SYMMETRY: A SYNOPSIS

F.A. Bais

Institute for Theoretical Physics
University of Amsterdam

Abstract:

The following pages form a synopsis of some lectures on symmetry which are part of the **1998-'99 course Quantum Physics II** (for second year students) at the University of Amsterdam. They are supplied as to supplement the treatment given in *Introduction to Quantum mechanics* of David J. Griffith, which is used as text book with the course.

The basic concepts and uses of symmetry in the classical and quantum description of simple dynamical systems in three dimensions are summarized.

In the first part we recall some basics of classical mechanics. In the following section the notion of symmetry and conserved quantities is discussed on the classical level. In section 3. the consequences of symmetry on the quantum level are discussed. The final section discusses as a sample problem the symmetries of two dimensional -isotropic as well as anisotropic - two-dimensional harmonic oscillator.

The symmetries that are really important in nature, are not the symmetries of things but the symmetries of laws.

Steven Weinberg

Symmetries in simple dynamical systems

Contents

Sections labeled with a * are included for the interested student, but not obligatory.

1 Classical dynamical systems (a reminder)

1.1 Particle in external potential

Think of a particle moving in a potential $V(q)$

- *Dynamical variables*:
  position q
  velocity $v = \dot{q}$ or momentum $p = mv$
  acceleration $a = \dot{v}$
  energy E etc. etc.

- *Parameters*:
  mass m
  height and width parameters in potential

- *State of the system*:
  at given time t the state is determined by $q(t)$ and $v(t)$

- *Time evolution*:
  The (Newtonian) equation of motion

  $$\boxed{F = ma \quad \textit{with} \quad F = -\frac{\partial V(q)}{\partial q}}$$

  Note that the equation is a second order differential equation, the solution is determined by specifying $q(0)$ and $v(0)$.

1.2 Lagrange formulation of dynamics

- *Lagrangian*:

  $L(t) = L(q, \dot{q})$

  $=$ *Kinetic Energy - Potential Energy*

  $= \frac{1}{2}mv^2 - V(q)$

- *The Action*:

  $S[q] = \int L(t)dt$

- *The variational principle* :

  The condition that the action S is stationary under small but otherwise arbitrary variations

  $q \rightarrow q + \delta q$

  implies that we impose that

  $\delta S = S[q + \delta q] - S[q] = 0$

- *Euler-Lagrange eqns* :

  This stationarity requirement of the Action implies the Euler-Lagrange eqns to hold

  $$\boxed{\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}}) - \frac{\partial L}{\partial q} = 0}$$

  for the case at hand we recover ofcourse Newtons equation:

  $m\dot{v} + \frac{\partial V}{\partial q} = 0$

1.3 Hamiltonian formulation of dynamics

- *Canonical momentum*:

  $p = \frac{\partial L}{\partial \dot{q}}$

- *Phase space*: (q,p)- space

- *Hamiltonian or Energy function*:

  $H(q, p)$

  $= p\dot{q} - L$

  $= p^2/2m + V(q)$

  $=$ *Kinetic Energy* $+$ *Potential Energy*

- *Hamilton equations*:

$$\boxed{\begin{array}{l} \dot{q} = \frac{\partial H}{\partial p} \\[2ex] \dot{p} = -\frac{\partial H}{\partial q} \end{array}} \begin{array}{l} \to \quad \dot{q} = \frac{p}{m} \\[2ex] \to \quad \dot{p} = -\frac{\partial V}{\partial q} \end{array}$$

1.4 Time evolution of functions on phase space

- *Time evolution*

  $\frac{df}{dt}$
  $= \frac{\partial f}{\partial q}\dot{q} + \frac{\partial f}{\partial p}\dot{p}$
  $= \frac{\partial f}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial H}{\partial q}$

