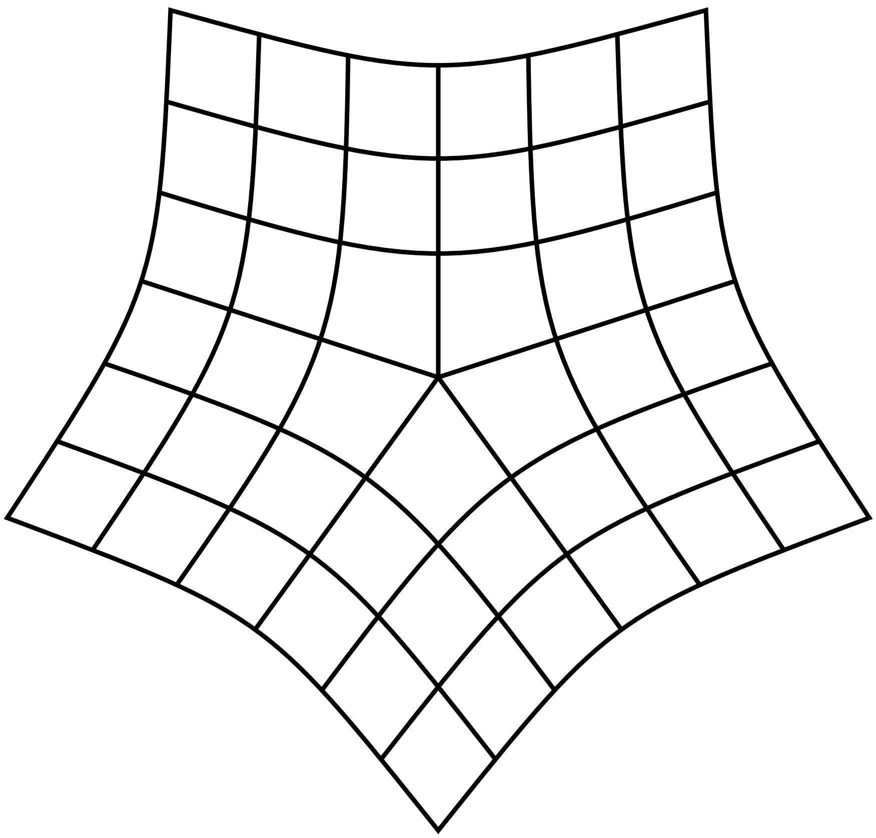
Kleine Werkgroep 1994–1995

Topological Defects



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Overview “Kleine Werkgroep”

The subject of this year's "Kleine Werkgroep" will be topological excitations in spontaneously broken gauge theories. Symmetries play an important role in a wide range of physical systems, and to a high extent they determine the properties of the fundamental excitations in a medium. If the ground state of a system is not invariant under the symmetry group of the Hamiltonian, we say that there is a spontaneously broken phase. Examples are the ferromagnet, the BCS superconductor, nematic crystals, superfluid helium, the theory of electroweak interactions, Grand Unified Theories, etc. These broken phases of a theory are characterised by the occurrence of topological excitations or defects: domain walls, vortices (flux tubes), magnetic monopoles, instantons and sphalerons.

We will start with a short introduction into field theory and gauge theories, which will be illustrated by some basic examples. Then we will turn to the topological structure of the excitations and their physical interpretation, where we will often return to the examples that were discussed before. Finally some quantum mechanical properties will be studied.

Although the notes and problems are rather self-contained, it is inevitable that the student has to consult the literature for further explanation or refreshment of knowledge. Every problem set will refer to a few books and lecture notes that can be used, but of course there are many more that treat the subjects.

1st afternoon, 3 April **Classical Field Theory**

- Introduction into field theory, Lagrange formalism and Noether's theorem for fields
- Electrodynamics and its symmetries: Maxwell action, principle of gauge invariance
- The Klein–Gordon equation: its action, symmetries, conserved currents
- Coupling of complex scalar field to electromagnetism: local symmetry, covariant derivative, gauge invariance

2nd afternoon, 10 April **Non-Abelian theories**

- Three-component scalar field and a non-abelian group: internal symmetry
- Yang–Mills theory: non-abelian field strength, Lagrangian, Georgi–Glashow model
- The Standard Model: some symmetries and some fields

3rd afternoon, 24 April **Symmetry breaking**

- The Hamiltonian in field theory
- Global symmetry breaking: vacuum degeneracy, Goldstone fields
- Local symmetry breaking, the abelian Higgs model: Higgs mechanism of mass generation
- Glashow–Weinberg–Salam model: Higgs field, masses of elementary particles

4th afternoon, 8 May **Topological Excitations**

- Topological conservation laws, sine-Gordon equation: solitons
- Nielsen–Olesen vortices: flux tubes in the abelian Higgs model, winding number, magnetic flux quantisation
- Homotopy groups: mathematical background of topological charge, fundamental group

5th afternoon, 15 May **Magnetic Monopoles**

- The broken Georgi–Glashow model: topological charge, the vacuum manifold
- Magnetic charge as a topological charge: identity mapping, asymptotics
- An exact monopole solution: Bogomolny argument, BPS-limit

6th afternoon, 22 May **Quantisation**

- Path integral in quantum mechanics: sum over histories
- Example, the harmonic oscillator
- The semi-classical approximation
- Double well: an introduction to instantons

7th afternoon, 29 May **Instantons**

- The lowest energy levels: energy split for the double well
- Final calculation of the energy split: K and λ_0
- Bounces, and the life-time of the neutron

8th afternoon, 12 June **Solitons**

- Semi-classical approximation for the soliton
- The quantum correction on the soliton mass

9th afternoon, 19 June

- Discussion of problem set of previous afternoon

I. Classical Field Theory

Abstract

We discuss some of the basic concepts of classical field theory. The equation of motion for the fields will be derived from the action functional, using the Lagrangian formalism.[1] We will consider the electromagnetic field, a real scalar field and a complex scalar field. Imposing local gauge invariance leads to the introduction of gauge fields.

References:
F.A. Bais, lecture notes "Inleiding Quantumveldentheorie".
J.W. van Holten, lecture notes "Symmetriebeschouwingen in de natuurkunde" (2 volumes).
D. Bailin & A. Love, "Introduction to Gauge Field Theory", Adam Hilger, Bristol 1986.

Fields

A field ϕ assigns to each point x in spacetime a quantity $\phi(x)$. This quantity may be a real or complex number, vector, tensor, etc. As examples one may think of temperature, velocity of flow in a liquid, stress in a solid body. After quantisation a large class of relativistic field theories describe particles.
Recall that under a spacetime coordinate transformation

$$x \to x' = f(x), \tag{1}$$

a general tensor field $\phi^{\alpha\beta\cdots}_{\mu\nu\ldots}(x)$ transforms according to

$$\phi^{\alpha\beta\ldots}_{\mu\nu\ldots}(x) \to \phi'^{\alpha\beta\ldots}_{\mu\nu\ldots}(x) = \frac{\partial x'^{\sigma}}{\partial x^{\mu}}\frac{\partial x'^{\tau}}{\partial x^{\nu}}\cdots\frac{\partial x^{\alpha}}{\partial x'^{\zeta}}\frac{\partial x^{\beta}}{\partial x'^{\eta}}\cdots\phi^{\zeta\eta\cdots}_{\sigma\tau\ldots}(x'). \tag{2}$$

In particular, the transformation of a scalar field ϕ is given by

$$\phi(x) \to \phi'(x) = \phi(x'). \tag{3}$$

The Lagrangian formalism

We assume that the action $S[\phi]$ of a field $\phi(x)$ is given by[2]

$$S[\phi] = \int \mathrm{d}^4x\, \mathcal{L}(\phi, \partial_\mu\phi), \tag{4}$$

where $\mathcal{L}$ is the Lagrangian density. This means that $\mathcal{L}$ is a local function depending only on ϕ and its first derivatives. In analogy to classical mechanics, the variational principle requires that the action is stationary with respect to infinitesimal variations in the field. In order to derive the equation of motion for the field from this condition, consider an infinitesimal variation $\delta\phi(x)$, which vanishes at infinity, of the field $\phi(x)$. This gives rise to a variation of the action:

$$\delta S \;=\; \int \mathrm{d}^4x\, \delta\mathcal{L}$$

[1]This you have encountered in the courses "Klassieke Mechanica" and "Symmetriebeschouwingen", where a classical, non-relativistic particle was considered.
[2]$\partial_\mu = \frac{\partial}{\partial x^\mu}$; in our conventions the signature of the metric on spacetime will be $(+---)$.

$$
\begin{aligned}
&= \int \mathrm{d}^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) \right) \\
&= \int \mathrm{d}^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu(\delta \phi) \right) \\
&= \int \mathrm{d}^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) + \left. \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right|^{\infty} \\
&= \int \mathrm{d}^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi. \qquad (5)
\end{aligned}
$$

Thus the action is stationary for any variation $\delta\phi$ if the Euler–Lagrange equation

$$
\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0 \qquad (6)
$$

is satisfied, which is the equation of motion for the field.

Noether's theorem

Noether's theorem states that for any 1-parameter group of invariances of the action, there is a corresponding conserved quantity. This conserved quantity can be found as follows. An infinitesimal transformation gives rise to a variation $\delta\phi$ of the field, hence a variation

$$
\delta S = \int \mathrm{d}^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) \right) \qquad (7)
$$

of the action. Invariance of the action under the transformation means that δS vanishes, which implies that the above integrand is a total derivative:

$$
\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) = \partial_\mu \Omega^\mu. \qquad (8)
$$

Using the Euler–Lagrange equation one finds that the current

$$
J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi - \Omega^\mu \qquad (9)
$$

is conserved (divergenceless):

$$
\partial_\mu J^\mu = 0, \qquad (10)
$$

so the corresponding charge

$$
Q = \int \mathrm{d}^3x \, J^0 \qquad (11)
$$

is conserved (time-independent). Thus the quantity Q is does not change in time, provided that the field evolves according to the field equations.

Problem 1 **Electrodynamics and its symmetries**

Recall that the electric field $\vec{E}$ and the magnetic field $\vec{B}$ can be written in terms of a vector potential $\vec{A}$ and a scalar potential ϕ:

$$\begin{aligned} \vec{B} &\equiv \vec{\nabla} \times \vec{A} \\ \vec{E} &\equiv -\vec{\nabla}\phi - \partial_t \vec{A}. \end{aligned} \tag{12}$$

In a manifestly Lorentz covariant notation, we introduce the four-vector potential $A^\mu = (\phi, \vec{A}\,)$ and the electromagnetic field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{13}$$

The six independent components of this anti-symmetric tensor field can be expressed in the components of the electric and magnetic field by

$$\begin{aligned} F_{ij} &= -\epsilon_{ijk} B^k \\ F_{0i} &= E^i \end{aligned} \tag{14}$$

where the units are chosen such that the constants c, ϵ, and μ are equal to 1. The action for the electromagnetic field is given by

$$S[A] = \int \mathrm{d}^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right). \tag{15}$$

(a) Determine the field equations from this action, using the Lagrangian formalism. Verify that you have regained the Maxwell equations.

(b) Under a spacetime coordinate transformation $x \to x'$, the field A^μ transforms as a vector, Eq. (2). Consider an infinitesimal coordinate transformation

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu, \tag{16}$$

where ξ may depend on x. Calculate

$$\delta A_\mu \equiv A'_\mu(x) - A_\mu(x). \tag{17}$$

(c) Calculate $\delta F_{\mu\nu}$ for the transformation Eq. (16). Show that $\mathcal{L}$ transforms as a scalar.

(d) The Maxwell action, Eq. (15), is invariant under translations, which are a special case of Eq. (16). Using Noether's theorem, show that the conserved currents $T^\mu{}_\nu$ corresponding to the translation invariance are given by

$$T^\mu{}_\nu = -F^{\mu\sigma} \partial_\nu A_\sigma - \delta^\mu{}_\nu \mathcal{L}. \tag{18}$$

$T^\mu{}_\nu$ is called the energy-momentum tensor. Calculate the corresponding conserved charges P_ν. Interpret P_0 and $\vec{P}$.

