocommutator is coboundary if it can be written

$$\delta(X) = [X, r] = (\mathbb{1} \otimes ad_X + ad_X \otimes \mathbb{1})(r) \quad \forall X \in \mathfrak{g} \tag{112}$$

where r is an element in $\mathfrak{g} \otimes \mathfrak{g}$ of which the symmetric part is an invariant in $\mathfrak{g} \otimes \mathfrak{g}$ and which satisfies the *classical Yang-Baxter equation*:

$$[[r, r]] = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0. \tag{113}$$

Here the notation can be clarified by writing down r in components:

$$r = \sum_i a_i \otimes b_i. \tag{114}$$

The meaning of $[r_{12}, r_{13}]$ is now:

$$[r_{12}, r_{13}] = \sum_i [a_i, a_i] \otimes b_i \otimes b_i. \tag{115}$$

One can show (Chari and Pressley [9]) that a Lie group G is Poisson-Lie if its algebra $\mathfrak{g}$ admits a Lie bialgebra structure. If the corresponding r-element is skew symmetric, it defines a Poisson-Lie bracket for functions on the group given by the following abstract notations which we will explain below:

$$\{f_1, f_2\} = \sum_{s,t} r^{st}(\langle \nabla_L f_1, X_s \rangle \langle \nabla_L f_2, X_t \rangle - \langle \nabla_R f_1, X_s \rangle \langle \nabla_R f_2, X_t \rangle) \tag{116}$$

if the symmetrised part of r corresponds to the invariant bilinear product on the algebra. In the above equation we have written the r-element in the algebra basis as

$$r = r^{st} X_s \otimes X_t$$

The action of an algebra element on the functions in this notation is similar to the previously described pairing, where now we distinguish between the action of left invariant and right invariant differential operators corresponding to the algebra elements. Explicitly:

$$\langle \nabla_L f, X\rangle(g) = \frac{d}{dt} f(e^{-Xt}g)|_{t=0}, \qquad \langle \nabla_R f, X\rangle(g) = \frac{d}{dt} f(ge^{Xt})|_{t=0} \tag{117}$$

This Poisson bracket is called a *Sklyannin bracket*. The classical Yang-Baxter equation, and the invariance of the symmetrised part are restrictions that guarantee the correct properties of Poisson-Lie groups. For a detailed discussion of these relations we again refer to Chari and Pressley [[9]].

4.5 Application to Chern Simons theory

In this subsection we will finally give a Poisson bracket for Chern Simons theory on a manifold $\Sigma \times \mathbb{R}$ where Σ has arbitrary genus and in addition N punctures with particles and thus sources of curvature inserted in them. The approach taken was described by Fock and Rosly [10] and later further investigated for the use of quantisation by Alekseev and Schomerus [11], [12] and [13]. Again in this approach, most of the observations are valid on a classical level, and we will give a description here:

4.5.1 Graph connections

On the full unreduced phase space, a canonical symplectic form is the one for Chern Simons theory often referred to as the Atiyah-Bott symplectic structure [11]. In a symbolic notation it reads:

$$\omega = \int \delta A \wedge \delta A \tag{118}$$

and coincides with the poisson structure (13) defined in section 2. A problem with this symplectic form is that it is defined for an unreduced phase space which is infinite dimensional and results in problems due to the singular nature of the bracket (see Alekseev and Schomerus [11], Fock and Rosly [10]). The advantage of dividing out the gauge symmetry before quantising the theory is that one doesn't have this problem, because phase space reduces to a relatively simple, finite dimensional space.

The approach taken by Fock and Rosly, inspired by lattice gauge theory, is to define poisson brackets on a lattice covering the manifold Σ, a triangulation of Σ, a finite dimensional space related to the reduced phase space as in (9), which, in a suitable limit that connects the finite dimensional with the infinite dimensional case, corresponds to the Atiyah-Bott symplectic structure. In particular, it should give the same answer for functions on reduced phase space, which in our case are conjugation invariant functions.

The route taken is to cover the manifold Σ by a lattice of links and vertices. To each link one can assign a group valued 'lattice gauge field' , defined by

$$L_I = P \exp(\int_{t_i}^{s_j} A_I) \tag{119}$$

where the gauge field A_I is taken in a certain representation I and t_i and s_j are endpoints of the link. Of course one will have to give an ordering of links to arrive at single-valued link variables. The reduced phase space of the theory, the moduli space of flat connections modulo gauge transformations (see the general discussion of Chern Simons theory in section 2), is obviously a subset of the space of all functions on the space of lattice gauge fields, as one can see by combining lattice gauge fields to closed (Wilson) loops. On the other hand, in the limit of infinitely small links, a Poisson bracket should correspond to the singular Atiyah-Bott symplectic structure.

4.5.2 Lie bialgebra structure and the Fock-Rosley bracket

In their article, Fock and Rosley show that there is a natural poisson bracket for functions on the moduli space of flat connections, corresponding to the Atiyah - Bott symplectic structure in the limit, where a suitable regularisation of the singular poisson bracket is chosen. The moduli space is the space of all flat connections on the Riemann surface, which was represented by homomorphisms of the fundamental group $\pi_1(\Sigma)$ into the gauge group as described in section 2 (9).
Noncontractible loops will arise due to *handles* of the surface and *punctures* with particles inserted in them. The fundamental group of the surface $\pi_1(\Sigma)$ thus depends on the genus g of the surface plus the number of punctures with particles N, and consists of $2g + N$ generators a_i, b_i, l_j, where $i = 1 \dots g, j = 1 \dots N$ (for example think of a torus on which one can define two homotopically inequivalent noncontractible loops to understand why there are $2g$ generators of $\pi_1(\Sigma)$ due to its genus).
These generators satisfy the relation:

$$a_1 b_1 {a_1}^{-1} {b_1}^{-1} \dots a_g b_g {a_g}^{-1} {b_g}^{-1} l_1 \dots l_N = \mathbb{1} \tag{120}$$

Before dividing out gauge equivalence, each generator is represented by a gauge group element, a Wilson loop observable (remember that dividing out the gauge equivalence means considering the conjugacy class in stead of the group element, which can be distinghuished by its trace) and thus we denote the representations:

$$\begin{aligned} \rho(a_i) &= U_i \\ \rho(b_i) &= V_i \\ \rho(l_j) &= M_j \end{aligned} \tag{121}$$

The homomorphism property of the representation implies the constraint

$$U_1 V_1 {U_1}^{-1} {V_1}^{-1} \dots U_g V_g {U_g}^{-1} {V_g}^{-1} M_1 \dots M_N = e \tag{122}$$

The group element M_j is forced to lie in a certain conjugacy class by assigning sources of curvature to the particle j inside the puncture. Thus we find an elegant way of describing the moduli space by introducing the momentum

mapping

$$\begin{aligned}&\mu(U_1, V_1, \dots U_g, V_g, M_1 \dots M_N)\\&= U_1 V_1 {U_1}^{-1} {V_1}^{-1} \dots U_g V_g {U_g}^{-1} {V_g}^{-1} M_1 \dots M_N\end{aligned} \tag{123}$$

and stating that phase space can be represented by

$$\mathcal{M}(G, g, \mathcal{C}_1, \dots \mathcal{C}_N) = \mu^{-1}(e)/\sim . \tag{124}$$

where the $\sim$ quotient means dividing out by simultaneous conjugation.

The Fock-Rosley bracket now defines a Poisson structure such that it corresponds to the Atiyah-Bott symplectic structure and which reduces to the space of functions on the moduli space invariant under simultanious conjugation of all holonomies. This Poisson-bracket is expressed in terms of an r-matrix that defines a bialgebra structure.

4.5.3 The Fock-Rosley bracket

Before introducing the Fock-Rosley bracket, we need some notations which will be very useful: Like before, left and right invariant differential operators that act on functions on the group and take values in the Lie algebra's dual are defined by:

$$\begin{aligned}\langle \nabla_L f(g), X\rangle &= \frac{d}{dt} f(e^{-tX} g)|_{t=0}\\ \langle \nabla_R f(g), X\rangle &= \frac{d}{dt} f(g e^{tX})|_{t=0}\end{aligned} \tag{125}$$

In addition to this, as we will be dealing with functions of many grouplike arguments, it will now be necessary to introduce an ordering of the differential operators:

$$\begin{array}{cccc}\nabla_{2j-1} = \nabla_R^{M_j} & \nabla_{2j} = \nabla_L^{M_j} & \text{for } j = 1, \dots N;\\ \nabla_{N+4i-3} = \nabla_R^{U_i} & \nabla_{N+4i-1} = \nabla_L^{U_i} & \text{for } i = 1, \dots g;\\ \nabla_{N+4i-2} = \nabla_R^{V_i} & \nabla_{N+4i} = \nabla_L^{V_i} & \text{for } i = 1, \dots g.\end{array} \tag{126}$$

As an example, the expression

$$\langle \nabla_{2j-1} f(M_1, \dots, M_N, U_1, \dots, U_g, V_1, \dots, V_g), X\rangle$$

can be written as

$$\frac{d}{dt} f(M_1, \dots, M_j e^{tX}, \dots, M_N, U_1, \dots, U_g, V_1, \dots, V_g)|_{t=0}$$

With these notations we can finally define the Poisson bracket, introduced in this form in [13]:

$$\{f, h\} = \frac{1}{2} \sum_i \langle r, \nabla_i f \wedge \nabla_i h\rangle + \sum_{i<j} \langle r, \nabla_i f \wedge \nabla_j h\rangle. \tag{127}$$

where $r \in \mathfrak{g} \otimes \mathfrak{g}$ is a solution of the classical Yang-Baxter equation, defining a bialgebra structure. If the symmetrised part $r_s = \frac{1}{2}(r + \sigma r)$ corresponds to the bilinear invariant form used to define the Chern-Simons action, this Poisson bracket will correspond to the Atiyah-Bott symplectic structure, for a certain regularisation.
The functions we are considering here are functions on reduced phase space, i.e. functions of the representatives for Wilson loops, M, U, V, invariant under simultaneous conjugation. Regarding matrix elements of group elements M, U and V in a certain representation as functions on the moduli space, one has the possibility to explicitly calculate poisson brackets between different matrix elements. Choosing representations τ^I and τ^J one denotes an element $W \in G$ in the representation τ^I by W^I and the r-element in the representation $\tau^I \otimes \tau^J$ by $(\tau^I \otimes \tau^J)(r) = r^{IJ}$. Using the tensor notation $W^1 = W \otimes \mathbb{1}, W^2 = \mathbb{1} \otimes W$ we can clarify this abstract story by giving an explicit calculation of the Poisson brackets for one of the possible combinations of matrix elements:

$$\begin{aligned}
&\{\overset{1}{M}{}_i^I, \overset{2}{M}{}_j^J\} \\
&= \frac{1}{2} \sum_k \Big(\langle r^{IJ}, \nabla_k \overset{1}{M}{}_i^I \otimes \nabla_k \overset{2}{M}{}_j^J \rangle - \langle r^{IJ}, \nabla_k \overset{2}{M}{}_j^J \otimes \nabla_k \overset{1}{M}{}_i^I \rangle \Big) \\
&+ \sum_{k<l} \Big(\langle r^{IJ}, \nabla_k \overset{1}{M}{}_i^I \otimes \nabla_l \overset{2}{M}{}_j^J \rangle - \langle r^{IJ}, \nabla_k \overset{2}{M}{}_j^J \otimes \nabla_l \overset{1}{M}{}_i^I \rangle \Big)
\end{aligned} \tag{128}$$