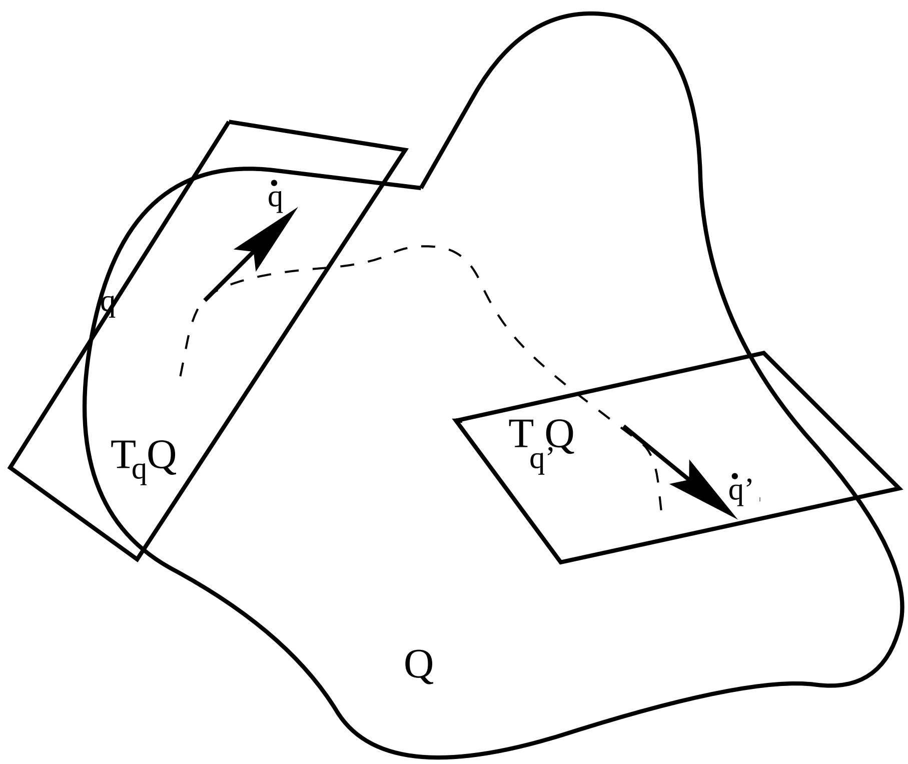
- *Poisson bracket* :
  $\{f, g\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}$
  so that

  $$\boxed{\frac{df}{dt} = \{f, H\}}$$

  The Hamiltonian generates the time evolution.

1.5 * Classical mechanics and symplectic geometry

- Configuration space Q (Diff. manifold) with $q \in Q$
- Tangent space of Q at the point q is T_qQ with $\dot{q} \in T_qQ$
- Tangent bundle TQ with $(q, \dot{q}) \in TQ$
  and $L : TQ \rightarrow \Re$
- Cotangent bundle T^*Q with $(q, p) \in T^*Q$
  and $H : T^*Q \rightarrow \Re$



Tangentbundle TQ

Tangent bundle TQ

$$\begin{array}{lccc} & & & \text{Bundle space} \\ \text{Fibre} & R^N & \longrightarrow & TQ^N \\ & & & \downarrow \\ & & & Q^N \\ & & & \text{Base manifold} \end{array}$$

1.6 * Poisson structure - Symplectic form

- The space of functions on phase space $P \simeq T^*Q$ is equipped with a Poisson structure. It is called symplectic because there exists a symplectic two-form

  $\omega : C^\infty(P) \to TP$

  explicitly

  $\omega = \omega_{ij} dx^i \wedge dx^j = dq^i \wedge dp_i$

  i.e. it maps a function f to a socalled Hamiltonian vectorfield X_f by

  $i_{X_f}\omega = df$

  if X_f is uniquely defined then w is called nondegenerate, so for the explicit choice above we obtain