The Maxwell action is also invariant under Lorentz transformations, where ξ^μ in Eq. (16) is given by

$$\xi^\mu = \omega^\mu_\nu x^\nu. \tag{19}$$

$\omega_{\mu\nu}$ is antisymmetric and independent of x. We shall not elaborate on the Lorentz invariance of electrodynamics. Translations and Lorentz transformations are called *global* transformations, because they are the same for each point in spacetime: for the former ξ^μ is constant, for the latter $\omega_{\mu\nu}$ does not depend on x.
We now return to Eq. (12) and note that for given $\vec{E}$ and $\vec{B}$ the vector and scalar potential are not uniquely determined. In fact we can perform a so-called *gauge transformation*

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu \alpha \qquad (20)$$

where $\alpha(x)$ is any function of spacetime coordinates. This transformation leaves the electric and magnetic field unchanged. Because $\alpha(x)$ depends on the point x in spacetime, the gauge transformation is called a *local* transformation.

(e) Show that the physical fields and the Maxwell action remain unchanged under the gauge transformation Eq. (20).

(f) The energy-momentum tensor $T^{\mu\nu}$ is not invariant under the gauge transformation Eq. (20). Show that

$$\hat{T}^\mu{}_\nu \equiv T^\mu{}_\nu + F^{\mu\sigma}\partial_\sigma A_\nu \qquad (21)$$

is a conserved current whose conserved charge is equal to P_ν. Show that $\hat{T}^\mu{}_\nu$ is gauge invariant.

(g) Show that $\hat{T}^\mu{}_\nu$ is symmetric and traceless, that is

$$\hat{T}^\mu{}_\mu = 0. \qquad (22)$$

(h) Express $\hat{T}^\mu{}_\nu$ in terms of $\vec{E}$ and $\vec{B}$. Interpret the result.

(i) Replace the action Eq. (15) with

$$S[A] = \int \mathrm{d}^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_\mu j^\mu\right), \qquad (23)$$

where j^μ is a gauge invariant four-vector. Determine the field equations for this action and interpret your findings. What constraint on j^μ is implied by gauge invariance of S?

Problem 2 **The Klein–Gordon equation**

First we will derive the Klein–Gordon equation as a relativistic generalisation of the Schrödinger equation, and then obtain the corresponding Lagrangian density. Demanding Lorentz invariance will give us conserved current and charges.
The basic equation for non-relativistic quantum mechanics is the Schrödinger equation[3]

$$i\partial_t\psi(\vec{x},t) = H\psi(\vec{x},t). \qquad (24)$$

A non-relativistic particle in potential $V(\vec{x})$ corresponds to the Hamiltonian

$$H = \frac{\vec{p}^{\,2}}{2m} + V(\vec{x}). \qquad (25)$$

[3]We set $\hbar = 1$.

Written like this, we are dealing with the coordinates and their corresponding momenta, which you should recognise from the Hamilton formalism. Since we want to use the Euler–Lagrange formalism to derive the equations of motion, we should substitute the coordinate representation for the momentum (*i.e.* rewrite the momentum operator), $\vec{p} = -i\vec{\nabla}$. Trying to make the equation relativistic, one could try to substitute the relativistic expression for the energy

$$E = \sqrt{\vec{p}^2 + m^2}. \tag{26}$$

The resulting equation is very difficult to solve, because of the higher order terms in the expansion of the square root. This can be circumvented by squaring both sides of the Schrödinger equation.

(a) Perform the substitution and squaring described above. Use the expression for the relativistic Laplace operator (d'Alembertian):

$$\partial_\mu \partial^\mu = \partial_t^2 - \vec{\nabla}^2 \tag{27}$$

and show that

$$(\partial_\mu \partial^\mu + m^2)\psi = 0. \tag{28}$$

This is the *Klein–Gordon* equation. The interpretation as a relativistic wave equation leads to difficulties we cannot go into here. Without further motivation we now interpret it not as a wave equation, but as an equation for a classical field ϕ:

$$(\partial_\mu \partial^\mu + m^2)\phi = 0. \tag{29}$$

Upon quantisation this field describes particles that satisfy the relativistic energy momentum relation Eq. (26).

(b) The equation describing the field may not depend on the inertial frame of the observer. Therefore the field equation should be Lorentz covariant. How should the field ϕ transform under Lorentz transformations?

(c) Find an action $S[\phi]$ for which Eq. (29) is the equation of motion. Which term in $S[\phi]$ gives rise to the $m^2\phi$ term in the field equation?

The mass term in the Lagrangian density for a general field (so not necessarily transforming as the field ϕ we discussed above) has the same form as the mass term for the Klein–Gordon field. This will return frequently during the course.

(d) Show that the action for the Klein–Gordon field is invariant under translations. Calculate the corresponding conserved current $T^{\mu\nu}$.

(e) As was already indicated under (b), the action for the Klein–Gordon field is also invariant under Lorentz transformations. Give the infinitesimal form of the Lorentz transformations and show that the corresponding conserved currents are given by

$$J^{\mu\sigma\rho} = x^\rho T^{\mu\sigma} - x^\sigma T^{\mu\rho}. \tag{30}$$

Problem 3 **Coupling of a complex scalar field to electromagnetism**

As an example of an interacting field theory, we consider a complex scalar field with Lagrangian density

$$\mathcal{L}(\phi, \phi^*) = (\partial_\mu \phi)(\partial^\mu \phi^*) - m^2 \phi\phi^*. \tag{31}$$

(a) Give the field equations.

(b) Show that the Lagrangian density is invariant under global phase transformations of the scalar field

$$\phi \to \phi' = e^{i\alpha}\phi \tag{32}$$

and give the corresponding Noether current. These phase transformations form the group $U(1)$.

It is interesting to see what happens if we demand that the theory is also invariant under local phase transformations, *i.e.*, α now depends on the spacetime coordinates:

$$\phi \to \phi' = e^{i\alpha(x)}\phi. \tag{33}$$

(c) Is the Lagrangian density invariant under these transformations?

In order to restore the invariance we may introduce a so-called *gauge potential* A_μ and define the *(gauge) covariant derivative* D_μ by

$$\begin{aligned} D_\mu \phi &= (\partial_\mu + ieA_\mu)\phi \\ D_\mu \phi^* &= (\partial_\mu - ieA_\mu)\phi^*. \end{aligned} \tag{34}$$

Note that we have to define the covariant derivative for each independent field. The constant e is the so-called *coupling constant*, because it arises in the term in the Lagrangian that couples the field and the gauge potential. It is crucial that D_μ transforms covariantly

$$D_\mu \phi \to (D_\mu \phi)' = e^{i\alpha(x)} D_\mu \phi. \tag{35}$$

(d) Using this transformation property of $D_\mu \phi$, calculate the transformation of the gauge potential A_μ.

In the Lagrangian density Eq. (31) we now replace the derivative by the covariant derivative.

(e) Verify that the new Lagrangian is invariant under local phase transformations.

Comparing the transformation of A_μ to the gauge transformation in Problem 1 suggests that the gauge potential from this problem can be identified with the vector potential that we have encountered before. If we want A_μ to describe independent, physical degrees of freedom, we have to add the Lagrangian of the free[4] Maxwell field. The resulting Lagrangian then describes *a complex scalar field that is coupled to the electromagnetic field in a gauge invariant way.* This way of introducing interactions can be generalised to construct non-abelian gauge theories, as we will show next time.

[4]Meaning: "not interacting".

II. Non-Abelian Theories

Abstract

This time we will concentrate on non-abelian gauge symmetries. We will elaborate on the concepts that we have encountered last time, so derive the equations of motion (for several fields) and couple them to gauge fields to make the theory locally invariant under a (now non-abelian) symmetry. First we will introduce the concept of non-abelian groups via the three-component scalar field, where we will discuss some group theory which will be useful later. Then we will treat the so-called Georgi–Glashow model for electrodynamics. Finally some aspects of Quantum Chromo Dynamics, the theory for the strong interaction, and the Standard Model will be discussed. Here you will get a flavour of the use of gauge theories.

References:
F.A. Bais, lecture notes "Inleiding Quantumveldentheorie".
J.W. van Holten, lecture notes "Symmetriebeschouwingen in de natuurkunde" (2 volumes).
D. Bailin & A. Love, "Introduction to Gauge Field Theory", Chapter 9, Adam Hilger, Bristol, 1986.
F.A. Bais, in "Geometric Techniques in Gauge Theories", Springer-Verlag, Berlin, 1982: "Symmetry as a clue to the physics of elementary particles".
C. Quigg, "Gauge theories of the strong, weak, and electromagnetic interactions", Benjamin/Cummings, Reading Mass., 1983.

Problem 4 **A three-component scalar field and a non-abelian group**

We will rewrite the the transformations from Problem 3 in such a way that a matrix representation is obtained. To this end we associate a two-component real field to the complex scalar field:[5]

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \leftrightarrow \vec{\phi} \equiv \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \tag{36}$$

which gives the Lagrangian in vector notation:

$$\mathcal{L}(\vec{\phi}) = \frac{1}{2}(\partial_\mu \vec{\phi}) \cdot (\partial^\mu \vec{\phi}) - \frac{1}{2} m^2 \vec{\phi} \cdot \vec{\phi} \tag{37}$$

Under a coordinate transformation $x \to x'$ the fields $\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ transform according to

$$\begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix} \to \begin{pmatrix} \phi_1'(x) \\ \phi_2'(x) \end{pmatrix} = \begin{pmatrix} \phi_1(x') \\ \phi_2(x') \end{pmatrix}. \tag{38}$$

The components transform independently of each other under coordinate transformation, but can be transformed *into* each other by a gauge transformation. In other words: this field ϕ transforms as a scalar under coordinate transformations and as a vector under internal transformations. The covariant derivative now has the following form:

$$D_\mu \vec{\phi} = (\partial_\mu + ie\hat{A}_\mu)\vec{\phi} \tag{39}$$

[5] This is nothing else than considering two independent real fields instead of one complex field.

with

$$\hat{A}_\mu = A_\mu T = A_\mu \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}. \tag{40}$$

A gauge transformation can now be written as

$$\begin{aligned} \Omega &= e^{i\alpha(x)T} \\ \vec{\phi} \to \vec{\phi}' &= \Omega\vec{\phi} \\ \hat{A}_\mu \to \hat{A}'_\mu &= \hat{A}_\mu + \frac{i}{e}(\partial_\mu\Omega)\Omega^{-1} \end{aligned} \tag{41}$$

A gauge transformation is, of course, an element of the gauge group. The way it is written in Eq. (41) you can recognise the exponentiation of elements of the Lie algebra corresponding to the group.[6] The matrix T therefore is the generator of the (Lie algebra of the) gauge group. This rewriting of the gauge transformation is nothing else than using the equivalence of phase transformations in the complex plane with rotations in the two-dimensional (ϕ_1, ϕ_2) plane. Mathematically spoken it shows the isomorphism between the group $U(1)$ and $SO(2)$, the rotation group in two dimensions.
Completely analogous to Problem 3, we will now consider a three-component real scalar field. The Lagrangian is the same as given in Eq. (37), but now a three-component vector is associated with the field $\vec{\phi}$. Under a symmetry transformation each term must be invariant, which means that the inner product that occurs in the second term may not change upon performing a gauge transformation.[7] By definition the group that leaves the inner product invariant is the group of orthogonal matrices, $O(3)$. It consists of the rotations and reflections.
Analogously to Eq.(41) general gauge transformations (so also from higher dimensional symmetry groups) can be written as

$$\Omega = e^{i\alpha(x)^b T_b}. \tag{42}$$

Note the i in the exponent, which is standard in physics. The parameters $\alpha(x)^b$ are real, and b runs from 1 to the dimension of the group, which is equal to the number of generators of its Lie algebra. The T_b form a representation of the generators of the Lie algebra. So in this case of a three component scalar field, the generators are 3×3 matrices.
By exponentiation of the generators like in Eq.(42) only the group elements that are "connected to the identity" can be obtained. For $O(3)$ this means that the reflections are no gauge transformations. What remains is $SO(3)$, the group of rotations in three dimensions.
Note that the dimension of the group and the dimension of the representation are completely different things, though in this case of gauge group $SO(3)$ (three dimensional) and three-component scalar field (which requires a three dimensional representation of $SO(3)$) these dimensions happen to be equal. Until question (d) we shall consider the 3×3 matrix representation of $SO(3)$, which is the so-called *adjoint representation.*
In general the commutation relations for a Lie algebra are of the form

$$[T_a, T_b] = i f_{ab}{}^c T_c, \tag{43}$$

[6] This you have seen in the course "Symmetriebeschouwingen".