Here $\nabla_k M_i$ only gives nonzero values for $k = 2i-1$ or $k = 2i$ as the above differential operator ordering tells us. In fact M_i^I is the function in $F(G, g, \mathcal{C}_1, \ldots \mathcal{C}_N)$ that gives the value of the i-th conjugacy class argument in the representation I Proceeding with the above calculation we find that only the first part of the sum for $k < l$ contributes to the answer:

$$\begin{aligned}
\{\overset{1}{M}{}_i^I, \overset{2}{M}{}_j^J\} = &\langle r^{IJ}, (\nabla_L^{M_i} \overset{1}{M}{}_i^I \otimes \nabla_L^{M_j} \overset{2}{M}{}_j^J + \nabla_R^{M_i} \overset{1}{M}{}_i^I \otimes \nabla_R^{M_j} \overset{2}{M}{}_j^J) \rangle \\
&+ \langle r^{IJ}, (\nabla_R^{M_i} \overset{1}{M}{}_i^I \otimes \nabla_L^{M_j} \overset{2}{M}{}_j^J + \nabla_R^{M_i} \overset{1}{M}{}_i^I \otimes \nabla_L^{M_j} \overset{2}{M}{}_j^J) \rangle.
\end{aligned} \tag{129}$$

From equations (125) we see that the evaluation of the pairing gives minus signs for every pairing with left invariant differential operators. The result in the representation I, J is:

$$\begin{aligned}
\{\overset{1}{M}{}_i^I, \overset{2}{M}{}_j^J\} \;=\; & r^{IJ} \overset{1}{M}{}_i^I \overset{2}{M}{}_j^J - \overset{1}{M}{}_i^I r^{IJ} \overset{2}{M}{}_j^J \\
& - \overset{2}{M}{}_j^J r^{IJ} \overset{1}{M}{}_i^I + \overset{1}{M}{}_i^I \overset{2}{M}{}_j^J r^{IJ} \quad \text{for } i < j.
\end{aligned} \tag{130}$$

4.5.4 Some more remarks on the classical r-matrix

We have seen that in this formalism, the problem for finding a poisson structure on the phase space of Chern Simons theory can be reduced to the problem of finding a bialgebra structure for the corresponding gauge algebra, and with it a classical r-matrix, satisfying the Classical Yang Baxter Equation, and of

which the symmetrised part coincides with the bilinear invariant form used to define the inner product on the algebra that becomes manifest in the Chern Simons action. For simple gauge groups a standard r-structure is available as described in books like [9] and [14], though it is not uniquely defined. The restriction that the symmetrised part of r coincides with the invariant bilinear form leaves open degrees of freedom form the antisymmetric part, as described in [11]. Nevertheless, we can imagine the r-matrix being uniquely determined if it must in addition satisfy the Classical Yang Baxter Equation, though this is not a priori clear.

In section 7 we return to the issue of finding solutions to the Classical Yang Baxter Equation and give a general description of classical double bialgebras, of which the $iso(3)$ bialgebra structure we will be considering and which is presented in the appendix is an example.

5 Quantisation

In this section we will at last make an attempt on describing quantisation theory. First we give a description of the missing parts to arrive at a quantum theory, where two approaches will be explained in more detail. The two approaches have already been mentioned in the previous section:
An approach taken by Witten [1] for the case of Chern Simons theory with $ISO(3)$ gauge group without particles is described as an example of geometric quantisation. In addition to this we have a description of a quantisation procedure for internal phase space of a single particle, the coadjoint orbit. This is in fact another example of geometric quantisation, but we will just give the result in this section.
The approach taken by Alekseev and Schomerus is in fact an example of deformation quantisation leading to quantum groups (although they call it combinatorial quantisation, which we will explain further on).
The advantage of the deformation quantisation approach is that it seems to provide a way to describe the quantisation of Chern Simons theory with particles.
In a way, the considerations in this section will be a setup to discuss the quantisation procedure found by Bais and Muller [2]. The results of the coadjoint orbit quantisation together with arguments related to the topological nature of the theory, fusion and braiding properties, were reason to propose the quantum symmetry of the *quantum double of* $SU(2)$ to be result of quantisation of $ISO(3)$ Chern Simons theory with particles. Very recently, this result has been shown to be correct according to the Alekseev and Schomerus procedure of combinatorial quantisation by Schroers [3]. We will describe this correspondence in the following section.

5.1 Quantisation and the quantum condition

Unfortunately, quantisation is not as straightforward as one may wish, and there is no evident canonical procedure which works for every classical theory or case one wants to describe. In fact, since Schrödinger, many different approaches have been developed and no one stands out in an absolute sense
Of course all the approaches should at least have many results in common, first of all, that they agree with the postulates of quantum mechanics. Here we give a summary of them:

In quantum mechanics every observable, in classical mechanics represented by a function on the phase space, is replaced by a hermitian operator. The operator does not act on phase space itself, but on a Hilbert space, which is usually a space of functions on part of the phase space in a certain representation.
In the Hamiltonian formalism, the observables, the functions on phase space form a Poisson algebra under poisson brackets: Upon quantisation this algebra is replaced by an operator algebra and the Poisson bracket is replaced by the

operator commutator. The famous example is that of the Heisenberg algebra:

$$\{x_i, p_j\} = \delta_{ij} \tag{131}$$

which is being replaced by:

$$[\mathcal{O}_{x_i}, \mathcal{O}_{p_j}] = -i\hbar\delta_{ij} \tag{132}$$

The general picture of which this is an example is:

$$[\mathcal{O}_f, \mathcal{O}_g] = -i\hbar\mathcal{O}_k \tag{133}$$

where

$$k = \{f, g\}. \tag{134}$$

This is often referred to as the quantum condition. Here we recognise the introduction of $\hbar$, Planck's constant as *a quantum parameter*. Conceivable inequivalent quantisations should in the limit $\hbar \to 0$ reproduce classical mechanics.
In general identifying phase space is an important preparation for the actual quantisation procedure.
The phase space one starts of with, is possibly still subject to constraints, or perhaps some symmetry still has to be divided out. One can choose to first quantise a system and afterwards restrict the Hilbert space to functions that obey these constraints (which will then be imposed by constraint operators) and are invariant under the residual symmetry. Poisson brackets must be compatible with these symmetry transformations.

5.2 Geometric quantisation

The whole subject of geometric quantisation is based on the possibility of defining symplectic structures on the initial phase space and possibly, if there are constraints or gauge equivalences, the reduced phase space. In particular, reduction to a subspace of the initial phase space should leave the symplectic structure intact. One now would like to define a Hilbert space: A space of square integrable functions on part of the phase space on which Hermitian operators, representing the observable functions on phase space, will act.
The wave functions are defined on half of the parameters of phase space: The coordinates parametrising configuration space, as opposed to the momenta. The symplectic structure provides a natural choice for this identification. Finding a representation of the Hilbert space is still then a challenging problem in most cases.
A symplectic space on which one has a locally defined one-form θ such that $d\theta = \omega$ allows the definition of *prequantum bundles*, which satisfy the $U(1)$ invariance for the expectation values of operaters. We will not go into the details here, but refer to [6] and proceed with the example of $ISO(3)$ Chern Simons theory without particles.

5.2.1 The example of $ISO(3)$ Chern Simons theory

We will now apply the method to $ISO(3)$ Chern Simons theory: We will first discuss the quantisation of the pure gauge theory, where there is no presence of matter terms in the action, following the remarks made by Witten [1]. After this we give the result of the coadjoint orbit quantisation.
Our first goal in quantising $ISO(3)$ Chern Simons theory would again be to identify a phase space, possibly with constraints.
The unreduced phase space (before imposing constraints) is the space of all gauge connections $\omega_\mu{}^a, e_\mu{}^a$. Through the Chern Simons action in the Hamiltonian formulation, we are already supplied with Poisson brackets, which were given for $ISO(3)$ in equation (60). These Poisson brackets seem suitable for identifying either the $\omega_\mu{}^a$ or the $e_\mu{}^a$ with coordinates on which wave functions can be defined. But here the phase space reduction becomes important: The constraint equations without curvature sources yield the curvature is zero everywhere and thus $\omega_\mu{}^a$ and $e_\mu{}^a$ must be *flat* connections. Besides this, we still have the gauge equivalence of the theory to be divided out.

Physical sectors of the theory should be gauge invariant. Our wavefunctions, which depend on the variables which we have chosen to be the *coordinates*, therefore after a gauge transformation still depend on coordinates only. The gauge transformations on the gauge fields acted as:

$$\begin{aligned} \delta\omega_\mu{}^a &= -\partial_\mu\tau^a - \epsilon^a{}_{bc}\omega_\mu{}^b\tau^c \\ \delta e_\mu{}^a &= -\partial_\mu\rho^a - \epsilon^a{}_{bc}\omega_\mu{}^b\rho^c - \epsilon^a{}_{bc}e_\mu{}^b\tau^c \end{aligned} \tag{135}$$

We see that the transformation of the $\omega_\mu{}^a$ contains only a term proportional to itself, while the $e_\mu{}^a$ after transformation contain a term proportional to $\omega_\mu{}^a$.
This means the only suitable candidates for the coordinates are the $\omega_\mu{}^a$, the flat $SO(3)$-connections, forming a *configuration space* $\mathcal{M}$
All coordinates $\omega_\mu{}^a$ that are connected through gauge transformations are equivalent, which reduces the configuration space to $(\mathcal{M}/\sim) = \mathcal{N}$, where $\sim$ stands for gauge equivalence.
We can now identify the phase space Γ of flat $ISO(3)$-connections modulo gauge transformations with the total space of the cotangent bundle of $\mathcal{N}$, which means a canonical symplectic structure corresponds to the Poisson brackets (60).

The reader might notice that the argumentation in this line of reasoning has come in in a different order than was the case while giving the description of phase space in terms of symplectic structures. Here we already used the fact that we had identified coordinates to define the symplectic structure, whereas in the formal discussion a symplectic structure is used to distinguish between configuration space and momentum space. This could be interpreted as a typical example of the disadvantage of formalised mathematical descriptions. It emphasizes that generalising a procedure does not allways make it easier.

If we do not consider particles, we will assume the Riemann surface Σ

will not be punctured, and noncontractible loops will only arise due to handles of the surface. The fundamental group of the surface $\pi_1(\Sigma)$ thus only depends on the genus g of the surface and consists of $2g$ generators a_i, b_j, where $i, j = 1 \dots g$ Very much like the case described at the end of section 4 these generators satisfy the relation:

$$a_1 b_1 {a_1}^{-1} {b_1}^{-1} \dots a_g b_g {a_g}^{-1} {b_g}^{-1} = \mathbb{1} \tag{136}$$

By the above identification of phase space with $T^*\mathcal{N}$ we can now identify our moduli space with the cotangent space of homomorphisms of $\pi_1(\Sigma)$ into $SO(3)$. The space on which wave functions will be defined, our configuration space paramatrised by coordinates, is the space of group elements $S_i, T_j \in SO(3)$ which represent elements $a_i, b_j \in \pi_1(\Sigma)$ and thus satisfy

$$S_1 T_1 {S_1}^{-1} {T_1}^{-1} \dots S_g T_g {S_g}^{-1} {T_g}^{-1} = \mathbb{1} \tag{137}$$

Along with this condition two representations are equivalent if they differ by a global gauge transformation by a fixed element $\Lambda \in SO(3)$:

$$\begin{aligned} S_i &\to \Lambda^{-1} S_i \Lambda \\ T_j &\to \Lambda^{-1} T_j \Lambda \end{aligned} \tag{138}$$

The Hilbert space consists of square integrable functions $\Psi(S_i, T_j)$, which are invariant under these transformations.