  $X_f = \frac{\partial f}{\partial p}\frac{\partial}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial}{\partial p}$

  in particular

  $X_p = \frac{\partial}{\partial q}$ and $X_q = -\frac{\partial}{\partial p}$

- Relation between Poisson bracket and symplectic form

  $\{.,.\} : C^\infty(P) \to C^\infty(P)$

  and

  $\{f, g\} = \omega(X_f, X_g) = \omega_{ij}\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j}$

  Furthermore we have the relation between the Lie-bracket of vector fields and the vector field of the Poisson bracket

  $[X_f, X_g] = X_{\{f,g\}}$

- The time evolution on phase space P is generated by X_H, because

  $$\boxed{\frac{df}{dt} = X_H f = \{f, H\}}$$

2 Symmetries

2.1 Relation between Symmetries and Conservation laws

- Conserved quantity Q_a in Lagrangian formulation:

  $\frac{d}{dt}Q_a\left(q_i(t), \dot{q}_i(t)\right) = 0$

- Noether's Theorem

  If the action functional $S[q_i]$ is invariant under an infinitesimal symmetry transformation

  $q_i \to {q_i}' = q_i + \delta_a q_i$

  i.e.

  $\delta S = S[{q_i}'] - S[q_i] = 0$

  then the Lagrangian can at most change by a total time derivative

  $\delta L = \frac{d}{dt}\Lambda.$

  On the other hand:

  $$\delta L = \frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i$$
  $$= \underbrace{\left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right]}_{= 0 \text{ because of equation of motion.}} \delta q_i + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right)$$

- Conserved charge:

  From the above it follows that for each one parameter family of symmetry transformations we obtain a conserved charge or constant of the motion:

  $$\boxed{\begin{aligned} Q_a &= \frac{\partial L}{\partial \dot{q}_i}\delta_a q_i - \Lambda \\ \frac{dQ_a}{dt} &= 0 \end{aligned}}$$

Examples:

a. Momentum conservation and translational invariance

Lagrangian of free particle: $L = \frac{1}{2}m\dot{q}^2$

Translation $q \to q + \varepsilon$:

$\delta_\varepsilon q = \varepsilon \longrightarrow \delta L = 0 \longrightarrow \Lambda = 0$

$Q_\varepsilon = \frac{\partial L}{\partial \dot{q}}\delta q \simeq p$

b. Energy conservation and time translation symmetry

Time translation $t \to t + \varepsilon$:

$\delta L(t) = \varepsilon \partial_t L \longrightarrow \Lambda = \varepsilon L$

On the other hand:

$\delta_\varepsilon q = \varepsilon \partial_t q$

$\longrightarrow \delta L = \partial_t(\frac{\partial L}{\partial \dot{q}}\varepsilon\dot{q} = \varepsilon\partial_t(p\dot{q})$

$\longrightarrow Q_\varepsilon \simeq p\dot{q} - L = H$

c. Spherical symmetry of central potential

$L = \frac{1}{2}m\dot{\mathbf{x}}^2 - V(|\mathbf{x}|)$

Rotation around $\hat{x}_i$-axis by angle $\delta\alpha$:

$\delta_i x_j = -\delta\alpha\varepsilon_{ijk}x_k \longrightarrow \delta L = 0 \longrightarrow \Lambda = 0$

Obtain:

$$\begin{aligned} Q_i &= \frac{\partial L}{\partial \dot{x}_j}\delta_i x_j \\ &\simeq -p_j\varepsilon_{ijk}x_k = (\mathbf{x}\times\mathbf{p})_i = L_i \end{aligned}$$

We see that the conserved charges Q_i correspond to the components of the angular momentum vector $\mathbf{L}$.

2.2 Conserved charges generate the corresponding symmetries

- In the Hamiltonian framework it is convenient to use Poisson brackets to discuss symmetries and conserved quantities:

  Consider a quantity $Q_a = Q_a(q_i, p_i)$

  then

  $\frac{dQ_a}{dt} = \{Q_a, H\}$

  so that

  $$\boxed{\{Q_a, H\} = 0 \longleftrightarrow \frac{dQ_a}{dt} = 0}$$

  Note that energy conservation is automatic i n this picture.