[7] Remember that this ϕ only has labels in the internal space, because it is a scalar under spacetime transformations, but a vector under internal transformations. So the vector arrow denotes components in internal space.

with $f_{ab}{}^c$ the structure constants, which completely determine the algebra. So in every representation Eq.(43) must hold. (What are the structure constants of an abelian group?)

(a) Show that the Lie algebra $so(3)$ consists of the hermitian, purely imaginary 3×3 matrices.

(b) Show that $(T_a)_{bc} \equiv -i\epsilon_{abc}$ is an explicit matrix representation for $so(3)$, with ϵ_{abc} the completely anti-symmetric Levi-Civita tensor, take $\epsilon_{123} = 1$.

(c) Using the adjoint representation, calculate the structure constants of $SO(3)$.

(d) Suppose we had taken $T_a \equiv \sigma_a/2$, where σ_a are the Pauli matrices, show that it is a (matrix) representation of $so(3)$.

The corresponding representation space (which is the space of vectors where the representation acts upon) for the representation with the Pauli matrices as generators, is complex and two-dimensional with elements

$$\chi = \begin{pmatrix} u \\ v \end{pmatrix} \quad \text{with} \quad (u, v) \in \mathbf{C}. \tag{44}$$

Under $SO(3)$ such an element transforms as

$$\chi \to \chi' = M(\alpha_a)\chi = e^{\frac{i}{2}\alpha^a \sigma_a}\chi. \tag{45}$$

Two-dimensional vectors which under a rotation transform as Eq.(45) are called *spinors.* The representation which is generated by the Pauli matrices therefore is called the *spinor-representation* of $SO(3)$. Consider now a rotation around the internal x-axis (it is better to call this the "1-direction"), $M(\alpha, 0, 0)$.

(e) Show that upon rotation over 2π the spinor transforms into its opposite: $\chi \to -\chi$.

So with every element of $SO(3)$ we can associate two different matrices, namely $M(\alpha, 0, 0)$ and $M(\alpha + 2\pi, 0, 0)$. Therefore the spinor representation of $SO(3)$ is called *double-valued.* It corresponds to a single-valued representation of $SU(2)$, the group of special unitary transformations in two-dimensional complex space.

(f) Show that indeed the M form the group of unitary (2×2) matrices with $\det M(\alpha_a) = 1$. Prove that the generators of unitary transformations are hermitian.

It is important to realise that we are here dealing with *internal* symmetries. The group $SO(3)$ now does not act on spatial indices, we are not dealing with rotations in space-time, but it acts on internal indices, which means that we consider rotations in the three dimensional internal space. The three independent directions come from the three components of the field. Conserved quantities that are associated with this kind of internal rotation symmetries are called *isospin*, where the "iso" refers to the internal character of the symmetry. Compare this to the spin that describes the effect of spatial rotations on the components of the wave function, Chapter 14 in "Symmetriebeschouwingen".

We will now end this little detour in group theory and return to physics. Let us take again the Lagrangian for a multi-component scalar field, Eq. (37).

(g) What can you say about the masses of the components of the scalar field?

Analogously to Eq. (40) we note that the gauge potential A_μ takes its values in the Lie algebra:

$$A_\mu = A_\mu^b T_b. \tag{46}$$

In other words: the gauge field has three (=dim $SO(3)$) internal components. Each of these components is a four-dimensional vector field in spacetime.
Remember that the covariant derivative transforms covariantly in order to ensure the local invariance of the Lagrangian.

(h) How should D_μ transform?

(i) Prove that the gauge field transforms as

$$A_\mu \to A'_\mu = \Omega A_\mu \Omega^{-1} + \frac{i}{e}(\partial_\mu \Omega)\Omega^{-1}. \tag{47}$$

Problem 5 **Yang–Mills theory**

Models that have a non-abelian gauge symmetry are called *Yang–Mills theories*. They are of great importance in (particle) physics, as we will see later.
Like in electrodynamics the gauge field (or better, its components) "carries the interaction". In electrodynamics the gauge field has one component, the well-known photon field. In the case of $SO(3)$ there are three vector fields carrying the interaction.

(a) How does the field strength as defined in Eq. (13) transform under a non-abelian gauge group?

In general the field strength is defined as

$$F_{\mu\nu} = F_{\mu\nu}{}^b T_b \equiv -\frac{i}{e}[D_\mu, D_\nu]. \tag{48}$$

(b) Calculate the transformation of the field strength Eq. (48) under the gauge transformation Eq. (42).

(c) Express the field strength in the gauge potential, and check that for the abelian case it simplifies to Eq. (13).

It is the non-linearity of $F_{\mu\nu}$ in A_μ, caused by the self-interactions of the gauge field, that makes the physics of non-abelian gauge theories very different from abelian theories like electrodynamics. The fact that the gauge field has self-interactions means that it carries charge corresponding to the gauge group, in contrast to the uncharged photon in electrodynamics.

The Lagrangian density for a free non-abelian gauge field reads:

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{2}\,\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}), \tag{49}$$

which is the generalisation of Eq. (15). It is customary to choose the generators T_b such that

$$\mathrm{tr}(T_a T_b) = \frac{1}{2}\delta_{ab}. \tag{50}$$

Now we will couple this to the three-component scalar field of Problem 4. So we add the "covariantised" Lagrangian of the scalar field to the Yang–Mills Lagrangian.

(d) Check the invariances of the resulting Lagrangian.

We can add potential terms to this Lagrangian, to describe self-interactions of the fields.

(e) What potentials are possible for the scalar field?

(f) Show that demanding local gauge invariance forbids the presence of mass terms for the vector fields.

The resulting model is the Georgi–Glashow model, which describes a three-dimensional scalar field, interacting via an $SO(3)$ gauge field. We shall return to this model later.

Problem 6 **The Standard Model**

(Non-)abelian gauge theories are the theoretical framework for the modern theories of elementary particles and their interactions. In Problem 5 you have encountered a rather simple example, where we showed how to couple a scalar field to a non-abelian gauge field. We will give yet another example of interacting fields, leading to the basic principles of the Standard Model, which describes three of the four fundamental forces in nature: the strong, weak and electromagnetic forces.
This time we will consider the gauge group $SU(2)$. The gauge field W_μ takes values in the Lie algebra $su(2)$ and the Lagrangian is given in Eq. (49). Take the explicit 2×2 matrix representation of $SU(2)$ generated by the Pauli matrices σ_i, as discussed in (d) of Problem 4.

(a) Show that the gauge transformations can be written as

$$\Omega(\vec{\alpha}) = \cos\frac{|\vec{\alpha}|}{2} + i\hat{\alpha}\cdot\vec{\sigma}\sin\frac{|\vec{\alpha}|}{2} \tag{51}$$

and give the explicit matrix form.

For the corresponding representation of a matter field we can now take a doublet of complex fields

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \tag{52}$$

To clarify the jargon a bit you should note the following. This doublet can be thought of as a complex "two-vector" under *internal* transformations. The names scalar, vector, spinor as described in the introduction of the first week, referred to the transformation properties under external spacetime transformations. These names can also be used to describe these properties under internal symmetries, which is indicated by the prefix "iso". Since we are now dealing with an internal symmetry, we could also call the doublet an isospinor.
In the Standard Model there is an additional $U(1)$ symmetry and a corresponding gauge field denoted by B_μ. The total gauge group now is $SU(2) \times U(1)_Y$, where Y refers to the *weak hypercharge*, corresponding to the generator of $U(1)$. Note that this generator commutes with the generators of $SU(2)$. (Remember the fact that in electromagnetism

the electric charge is the conserved quantity that generates the gauge transformations.) The scalar field transforms also under this $U(1)_Y$ symmetry, in a representation that is labelled by Y. So the scalar doublet transforms in different ways under different groups.

(b) How does $U(1)_Y$ act on the field ϕ, in other words: what is the explicit matrix form of the generator Y? Hint: use Schur's lemma.

(c) Give the general form of a gauge transformation in the group $SU(2) \times U(1)_Y$.

(d) Give the general form of the covariant derivative for the scalar field, which respects this gauge symmetry. Explain the terms and factors that you use.

(e) Give the transformations of all the fields under a general gauge transformation.

Until now we have only looked at scalar fields for the non-gauge fields. Most of matter in nature is represented by another type of fields, so-called *fermion fields*, usually denoted by ψ, which have half-integer spin,[8] reflecting the fact that they transform as half-integer representations under rotations. Typically they change sign under rotation over 2π, so the fields associated with these fermion fields are the spinor fields that we have mentioned briefly in Problem 4. See also the course "Symmetriebeschouwingen".
The matter fields together with the $SU(2) \times U(1)_Y$ gauge fields form the so-called *electro-weak theory*. The symmetry in the *Standard Model*, which besides the electromagnetic and the weak interactions also describes the strong interactions, is even larger: there also is an $SU(3)_C$ symmetry. The associated charge is called *colour*. The theory that describes the strong force is called *Quantum Chromo Dynamics*, QCD, where the "chromo" refers to the nature of the charge, like "electro" in QED, which is the quantised version of the Maxwell theory that we discussed in Problem 1. This renders the total local symmetry group of the Standard Model as $SU(3)_C \times SU(2) \times U(1)_Y$.

(f) How many gauge fields mediate the strong interactions? They are called the *gluon* fields, and bind the quarks within a nucleon.

[8] Scalar fields and gauge fields have spin 0 and spin 1, respectively.

III. Symmetry Breaking

Abstract

In the previous sessions we have studied the symmetries of the Lagrangian of certain theories. Here we will consider the case where that symmetry is no longer manifest: it is "broken". We will explain how this can arise, and what the consequences are. We start with a broken global symmetry and then study a broken local symmetry. In these two cases the (broken) group is taken to be abelian. The third problem deals with a broken non-abelian symmetry.