5.2.2 Hilbert space representations obtained by coadjoint orbit quantisation of $ISO(3)$

The quantisation of the coadjoint orbit is described in [8] and [15]. Here we just give some crucial results: The representations of $ISO(3)$ can be labeled by the invariants μ, s of the orbit, where μ is interpreted as the Euclidean mass, and s is restricted to certain values as a result of the integrability condition mentioned above. For $\mu > 0$, s has to be an element of $\frac{1}{2}\mathbb{Z}$ and if $\mu = 0$ in addition we have the restriction $s > 0$. The Hilbert space is now the tensor product of square integrable functions on the orbit of p_0 under $SU(2)$, denoted by $\mathcal{O}_\mu$ and the Hilbert space of representations of the centraliser group N_s, labeled by $s \in \frac{1}{2}\mathbb{Z}$:

$$\mathcal{H}^{\mu,s} = L^2(\mathcal{O}_\mu) \otimes \mathcal{H}_s. \tag{139}$$

The centraliser is the subgroup in $SU(2)$ that leaves the element p_0 invariant, so it is isomorphic to $U(1)$. Clearly one observes that the choice of configuration space on which one has defined the wave functions agrees with the cotangent bundle geometry of the coadjoint orbit as described in section 4 and the appendix***. The above space can be shown to be identical to the space:

$$L^2_s(SU(2), \mathcal{H}_s) = \{\phi : SU(2) \to \mathcal{H}_s | \phi(xh) = \Pi_s(h^{-1})\phi(x), \forall h \in N_s\} \tag{140}$$

This is a result we will feature again later on.
These representations could be viewed as correct representations of the single

particle case of the $ISO(3)$ Chern Simons theory. However, the fact that we have quantised a coadjoint orbit and not a conjugacy class, will become clear in particular when it comes to constructing tensor products and considering multiparticle Hilbert spaces.
The multiplication of two representing Wilson loops for different particles should give a fusion rule for 2 particle representations which is compatible with the group multiplication as discussed in equations (8). One can imagine that these rules are not obeyed by tensor product representations of the quantised coadjoint orbit, so the multi-particle case needs a different approach.

5.3 Quantum groups

In this section we will describe a different class of approaches to quantisation. An approach close to this will be used in section 5 to justify a description of quantisation of $ISO(3)$ Chern Simons theory.
The ideas of deformation quantisation lead to the definition of quantum groups. These quantum groups are mathematical objects which can actually be recognized as Hopf algebras rather than groups. The key ideas that lead to these objects will be described next.

5.3.1 Deformation quantisation

From the quantum condition described by (133) we see that the main problem of quantisation is the change of functions on phase space representing observables to operators on a Hilbert space. These operators should be elements of a Lie algebra isomorphic to the poisson algebra under Poisson brackets. Thus the poisson bracket is replaced by the commutator, which is the algebra product on the Lie algebra.
A way to do this is to consider the algebra of functions on phase space $F(\mathcal{M})$. It is equipped with the natural product of pointwise multiplication:

$$m(f_1, f_2)(x) = f_1(x)f_2(x) \tag{141}$$

This algebra does obviously not coincide with the Lie-algebra we are lookin for, because functions on phase space commute with respect to the product defined by pointwise multiplication. The way one achieves the right commutator is by deforming the product to a new one, $*_h$ such that

$$\frac{f_1 *_h f_2 - f_2 *_h f_1}{h} = \{f_1, f_2\} \mod(h). \tag{142}$$

Here we have introduced a deformation parameter h, which will play the role of Planck's constant. The classical limit again should correspond to letting $h \to 0$. Representations $\Phi(f)$ for operators corresponding to f must satisfy

$$\Phi(f_1 *_h f_2) = \Phi(f_1)\Phi(f_2). \tag{143}$$

The above deformation is known as *Moyal's deformation* (see [9]).

5.3.2 Deforming Poisson-Hopf algebras

Similar to the Moyal deformation (142), we can give a quantisation of a Poisson-Hopf algebra $\mathcal{A}$ by deforming the product on it, making it noncommutative (assuming it was commutative before) such that:

$$\{x_1, x_2\} = \frac{a_1 a_2 - a_2 a_1}{h} (\text{mod } h) \tag{144}$$

for $x_1, x_2 \in \mathcal{A}, a_1, a_2 \in \mathcal{A}_h$ where $\mathcal{A}_h$ denotes the deformed Poisson Hopf algebra and a_1, a_2 reduce to x_1, x_2 in the classical limit. If The Poisson-Hopf algebra $\mathcal{A}$ corresponds to a Hopf algebra of functions on a Poisson-Lie group $F(G)$, its deformation $F_h(G)$ is called the quantization of the Poisson-Lie group $(G, \{\ ,\ \})$. The dual picture that corresponds to this is the following: A quantisation of a Co-Poisson-Hopf algebra $\mathcal{B}$ is a deformation of the coproduct, such that

$$\delta(y) = \frac{\Delta_h(b) - \Delta_h^{op}(b)}{h} (\text{mod } h) \tag{145}$$

where, as before, $y \in \mathcal{B}, b \in \mathcal{B}_h$ and b reduces to y in the classical limit. By Δ_h^{op} we mean the deformed coproduct followed by a flip operation, i.e.

$$\Delta_h^{op} = \sigma_{12} \Delta_h. \tag{146}$$

We now see the Co-Poisson-Hopf algebra is no longer cocommutative. Well-studied examples of this kind of deformation are quanised universal enveloping algebra's $U_q(\mathfrak{g})$. As we have seen in equation (108), a coproduct defines a product on the dual algebra, and vice-versa, which leads us to the suggestion that a *deformed coproduct* on a universal enveloping algebra defines a deformation of the product on its dual, the algebra of function on the group as we argued before (where actually we should emphasise that this is not true for every Lie group and if we want to be mathematically correct we should discuss some questions of completeness). Thus instead of looking for a deformed product for the algebra of functions on the group, one could as well give a description of the quantised universal enveloping algebra. For the details and explicit example for the case $F(SL(2, \mathbb{C}))$ versus $U(sl(2, \mathbb{C}))$ we refer to Chari and Pressley [9].

5.3.3 The universal $\mathcal{R}$-matrix and quasitriangularity

The noncocommutativity of a Hopf algebra as in (145) can be described by an invertible element of $\mathcal{A} \otimes \mathcal{A}$, called the universal $\mathcal{R}$-matrix. The defining relation is

$$\Delta^{op}(a)\mathcal{R} = \mathcal{R}\Delta(a) \tag{147}$$

for all $a \in \mathcal{A}$. The algebra $\mathcal{A}$ is obviously cocommutative if the universal $\mathcal{R}$-element is equal to $(\mathbb{1} \otimes \mathbb{1})$. In our applications it will be close to this, which turns the Hopf algebra $\mathcal{A}$ into an *almost cocommutative* one.
In addition to this, a Hopf algebra is said to be quasitriangular if the $\mathcal{R}$-element satisfies the conditions:

$$(\Delta \otimes \mathbb{1})(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}, \tag{148}$$

$$(\mathbb{1} \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}. \tag{149}$$

Note that in this case the coproduct explicitly acts on the $\mathcal{R}$-element, and makes the left hand side of these equations into an element of $\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$, while in equation (147) Δ acts on an element a and $\mathcal{R}$ then acts through the multiplication rule defined on $\mathcal{A} \otimes \mathcal{A}$.
The equations (147) and (148, 149) lead us to another important equation: If we use equation (147) to act on $\mathcal{R}$ itself, we find:

$$\mathcal{R}_{12}.(\Delta \otimes \mathbb{1})(\mathcal{R}) = (\Delta^{op} \otimes \mathbb{1})(\mathcal{R}).\mathcal{R}_{12} \tag{150}$$

where we only used equation (147) for acting on the first element of $\mathcal{R}$. We now apply σ_{12} to both sides of (148) and find:

$$(\Delta^{op} \otimes \mathbb{1})(\mathcal{R}) = \mathcal{R}_{23}\mathcal{R}_{13} \tag{151}$$

Substituting this in the previous equation, together with equation (148), we find:

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12} \tag{152}$$

This relation is referred to as the quantum Yang-Baxter equation (QYBE). We can conclude that quasitriangularity is equivalent to satisfying the QYBE.
One can now check that the algebra $\mathcal{A}^{op}$ is coassociative, which means

$$\mathcal{R}_{12}(\Delta \otimes \mathbb{1})(\mathcal{R}) = \mathcal{R}_{23}(\mathbb{1} \otimes \Delta)(\mathcal{R}) \tag{153}$$

The 'almost cocommutativity' of the quasitriangular Hopf algebra is in our case reflected in the relation between the universal $\mathcal{R}$-element and the classical r-matrix defined in section 4. In the limit in which the deformation parameter h goes to 0, our $\mathcal{R}$-element will become the identity, whereas, the first order part in h will give rise to the Poisson-like structure through the r-element. Quantising universal enveloping algebras can be achieved by finding an appropriate $\mathcal{R}$-element which in a certain representation can be written

$$\mathcal{R} = (\mathbb{1} \otimes \mathbb{1}) + hr + O(h^2) \tag{154}$$

In this equation the noncocommutative part represents the Poisson structure corresponding to the classical r-element (see equation (116)).
Quantised universal enveloping algebra's are related to the deformed algebra's of functions on a group by the same equation (116), or in the Chern Simons case, (127). In applications one can express this with help of representations. This is done by regarding matrix elements of matrix representations of the group G as functions on the group and writing the bracket in this representation (which also yields writing r in this representation). The deformation of the product on thes matrix elements can now be written in terms of a deformation of the coproduct on $U(\mathfrak{g})$ and in terms of a universal $\mathcal{R}$-element. This will define a new product on the algebra of matrix elements of the matrix group representation, which now cease to commute.
Instead of labelling the quantised universal enveloping algebra by the parameter h also used in the expansion of $\mathcal{R}$, the usual convention to introduce the

parameter $q = e^h$. With this convention a coproduct on the quantised universal enveloping algebra can often easily be written down, as is the case for $U_q(sl_2)$ (see Chari and Pressley [9]), where the coproduct is given by:

$$\begin{aligned} \Delta(H) &= H \otimes \mathbb{1} + \mathbb{1} \otimes H \\ \Delta(X^+) &= \mathbb{1} \otimes X^+ + X^+ \otimes q^H \\ \Delta(X^-) &= q^{-H} \otimes X^- + X^- \otimes \mathbb{1} \end{aligned} \tag{155}$$

In this example X^+ and X^- are deformed basis elements of the positive and negative roots respectively and H is the simple root. The limit $h \to 0$ obviously corresponds to $q \to 1$.