- The conserved quantity Q_a *"generates"* the symmetry transformations on the phase space through the Poisson brackets:

  $\delta_a q^i = \{q^i, Q_a\} = \frac{\partial Q_a}{\partial p_i}$

  $\delta_a p_i = \{p_i, Q_a\} = -\frac{\partial Q_a}{\partial q_i}$

2.3 Algebraic structure

- Consider a set $\{Q_a\}$ of independent conserved quantities (i.e. $\{\partial_{q_i} Q_a, \partial_{p_i} Q_a\}$ are independent vectors),

  then

  $\{Q_a, Q_b\}$

  will satisfy (because of Jacobi identity)

  $\{\{Q_a, Q_b\}, H\} = 0$

  i.e. is also conserved.

- In general we obtain a closed (polynomial) algebra

  $\{Q_a, Q_b\} = P_{ab}(\{Q_c\})$

  (finitely generated - (non)linear)

- Lie Algebra

  Most relevant is the linear case, in which case we speak of a Lie algebra:

  $\{Q_a, Q_n\} = f_{ab}{}^c Q_c \quad ,$

  with the $f_{ab}{}^c$ being the structure constants.

  Example:

  The components of the angular momentum vector span the following algebra:

  $\boxed{\{L_i, L_j\} = \varepsilon_{ijk} L_k}$

  where the structure constants correspond to the totally antisymmetric ε symbol, which is completely fixed by specifying $\varepsilon_{123} = 1$ and its antisymmetry $(\varepsilon_{ijk} = -\varepsilon_{jik} = -\varepsilon_{ikj})$ This is the algebra of the generators of rotations in three dimensions denoted as SO_3.

3 Quantum mechanics

3.1 Quantum versus classical

Classical	Quantum
State: $(q(t), p(t))$	State vector: $\lvert\psi(t)\rangle \in \mathcal{H}$
Dynamical variables: $A(q,p)$	Operators: $\hat{A}(\hat{p},\hat{q}) = \hat{A}^\dagger$
Poisson brackets: $\{A,B\} \equiv \frac{\partial A}{\partial q}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial q}$ $\{q,p\} = 1$	Commutators: $\left[\hat{A},\hat{B}\right] \equiv \hat{A}\hat{B} - \hat{A}\hat{B}$ $[\hat{q},\hat{p}] \equiv i\hbar$
Time evolution of the state: $\dot{q} = \frac{\partial H}{\partial p}$ $\dot{p} = -\frac{\partial H}{\partial q}$ (Hamilton equations)	$i\hbar\partial_t \lvert\psi(t)\rangle = \hat{H}\lvert\psi(t)\rangle$ (Schrödinger equation)
Description of dynamics: $\boxed{\frac{dA}{dt} = \{A,H\}}$	$\boxed{i\frac{d\hat{A}(t)}{dt} = [\hat{A}(t),\hat{H}]}$ (Heisenberg equation) $\hat{A}(t) \equiv \hat{U}(t)\hat{A}\hat{U}^\dagger(t)$ $\lvert\psi(t)\rangle = \hat{U}(t)\lvert\psi(0)\rangle$ $\hat{U}(t) = e^{-i\hat{H}t/\hbar}$

3.2 Symmetries and Quantum degeneracies

Symmetries lead on the quantum level in general to degeneracies.

- The symmetry operators Q_a satisfy

  $$\begin{array}{lll} [Q_a, H] = 0 & \longrightarrow Q_a & \text{is time independent} \\ & \longrightarrow [Q_a, Q_b] & \text{is also symm.op.} \end{array}$$

- Spectral degeneracy:

  $H |u_n\rangle = E_n |u_n\rangle$

  $|\psi(t)\rangle = \sum_n c_n(t) |u_n\rangle$ (?)