References:
F.A. Bais, lecture notes "Inleiding Quantumveldentheorie".
F.A. Bais, in "Geometric Techniques in Gauge Theories", Springer-Verlag, Berlin, 1982: "Symmetry as a clue to the physics of elementary particles".
S. Coleman, "Aspects of symmetry", Cambridge University Press, Cambridge, 1985: Chapter 5.
C. Quigg, "Gauge theories of the strong, weak, and electromagnetic interactions", Benjamin/Cummings, Reading Mass. 1983.
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The Hamiltonian in field theory

The notion of a Hamiltonian as a function of the generalised coordinates and canonical momenta can be generalised to field theories. The generalised coordinates are the fields $\phi_i(x)$, where x denotes a point in spacetime that can be interpreted as a continuous index which labels the degrees of freedom. The Hamilton density is, in analogy to the particle case, given by

$$\mathcal{H} = -\mathcal{L} + \sum_i \pi_i \dot{\phi}_i, \tag{53}$$

where $\dot{\phi}_i = \partial_t \phi_i$ and the canonically conjugate momenta $\pi_i(x)$ are given by

$$\pi_i = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_i}. \tag{54}$$

$\mathcal{H}$ can be regarded as a function depending on ϕ_i, π_i and the space-derivatives $\partial_k \phi_i$, but not on $\dot{\phi}_i$. The Hamiltonian is given by

$$H(\phi_i, \pi_i) = \int \mathrm{d}^3x \, \mathcal{H}(\phi_i, \pi_i, \partial_k \phi_i). \tag{55}$$

Comparing this with Eq. (18) one sees that $\mathcal{H} = T^{00}$, as was to be expected.
The classical *ground state* of the system is the field configuration $\phi(x)$ for which the Hamiltonian (density) is minimised.
In case of a real scalar field $\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - V(\phi)$, so

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla \phi)^2 + V(\phi). \tag{56}$$

The first two terms are positive definite, so to minimise $\mathcal{H}$ we have to demand

$$\pi(x) = 0 \quad \text{and} \quad \nabla \phi(x) = 0. \tag{57}$$

Denoting the field value for which $\mathcal{H}$ is minimal by $\phi_0(x)$, we see that $\phi_0(x)$ must be a constant, ϕ_0, which is determined by

$$\left.\frac{\partial V}{\partial \phi}\right|_{\phi=\phi_0} = 0, \qquad \left.\frac{\partial^2 V}{\partial \phi^2}\right|_{\phi=\phi_0} > 0. \tag{58}$$

Problem 7 **Global symmetry breaking**

Saying that the Lagrangian density of a theory is invariant under a certain group is equivalent to stating that the dynamics of the theory have a certain symmetry. The key notion now is that the ground state of the system (so the state with the lowest energy) does not have to possess this symmetry. A good example is a magnet. For an observer inside a magnet it is far from obvious that the equations governing the interactions of the spins which make up the overall magnetisation, are rotationally symmetric. We say that the (rotation) symmetry is *spontaneously broken*, or rather, that it is *hidden*, because it is not absent but merely realised in a different way. Note that it is a statement about the ground state rather than about the dynamics, so a broken symmetry can still be exact.
Consider again a Lagrangian density of a complex scalar field

$$\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - V(\phi^* \phi). \tag{59}$$

Assume that the potential that describes the self-interactions of the scalar field has the form

$$V(\phi^*\phi) = \frac{\lambda}{4}(\phi^*\phi)^2 + \frac{\mu^2}{2}\phi^*\phi, \tag{60}$$

where λ is a positive number and μ^2 can be either positive or negative.

(a) Under what symmetry transformation is this potential invariant? Sketch the potentials for positive as well as negative μ^2.

Rewrite the potential such that the minima become clear for negative μ^2:

$$V(\phi^*\phi) = \frac{\lambda}{4}(\phi^*\phi - f^2)^2 + \text{const.} \tag{61}$$

(What is the value of f^2?) The collection of points that are the minima of the potential is called the *vacuum manifold*, in other words: the vacuum is *degenerate*. We now have to make a choice which point of this manifold we call the vacuum. It makes no difference for the physics, so each choice spontaneously breaks the $U(1)$ invariance of $\mathcal{L}$. Note that the vacuum manifold is obtained by letting the symmetry group act on a chosen vacuum. This degenerate set of ground states, of which each can be transformed into any other ground state by the action of the symmetry group, is an example of an *orbit* of the group action.
To exhibit the physical implications we expand the field ϕ around a value in the degenerate ground state

$$\phi(x) = f + \frac{1}{\sqrt{2}}(\chi(x) + i\sigma(x)) \tag{62}$$

where χ, σ are real fields. We now want to study the spectrum of small excitations around the non-trivial vacuum state.

(b) Calculate the Lagrangian to second order in the fields.

(c) What are the masses of the fields χ and σ?

The massless field is called the *Goldstone field* and its occurrence is a consequence of the vacuum degeneracy. It arises upon spontaneous breaking of a continuous global symmetry by a choice of the vacuum. In general the number of massless excitations on such a ground state equals the dimension of the vacuum manifold.
We will now make this more visible by returning to real fields, like in Problem 4. Introduce the angular variables ρ and θ by

$$\begin{aligned} \phi_1 &= \rho\cos\theta, \\ \phi_2 &= \rho\sin\theta. \end{aligned} \tag{63}$$

(d) Give the gauge transformations from (a) in terms of these variables.

(e) Write down the Lagrangian density in these variables. How can you recognise the $U(1)$, or rather, for these real fields the $SO(2)$, invariance?

(f) Choose a vacuum state and, in analogy to Eq. (62), perform an expansion around the vacuum values of ρ and θ. Calculate the Lagrangian density. Explain why there is a massless field.

Problem 8 **Local symmetry breaking, the abelian Higgs model**

In this problem we will assume that the ground state of a certain system doesn't admit the *local* symmetry of the action (dynamics), and thus breaks this symmetry. We consider the abelian Higgs model[9] with the Lagrangian density

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^*(D^\mu\phi) - V(\phi^*\phi), \tag{64}$$

with

$$V(\phi^*\phi) = \frac{\lambda}{4}(\phi^*\phi - f^2)^2, \tag{65}$$

where f is real. The system has a local $U(1)$ symmetry, and symmetry transformations can be written as in Eq. (33).
Like in Problem 7 the ground state breaks the symmetry and we should expand around a chosen vacuum. Instead of Eq. (62) it turns out to be more convenient to take an expansion of the form

$$\phi = (f + \chi(x))\, e^{-i\sigma(x)}. \tag{66}$$

(a) Show that the field $\sigma(x)$ in this theory can be removed by a suitable gauge transformation. This is an example of *choosing a gauge.*

(b) Perform this transformation and expand the resulting Lagrangian density up to second order in the remaining fields. Discuss the mass spectrum.

(c) Explain the paraphrase of this *Higgs mechanism*: "Gauge bosons eat the Goldstone bosons and thereby grow heavy."

[9]This is the relativistic generalisation of the phenomenological model of a superconductor and we will return to it later

Problem 9 **The Glashow–Weinberg–Salam model**

Let us return to the electroweak theory as discussed in Problem 6. In Problem 5 we concluded that the local gauge invariance of the theory forbids the presence of mass terms for the vector fields. However, experimentally the weak force is not just characterised by its weakness, but also by the fact that it acts only over very small distances ($< 10^{-15}$ cm). This in fact suggests that the vector particles (that arise upon quantisation of the gauge fields) have a mass m. This namely would give the *Yukawa* potential

$$V(r) \sim \frac{e^{-mr}}{r} \tag{67}$$

between two particles that carry a weak charge[10] and are separated by a distance r. The range of the force is then of order m^{-1}.

(a) To understand this we go back to electromagnetism where the gauge field consists of the scalar potential A_0 and the vector potential $\vec{A}$. Make the Ansatz

$$\vec{A}(\vec{x},t) = 0 \quad \text{and} \quad A_0(\vec{x},t) = A_0(r), \tag{68}$$

and solve the Maxwell equations. What is the physical interpretation of this *Coulomb* solution? We now assume that for the gauge field carrying the weak interaction we can make a similar Ansatz, such that the equations of motion for the gauge field simplify to an equation for a Lie algebra valued scalar potential, which now is the Klein–Gordon equation. Solve this equation, assuming that the field is static and spherically symmetric. Interpret the solution and explain Eq. (67).

Recall that the symmetry group of the electroweak theory has four generators: the three generators T_a of $SU(2)$ and the generator Y of $U(1)_Y$. The corresponding gauge fields are denoted by W^a_μ and B_μ, with coupling constants g and $\frac{1}{2}g'$, respectively. The aim is now to give mass to three of the four vector fields such that there remains only one massless vector particle: the photon, which carries the interactions of the unbroken $U(1)$ electromagnetic sector. To that end we again take the complex doublet

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \tag{69}$$

with Lagrangian density

$$\mathcal{L} = (D_\mu\phi)^\dagger(D^\mu\phi) - V(\phi^\dagger\phi), \tag{70}$$

where

$$V(\phi^\dagger\phi) = \frac{m^2}{2}\phi^\dagger\phi + \frac{\lambda}{4}(\phi^\dagger\phi)^2. \tag{71}$$

In order to break the symmetry we take $m^2 < 0$. The representation of $SU(2)\times U(1)_Y$ in internal space is given by

$$T_a = \frac{\sigma_a}{2} \quad \text{and} \quad Y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{72}$$

(b) Calculate $|\phi_0|$.

[10] Do not confuse this with a weak *electric* charge.

Write

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \eta_1 + i\eta_2 \\ \eta_3 + i\eta_4 \end{pmatrix} \tag{73}$$

with η_i $(i = 1, ..., 4)$ real fields, and choose the vacuum in the η_3 direction, so

$$(\eta_3)_0 > 0, \qquad (\eta_j)_0 = 0 \text{ for } j = 1, 2, 4. \tag{74}$$

(c) What does the vacuum manifold look like?

(d) Expand around the chosen vacuum, in analogy to Eq. (66), such that upon picking a suitable gauge, three of the four scalar field degrees of freedom vanish. Explain why there is only one matter field left (this can be done in one line!).

(e) Show that the unbroken $U(1)$ symmetry is generated by $Q = T_3 + \frac{1}{2}Y$.

We will now calculate the masses of the massive vector fields that correspond to the broken $SU(2)$ symmetry.

(f) Substitute the gauge fixed field that you have found under (d) in the scalar potential and show that the remaining matter field, the *Higgs field*, has the correct sign for the mass term.

(g) Substitute the vacuum field in the kinetic term for the scalar field, $(D_\mu\phi)^\dagger(D^\mu\phi)$, with

$$D_\mu = \partial_\mu - igW_\mu^a T_a - ig' B_\mu \frac{Y}{2}. \tag{75}$$

Upon this substitution you will find terms quadratic in the vector fields, some of them mixed (so cross terms like $B_\mu W_\mu^a$ can arise). Rewrite the result as

$$\begin{pmatrix} W_\mu^1 & W_\mu^2 & W_\mu^3 & B_\mu \end{pmatrix} \mathcal{M} \begin{pmatrix} W_\mu^1 \\ W_\mu^2 \\ W_\mu^3 \\ B_\mu \end{pmatrix}, \tag{76}$$

where $\mathcal{M}$ is a symmetric matrix, called the *mass matrix*.

(h) Diagonalise the mass matrix by putting

$$W_\mu^\pm = \frac{1}{\sqrt{2}}(W_\mu^1 \mp W_\mu^2), \tag{77}$$

$$Z_\mu^0 = B_\mu \sin\theta_{\rm W} - W_\mu^3 \cos\theta_{\rm W}, \tag{78}$$

$$A_\mu = B_\mu \cos\theta_{\rm W} + W_\mu^3 \sin\theta_{\rm W}, \tag{79}$$

where $\theta_{\rm W}$ is the *mixing angle* or *Weinberg angle*, given by

$$\tan\theta_{\rm W} = \frac{g'}{g}. \tag{80}$$

Calculate the masses $m_{W^\pm}$, m_{Z^0} and m_A. Interpret the A_μ field.

IV. Topological Excitations

Abstract

There are particular classes of solutions in field theories that are characterised by charges which are conserved for topological reasons. In classical theories they are solitary waves (*solitons*), in quantum theories they correspond to collective excitations in the sense that many particles together form a kind of coherent state. This type of solution occurs in (many) theories that have a broken symmetry and thus a degenerate vacuum. This time we will discuss two examples, the sine–Gordon equation and the abelian Higgs model. In the end we turn to the underlying mathematics which is closely related to homotopy groups.

References:
F.A. Bais, in "Geometric Techniques in Gauge Theories", Springer-Verlag, Berlin, 1982: "Topological excitations in gauge theories; an introduction from the physical point of view".
S. Coleman, "Aspects of symmetry", Cambridge University Press, Cambridge, 1985: Chapter 6.
J. Preskill, in the proceedings of the Les Houches Summer School 1985 "Architecture of fundamental interactions at short distances", Elsevier Science Publishers B.V., 1987: Course 3, Vortices and Monopoles.

Problem 10 **Topological conservation laws, the sine–Gordon equation**

Throughout this problem we work in one space dimension (and one time dimension). The one space variable will be denoted by x. Consider a real scalar field ϕ with Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{a}{b}(1 - \cos b\phi), \tag{81}$$

where a and b are positive real parameters. The Euler–Lagrange equation for this field is

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} + a \sin b\phi = 0, \tag{82}$$

called the *sine–Gordon* equation. We shall consider particular solutions of this equation which have finite energy.