5.3.4 The example of the combinatorial quantisation of Chern Simons theory

The nomenclature of combinatorial quantisation of Alekseev and Schomerus is inspired by the concept of combining lattice gauge fields as described in section 4. The Quantisation procedure is actually a deformation quantisation as described above. As the Poisson bracket is already given in terms of the Fock-Rosley bracket, all one searches for is actually a suitable $\mathcal{R}$-element, if one has a Lie bialgebra defining classical r-element, as we describe next:
We pick up the classical story of section 4 from the final equation (130): We see that the Poisson brackets of matrix elements are quadratic in these matrix elements and are indeed explicitly determined by the Lie bialgebra structure defining r-element. In the spirit of deformation quantisation, one can deform the product on the algebra of functions on the group, by introducing an $\mathcal{R}$-element that defines the *quadratic exchange relations* between the matrix elements of M_i, in a manner such that the first order in the deformation parameter of the commutator of two elements coincides with the Poisson bracket.
This means the $\mathcal{R}$-element in the representation space of r can be expanded as:

$$R^{IJ}|_{h\to 0} = \mathbb{1} \otimes \mathbb{1} + hr^{IJ} + \mathcal{O}(h^2) \tag{156}$$

This $\mathcal{R}$ element satisfies the Quantum Yang-Baxter equation (at least up to order h^2) as r satisfies the Jacobi-identity. This leads us to the observation that the framework we are dealing with is that of the quasi-triangular Hopf algebra's. Examples of these are the deformed universal enveloping algebra's $U_q(\mathfrak{g})$. In ***representation theory etc.. quadratic exchange relations are given that follow by introducing a suitable $\mathcal{R}$-element. Those corresponding to the above poisson relation are:

$$(\mathcal{R}^{-1})^{IJ} \overset{1}{M}{}_i^I \mathcal{R}^{IJ} \overset{2}{M}{}_j^J = \overset{2}{M}{}_j^J (\mathcal{R}^{-1})^{IJ} \overset{1}{M}{}_i^I \mathcal{R}^{IJ} \quad \text{for } i < j. \tag{157}$$

One can check that the expansion of $\mathcal{R}$ as above in equation (156), and with $(\mathcal{R}^{-1})^{IJ} = \mathbb{1} \otimes \mathbb{1} - hr^{IJ} + O(h^2)$, yields a commutator of $\overset{1}{M}{}_i^I$ and $\overset{2}{M}{}_j^J$ which is equal to the Poisson brackets in the first power of h:

$$\begin{aligned} &(\mathbb{1} \otimes \mathbb{1} - hr^{IJ} + O(h^2)) \overset{1}{M}{}_i^I (\mathbb{1} \otimes \mathbb{1} + hr^{IJ} + O(h^2)) \overset{2}{M}{}_j^J \\ &= \overset{2}{M}{}_j^J (\mathbb{1} \otimes \mathbb{1} - hr^{IJ} + O(h^2)) \overset{1}{M}{}_i^I (\mathbb{1} \otimes \mathbb{1} + hr^{IJ} + O(h^2)) \end{aligned} \tag{158}$$

so

$$\begin{aligned} \overset{1}{M}_i^I \overset{2}{M}_j^J - \overset{2}{M}_j^J \overset{1}{M}_i^I &= h(r^{IJ} \overset{1}{M}_i^I \overset{2}{M}_j^J - \overset{1}{M}_i^I r^{IJ} \overset{2}{M}_j^J \\ &\quad - \overset{2}{M}_j^J r^{IJ} \overset{1}{M}_i^I + \overset{1}{M}_i^I \overset{2}{M}_j^J r^{IJ}) + O(h^2) \end{aligned} \tag{159}$$

Where the Fock-Rosley bracket was defined for conjugation invariant functions, all quadratic exchange relations following from it through this procedure should be invariant under a new quantum algebra symmetry.

5.3.5 The $ISO(3)$-case

The framework above suggests that in the case of $ISO(3)$ Chern Simons theory the corresponding quantum symmetry algebra will become the universal enveloping algebra $U_q(iso(3))$. The crucial point in arriving at the quantum double symmetry, to be discussed in the next section, is the choice of bialgebra structure and the particular deformation chosen.
In the appendix we show that $ISO(3)$ has a Lie bialgebra structure, given by the r-element $P_a \otimes J_a$. Thus $ISO(3)$ corresponds to a Poisson Lie group and there is a direct relation between this r-element and the Poisson bracket defined on the functions on our phase space.
The symmetrised part of the r-element $P_a \otimes J_a$ obviously corresponds to the invariant bilinear form introduced to define the action for Euclidean gravity with zero cosmological constant. An important observation (Schroers [3]) is that this r-matrix is obtained by regarding the algebra $iso(3)$ as a classical double of the algebra $su(2)$. The action of the $\mathcal{R}$-element of the quantum double $D(SU(2))$ is found to admit an expansion as in equation (156), where the classical r-element is the r-element given above. We will give the procedure for this quantisation in the next section. What we will not show, but only mention here, is that the classical description of a single particle with the help of the Fock-Rosly bracket is identical to the coadjoint orbit description, which means the Kirillov-Kostant bracket on this orbit coincides with the Fock-Rosly bracket determined by the above r-element (see again [3]). We should emphasize that the way of quantising $ISO(3)$ Chern Simons theory might not be uniquely determined. The Lie bialgebra structure leading to the interpretation as a classical double is not the only possible Lie bialgebra structure, as noted in [3]. Besides this, different deformation schemes could be available for one bialgebra structure. We consider these questions beyond the scope of this thesis.

6 Quantum double quantisation

In this section we return to the main subject of this thesis, i.e. the quantisation of $ISO(3)$ Chern Simons theory with particles. In section 4 and 5 we concluded that the multi-particle sector of the quantum theory could not coincide with tensor products of $ISO(3)$ representations obtained by coadjoint orbit quantisation, and Bais and Muller [2] have shown that the object $D(SU(2))$, i.e. the quantum double of the group $SU(2)$ is a better candidate for describing Euclidean quantum gravity without cosmological constant in 2+1 dimensions. The argumentation was based on the observation that the single particle representation of $ISO(3)$ is contained in the representations of $D(SU(2))$ and the fact that the $D(SU(2))$ representations obey the correct fusion and braiding properties on the classical level. Most importantly their method of quantisation was suggested by a strict analogy with the discrete group case, as discussed for example in [16].
Very recently, Schroers [3] has shown that the quantum double $D(SU(2))$ is a deformation of the algebra of functions on the group $ISO(3)$ and thus indeed corresponds to a correct quantisation of the theory. In this section we will give a description of this correspondence.

6.1 The quantum double of $SU(2)$

The general definition of a quantum double can be found in various books about quantum groups. We refer to Majid [14]. The main features are the following: The quantum double of a finite group H can be constructed by the tensor product of the Hopf algebra of functions on the group $F(H)$ and it's canonical dual, the group Hopf algebra $\mathbb{C}(H)$ (both described in section 4). The quantum double $D(H)$ is itself a quasi-triangular Hopf algebra, and for detailed descriptions of its representations and constructions in the case which is of particular relevance for us and in which H is replaced by $SU(2)$ we refer to Bais, Koornwinder and Muller (BKM) [2], [17] and [18]. We will just use the results and give a summary of the paper in which Schroers describes his deformation of $ISO(3)$. The formalism used here will be based on the linear bijection

$$F(G) \otimes \mathbb{C}(G) \Longleftrightarrow F(G \times G) \tag{160}$$

described by BKM [2], which endows $F(G \times G)$ with the Hopf* structure inherited from the quantum double. To proceed, we will first have to give this structure:

6.1.1 Quasi-triangular Hopf* structure on $F(G \times G)$ inherited from $D(G)$

i) Multiplication:

$$m(F_1 \otimes F_2)(x, y) = \int_G dz F_1(x, z) F_2(z^{-1}xz, z^{-1}y) \tag{161}$$

ii) Comultiplication:

$$(\Delta F)(x_1, y_1; x_2, y_2) = F(x_1 x_2, y_1)\delta_e(y_1^{-1} y_2) \tag{162}$$

iii) The unit:

$$\eta(x, y) = \delta_e(y) \tag{163}$$

iv) The co-unit:

$$\epsilon(F) = \int_G F(e, y)dy \tag{164}$$

v) The antipode:

$$(S(F))(x, y) = F(y^{-1}x^{-1}y, y^{-1}) \tag{165}$$

Additional structures making it a quasi-triangular Hopf* algebra:
vi) Universal $\mathcal{R}$-element:

$$\mathcal{R}(x_1, y_1; x_2, y_2) = \delta_e(x_1 y_2^{-1})\delta_e(y_1) \tag{166}$$

vii) *-operation:

$$F^*(x, y) = \overline{F(y^{-1}xy, y^{-1})} \tag{167}$$

6.1.2 Representations of $D(SU(2))$

In this formalism, BKM describe the irreducible representations of $D(SU(2))$. They are labeled by pairs of quantum numbers (μ, s), where $\mu \in [0, 2\pi]$ is the label of a conjugacy class in $SU(2)$ and s labels an irreducible representation of the centraliser of an element of this conjugacy class (all centralisers in a conjugacy class are isomorphic, thus one can choose a single element of the conjugacy class). Conjugacy classes in $SU(2)$ are topologically equivalent to 2-spheres as one can show by choosing the Euler angle parametrisation. The quantum number μ corresponds to the radius of these 2-spheres.
For centraliser representations one can distinguish between two different cases: In the generic case, for which $\mu > 0$, the centraliser is isomorphic to $U(1)$, and thus the label s corresponds to an irreducible $U(1)$ representation. If $\mu = 0, 2\pi$, corresponding to conjugacy classes $\{\mathbb{1}\}$ or $\{-\mathbb{1}\}$, the centraliser is the whole $SU(2)$ group and we can think of s as the ordinary j-quantum number for $SU(2)$.
Explicitly we can write an element $h \in SU(2)$ in axis-angle parametrisation as

$$h(\mu, \mathbf{n}) = \exp \mu(n_a J_a) \tag{168}$$

where $\mathbf{n}$ is a unit vector parametrising the 2-sphere, $J_a = \frac{i}{2}\sigma_a$ and σ_a are the Pauli matrices. We will restrict ourselves to the generic case in which $\mu > 0$ for the moment.
The representing element in the conjugacy class can be chosen to be the intersection of the unit sphere and the positive 3-axis. Elements $h_\mu = \exp \mu J_3$, rotations around the 3-axis now form the $U(1)$ centraliser and representations

π_s are labeled by elements $s \in \mathbb{Z}/2$.
The carrier space of representations $\Pi_{\mu s}$ is given by:

$$H_s = \{\phi \in L^2(SU(2), \mathbb{C}) | \phi(xh_\mu) = \pi_s(h_\mu^{-1})\phi(x)\}. \tag{169}$$

Notice the similarity with the single particle representations for coadjoint orbit quantisation of $ISO(3)$ described in section 4.
The action of an element $F \in D(SU(2))$ in this representation becomes [2]

$$(\Pi_{\mu s}(F)\phi)(x) = \int_{SU(2)} dw F(xh_\mu x^{-1}, w)\phi(w^{-1}x). \tag{170}$$

With help of these expressions and the defining relations for the product and the $\mathcal{R}$-element (see appendix), one can derive the action of the $\mathcal{R}$-element in a representation $\Pi_{\mu_1,s_1} \otimes \Pi_{\mu_2,s_2}$:

$$(\Pi_{\mu_1,s_1} \otimes \Pi_{\mu_2,s_2}(\mathcal{R})\Phi)(x_1, x_2) = \Phi(x_1, x_1 h_\mu x_1^{-1} x_2). \tag{171}$$

For some state $\Phi \in H_{s_1} \otimes H_{s_2}$.