  Consider the state $Q_a |u_n\rangle$

  $$\begin{aligned} H(Q_a |u_n\rangle) &= [H, Q_a] |u_n\rangle + Q_a H |u_n\rangle \\ &= Q_a E_n |u_n\rangle = E_n (Q_a |u_n\rangle) \end{aligned}$$

  $\longrightarrow Q_a |u_n\rangle$ is also eigenstate of H with same eigenvalue

- Eigenspaces of H "carry" *unitary* $\left(because\ for\ an\ observable : Q_a = Q_a^\dagger\right)$ *representations* of the *symmetry algebra* $([Q_a, Q_b] = f_{ab}{}^c Q_c)$

  $\longrightarrow \left[\text{Associated Lie Group: } U(x^a) = e^{ix^a Q_a}\right]$

3.3 Labelling of states in Quantum Mechanics

- To label the states we choose a maximal set of *mutually commuting* independent operators $\{H_i\}$ (one of which is the Hamiltonian)

  $[H_i, H_j] = 0$ for all i, j

  As these operators H_i mutually commute they can be simultaneously diagonalized and we can choose a basis for the Hilbert space consisting of simultaneous eigenvectors of $\{H_i\}$

  $H_i \,|\{a_k\}\rangle = a_i \,|\{a_k\}\rangle$

- The other $Q_a \notin \{H_i\}$ will mix these states to a certain extent, leading to irreducible subspaces $\longleftrightarrow$ irred.reps of symm.algebra

  Example

  Traslational invariance $x \in \Re$:

  $H_0 = H = \frac{p^2}{2m} \quad H_1 = p = -i\hbar\frac{d}{dx}$

  Here p is the generator of translations.

  Eigenstates:

  $p|k\rangle = \hbar k|k\rangle$

  $H|k\rangle = \frac{\hbar^2 k^2}{2m}|k\rangle$

  with wavefunctions

  $\psi_k(x) = \langle x|k\rangle = \sqrt{\frac{1}{2\pi\hbar}}e^{ikx}$

  The representations are one-dimensional and labeled by k.

  Remark:

  Suppose that we would have quantised the particle on a circle, implying that

  $x \to x + 2\pi R$,

  then we have to demand in addition that

  $\psi_k(x) = \psi_k(x + 2\pi R)$.

  From this condition one derives that k has to satisfy the condition

  $k = \frac{n}{R}$ for any integer n. The momenta would be quantised.

3.4 The Hydrogen atom

- Hamiltonian: $H = \frac{\mathbf{p}^2}{2m} - \frac{\mu}{r} \quad r = |\mathbf{x}|$

  Where the operator $\frac{\mathbf{p}^2}{2m} = \frac{-\hbar^2}{2m}\nabla^2$ is up to a constant the three-dimensional Laplace operator.

- The Hamiltonian is invariant under rotations because it only depends on the 'length' of the vectors $\mathbf{p}$ and $\mathbf{x}$. The rotational symmetry is generated by the components of the angular momentum operator

  $L_i = \varepsilon_{ijk} x_j p_k \quad [L_i, H] = 0$

  SO_3 algebra*: $[L_i, L_j] = i\varepsilon_{ijk} L_k$

  Important is the socalled Casimir operator $\mathbf{L}^2 = \sum_i L_i^2$: it is an invariant operator which does commute with all components L_i

  $[\mathbf{L}^2, L_i] = 0$

- Considering the Schrödinger equation as a second order differential equation the presence of symmetry will lead to a preferred system of coordinates for which the equation becomes seperable. For the case of spherical (cylindrical) symmetry for example one may choose spherical (cylindrical) coordinates. See section **4.1** in the book of Griffith. The connection with the symmetry operators is as follows:

  $H = \frac{-\hbar^2}{2m} \frac{\partial}{\partial r}^2 - \frac{\mu}{r} + \frac{\hbar^2}{2mr^2}\mathbf{L}^2$

  where $\mathbf{L}^2$ only depends on the angular variables (θ, ϕ)

  $\mathbf{L}^2 = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$

  To solve the Schrödinger equation the wavefunction ψ can be written as a product

  $\psi(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi)$

  and the three-dimensional partial differential equation can be separated in three ordinary differential equations.