(a) Determine the vacuum manifold and the vacuum energy.

(b) Calculate the Hamiltonian density for the field. Let ϕ be a regular solution of Eq. (82) of finite energy. Show that the finiteness of the energy implies that for any time t, $\lim_{x\to-\infty} \phi(x,t)$ and $\lim_{x\to+\infty} \phi(x,t)$ exist and lie on the vacuum manifold. Explain that these limits do not change with t. They will be denoted by $\phi(-\infty)$ and $\phi(+\infty)$.

The statement that $\phi(-\infty)$ and $\phi(+\infty)$ do not change in time is an elementary example of a *topological conservation law.* It arises from the topological properties of one-dimensional space, which has two "ends", $-\infty$ and $+\infty$, and of the vacuum manifold, which consists of discrete points. It is not a consequence of any symmetry of the action, and the conserved quantities $\phi(-\infty)$ and $\phi(+\infty)$ are not Noether charges.
We shall now determine explicit solutions to Eq. (82) which have finite energy. (Of course there are trivial solutions where $\phi(x,t)$ is constant.) Let us begin with time-independent

solutions. These can be obtained by minimising the total energy of the field. The following procedure to achieve this is known as the *Bogomolny* argument. The total energy can be written as

$$H = \frac{1}{2} \int \mathrm{d}x \left(\frac{\mathrm{d}\phi}{\mathrm{d}x} \mp \sqrt{2\frac{a}{b}(1 - \cos b\phi)} \right)^2 \pm \int_{\phi(-\infty)}^{\phi(+\infty)} \mathrm{d}\phi \sqrt{2\frac{a}{b}(1 - \cos b\phi)} \tag{83}$$

(verify this). The second integral on the right-hand side depends only on $\phi(-\infty)$ and $\phi(+\infty)$.

(c) Minimise the first integral on the right-hand side. Solve the resulting differential equation for ϕ. What values for $\phi(-\infty)$ and $\phi(+\infty)$ are possible? Pick a non-trivial solution and draw it.

(d) Calculate the energy of the solutions ϕ obtained in (c). Draw the Hamiltonian density of the solution you picked there.

(e) Note that the sine–Gordon equation is Lorentz invariant. Apply a suitable Lorentz transformation to the (time-independent) solutions from (c) in order to obtain time-dependent solutions of Eq. (82).

The (non-trivial) solutions of Eq. (82) found in above can be seen as solitary waves. Such solitary waves are known as *solitons*. They are stable particle-like “lumps” containing a finite amount of energy and having a non-zero spatial extent.

Problem 11 **Nielsen–Olesen vortices**

We will now look at a more complicated example of non-trivial finite energy solutions: we consider a scalar field coupled to a gauge field. This turns out to be a good description of certain line-shaped defects in condensed matter. They can be regarded as two-dimensional defects that are extended into three dimensions. These two-dimensional defects are stable solutions of finite energy to a set of classical field equations and are called *vortices*. One can think for instance of the Red Spot on Jupiter, a hurricane, etc.

We return to the abelian Higgs model of Problem 8 and will try to find such vortices. Note that in principle a vacuum is a vortex, but it is a trivial solution; we want to investigate whether there is more. To that aim it will be necessary to focus on the asymptotical behaviour of the fields and study the topology of space and of the vacuum manifold. At first the precise form of the solutions will not be of particular interest, only the integrability of their energy density and the single-valuedness of the field will be important.

(a) Using Eqs. (64) and (53), show that the energy of a time-independent field configuration is a sum of three non-negative terms:

$$E = \int \mathrm{d}^2x \left(\frac{1}{2}(E_i E^i + B_i B^i) + D_i\phi D^i\phi^\dagger + V(\phi) \right). \tag{84}$$

In order for the total energy to be finite, each term must be finite. For the third term this implies that $\lim_{r\to\infty} V(\phi) = 0$, where we have transformed to polar coordinates.

(b) Explain that for the abelian Higgs model

$$\phi(r \to \infty, \theta) = f e^{i\alpha(\theta)}, \tag{85}$$

with $e^{i\alpha(\theta)}$ a phase factor. Which space does θ parametrise and which space does $\alpha(\theta)$ parametrise?

(c) Show that for ϕ to be single valued we should have $\alpha(\theta + 2\pi) - \alpha(\theta) = 2\pi n$, with $n \in \mathbf{Z}$.

The boundary of two-dimensional space can be thought of as a circle[11] (at infinity), S^1_r. The vacuum manifold is also a circle, S^1_ϕ. The essential notion now is that with every field configuration of finite energy at some time t, we associate a mapping from spatial infinity to the vacuum manifold,

$$\phi_{r\to\infty} \quad : \quad S^1_r \; \to \; S^1_\phi, \tag{86}$$

defined as the limit of the field at spatial infinity:

$$\phi_{r\to\infty} \quad : \quad \theta \mapsto \phi(r \to \infty, \theta). \tag{87}$$

Now recall that any mapping from a circle to a circle has a *winding number*, which may be defined as

$$n = \frac{1}{2\pi}\left(\alpha(\theta = 2\pi) - \alpha(\theta = 0)\right), \tag{88}$$

and has the important property that it is an integer.

(d) What is the winding number of the vacuum? Draw a field configuration at $r \to \infty$ for $n = 0$, where you can think of the field at infinity as represented by a unit-vector denoting the phase $\alpha(\theta)$ and living on a circle that denotes the phase θ. Same question for $n = 1$ and $n = 2$.

The fact that the winding number is an integer assures that it is preserved by smooth deformations of the field[12] and by time evolution. Therefore the winding number is a *topological invariant*, and the fact that it doesn't change in time is a topological conservation law. Just as the topological conservation from Problem 10, is not directly associated with any symmetry of the action.
The behaviour of the gauge field at infinity follows from the vanishing of the second term in Eq. (84), given the behaviour of ϕ at infinity.

(e) A gauge field is a *pure* gauge if it can be gauged to zero by choosing an appropriate gauge transformation. Show that at infinity A_μ locally approaches a pure gauge, *i.e.*, it can locally be written as a gradient, and that the energy contribution of the first term in Eq. (84) is finite. Calculate the (physical) electric and magnetic fields, $\vec{E}$ and $\vec{B}$, respectively, at infinity.

It can indeed be shown that there exist classes of solutions with finite total energy that asymptotically behave in the way described above, that is, the scalar field takes values in the vacuum manifold and the gauge field is a pure gauge. We will not look at the exact form of particular solutions, but turn to a remarkable property of these classes of solutions. Suppose that the two-dimensional space that we considered is taken to be the xy-plane in a three dimensional space. Let us calculate the magnetic flux in the z-direction.

[11] Mathematicians write S^1 for a circle, because it is a one-dimensional sphere. Similarly, an "ordinary" (*i.e.*, two-dimensional) sphere is denoted by S^2.

[12] As long as the finiteness of the total energy is preserved.

(f) Show that for the abelian Higgs model the magnetic flux through the plane is given by

$$\Phi = \frac{2\pi n}{e}. \tag{89}$$

Hint: use Stokes theorem.

This indicates that despite the fact that A_μ locally is a pure gauge and could be gauged away, globally it can not be removed, resulting in a singular magnetic flux. It seems we now have the rather paradoxical situation that a *singular* gauge transformation may affect a gauge invariant quantity like the magnetic flux. Therefore these transformations are not admissible as gauge transformations, but they may certainly be used to construct new solutions to the field equations. The conclusion is that if there exists a solution that is independent of time and of the z-direction and that has finite energy per unit length in the z-direction, then it has a magnetic flux which is quantised in units of $\frac{2\pi}{e}$. But: we are working in a classical theory!

The model discussed above is the so-called Landau–Ginzburg theory, which is a good effective description of Type II superconductors. The topological excitations that are found are known as Nielsen–Olesen (or Abrikosov) flux tubes, and they have been observed experimentally.

Problem 12 **Homotopy groups**

In the previous problem we considered the asymptotic ($r \to \infty$) behaviour of the fields of the abelian Higgs model in two space dimensions, and concluded that the total magnetic flux through a superconductor is quantised. For this conclusion only the finiteness of the total energy and the single-valuedness of the field at spatial infinity were important, whereas the behaviour of the fields for finite r was irrelevant. Here we will consider the general mathematical setting of topological excitations and classify them according to topological properties of spatial infinity and of the vacuum manifold of the theory.
Like in Problems 10 and 11 we start from a spontaneously broken theory to have a degenerate vacuum. Let us choose a vacuum ϕ_0. We will now assume that there is no other degeneracy, so $V(\phi)$ is minimal iff[13] $\phi = g\phi_0$ for some $g \in G$, where G is the gauge group (note that this condition was not fulfilled in Problem 10). This is nothing but saying that the vacuum manifold should be equal to the *orbit* of ϕ_0 under G:

$$\mathcal{V} = G\phi_0 = \{\, g\phi_0 \mid g \in G \,\}. \tag{90}$$

The *residual symmetry group* H consists of those symmetries that are not broken by the vacuum degeneracy. Mathematically spoken, it is the *stabiliser* of the chosen vacuum:

$$H = \{\, h \in G \mid h\phi_0 = \phi_0 \,\}. \tag{91}$$

(a) Justify the term "residual symmetry group" by proving (from the formal definition Eq. (91)) that H is indeed a subgroup of G.

We define an equivalence relation $\sim$ on G by

$$g_1 \sim g_2 \quad \text{iff} \quad g_1\phi_0 = g_2\phi_0. \tag{92}$$

[13] This is the standard abbreviation for: "if and only if".

(b) Let $g_1, g_2 \in G$. Show that $g_1 \sim g_2$ iff $g_1 = g_2 h$ for some $h \in H$.

This leads us to considering cosets of H. By definition, a left coset of H in G is a subset of G of the form

$$gH = \{ \, gh \mid h \in H \, \}, \tag{93}$$

with g some element of G. The collection of all left cosets of H in G is denoted by G/H:

$$G/H = \{ \, gH \mid g \in G \, \}. \tag{94}$$

Note that these definitions are valid for any subgroup H of any group G.

(c) Let $g_1, g_2 \in G$. Prove that the left cosets $g_1 H$ and $g_2 H$ are either disjoint or equal. Prove that $g_1 \sim g_2$ iff $g_1 H = g_2 H$.

This shows that the equivalence classes for $\sim$ are just the left cosets of H in G. Thus G/H is the quotient of G with respect to the equivalence relation $\sim$. Also we have established a one-to-one correspondence between G/H and the vacuum manifold:

$$gH \;\leftrightarrow\; g\phi_0 \tag{95}$$

for $g \in G$. The above results can be summarised as

$$\mathcal{V} \;=\; G\phi_0 \;\cong\; G/\!\sim \;\cong\; G/H. \tag{96}$$

After this digression in group theory, we turn to topology. Eq. (86) can be generalised to the statement that for a (spontaneously broken) gauge theory in Euclidean space $M = \mathbf{R}^d$, one can associate to every solution of finite energy, for any time t, a (continuous) mapping from the boundary of space to the vacuum manifold:

$$\phi_{r\to\infty} \;:\; \partial M \;\to\; G/H. \tag{97}$$

These mappings fall into so-called *homotopy classes*: two mappings are *homotopic* (in the same homotopy class, that is) if they can be continuously deformed into each other. The homotopy class of any mapping α is denoted by $[\alpha]$. Clearly the mapping $\phi_{r\to\infty}$ stays in the same homotopy class when the field ϕ evolves in time, so $[\phi_{r\to\infty}]$ is a conserved quantity. This is a topological conservation law.

(d) Can you think of a formal definition of “continuously deforming two mappings into each other”?

We shall endow the set of all homotopy classes of mappings $\partial M \to G/H$ with a group structure. First we consider a restricted set of such mappings, and a restricted notion of homotopy. We have already picked a point ϕ_0 on the vacuum manifold, we now also pick a point s_0 on ∂M. These points are called *base points*. We only consider mappings $\partial M \to G/H$ that map the base point s_0 of ∂M to the base point ϕ_0 of G/H. The same restriction is imposed on homotopies: throughout the continuous deformation, s_0 must be mapped to ϕ_0.