6.2 Functions on the group $\widetilde{ISO(3)}$ and the Hopf algebra A_0

A crucial point in the observation of the correspondence between the quantum double $D(SU(2))$ and the algebra of functions on $ISO(3)$ by Schroers was the formalism in which he presented the latter (see [3]).
A Hopf algebra A_0 is introduced: The defining relations of the cocommutative Hopf algebra A_0, which consists of functions on $((\mathbb{R}^3)^* \times SU(2))$ are:
The multiplication:

$$m(f_1 \otimes f_2)(\mathbf{a}, u) = \int_{SU(2)} dw f_1(\mathbf{a}, w) f_2(\mathrm{Ad}(w^{-1})\mathbf{a}, w^{-1}u). \tag{172}$$

The comultiplication:

$$(\Delta f)(\mathbf{a}_1, u_1, \mathbf{a}_2, u_2) = f(\mathbf{a}_1 + \mathbf{a}_2, u_1)\delta_e(u_1^{-1}u_2). \tag{173}$$

The unit:

$$\eta(\mathbf{a}, u) = \delta_e(u) \tag{174}$$

The counit:

$$\epsilon(f) = \int_{SU(2)} f(0, u) \tag{175}$$

The antipode:

$$Sf(\mathbf{a}, u) = (-\mathrm{Ad}(u^{-1})\mathbf{a}, u^{-1}). \tag{176}$$

The functions on $(\mathbb{R}^3)^* \times SU(2)$ are obtained by Fourier transforming the first argument of a function on $(\mathbb{R}^3) \times SU(2)$, which as a manifold is isomorphic to $\widetilde{ISO(3)}$.

[2] An explicit basis is furnished by the Wigner functions ${}^{\mu}D^j_{mn}$ for the case of $0 < \mu < 2\pi$ and fixed n, see [18]

Identifying representations of this Hopf algebra with representations of $\widetilde{ISO(3)}$ one can deduce the action of the classical r-element in this representation. Schroers observed that the action of the universal $\mathcal{R}$-element in $D(SU(2))$ is a deformation of this action, in the sense of equation (156).
Irreducible representations of $\widetilde{ISO(3)}$ are indeed very similar to the irreps of $D(SU(2))$. The labels again correspond to centraliser representations and orbits under $SU(2)$, the latter now being the orbit of an element of $(\mathbb{R}^)3^*$ in stead of a conjugacy class. Again the radius of the orbit, which we will now denote by M for a representing element $\mathbf{d}_3^M = (0, 0, M)$, is it's label, and the centraliser representation is labelled by an element $s \in \mathbb{Z}/2$.
The carrier space of these representations is the same H_s as in (169). The action of an element $f \in A_0$ is now given by

$$(\Pi_{Ms}(f)\phi)(x) = \int_{SU(2)} dw f(\mathrm{Ad}(x)\mathbf{d}_3^M, w)\phi(w^{-1}x). \tag{177}$$

To recover the familiar $\widetilde{ISO(3)}$ representations one picks out a function corresponding to an element $(\mathbf{a}, u)$:

$$f_{\mathbf{a},u}(\mathbf{k}, v) = \exp(i\mathbf{a}\cdot\mathbf{k})\delta_u(v) \tag{178}$$

and defining

$$\mathbf{k}(x, M) = \mathrm{Ad}(x)\mathbf{d}_3^M \tag{179}$$

one finds:

$$(\Pi_{M,s}(\mathbf{a}, u)\phi)(x) = \exp(i\mathbf{a}\cdot\mathbf{k}(x, M))\phi(u^{-1}x) \tag{180}$$

Now that we know how group elements act on this representation, we can use a pairing similar to the one described by equation (109) to find the action of algebra elements. For generators of translations P_a we find:

$$(\Pi_{Ms}(P_a)\phi)(x) = \frac{d}{d\epsilon}(\Pi_{Ms}(\epsilon\mathbf{d}_a, 1)\phi)(x)|_{\epsilon=0} = ik_a(x, M)\phi(x). \tag{181}$$

And the generators of rotations J_a act as:

$$(\Pi_{Ms}(J_a)\phi)(x) = \frac{d}{d\epsilon}(\Pi_{Ms}(0, \exp(\epsilon J_a))\phi)(x)|_{\epsilon=0} = -J_a\phi(x). \tag{182}$$

Where we should emphasize that this is still a shorthand notation for a left invariant differential operator generated by J_a on the group $SU(2)$. We can now write down the action of the r-element on tensor product representations of Φ on $H_{s_1} \otimes H_{s_2}$:

$$((\Pi_{M_1 s_1} \otimes \Pi_{M_2 s_2})(r)\Phi)(x_1, x_2) = -i(k_a(x_1, M_1)J_a^{(2)})\Phi(x_1, x_2). \tag{183}$$

The generator J_a of course acts on the second argument of Φ, as the superscript $^{(2)}$ is meant to indicate.

6.3 The deformation $A_0 \to D(SU(2))$

Functions on $(\mathbb{R}^3)^* \times SU(2)$ can be extended to functions on $SU(2) \times SU(2)$ if one knows a map from $(\mathbb{R}^3)^*$ to $SU(2)$ that preserves underlying Hopf-structures when attached to another $SU(2)$-copy (through the Cartesian product). The pull back of such a map should give an action on A_0 corresponding to the action of the $\mathcal{R}$-element on $D(SU(2))$. This is exactly the action we are interested in. The map introduced by Schroers is in fact a family of exponential maps, labeled by a parameter κ. $SU(2)$ generators $J_a^\kappa = \kappa J_a$ obeying the relations

$$[J_a^\kappa, J_b^\kappa] = \kappa \epsilon_{abc} J_c^\kappa \tag{184}$$

are introduced, and the family of exponential maps is given by:

$$\exp_\kappa : (\mathbb{R}^3)^* \to SU(2), \quad \exp_\kappa(\mathbf{k}) = \exp(k_a J_a^\kappa) \tag{185}$$

For invertibility of the pull-back map, a special subset V_κ of functions on $(\mathbb{R}^3)^*$ is defined:

$$V_\kappa = \{f \in C((\mathbb{R}^3)^*) | f|_{S^2_{2\pi/\kappa}} = \text{const.}, f(\mathbf{k} - (4\pi/\kappa)\hat{\mathbf{k}}) = f(\mathbf{k}) \forall \mathbf{k} \in \mathbb{R}^3 \setminus \{0\}\}. \tag{186}$$

Here $S^2_{2\pi/\kappa}$ denotes the 2-spere in $(\mathbb{R}^3)^*$ with radius $2\pi/\kappa$, and $\hat{\mathbf{k}} = \mathbf{k}/|\mathbf{k}|$. We now have the pull-back map:

$$\exp_\kappa^* : C(SU(2)) \to C((\mathbb{R}^3)^*) \tag{187}$$

which is invertible when restricted to the subset V_κ of $C((\mathbb{R}^3)^*)$. Thus the inverse map is denoted by:

$$\log_\kappa^* : V_\kappa \to C(SU(2)). \tag{188}$$

We now want to make the connection with functions on $(\mathbb{R}^3)^* \times SU(2)$. The subset of these functions for which the functions are in V_κ as a function of the first argument is denoted by A_κ. As A_κ is a subset of A_0, it can inherit a few properties. In fact it inherits multiplication and unit, but does not inherit the comultiplication in a consistent way. This means the comultiplication (Δ) given in (173) does not map A_κ into $A_\kappa \otimes A_\kappa$. In order to get a consistent comultiplication, the comultiplication (173) must be deformed to one which is consistent. To achieve this, the complete Hopf algebra structure of $D(SU(2))$ should be mapped to A_κ through extension of the above pull-back maps. The extended maps are defined by:

$$\text{EXP}_\kappa^* : D(SU(2)) \to A_\kappa \tag{189}$$

and its inverse:

$$\text{LOG}_\kappa^* : A_\kappa \to D(SU(2)). \tag{190}$$

Both maps are obviously obtained through the pull-back on the first argument. The multiplication on A_κ now is obtained by mapping the arguments of $f_1, f_2 \in A_\kappa$ to $SU(2) \times SU(2)$ and applying the multiplication (161):

$$m(f_1 \otimes f_2)(\mathbf{a}, u) = \text{LOG}_\kappa^* \Big(\int_{SU(2)} dz f_1(\exp_\kappa(\mathbf{a}), z) f_2(z^{-1} \exp_\kappa(\mathbf{a}) z, z^{-1} u) \Big). \tag{191}$$

This multiplication rule does not depend on κ, in fact it is exactly the multiplication rule already existing on A_0. The comultiplication is found through the same procedure:

$$(\Delta_\kappa f)(\mathbf{a}_1, u_1, \mathbf{a}_2, u_2) = \mathrm{LOG}^*_\kappa \left(f(\exp_\kappa(\mathbf{a}_1) \exp_\kappa(\mathbf{a}_2), u_1) \right) \delta_e(u_1^{-1} u_2). \tag{192}$$

This structure does depend on κ, which can be made clear by writing the product $\exp_\kappa(\mathbf{a}_1) \exp_\kappa(\mathbf{a}_2)$ as a single exponent with help of the Campbell-Baker-Hausdorff formula:
One arrives at:

$$\exp_\kappa(\mathbf{a}_1) \exp_\kappa(\mathbf{a}_2) = \exp_\kappa \left(\mathbf{a}_1 + \mathbf{a}_2 + \frac{\kappa}{2} \mathbf{a}_1 \wedge \mathbf{a}_2 + O(\kappa^2) \right) \tag{193}$$

The fact that we have only continued the computation up to order κ already indicates that we are interested in the small κ limit of this deformed coproduct. If one views the new Hopf algebra A_κ as the Hopf algebra of functions on the group $(\mathbb{R}^3)^* \times SU(2)$, one finds that the deformed coproduct yields a deformed product on $(\mathbb{R}^3)^*$, given by the argument of the exponent above. With the deformed coproduct, A_κ is turned into a noncocommutative quasitriangular Hopf algebra. In the limit $\kappa \to 0$ we recover the undeformed coproduct, with usual addition of vectors in $(\mathbb{R}^3)^*$, yielding the A_0 Hopf algebra structure. The $\mathcal{R}$-element that describes the quasitriangular Hopf-structure on A_κ is again the one inherited from $D(SU(2))$. As the $\mathcal{R}$-element acts on the arguments of $F(SU(2) \times SU(2)) \otimes F(SU(2) \times SU(2))$, it is mapped to $A_\kappa \otimes A_\kappa$ by:

$$\mathcal{R}_\kappa = (\mathrm{EXP}^*_\kappa \otimes \mathrm{EXP}^*_\kappa)(\mathcal{R}) \tag{194}$$

This yields:

$$\mathcal{R}_\kappa(\mathbf{a}_1, u_1, \mathbf{a}_2, u_2) = \delta_e(\exp_\kappa(\mathbf{a}_1) u_2^{-1}) \delta_e(u_1). \tag{195}$$

To investigate the action of $\mathcal{R}$-element on $A_\kappa \otimes A_\kappa$ it is calculated for the element $\Phi \in H_{s_1} \otimes H_{s_2}$ in the irreducible representation $\Pi_{M_1 s_1} \otimes \Pi_{M_2 s_2}$, for small κ:

$$\left((\Pi_{M_1 s_1} \otimes \Pi_{M_2 s_2})(\mathcal{R}_\kappa) \Phi \right)(x_1, x_2) = \Phi(x_1, \exp_\kappa \left(\mathbf{k}(x_1, M_1) \right) x_2), \tag{196}$$

Keeping in mind the meaning of the expression $\mathbf{k}(x_1, M_1)$ (see equation (179)), we find in the limit for $\kappa \to 0$ that

$$\begin{aligned} &((\Pi_{M_1 s_1} \otimes \Pi_{M_2 s_2})(\mathcal{R}_\kappa)\Phi)(x_1, x_2) \\ &\quad = (\mathbb{1} \otimes \mathbb{1} + \kappa k^a(x_1, M_1) J_a^{(2)}) \Phi(x_1, x_2) + O(\kappa^2). \end{aligned} \tag{197}$$

where it is made explicitly clear that the Hopf algebra A_κ is a deformation of the Hopf algebra A_0 and that the noncommutativity of the coproduct is described by the $\mathcal{R}_\kappa$-element in which we recognise:

$$\mathcal{R}_\kappa = \mathbb{1} \otimes \mathbb{1} + i\kappa r + O(\kappa^2). \tag{198}$$

6.3.1 Conclusions on $D(SU(2))$

This observation shows that the deformation of the algebra A_0 to $D(SU(2))$ is a deformation quantisation of the type described in section 5. One can as well conclude that the dual picture is a deformation of the algebra $U(iso_3)$, regarded as the classical double of su_2. This matter will be speculated on in the last section.