  Up to normalisation the solutions can be written as

  $\psi_{n,l,m}(r, \theta, \phi) = \langle r, \theta, \phi | n, l, m\rangle = j_l(k_n r) Y_l^m(\theta, \phi)$

- As a maximal set of mutually commuting operators one may choose $\{H, \mathbf{L}^2, L_3\}$ with simultaneous eigenstates $|n, l, m\rangle$

  $H\,|n, l, m\rangle = \frac{c}{n^2}\,|n, l, m\rangle$

  $\mathbf{L}^2\,|n, l, m\rangle = l(l+1)\,|n, l, m\rangle$, with $0 \leq l \leq n-1$

  $L_3\,|n, l, m\rangle = m\,|n, l, m\rangle$, with $-l \leq m \leq l$

  The eigenvalues l and m are completely determined by the structure of the symmetry algebra SO_3 and the hermiticity of the L_i operators (as is explained in section **4.3** in Griffith).

- Unitary Irreducible Reps of SO_3: UIR's of SO_3, labeled by $l \in Z_+$, are $(2l+1)$ dimensional with basis $\{|l, m\rangle\}$, $-l \leq m \leq l$

  $\longrightarrow$ Eigenspaces of H are not irreducible SO_3 rep's (the energy eigenvalues are not only independent of m, as it should, but also independent of l).

  $\longrightarrow$ is this additional degeneracy due to some extra, accidental (dynamical) symmetry?

3.5 Symmetries of the H-atom (continued)

- Indeed, for the special case $V(r) = -\frac{\mu}{r}$ there is an additional (dynamical) symmetry generated by the Runge-Lenz vector $\mathbf{R}$:

  $\mathbf{R} = \mathbf{L} \times \mathbf{p} + \mu \frac{\mathbf{r}}{r}$

  with $[R_i, H] = 0$

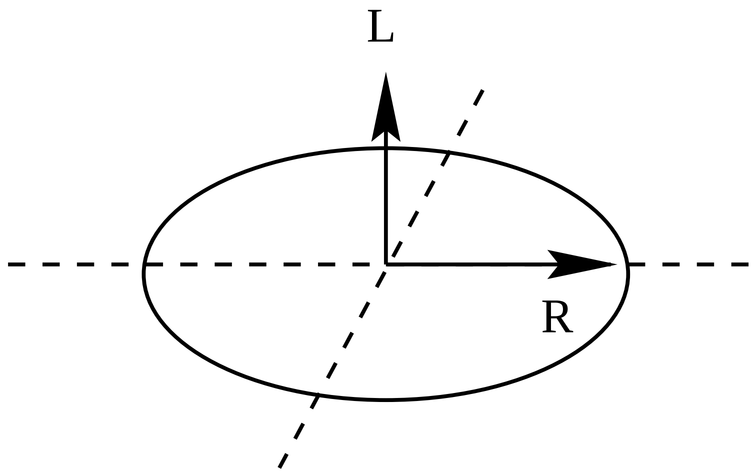


Figure 1: *On the classical level the Keppler orbit is indeed closed (and fixed) if $V = -\frac{\mu}{r}$, its orientation in space is fixed by the conserved vectors* $\mathbf{L}$ *and* $\mathbf{R}$.

- The total symmetry algebra now becomes:

  $[L_i, L_j] = i\varepsilon_{ijk} L_k$

  $[L_i, R_j] = i\varepsilon_{ijk} R_k$

  $[R_i, R_j] = -2\varepsilon_{ijk} \underbrace{H L_k}$

  $\longrightarrow$ Algebra is non-linear!