We now focus on the case that M is two-dimensional. The boundary of space, ∂M, is then a circle S^1. When we trace this circle in a prescribed direction, beginning and ending at s_0, a (restricted) mapping $S^1 = \partial M \to G/H$ describes a loop (a closed path) in G/H, beginning and ending at ϕ_0. Two such loops are composed as follows: begin at ϕ_0, run through the first loop, which brings you back to ϕ_0, then run through the second loop, and you have ended up in ϕ_0 again. The resulting mapping $S^1 \to G/H$ is pictured in Figure 1. Note that for this composition of loops, the first loop should end at the point where the second loop begins; we required the mappings to respect the base points in order to force this. It is not difficult to show that this composition of loops gives rise to a group structure on the set of (restricted) homotopy classes of (restricted) mappings $S^1 \to G/H$, as follows. In order to multiply two (restricted) homotopy classes, one chooses representative mappings $S^1 \to G/H$ from each class, composes these loops as described above, and takes the (restricted) homotopy class of the mapping $S^1 \to G/H$ obtained. This group is called the *first homotopy group* or *fundamental group* of the space G/H with base point ϕ_0. It is denoted by $\pi_1(G/H, \phi_0)$.

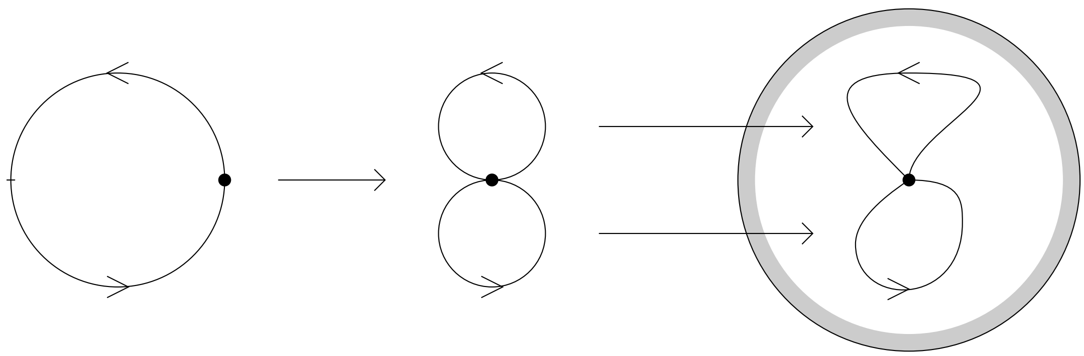

Figure 1: Schematic of the composition law for two loops. The circle S^1 representing the boundary of space, ∂M, (left) is first divided into two circles (middle). Each of them is mapped to the vacuum manifold G/H (right). These mappings are the loops to be composed. The resulting mapping $S^1 = \partial M \to G/H$ is the composite of the two loops. The base points are indicated by dots.

(e) Describe the unit element of the fundamental group. Describe the inverse of any element of the fundamental group.

A simple, but non-trivial example is the abelian Higgs model in two space dimensions. Here $G = U(1)$ and $H = \{bf1\}$, and the vacuum manifold G/H is a circle $U(1)$. Two loops on the circle are homotopic iff they have the same winding number. Thus $\pi_1(U(1))$ is the set of winding numbers, hence $\mathbf{Z}$. The group operation is addition. Note that we have omitted any reference to the base points; this will be justified below.

The case that M has a higher dimension, $d > 2$, is fully analogous. The boundary of space, ∂M, is a $(d-1)$-dimensional (hyper)sphere S^{d-1}. The composition of two (restricted) mappings is pictured in Figure 2. This again yields a group structure on the set of (restricted) homotopy classes of (restricted) mappings $S^{d-1} \to G/H$. This group is known as the $(d-1)$th *homotopy group*, and is denoted by $\pi_{d-1}(G/H, \phi_0)$. For $d > 2$ the homotopy group π_{d-1} is always abelian, but π_1 need not be. Table 1 gives a list of homotopy groups of hyperspheres.

So far we have restricted ourselves to mappings $\partial M \to G/H$ and homotopies that respect

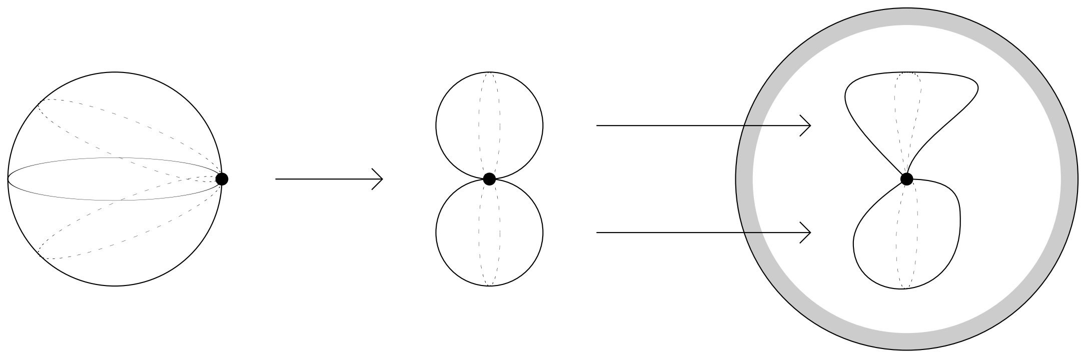

Figure 2: Schematic of the composition law for two mappings $S^{d-1} = \partial M \to G/H$. The hypersphere S^{d-1} representing the boundary of space, ∂M, (left) is divided into two such hyperspheres (middle) by shrinking its equator to a single point. Each of these two hyperspheres is mapped to the vacuum manifold G/H (right). These are the mappings to be composed. The resulting mapping $S^{d-1} = \partial M \to G/H$ is the composite of the initial two mappings. The base points are indicated by dots.

Table 1: Homotopy groups of hyperspheres

	π_1	π_2	π_3
S^1	**Z**	0	0
S^2	0	**Z**	**Z**
S^3	0	0	**Z**

the base points. Getting rid of this restriction involves some mathematical subtleties we cannot go into. We state without proof that when G and H are connected, there is a natural one-to-one correspondence between the homotopy classes of mappings $S^{d-1} \to G/H$ and the restricted homotopy classes of restricted mappings $S^{d-1} \to G/H$.
After this topological exposition we return to physics. We already considered the mapping Eq. (97) and noted that $[\phi_{r\to\infty}]$ is a conserved topological charge. This charge is an element of the homotopy group $\pi_{d-1}(G/H)$. Thus if this homotopy group is trivial, then so is the topological conservation law. However, if the group is non-trivial, this does not yet imply the existence of non-trivial solutions of finite energy.

(f) As an example, consider the Georgi–Glashow model from Problem 5, and take space to be two-dimensional. What are G, H and G/H? Explain why there is no non-trivial topological conservation law.

(g) Another example is the Weinberg–Salam model in three spatial dimensions, which has a two-component complex scalar field ϕ which transforms as an isospinor (see Problem 6) under the gauge group $G = SU(2)$. The vacuum manifold is given by $\phi^*\phi = f^2$, where $f^2 > 0$. Can there be topological charges? Explain.

Let there be given two field configurations $\phi(\vec{x})$ and $\psi(\vec{x})$, each of which forms a "lump" of finite energy (see Problem 10), and assume that these two lumps are far apart in space. Suppose that we can patch these configurations together, so as to get a new configuration $\chi(\vec{x})$ of finite energy, forming two lumps. Near the lump of ϕ, χ looks like ϕ, whereas near the lump of ψ, it looks like ψ. Now consider the topological charges of these field configurations. It turns out that $[\chi_{r\to\infty}]$ is the product of $[\phi_{r\to\infty}]$ and $[\psi_{r\to\infty}]$ in the appropriate homotopy group. So topological charges are composed according to the multiplication of this homotopy group.

(h) As an example, consider the sine–Gordon equation (Problem 10). It is clear that $\phi(+\infty) - \phi(-\infty)$ is a conserved topological charge. Draw pictures to show how distant lumps can be patched together. Explain the composition of topological charges.

V. Magnetic Monopoles

Abstract

Magnetic monopoles, like vortices, arise as time-independent solutions with finite energy to the classical field equations of a spontaneously broken gauge theory. But a monopole has finite energy in three spatial dimensions, instead of two. And, unlike a vortex, a monopole has a long-range (magnetic) gauge field, from which it gets its name. Monopoles can be classified topologically, analogously to the classification of vortices in Problem 11. First we will consider the simplest model in which monopoles can occur, but for the topologically trivial case. Then we turn to the solution with minimal topological charge and show that it corresponds to a magnetic monopole. Its physical properties will be discussed, and finally a full solution will be calculated (so not only asymptotically) in the Bogomolny–Prasad–Sommerfield limit.

References:
F.A. Bais, in "Geometric Techniques in Gauge Theories", Springer-Verlag, Berlin, 1982: "Topological excitations in gauge theories; an introduction from the physical point of view".
S. Coleman, "Aspects of symmetry", Cambridge University Press, Cambridge, 1985: Chapter 6.
J. Preskill, in the proceedings of the Les Houches Summer School 1985 "Architecture of fundamental interactions at short distances", Elsevier Science Publishers B.V., 1987: Course 3, Vortices and Monopoles.

Problem 13 **Broken Georgi–Glashow model**

The simplest model in which monopoles occur is the Georgi–Glashow model (see Problem 5). The gauge group $G = SO(3)$ and the scalar field Φ is in the triplet representation. The Lagrangian density is

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a + \frac{1}{2}D_\mu\Phi^a D^\mu\Phi_a - V(\Phi), \tag{98}$$

with

$$V(\Phi) = \frac{1}{8}\lambda(\Phi^a\Phi_a - f^2)^2, \tag{99}$$

$$D_\mu\Phi_a = \partial_\mu\Phi_a + e\epsilon_{abc}A^b_\mu\Phi^c \tag{100}$$

and $a = 1, 2, 3$. The potential is minimised by

$$|\Phi|^2 = f^2, \tag{101}$$

which breaks the $SO(3)$ symmetry.

(a) Give the vacuum manifold and the residual symmetry group.

(b) Use Table 1 to explain the labelling of the topological charges.

We will now investigate the asymptotical behaviour of the solutions for two cases. In this problem the *trivial mapping* from $S^2_x \to S^2_\Phi$, corresponding to zero topological charge, will be considered.[14] In the next problem we will study the simplest non-trivial case, the *identity mapping*. To simplify calculations it is convenient to use a vector notation, so the internal components are denoted by a vector arrow:

$$\{\Phi^a\} \quad \leftrightarrow \quad \vec{\Phi}, \tag{102}$$

$$\{A^a_\mu\} \quad \leftrightarrow \quad \vec{A}_\mu. \tag{103}$$

Note that the directions of these vectors in internal space have *a priori* nothing to do with the spatial directions. The trivial mapping is $\vec{\Phi}(r \to \infty, \theta, \varphi) = \text{constant}$, such that $V(\Phi)$ is zero. This mapping corresponds to $n = 0$ in $\pi_2(SO(3)/SO(2))$. The choice of the direction of the vacuum is arbitrary, we can just choose

$$\vec{\Phi}_0 = f\hat{n}_3 \equiv \vec{f} \tag{104}$$

with $\hat{n}_3$ the unit vector in internal z-direction. In Figure 3 the large sphere denotes spatial infinity and in each point there is a small sphere denoting the internal space where Φ lives, in which the direction of $\vec{\Phi}_0$ is indicated.