7 Discussion and outlook

7.1 Different bialgebra structures and r-matrices

The bialgebra structure used in the previous section is one of the structures one can obtain when searching for solutions of the classical Yang Baxter equation. As mentioned at the end of section 5, the r-matrix $J_a \otimes P_a$ is obtained by regarding $iso(3)$ as the *classical double* of $su(2)$. We will give some details here:

7.1.1 The classical double

Let $(\mathfrak{g}, \mathfrak{g}^*)$ be a Lie bialgebra, then the vector space $\mathfrak{h} = \mathfrak{g} \oplus \mathfrak{g}^*$, has a natural inner product. The usual pairing $\langle , \rangle$ between elements $X \in \mathfrak{g}$ and $\xi \in \mathfrak{g}^*$ is used:

$$(X, \xi)_{\mathfrak{h}} = \langle X, \xi \rangle, \quad (X, X)_{\mathfrak{h}} = (\xi, \xi)_{\mathfrak{h}} = 0 \tag{199}$$

We define the basis $\{X_i\}$ of $\mathfrak{g}$ and the dual basis $\{\xi^i\}$ of $\mathfrak{g}^*$ to give the commutation relations and corresponding structure constants:

$$[X_i, X_j] = c_{ij}{}^k X_k, \quad [\xi^i, \xi^j] = \gamma^{ij}{}_k \xi^k \tag{200}$$

Demanding the bilinear form $(,)_{mathfrakh}$ to be invariant under the action of $\mathfrak{g}$ and $\mathfrak{g}^*$ yields:

$$([\xi^i, X_j], \xi^k) = -(X_j, [\xi^i, \xi^k]) \tag{201}$$

and

$$([X_i, X_j], \xi^k) = -(X_j, [X_i, \xi^k]). \tag{202}$$

This uniquely determines the commutator between an element of $\mathfrak{g}$ and an element of $\mathfrak{g}^*$:

$$[\xi^i, X_j] = c_{jk}{}^i \xi^k - \gamma^{ik}{}_j X_k \tag{203}$$

The proof that this construction only applies to Lie bialgebra's $(\mathfrak{g}, \mathfrak{g}^*)$ can be found in Chari and Pressley [9].
One can easily verify that the cocommutator for Lie bialgebra's in the above notation must be:

$$\delta_{\mathfrak{g}}(X_i) = \gamma^{jk}{}_i X_j \otimes X_k \tag{204}$$

The Lie bialgebra structure can be extended to the full vector space $\mathfrak{g} \oplus \mathfrak{g}^*$ through a canonical construction. The result is the classical double, denoted by $\mathcal{D}(\mathfrak{g})$, which is in fact a self-dual Lie bialgebra. The cocommutator is given by:

$$\delta_{\mathcal{D}(\mathfrak{g})}(u) = (\mathrm{ad}_u \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{ad}_u)(r) \tag{205}$$

where $r \in \mathfrak{g} \otimes \mathfrak{g}^* \subset \mathcal{D}(\mathfrak{g}) \otimes \mathcal{D}(\mathfrak{g})$ is the identity element $r : \mathfrak{g} \to \mathfrak{g}$. In the above notation it is obviously given by:

$$r = X_i \otimes \xi^i. \tag{206}$$

We can calculate:

$$\begin{aligned}\delta_{\mathcal{D}(\mathfrak{g})}(X_i) &= [X_i, X_j] \otimes \xi^j + X_j \otimes [X_i, \xi^j] \\ &= c_{ij}^k X_k \otimes \xi^j + \gamma^{kj}{}_i X_j \otimes X_k - c_{ik}{}^j X_j \otimes \xi^k \\ &= \gamma^{kj}{}_i X_j \otimes X_k, \end{aligned} \tag{207}$$

the same answer as for the cocommutator on $\mathfrak{g}$, which proves that the inclusion $\mathfrak{g} \hookrightarrow \mathfrak{g} \oplus \mathfrak{g}^*$ is a Lie algebra homomorphism. The same accounts for the inclusion $\mathfrak{g} \oplus \mathfrak{g}^* \hookleftarrow (\mathfrak{g}^*)^{\mathrm{op}}$.

7.1.2 Bialgebra structure of $U(sl_2)$

One of the advantages of the classical double is the fact that it provides a way to construct bialgebra structures from *-structures. For example the presumably best studied example of a universal enveloping algebra, the $U(sl_2)$, has a standard bialgebra structure which can be derived with help of a classical double construction, the Lie bialgebra $\mathcal{D}(\mathfrak{q}_+)$. $\mathfrak{q}_+$ consists of pairs (x, h) with x lying in the subspace spanned by simple and positive roots of sl_2, while h is the simple-root component of x. The inner product on $\mathcal{D}(\mathfrak{q}_+)$ is given by

$$((x, h), (x', h'))_{\mathfrak{q}} = (x, x')_{sl_2} - (h, h')_{sl_2} \tag{208}$$

and projection on the algebra sl_2 of the canonical bialgebra structure on this space gives the bialgebra structure

$$\delta(X^{\pm}) = \frac{1}{2} X^{\pm} \wedge H, \quad \delta|_{\mathfrak{h}} = 0 \tag{209}$$

where $X^{\pm}$ is an element of the positive or negative root space respectively, and $H \in \mathfrak{h}$ by, which we denote the simple root space. The above bialgebra structure multiplied by 2 is known as the standard bialgebra structure for sl_2 and can equivalently be described by the r-matrix :

$$r = \frac{1}{4} H \otimes H + X^- \otimes X^+ \tag{210}$$

7.1.3 Bialgebra structures for $iso(3)$

In the case of $iso(3)$, as we can now conclude, the problem of searching for a bialgebra structure is easily solved by observing the fact that with the inner product used throughout this whole thesis, we can regard $iso(3)$ as the classical double of $su(2)$. The r-element $J_a \otimes P_a$ is obviously nothing but the r-element (206).

7.1.4 Different *-structures and other considerations

However, the classical double of $su(2)$ is not a uniquely defined object if we do not specify the inner product. One can imagine there being a different *-structure, for which the dual vector space of su_2 is not spanned by generators

of translations as in our case.
A construction of the classical double of $su(2)$ described by Majid [14] starts by considering the classical double $\mathcal{D}(sl_2(\mathbb{C}))$. The standard structure described above determines the dual sl_2^*. One can show that its commutation relations should read:

$$[\psi^{\pm}, \phi] = \frac{1}{2}\psi^{\pm}, \quad [\psi^{+}, \psi^{-}] = 0 \tag{211}$$

where the *-structure is given by the nonzero pairings:

$$\langle X^{+}, \psi^{+} \rangle = 1, \quad \langle X^{-}, \psi^{-} \rangle = 1, \quad \langle H, \phi \rangle = 1. \tag{212}$$

The algebra $sl_2(\mathbb{C})$ has 2 real forms, $sl_2(\mathbb{R})$ and $su(2)$, of which $su(2)$ is the compact one. The restriction of the bialgebra to these real forms gives the real form $\mathcal{D}(sl_2(\mathbb{R}))$ and the *half real form* $\mathcal{D}(su_2)$. $\mathcal{D}(su_2)$ is called a half real form because one can choose a basis in which the algebra su_2 is real, but its dual has imaginary structure constants unless one allows the inner product to become imaginary.
As described by Majid, there is an isomorphism $\mathcal{D}(su_2) \simeq su_2 \oplus su_2^* \simeq so(3,1)$. It would be interesting to investigate what happens to this bialgebra when performing a Wigner contraction as in section 3. Besides this, following the argumentation of [2], where they suggested the procedure later on carried out by Schroers and which we described in section 6, a deformation of the classical double could be regarded as the dual picture of the quantum double construction. This suggests that with the different bialgebra structure for $D(su_2)$, the quantum double would correspond to a dual picture of deformation of $so(3,1)$. It would be very interesting to investigate if this reasoning makes sense, whereas, it would probably lead to a very different interpretation of $D(SU(2))$ representations. We must however admit that this idea has so far not been developed beyond this rather speculative stage.
The Wigner contraction of a deformed $so_4 \simeq so_3 \oplus so_3$ algebra has led Celeghini et al [19] to a quantum group version of $ISO(3)$. The remark made by Schroers [3] about this, is that it gives a different result than the quantum double quantisation a priori because here a different r-structure was taken, a double copy of the canonical su_2 r-matrix. Another effort in deforming the theory of Chern Simons gravity was made by Bimonte et al. [20] and it would be interesting to investigate the relation of the quantum double quantisation to this result.
The different gauge groups in section 3 result in different quantum groups as one may expect. For the case of $ISO(2,1)$ one could expect, through a similar kind of reasoning as in the $ISO(3)$ case, that the quantum symmetry is that of the quantum double $D(SU(1,1))$, the bialgebra structure of $iso(2,1)$ again being regarded as the classical double of $su(1,1)$.
For the $SO(4-p,p)$ we distinguish between the case in which the gauge algebra is a direct sum of 2 real algebra's and the case in which it is not (the latter in fact only occurs for $SO(3,1)$.
For $so(4) \simeq so(3) \oplus so(3)$ one expects a direct sum of 2 $U_q(su_2)$ algebras, and similarly one would find two copies of $U_q(sl_2(\mathbb{R})$ for $so(2,2) \simeq so(2,1) \oplus so(2,1) \simeq sl(2,\mathbb{R}) \oplus sl(2,\mathbb{R})$ in this reasoning, both cases which we did not make explicitly clear.

The case of $SO(3,1)$ is somewhat different, and we discussed a possible approach above. For compact gauge group, as is the case for $SO(4)$, there is a restriction on the deformation parameter as we will discuss below.

7.2 Remarks on the deformation parameter and the coupling constant

In this section we will give a few additional remarks on the deformation parameter. We will be considering cases in which the gauge group of a Chern Simons theory is compact and in which the coupling constant turns out to give a restriction on the possible deformation parameter values when performing combinatorial quantisation.