- On an eigenspace of H, the Hamiltonian is constant and we are left with the (linear) algebra $SO_4 \simeq SO_3 \times SO_3$ (as the notation suggests this algebra is indeed isomorphic to the one which generates rotations in 4 euclidean dimensions)

  $\rightarrow$ The SO_4 symmetry accounts for the additional degeneracy w.r.t the quantum number l.

3.6 * Beyond Symmetry: Spectrum generating algebra's

- In the operator algebra a special role is played by the operators which satisfy

  $[H, A_i] = a_i A_i$

  These operators transform eigenvectors of different eigenspaces into each other (hence they are called 'step' operators)

  $H\left(A_i\left|n\right\rangle\right) = [H, A_i]\left|n\right\rangle + A_i H\, n\rangle$

  $= (a_i + E_n)\left(A_i\left|n\right\rangle\right)$

- \- The symmetry algebra is a subalgebra of the spectrum generating algebra
  \- For the Hydrogen atom this spectrum generating algebra is the algebra $SO_{4,2}$
  (See for example the book *Classical Groups for Physicists* by B.G. Wybourne)

4 A sample problem: Symmetries of 2-dim. harmonic oscillators

Harmonic oscillators play a central role in the description of basically every physical system, this is so because small perturbations around a stable configuration in most cases lead lead to harmonic oscillations. This system is ofcourse trivial to solve, but here we study it mainly as an interesting playground to illustrate basic symmetry concepts in quantum mechanics.

So we consider in the following two problems a particle of mass M moving in the (x_1, x_2) - plane in a harmonic potential.

4.1 The isotropic case

1. Consider the Hamiltonian

$$H = \frac{1}{2M}(p_1^2 + p_2^2) + \frac{1}{2}M\omega^2(x_1^2 + x_2^2) \tag{1}$$

with ω the 'angular' frequency of the oscillator. Give the Hamilton equations.

2. These equations decouple if one considers the linear combinations

$$\begin{aligned} a^{\pm} &= \sqrt{\frac{M\omega^2}{2}}x_1 \pm i\sqrt{\frac{1}{2M}}p_1 \\ b^{\pm} &= \sqrt{\frac{M\omega^2}{2}}x_2 \pm i\sqrt{\frac{1}{2M}}p_2 \end{aligned}$$

Give the equations for $a^{\pm}$ and $b^{\pm}$.

3. Solve these equations and check that they also yield the expected solutions for $x_1(t)$ and $x_2(t)$.

4. In the corresponding quantummechanical problem we should interpret $x_i$and p_i as operators satisfying the canonical commutation relations

$$\begin{aligned} [x_i, p_j] &= i\hbar\delta_{ij} \\ [x_i, x_j] &= [p_i, p_j] = 0. \end{aligned}$$

Derive the commutation relations for the 'raising' and 'lowering' operators $a = a^+, a^\dagger = a^-, b = b^+$ and $b^\dagger = b^-$.

5. Write the Hamiltonian in terms of the 'raising' and 'lowering' operators.

6. The ground- or lowest energy state $|\Omega >$ is defined by the property that it is annihilated by the 'lowering' operators:

$$a|\Omega >= b|\Omega >= 0 \tag{2}$$

The excited states are obtained by acting with the 'raising' operators on the groundstate. Show that the state

$$|p,q>=(a^\dagger)^p(b^\dagger)^q|\Omega> \tag{3}$$

is an eigenstate of the Hamiltonian. Clearly the operator algebra of the 'raising' and 'lowering' operators is the 'spectrum generating' algebra of the problem at hand. Determine the energy eigenvalue of $|p,q>$.

7. Which states are degenerate and how many are there on a given level.

8. The degeneracy in the spectrum suggests the existence of synmetries. Consider the operators

$$\begin{aligned} S_+ &= ab^\dagger, \\ S_- &= a^\dagger b, \\ S_0 &= b^\dagger b - a^\dagger a. \end{aligned}$$

Show that these commute with H and are indeed time independent.