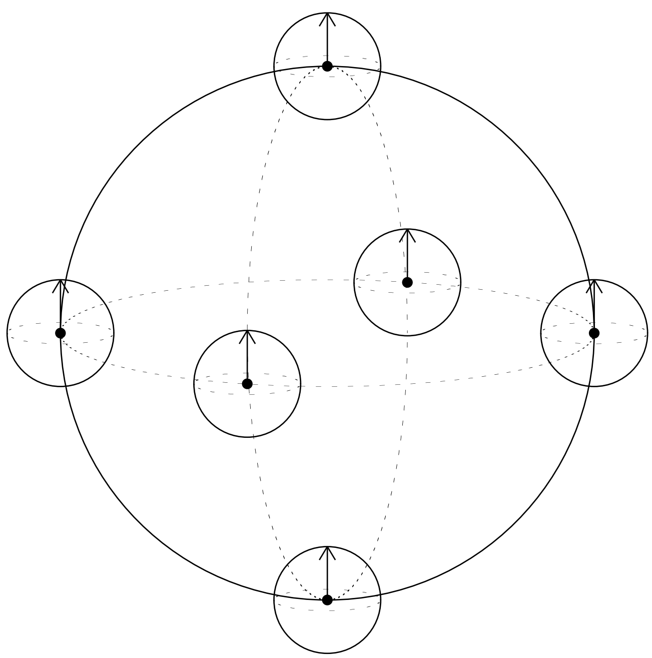

Figure 3: The trivial mapping $S^2_x \to S^2_\Phi$.

(c) Expand the Higgs field around the chosen vacuum up to second order. Calculate the masses of the vector fields and the Higgs field and explain the results.

[14] x indicates spatial coordinates.

Problem 14 **Magnetic charge as a topological charge**

We now turn to the simplest non-trivial mapping $S_x^2 \to S_\Phi^2$, which is the identity map and is the first excited solution. In components a general mapping can be written as

$$\Phi_{r\to\infty} \,:\, (\theta,\varphi) \;\mapsto\; (\alpha(\theta,\varphi),\beta(\theta,\varphi)) \tag{105}$$

(a) Give the identity mapping. What are $\alpha(\theta,\varphi)$ and $\beta(\theta,\varphi)$ for the trivial mapping and for the identity mapping? Draw the field configuration for $r\to\infty$ for the identity mapping, analogous to Figure 3 and explain why Alexander Polyakov, who discovered this solution simultaneously with Gerard 't Hooft (1973), called it a "hedgehog".

Now we will show that this class of solutions of the scalar field corresponds to a magnetic monopole, by which we mean that asymptotically it behaves like a magnetic monopole. In the next problem we will study the finite-r behaviour and explicitely show that there are non-singular solutions.

(b) For finite energy the second term in Eq. (98) should vanish asymptotically. Take $\vec{A}_0 = 0$ and assume time independence. Calculate the components of the gauge field in terms of $\vec{\Phi}_0$ and its derivatives, asymptotically. Note that for each component this solution contains one free function, which will be set to zero.

(c) Give the asymptotic form of the components of the gauge field for the identity mapping.

For low energies, $E \ll m_{A_\mu} = ef$, so when the $SO(3)$ symmetry is broken to $U(1)$, the theory described above is Maxwell's theory for electromagnetism, but including magnetic monopoles, as we will see. To calculate the magnetic field we need the field strength associated with the massless ($U(1)$) gauge field, but its definition should also be invariant under the full gauge group $SO(3)$.[15]
The massless $\vec{A}_\mu$ component is in the direction of the scalar field, so we should also consider the field strength in that direction:

$$f_{\mu\nu}(r\to\infty) \equiv \frac{1}{|\Phi_0|}\Phi_0^a F_{\mu\nu a} = \hat{\Phi}_0 \cdot \vec{F}_{\mu\nu}. \tag{106}$$

(d) Show that at spatial infinity the gauge field as you have found it under (c) gives the ordinary abelian field strength (so, for the identity mapping). Argue that for the trivial mapping definition Eq. (106) cannot be correct, but that it is for the identity mapping.

(e) Calculate the magnetic field, $B_i = \frac{1}{2}\epsilon_{ijk} f^{jk}$, at infinity.

(f) Calculate the magnetic charge.

Mappings corresponding to higher charges are labelled by an integer (see Problem 13). This yields a quantisation of the magnetic charge in units that you have found under (f), where we had $n = 1$, analogously to the quantisation of magnetic flux in Problem 11.

For reasons that we will not go into here, magnetic monopoles are directly related to the quantisation of electric charge. The latter is not explained by the Standard Model, but of course well-known from experiments. The magnetic monopoles are the remnants of the

[15] Remember that this is the symmetry of the system, only the ground state is not invariant under it.

non-abelian structure of the theory we started with, $SO(3)$, after taking the low energy limit to retrieve Maxwell theory. Thus they arise as a natural consequence of spontaneous symmetry breaking and are not inserted “by hand”. This would be an elegant way to obtain electric charge quantisation, but until so far magnetic monopoles haven't been found experimentally, probably for reasons that we will discuss in the next problem.

(g) Summarise and explain in your own words how magnetic monopoles occur in the Standard Model, in the way you have calculated it in this problem.

Problem 15 **An exact monopole solution**

One could ask whether the magnetic monopole solutions as found in Problem 14 really do exist. So whether the energy is finite, whether the finite-r behaviour is regular and what the physical properties are.

(a) Explain that the identity mapping represents a non-trivial homotopy class. Also show that for a regular solution of non-zero topologican charge, the Higgs field should vanish in at least one point.

We will now analyse the charge 1 monopole solution whose asymptotics we studied in Problem 14, and show the existence of completely regular solutions.

(b) Derive the following expression for the energy, which is called the Bogomolny decomposition:

$$E = \int \mathrm{d}^3x \, \left(\frac{1}{2}\left(\vec{B}_i \mp D_i\vec{\Phi}\right)^2 \pm \vec{B}_i \cdot D_i\vec{\Phi} + V(\Phi)\right) \tag{107}$$

Again for finiteness each term of Eq. (107) must be integrable.

(c) Show that for general (non-abelian) gauge fields $D_i\vec{B}_i = 0$. Use this to show that the second term in the expression for the energy is a topological invariant, equal to the magnetic charge times the vacuum expectation value of the Higgs field.

Minimising the energy and solving for the scalar and gauge field is simplified when we take the so-called Bogomolny–Prasad–Sommerfield limit: $\lambda \to 0$ in the potential.

(d) What does this limit mean physically? Discuss the mass of the monopoles and explain why monopoles haven't been found yet in accelerators.

For minimal energy the fields have to satisfy the Bogomolny equations:

$$\vec{B}_i = \pm D_i\vec{\Phi}. \tag{108}$$

For general n the general solutions are rather difficult to find. We restrict ourselves again to $n = 1$ and take the following simple Ansatz for the fields, where we just multiply the asymptotic solution for the gauge field and Higgs field by functions of only r:

$$A_i^a = \frac{1}{e}\epsilon^a{}_{ib}\frac{x^b}{r^2}\left(1 - r\mathcal{F}_1\right) \tag{109}$$

$$\Phi^a = \frac{x^a}{r}\left(1 + r\mathcal{F}_2\right), \tag{110}$$

where $\mathcal{F}_{1,2}$ are only functions of the distance $r = |x|$.

(e) Show that the Bogomolny equations take the form

$$\begin{aligned} \partial_r \mathcal{F}_1 &= \mathcal{F}_1 \mathcal{F}_2 \quad (111) \\ \partial_r \mathcal{F}_2 &= (\mathcal{F}_1)^2. \quad (112) \end{aligned}$$

and that

$$\begin{aligned} \mathcal{F}_1 &= (\sinh r)^{-1} \quad (113) \\ \mathcal{F}_2 &= -\coth r \quad (114) \end{aligned}$$

are solutions to these equations. Sketch the radial dependence of the components of $\vec{\Phi}$ and $\vec{A}$ and check that this solution is regular at $r \to 0$.

We have found an explicit solution to the field equations corresponding to the Georgi–Glashow model and the lowest topological charge. We assumed time independence and spherical symmetry and took the limit $\lambda \to 0$. More general exact solutions are much more complicated to find.

VI. Quantum Aspects

Abstract

In order to calculate some quantum mechanical properties of systems with degenerate ground states, we briefly discuss the notion of integration over paths, followed by an explicit example. Subsequently the (Euclidean) path integral will be approximated by the first terms in the expansion in $\hbar$, because we cannot calculate the path integral in general. This approximation will be used in a rather qualitative introduction to instantons and their relation to tunnelling processes.

References:
R.P. Feynman & A.R. Hibbs, "Quantum Mechanics and Path Integrals", McGraw-Hill, New York, 1965.
B. Felsager, "Geometry, Particles and Fields", Odense University Press, 1981: Chapter 5.
S. Coleman, "Aspects of symmetry", Cambridge University Press, Cambridge, 1985: Chapter 6.

Problem 16 **Path Integrals in Quantum Mechanics**

The path integral formalism is a quantisation method developed by Richard Feynman. It is also known as Feynman's "sum over histories" approach. It maintains manifest Lorentz covariance, contrary to the canonical quantisation method which explicitely separates the space and time variables. At the end of this problem we will discuss the link to the operator formulation of quantum mechanics which is used in canonical quantisation.

An important quantity in quantum mechanics is the probability amplitude for the transition from an initial state $|q_\mathrm{i}, t_\mathrm{i}\rangle$ to a final state $|q_\mathrm{f}, t_\mathrm{f}\rangle$:

$$\langle q_\mathrm{f}, t_\mathrm{f}|q_\mathrm{i}, t_\mathrm{i}\rangle = \langle q_\mathrm{f}|e^{-\frac{i}{\hbar}\hat{H}(t_\mathrm{f}-t_\mathrm{i})}|q_\mathrm{i}\rangle. \tag{115}$$

Here $\hat{H}$ denotes the Hamilton operator. The states are eigenstates of the position operator (in the Schrödinger picture):

$$\hat{q}|q,t\rangle = q|q,t\rangle. \tag{116}$$

Write

$$T = t_\mathrm{f} - t_\mathrm{i} \tag{117}$$

and split the time interval from t_i to t_f into n pieces of equal length T/n. Now use the completeness relation

$$\int_{-\infty}^{\infty} \mathrm{d}q(t)\, |q,t\rangle\langle q,t| = 1 \tag{118}$$

to write Eq. (115) as

$$\langle q_\mathrm{f}, t_\mathrm{f}|q_\mathrm{i}, t_\mathrm{i}\rangle = \int \mathrm{d}q_1 \ldots \mathrm{d}q_{n-1}\, \langle q_\mathrm{f}|e^{-\frac{i}{\hbar}\hat{H}\frac{T}{n}}|q_{n-1}\rangle\langle q_{n-1}|e^{-\frac{i}{\hbar}\hat{H}\frac{T}{n}}|q_{n-2}\rangle \ldots \langle q_1|e^{-\frac{i}{\hbar}\hat{H}\frac{T}{n}}|q_\mathrm{i}\rangle. \tag{119}$$

Thus we have divided the transition from $|q_\mathrm{i}, t_\mathrm{i}\rangle$ to $|q_\mathrm{f}, t_\mathrm{f}\rangle$ in n steps, each of which has its own transition matrix element $\langle q_l|e^{-\frac{i}{\hbar}\hat{H}\frac{T}{n}}|q_{l-1}\rangle$. All intermediate states are possible (in principle) so we have to integrate over all these states. This is called "integrating over all possible paths" between q_i (at time t_i) and q_f (at time t_f).

If we take n large enough, we can expand the exponent in T/n. Take the Hamiltonian

$$\hat{H}(\hat{p},\hat{q}) = \frac{\hat{p}^2}{2m} + V(\hat{q}) \tag{120}$$

where the hats denote the usual operator valued position and momentum operators corresponding to the classical position and momentum. The potential is a smooth function:

$$V(\hat{q})|q\rangle = V(q)|q\rangle \tag{121}$$

and so

$$\langle q_l|V(\hat{q})|q_{l-1}\rangle = V(q_l)\delta(q_l - q_{l-1}), \tag{122}$$

where δ denotes the Dirac delta function.