7.2.1 Deformation parameters for compact gauge group

In the work of Alekseev and Schomerus [11], before introducing the Fock and Rosley Poisson structure for the matrix elements U_i, V_i and M_i (see section 5), a Poisson structure for the link variables (lattice gauge fields) L_I defined in (119) is introduced, an intermediate step in taking the limit towards the Atiyah-Bott symplectic structure.
For the basic Poisson relations one distinguishes between different cases:

i) The links corresponding to the two different lattice gauge fields have no common endpoints:

$$\{\overset{1}{L}(i), \overset{2}{L}(j)\} = 0 \tag{213}$$

where i and j label the links.
ii) The links corresponding to different lattice gauge fields have one common endpoint, where the ordering is such that $i < j$:

$$\{\overset{1}{L}(i), \overset{2}{L}(j)\} = \frac{2\pi}{k}\overset{1}{L}(i)\overset{2}{L}(j)r \tag{214}$$

iii) The links corresponding to different lattice gauge fields have one common endpoint, where the ordering is such that $i > j$:

$$\{\overset{1}{L}(i), \overset{2}{L}(j)\} = -\frac{2\pi}{k}\overset{1}{L}(i)\overset{2}{L}(j)r' \tag{215}$$

where r' is the 'flipped' r-element σr.
iv) The links coincide:

$$\{\overset{1}{L}(i), \overset{2}{L}(i)\} = \frac{2\pi}{k}(r\overset{1}{L}(i)\overset{2}{L}(i) - \overset{1}{L}(i)\overset{2}{L}(i)r') \tag{216}$$

These poisson relations are constructed in such a way that they give the Atiyah Bott symplectic structure in the limit for infinitesimally small links, and on the other hand correspond to the poisson structure on the moduli space described in section 5. The factor of $\frac{2\pi}{k}$ is a consequence of the desired property that

in the limit the Atiyah bott structure is recovered. Let us recall the poisson structure in this limit:

$$\{A_i^a(z_1), A_j^b(z_2)\} = -\frac{2\pi}{k}\delta^{ab}\epsilon_{ij}\delta^{(2)}(z_1 - z_2) \tag{217}$$

It's straightforward quantisation, that is, replacement by an operator commutation relation, is:

$$[A_i^a(z_1), A_j^b(z_2)] = -\frac{2\pi}{k}\delta^{ab}\epsilon_{ij}\delta^{(2)}(z_1 - z_2) \tag{218}$$

Writing the quantisation in terms of quadratic exchange relations

$$\overset{1}{L}\overset{2}{L} = \overset{2}{L}\overset{1}{L}\mathcal{R} \tag{219}$$

where $\mathcal{R}$ is again expanded as

$$\mathcal{R} = \mathbb{1} \otimes \mathbb{1} + hr \tag{220}$$

we identify the deformation parameter

$$h = \frac{2\pi}{k}. \tag{221}$$

This result is of course only relevant in the case of compact gauge group, in which we cannot scale away the coupling constant k. Another remark that should be made is that on renormalisation we should take possible one-loop corrections into account in the coupling constant. This, as remarked in [11], in the standard scheme replaces k by $k + h^*$, where h^* is the 'dual Coxeter number'. The corresponding Hopf algebra symmetry is the $U_q(\mathfrak{g})$, where now

$$q = \exp(\frac{2\pi}{k + h^*}). \tag{222}$$

The restriction to the value of the coupling constant as discussed in section 2 has as a consequence that the parameter q is a *root of unity*.
Through a different route, as described by Witten et al. in the articles [21] and [22], a quantisation for compact gauge groups results in the level k conformal blocks of Wess-Zumino-Witten models, where for instance the $SU(2)$ case results in the Kac-Moody algebra $\widehat{SU(2)}_k$. The representation space is equivalent to that of the quantum group $U_q(sl_2)$, a relation which we did not investigate any further, but is interesting enough to try to give explicitly.

7.3 Conclusions

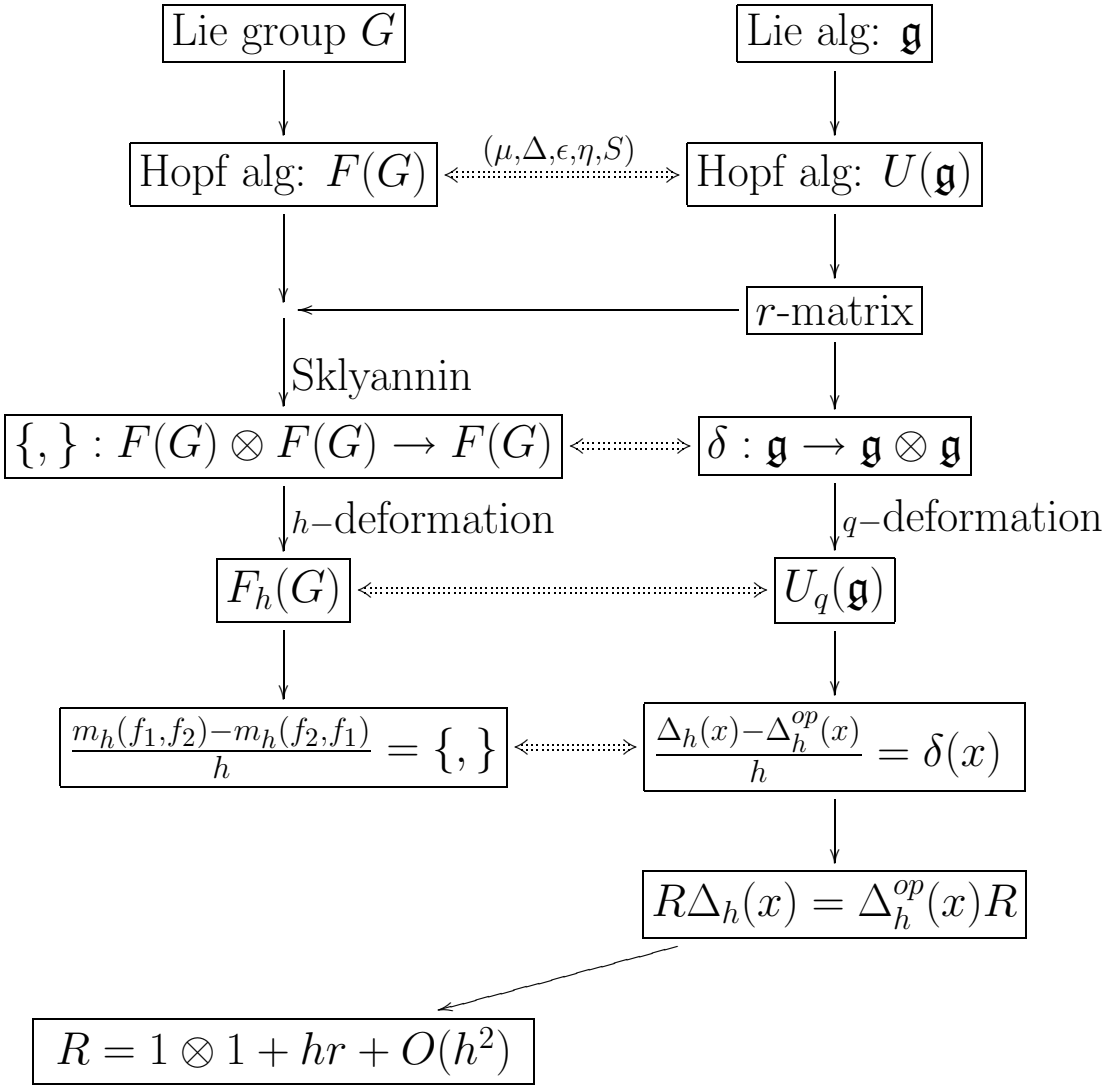
Summarising this thesis, we come to the following line of reasoning:
The general Chern Simons theory which is known for already quite some time is interesting from different points of view, one being the identification of Chern Simons theory for gauge groups $ISO(3-p,p)$ and $SO(4-p,p)$ with gravity theory in 2+1 dimensions, and specifying it to a Riemann surface with

time gives the theory a topological nature.
That is, the curvature form constructed from the gauge connection is zero everywhere, which turns our theory into a trivial one, unless the topology of the Riemann surface is non-trivial.
The observables of the theory are identified with Wilson loop observables, associated to generators of the fundamental group of the surface, which are in fact functions of gauge group elements, which have to obey group multiplication and should be overall conjugation invariant.
The Poisson structure on this phase space is however rather difficult to identify. Though we had a canonical Poisson bracket on our unreduced phase space, identifying two spatial components of the gauge field to be each others canonically conjugate variables, it is not clear how this structure reduces to the space of functions on the gauge group or multiple copies of gauge group arguments, where the reduction of the bracket should be compatible with the group multiplication.
With help of the formalism of Hopf algebra's, identifying the algebra of functions on a group manifold with a Hopf algebra and regarding the dual picture, the Universal enveloping algebra corresponding to the algebra of functions of the group, we find a framework in which a Lie bialgebra defining element called the classical r-matrix determines a Poisson bracket.
This result is first given for functions on a single group element, and later generalised for functions of multiple group-valued arguments, the latter also being connected to the application of Chern Simons theory.
Where a geometrical quantisation procedure seemed to be possible for a single particle sector of this theory, the Poisson bracket, constructed through a coadjoint orbit procedure, is not compatible with group multiplication, making it incapable of describing a multi-particle phase space. The Hopf algebra formalism, which gave us the group-compatible Poisson structure, is in particular suitable for deformation quantisation, which could be applied to Chern Simons theory.

We can conclude that, although we did not come to giving relations between different cases of Chern Simons gravity very explicitly, the mathematical framework one can use to investigate the different gauge groups and deformation schemes has been clarified to a certain extent. From this point, one can formulate a number of interesting questions concerning deformations of the different gravity-gauge groups and possible quantum double symmetries, or (contraction?) links between the $SO(4-p,p)$ groups and the $ISO(3-p,p)$ groups. The objects one should investigate are the Lie bialgebra structures, their r-matrices and the possible deformations. In literature a lot of these mathematical structures are described, but little of them have, as far as we know, explicitly been used in the physical applications, of which Chern Simons theory is an example. These physical applications often turn out to be a source of inspiration for new relations between mathematical structures. An example of this is the quantum double symmetry which was proposed as the correct quantisation from a different point of view as that of deformation quantisation. From the point of view of the framework described in this

thesis, the correctness of the $D(SU(2))$ result has been confirmed, though the deformation is not as straightforward as the diagram on the next page, describing a general mathematical procedure for deformation quantisation, suggests.

7.4 Diagram of mathematical framework for quantising phase spaces of functions on a group



8 Appendix

8.1 Generators for $SO(4-p,p)$

The group $SO(4-p,p)$ is the isometric group in a manifold with 4 real dimensions and a metric with $4-p$ plusses and p minusses. From this we derive:

$$\tilde{Y}^i\tilde{Y}_i = \eta_{ij}\Lambda^j{}_k Y^k \Lambda^i{}_l Y^l = Y^i Y_i \tag{223}$$

This simply yields the relations:

$$\eta_{ij}\Lambda^j{}_k\Lambda^i{}_l = \eta_{kl} \tag{224}$$

or

$$\Lambda^T\eta\Lambda = \eta. \tag{225}$$

Expanding Λ around the identity $\mathbb{1}$ yields

$$(\mathbb{1}+X^T)\eta(\mathbb{1}+X) = \eta \tag{226}$$

which up to first order in the generators X gives us

$$X^T\eta = -\eta X \tag{227}$$

And thus

$$X_{ij} = -X_{ji}. \tag{228}$$

A basis for the vector space of these matrices in n dimensions is obviously $n(n-1)/2$-dimensional. For starting off with 4 dimensions this gives a 6-dimensional algebra. A basis can be chosen by:

$$\begin{aligned}
(J_1)_{ij} &= \delta_{i,1}\delta_{j,2} - \delta_{i,2}\delta_{j,1}\\
(J_2)_{ij} &= \delta_{i,2}\delta_{j,3} - \delta_{i,3}\delta_{j,2}\\
(J_3)_{ij} &= \delta_{i,3}\delta_{j,1} - \delta_{i,1}\delta_{j,3}\\
(P_1)_{ij} &= \delta_{i,3}\delta_{j,4} - \delta_{i,4}\delta_{j,3}\\
(P_2)_{ij} &= \delta_{i,2}\delta_{j,4} - \delta_{i,4}\delta_{j,2}\\
(P_3)_{ij} &= \delta_{i,1}\delta_{j,4} - \delta_{i,4}\delta_{j,1}
\end{aligned} \tag{229}$$

The commutation relations can be calculated to give:

$$\begin{aligned}
[J_a, J_b] &= \epsilon_{ab}{}^c J_c\\
[J_a, P_b] &= \epsilon_{ab}{}^c P_c\\
[P_a, P_b] &= \epsilon_{ab}{}^c J_c.
\end{aligned} \tag{230}$$

To discriminate between the different metrics we should keep in mind that before calculating the commutation relations, we raise an index of the generators in (229). This leads to the raised index c in the commutation relations.