9. Now determine the algebra of the S-operators. Which well known algebra do you get. Note that this symmetry is 'accidental' or 'dynamical' as it does not immediately follow from the geometric symmetry in the problem (i.e. just the SO_2 symmetry of rotations in the plane generated by S_0).

10. The states $|p,q>$ are eigenstates of S_0, determine the eigenvalues s_0.
A highest weight state $|l>$ is one which satisfies

$$\begin{aligned} S_0|l> &= l|l> \\ S_+|l> &= 0. \end{aligned}$$

Which states $|p,q>$ are highest weight states and what are their eigenvalues s_0.

11. Act repeatedly with the operator S_- on $|l>$. What are the H and S_0 eigenvalues E and s_0 you subsequently obtain. What is the smallest value for s_0 that occurs. Draw the spectrum as dots in the (E,s_0)-plane.

12. Interprete the spectrum in terms of the representation theory of the symmetry algebra.

4.2 * The anisotropic case

We now consider a Hamiltonian

$$H = \frac{1}{2M}(p_1^2 + p_2^2) + \frac{1}{2}M(\omega_1^2 x_1^2 + \omega_2^2 x_2^2). \tag{4}$$

As we are interested in the case where degeneracies occur we choose $\omega_1 = 1/m$ and $\omega_2 = 1/n$, with m and n positive integers. For convenience we set $M = \hbar = 1$.

1. Check that the folllowing 'raising' and 'lowering' operators:

$$\begin{aligned} a^\dagger &= \frac{1}{\sqrt{2}}(\frac{x_1}{\sqrt{m}} - i\sqrt{m}p_1) \\ a &= \frac{1}{\sqrt{2}}(\frac{x_1}{\sqrt{m}} + i\sqrt{m}p_1) \\ b^\dagger &= \frac{1}{\sqrt{2}}(\frac{x_2}{\sqrt{n}} - i\sqrt{n}p_2) \\ b &= \frac{1}{\sqrt{2}}(\frac{x_2}{\sqrt{n}} + i\sqrt{n}p_2), \end{aligned}$$

   satisfy the standard commutation relations (as obtained in 1.e)

2. Give the Hamiltonian in terms of the 'raising' and 'lowering' operators.

3. Check that the operators $S_\pm$ defined in 1.h are no longer symmetry operators.

4. Determine the degeneracies in the spectrum of $|p, q >$ states.

5. Verify that the following operators do commute with H

$$\begin{aligned} \$_+ &= a^m (b^\dagger)^n \\ \$_- &= (a^\dagger)^m b^n \\ \$_0 &= \frac{1}{n}(b^\dagger b + \frac{1}{2}) - \frac{1}{m}(a^\dagger a + \frac{1}{2}). \end{aligned}$$

   What is the logic behind this.

6. Let us from now on restrict ourselves to the simplest nontrivial case where $m = 2$ and $n = 1$. Consider the following symmetry operators

$$\begin{aligned} j_+ &= \frac{1}{\sqrt{3}} a^2 b^\dagger \\ j_- &= \frac{1}{\sqrt{3}} (a^\dagger)^2 b \\ j_0 &= \frac{2}{3}(b^\dagger b - a^\dagger a). \end{aligned}$$

   Write down the algebra of these operators including also H. One of the commutators is nonlinear and contains terms proportional to j_0^2 and H^2. This nonlinear algebra is called $W_3^{(2)}$

7. We expect this algebra to have finite dimensional unitary representations. To find out about this, study the action of the symmetry operators on the low lying states of the spectrum. Draw the dots in the (E, j_0)-plane.

8. Determine which $|p, q>$ states are of highest weight, and which are of lowest weight.

9. Compare the spectra of the isotropic and anisotropic case in the (p, q)-plane and indicate the various quantum numbers as well as the actions of the various symmetry operators.