8.2 Coadjoint orbits of $iso(3)$

A group element of $ISO(3)$ is denoted by $(u, \mathbf{a})$, where $u \in SO(3)$ and $\mathbf{a}$ is a translation 3-vector. The group multiplication of two elements $(u, \mathbf{a}), (v, \mathbf{b}) \in ISO(3)$ is given by:

$$(u, \mathbf{a}).(v, \mathbf{b}) = (uv, \mathbf{a} + u\mathbf{b}) \tag{231}$$

This determines the inverse:

$$(u, \mathbf{a})^{-1} = (u^{-1}, -u^{-1}\mathbf{a}) \tag{232}$$

We will make use of a 4×4 matrix representation in which the element $(u, \mathbf{a})$ is represented as:

$$(u, \mathbf{a}) = \begin{pmatrix} u & \mathbf{a} \\ 0 & 1 \end{pmatrix} \tag{233}$$

Basis elements (J_a, P_a) in the $iso(3)$ algebra vector space can now be written:

$$(J_a, P_a) = \begin{pmatrix} J_a & P_a \\ 0 & 0 \end{pmatrix} \tag{234}$$

We now want to give the coadjoint orbit of a vector $\xi^* \in iso(3)$ under the action of the group $ISO(3)$. For this purpose we write $\xi^* \in iso(3)^*$ relative to its basis:

$$\xi^* = j^a J_a^* + p^a P_a^* \tag{235}$$

We find the corresponding algebra element ξ by demanding

$$\langle \xi, (J_a^*, P_a^*) \rangle = \langle (J_a, P_a), \xi^* \rangle \tag{236}$$

where we use the inner product on the algebra to identify coalgebra elements. Thus we find

$$\xi = j^a P_a + p^a J_a. \tag{237}$$

The coadjoint action is determined by the definition:

$$\langle x, Co_g Y \rangle = \langle Ad_g x, Y \rangle = \langle gxg^{-1}, Y \rangle \tag{238}$$

for $x \in \mathfrak{g}, Y \in \mathfrak{g}^*, g \in G$. The adjoint action of Lie-groups on the algebra vector space has a unitary representation, which means we can write

$$\langle x, Co_g Y \rangle = \langle x, Ad_{g^{-1}} Y \rangle = \langle x, g^{-1} Y g \rangle \tag{239}$$

The hamiltonian vector fields on the coalgebra generate the coadjoint action of g^{-1}, using the conventions of section 4. It can be confusing when one tries to keep using just one convention, where different conventions are used in literature. It is important to keep in mind that for calculating the *orbit* of an element in $\mathfrak{g}^*$ one can use both the coadjoint actions of g and g^{-1}. To find the coadjoint orbit we derive the transformation of the elements j^a and p^a under the action

of $(u, \mathbf{a})^{-1}$. The adjoint action of $(u, \mathbf{a})^{-1}$ on this element is calculated in the matrix representation:

$$\begin{pmatrix} u^{-1} & -u^{-1}\mathbf{a} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p^a J_a & j^a P_a \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u & \mathbf{a} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p^a u^{-1} J_a u & p^a u^{-1} J_a \mathbf{a} + j^a u^{-1} P_a \\ 0 & 0 \end{pmatrix} \tag{240}$$

In components relative to the co-algebra basis this is:

$$\begin{aligned} p^a &\rightarrow p^b (u^{-1})_b{}^a \\ j^a &\rightarrow j^b (u^{-1})_b{}^a + \epsilon_a{}^{bc} a^b (u^{-1})_d{}^c p^d \end{aligned} \tag{241}$$

Invariants of this action can be shown to be:

$$\begin{aligned} p^a p_a &= \mu^2 \\ p^a j_a &= \mu s \end{aligned} \tag{242}$$

Therefore, μ and s will label different orbits. If we choose to start with the element $(j_0 = (s, 0, 0), p_0 = (\mu, 0, 0))$, we can calculate the action for an explicit orbit, where we use the canonical one-form on the orbit:

$$\theta_{\mu,s}((u, \mathbf{a}), (j_0, p_0)) = \langle r J_0 + s P_0, (u^{-1}, -u^{-1}\mathbf{a}) d(u, \mathbf{a}) \rangle \tag{243}$$

The element $(u^{-1}, -u^{-1}\mathbf{a})d(u, \mathbf{a})$ takes values in the Lie algebra, which we make explicit in the matrix representation:

$$\begin{pmatrix} (u^{-1}) & -u^{-1}\mathbf{a} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} du & d\mathbf{a} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} (u^{-1}du)_{ab} J^{ab} & (u^{-1}da)_a P^a \\ 0 & 0 \end{pmatrix} \tag{244}$$

The inner product on the algebra between an element $X_{ab}J^{ab}$ and an element $Y_c P^c$ can be calculated by changing to the vector representation of J^{ab}, but we can make use of the inner product on the algebra $so(3)$ (which is a trace in 3-dimensional representation) for calculating explicit results. We thus get:

$$\langle s P_0, (u^{-1}du)_{ab} J^{ab} \rangle = -\frac{s}{2} tr(u^{-1} du J_0). \tag{245}$$

For the other part of the expression (243) we get:

$$\langle r J_0, (u^{-1}da)_a P^a \rangle = r (u^{-1})_a{}^b da_b \delta^a{}_0. \tag{246}$$

Using the orthogonality of $SO(3)$-elements we deduce:

$$r (u^{-1})_a{}^b = r (u^T)_a{}^b = r u^b{}_a \tag{247}$$

And thus

$$\theta_{\mu,s}((u, \mathbf{a}), (j_0, p_0)) = r u^a{}_0 da_a - \frac{s}{2} tr(u^{-1} du J_0). \tag{248}$$

From the first term we can deduce Poisson brackets, identifying coordinates a^a and momenta $ru^a{}_0$. Renaming momenta :

$$p^a = ru^a{}_0 \tag{249}$$

and introducing

$$j^a = \epsilon^{abc} a_b p_c + \frac{s}{r} p^a \tag{250}$$

which gives us an explicit version of (241) for the orbit of (j_0, p_0), we calculate the poisson brackets for these parameters:

$$\begin{aligned} \{j_a, j_b\} &= \epsilon_{ab}{}^c j_c \\ \{j_a, p_b\} &= \epsilon_{ab}{}^c p_c \\ \{p_a, p_b\} &= 0 \end{aligned} \tag{251}$$

8.3 Lie bialgebra structure of $iso(3)$

The r-element

$$r = P_a \otimes J_a \tag{252}$$

defines a Lie bialgebra structure on $iso(3)$, which has commutation relations

$$[J_a, J_b] = \epsilon_{abc} J_c, \quad [J_a, P_b] = \epsilon_{abc} P_c, \quad [P_a, P_b] = 0 \tag{253}$$

where the dual of iso(3) is defined by the inner product

$$\langle J_a, P_b \rangle = \delta_{ab}, \quad \langle J_a, J_b \rangle = \langle P_a, P_b \rangle = 0. \tag{254}$$

Proof: r satisfies the classical Yang Baxter equation:

$$\begin{aligned} [[r, r]] &= \Big([P_a, P_b] \otimes J_a \otimes J_b + P_a \otimes [J_a, P_b] \otimes J_b + P_a \otimes P_b \otimes [J_a, J_b] \Big) \\ &= \epsilon_{abc} \Big(P_a \otimes P_c \otimes J_b + P_a \otimes P_b \otimes J_c \Big) = 0. \end{aligned} \tag{255}$$

Secondly, the symmetrised part is an invariant of the algebra:

$$\begin{aligned} X.\frac{1}{2}(r + \sigma r) &= (\mathbb{1} \otimes \mathrm{ad}_X + \mathrm{ad}_X \otimes \mathbb{1}) \frac{1}{2}(P_b \otimes J_b + J_b \otimes P_b) \\ &= \frac{1}{2}(P_b \otimes [X, J_b] + [X, P_b] \otimes J_b + J_b \otimes [X, P_b] + [X, J_b] \otimes P_b) \\ &= \frac{1}{2}\Big[\delta_{X, J_a} \epsilon_{abc} \Big(P_b \otimes J_c + P_c \otimes J_b + J_b \otimes P_c + J_c \otimes P_b \Big) \\ &\quad + \delta_{X, P_a} \epsilon_{abc} \Big(P_b \otimes P_c + P_c \otimes P_b \Big) \Big] \\ &= 0 \end{aligned} \tag{256}$$

The cocommutator now becomes:

$$\begin{aligned} \delta(J_a) &= (\mathbb{1} \otimes \mathrm{ad}_{J_a} + \mathrm{ad}_{J_a} \otimes \mathbb{1})(P_b \otimes J_b) \\ &= P_b \otimes [J_a, J_b] + [J_a, P_b] \otimes J_b \\ &= \epsilon_{abc}(P_b \otimes J_c + P_c \otimes J_b) = 0 \end{aligned} \tag{257}$$

and

$$\begin{aligned}\delta(P_a) &= (\mathbb{1}\otimes \mathrm{ad}_{P_a} + \mathrm{ad}_{P_a}\otimes \mathbb{1})(P_b\otimes J_b)\\ &= P_b\otimes [P_a, J_b] + [P_a, P_b]\otimes J_b\\ &= \epsilon_{abc} P_b\otimes P_c \end{aligned} \tag{258}$$

With this cocommutator and the relation:

$$\langle \delta(X), Y\otimes Z\rangle = \langle X, \delta^*(Y\otimes Z)\rangle \tag{259}$$

where $X\in\mathfrak{g}$ and $Y, Z\in\mathfrak{g}^*\otimes\mathfrak{g}^*$, we find the commutator δ^* for the dual algebra $\mathfrak{g}^*$:

$$\begin{aligned}\langle \delta(X), Y\otimes Z\rangle &= \delta_{X,P_a}\langle P_a, \delta^*(X,Y)\rangle\\ &= \delta_{X,P_a}\epsilon_{abc}\langle P_b\otimes P_c, Y\otimes Z\rangle\\ &= \epsilon_{abc}\delta_{Y,J_b}\delta_{Z,J_c}\delta_{J_a,\delta^*(X,Y)}\end{aligned} \tag{260}$$

Thus

$$\delta^*(J_a\otimes J_b) = \epsilon_{abc}J_c,\quad \delta^*(P_a\otimes P_b) = 0,\quad \delta^*(P_a\otimes J_b) = 0 \tag{261}$$

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