(a) Show that

$$\langle q_l|\hat{H}(\hat{p},\hat{q})|q_{l-1}\rangle = \int \frac{\mathrm{d}p_l}{2\pi\hbar} e^{\frac{i}{\hbar}p_l(q_l-q_{l-1})} H\left(p_l, \frac{q_l+q_{l-1}}{2}\right), \tag{123}$$

where the operators have disappeared from the right hand side. Recall that

$$\langle q|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar}pq} \tag{124}$$

For $n \to \infty$ the time steps T/n become infinitesimal, and we can write for each transition matrix element:

$$\begin{aligned}
\langle q_l|e^{-\frac{i}{\hbar}\hat{H}\frac{T}{n}}|q_{l-1}\rangle &= \int \frac{\mathrm{d}p_l}{2\pi\hbar} e^{\frac{i}{\hbar}p_l(q_l-q_{l-1})} \left(1 - \frac{i}{\hbar}H\left(p_l, \frac{q_l+q_{l-1}}{2}\right)\frac{T}{n} + \mathcal{O}\left((\frac{T}{n})^2\right)\right) \\
&= \int \frac{\mathrm{d}p_l}{2\pi\hbar} e^{\frac{i}{\hbar}p_l(q_l-q_{l-1})}\, e^{-\frac{i}{\hbar}H(p_l,\frac{q_l+q_{l-1}}{2})\frac{T}{n}}.
\end{aligned} \tag{125}$$

Substituting this in Eq. (119) gives

$$\begin{aligned}
\langle q_\mathrm{f}, t_\mathrm{f}|q_\mathrm{i}, t_\mathrm{i}\rangle &= \lim_{n\to\infty} \int \mathrm{d}q_1 \ldots \int \mathrm{d}q_{n-1} \int \frac{\mathrm{d}p_1}{2\pi\hbar} \cdots \int \frac{\mathrm{d}p_n}{2\pi\hbar} \\
&\qquad \exp\left(\sum_{l=1}^{n} \frac{i}{\hbar}\left[p_l \frac{q_l - q_{l-1}}{T/n} - H(p_l, \frac{q_l+q_{l-1}}{2})\right]\frac{T}{n}\right) \\
&= \lim_{n\to\infty} \left(\prod_{k=1}^{n-1}\int \mathrm{d}q_k\right)\left(\prod_{l=1}^{n}\int\frac{\mathrm{d}p_l}{2\pi\hbar}\right) \exp\left[\frac{i}{\hbar}\int_{t_\mathrm{i}}^{t_\mathrm{f}} \mathrm{d}t\,(p\dot{q} - H(p,q))\right] \\
&\equiv \int \mathcal{D}q \int \mathcal{D}p \exp\left[\frac{i}{\hbar}\int_{t_\mathrm{i}}^{t_\mathrm{f}} \mathrm{d}t\,(p\dot{q} - H(p,q))\right].
\end{aligned} \tag{126}$$

The last equivalence is the definition of the symbols $\mathcal{D}q$ and $\mathcal{D}p$. This result holds for general Hamiltonians, although we have derived it only for the form Eq. (120). In this case it is also possible to perform the Gaussian integral over p.

(b) Perform this integral. To that aim go back to the discrete form of the path integral, so before we let $n \to \infty$, and show that

$$
\begin{aligned}
\langle q_\mathrm{f}, t_\mathrm{f} | q_\mathrm{i}, t_\mathrm{i} \rangle &= \lim_{n\to\infty} \frac{1}{\sqrt{2\pi i \hbar T/nm}} \left(\prod_{k=1}^{n-1} \int \frac{\mathrm{d}q_k}{\sqrt{2\pi i \hbar T/nm}} \right) \\
&\qquad \exp\left[\frac{i}{\hbar} \int_{t_\mathrm{i}}^{t_\mathrm{f}} \mathrm{d}t \, \left(\frac{1}{2} m \dot{q}^2 - V(q) \right) \right] \\
&\equiv \int \mathcal{D}q \, \exp\left(\frac{i}{\hbar} S[q] \right) , \qquad (127)
\end{aligned}
$$

with

$$
S[q] = \int_{t_\mathrm{i}}^{t_\mathrm{f}} \mathrm{d}t \, \left(\frac{1}{2} m \dot{q}^2 - V(q) \right) . \qquad (128)
$$

Summarising: to obtain the transition amplitude between quantum states $|q_\mathrm{i}, t_\mathrm{i}\rangle$ and $|q_\mathrm{f}, t_\mathrm{f}\rangle$ one has to sum over all possible paths from q_i at time t_i to q_f at time t_f. This is denoted by $\mathcal{D}q$. The weight of a particular path is given by $\exp(\frac{i}{\hbar} S[q])$, where S is the *classical* action of the path.
Obviously, the path integral Eq. (127) is not at all well-defined, as the integrand is a strongly oscillating function. This is remedied by the following trick, which in quantum field theory is known as *Wick rotation.* The time variable t is replaced everywhere with $-it$. The metric on spacetime changes from

$$
\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 \qquad (129)
$$

into

$$
\mathrm{d}(-it)^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 = -\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 . \qquad (130)
$$

It becomes (negative) definite so we are now working in Euclidean rather than Minkowskian spacetime. The exponent occurring in the path integral,

$$
\frac{i}{\hbar} S[q] = \frac{i}{\hbar} \int \mathrm{d}t \, \left(\frac{1}{2} m \dot{q}^2 - V(q) \right) , \qquad (131)
$$

transforms into

$$
\begin{aligned}
\frac{i}{\hbar} \int \mathrm{d}(-it) \, \left(\frac{1}{2} m (i\dot{q})^2 - V(q) \right) &= -\frac{1}{\hbar} \int \mathrm{d}t \, \left(\frac{1}{2} m \dot{q}^2 + V(q) \right) \\
&\equiv -\frac{1}{\hbar} S_\mathrm{E}[q] . \qquad (132)
\end{aligned}
$$

The transition amplitude in Euclidean time is given by

$$
\langle q_\mathrm{f}, t_\mathrm{f} | q_\mathrm{i}, t_\mathrm{i} \rangle_\mathrm{E} = \int_{q(t_\mathrm{i})=q_\mathrm{i}}^{q(t_\mathrm{f})=q_\mathrm{f}} \mathcal{D}q \, \exp\left(-\frac{1}{\hbar} S_\mathrm{E}[q] \right) \qquad (133)
$$

and now the integrand tends rapidly to 0 when the Euclidean action $S_\mathrm{E}[q]$ increases. After calculating the Euclidean path integral one rotates back to obtain the transition amplitude in Minkowskian time.

Two remarks:

$\mathcal{D}q$ contains a factor $1/\sqrt{T/n}$ which in the limit diverges. However, the expression in the exponent is not well-defined either and also diverges in this limit. These infinities can be shown to cancel, leaving a meaningful expression.
Note that we started with a Hamiltonian, but ended up with a Lorentz covariant expression. This makes it possible to quantise the theory in a way which is manifestly Lorentz covariant.

Now return to the ordinary operator treatment of quantum mechanics. In the energy picture, one considers energy eigenstates. Let k run through a complete orthonormal set of such eigenstates,

$$\hat{H}|k\rangle = E_k|k\rangle. \tag{134}$$

(c) Express the transition amplitude $\langle q_\mathrm{f}, t_\mathrm{f}|q_\mathrm{i}, t_\mathrm{i}\rangle$ in terms of the wave functions ψ_k and the energies E_k. To what extent can the wave functions ψ_k and the energy spectrum be reconstructed from the transition amplitudes?

Problem 17 **The Harmonic Oscillator**

As a basic example of the method described in the former problem, we consider a particle of mass m in a harmonic potential, without external force, in one space dimension. Its Lagrangian is given by

$$L(q,\dot{q}) = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2 q^2. \tag{135}$$

We shall calculate the transition amplitude $\langle q_\mathrm{f}, t_\mathrm{f}|q_\mathrm{i}, t_\mathrm{i}\rangle$ in two different ways: as a Euclidean path integral, and by means of Schrödinger quantum mechanics. Clearly this amplitude depends on t_i and t_f only through their difference $T = t_\mathrm{f} - t_\mathrm{i}$.
The transition amplitude in Euclidean time is given as a path integral by

$$\langle q_\mathrm{f}, t_\mathrm{f}|q_\mathrm{i}, t_\mathrm{i}\rangle_\mathrm{E} = \int_{q(t_\mathrm{i})=q_\mathrm{i}}^{q(t_\mathrm{f})=q_\mathrm{f}} \mathcal{D}q \, \exp\left(-\frac{1}{\hbar}S_\mathrm{E}[q]\right). \tag{136}$$

(a) Calculate the Euclidean action, obtained after Wick rotation. Derive the equations of motion in Euclidean time. Calculate $q_\mathrm{cl}(t)$, the classical path (in Euclidean time) from $(q_\mathrm{i}, t_\mathrm{i})$ to $(q_\mathrm{f}, t_\mathrm{f})$.

We write

$$q(t) = q_\mathrm{cl}(t) + \eta(t), \tag{137}$$

where $\eta(t)$ are the quantum fluctuations.

(b) Show that

$$\langle q_\mathrm{f}, t_\mathrm{f}|q_\mathrm{i}, t_\mathrm{i}\rangle_\mathrm{E} = \exp\left(-\frac{1}{\hbar}S_\mathrm{E}[q_\mathrm{cl}]\right) \int_{\eta(t_\mathrm{i})=0}^{\eta(t_\mathrm{f})=0} \mathcal{D}\eta \, \exp\left(-\frac{1}{\hbar}S_\mathrm{E}[\eta]\right). \tag{138}$$

Calculate $\exp(-\frac{1}{\hbar}S_\mathrm{E}[q_\mathrm{cl}])$.

In order to calculate the path integral that occurs in the right-hand side of Eq. (138), the time interval from t_i to t_f is divided into n pieces of equal length T/n. After putting

$$\eta_l = \eta(t_\mathrm{i} + l\frac{T}{n}) \quad \text{for} \quad l = 0, 1, \ldots, n, \tag{139}$$

the path integration becomes

$$\int \mathcal{D}\eta = \lim_{n\to\infty} \frac{1}{\sqrt{2\pi\hbar T/nm}} \prod_{l=1}^{n-1} \int \frac{\mathrm{d}\eta_l}{\sqrt{2\pi\hbar T/nm}} \tag{140}$$

and the discretised form of the action is

$$S_{\mathrm{E}}[\eta] = \sum_{l=1}^{n} \frac{T}{n} \left(\frac{1}{2} m \left(\frac{\eta_l - \eta_{l-1}}{T/n} \right)^2 + \frac{1}{2} m\omega^2 \left(\frac{\eta_l + \eta_{l-1}}{2} \right)^2 \right). \tag{141}$$

(c) Rewrite the action as

$$S_{\mathrm{E}}[\eta] = \frac{1}{2T/nm} \left(\eta_1 \quad \ldots \quad \eta_{n-1} \right) M \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_{n-1} \end{pmatrix}, \tag{142}$$

where M is a symmetric $(n-1) \times (n-1)$ matrix. Prove that

$$\prod_{l=1}^{n-1} \int \frac{\mathrm{d}\eta_l}{\sqrt{2\pi\hbar T/nm}} \exp\left(-\frac{1}{\hbar} S_{\mathrm{E}}[\eta] \right) = \frac{1}{\sqrt{\det M}}. \tag{143}$$

Calculate $\det M$.

(d) Calculate the path integral that occurs in the right-hand side of Eq. (138), that is,

$$\lim_{n\to\infty} \frac{1}{\sqrt{2\pi\hbar T/nm}} \prod_{l=1}^{n-1} \int \frac{\mathrm{d}\eta_l}{\sqrt{2\pi\hbar T/nm}} \exp\left(-\frac{1}{\hbar} S_{\mathrm{E}}[\eta] \right). \tag{144}$$

Calculate the transition amplitude in Euclidean time, $\langle q_{\mathrm{f}}, t_{\mathrm{f}} | q_{\mathrm{i}}, t_{\mathrm{i}} \rangle_{\mathrm{E}}$. Finally calculate the transition amplitude in real time, $\langle q_{\mathrm{f}}, t_{\mathrm{f}} | q_{\mathrm{i}}, t_{\mathrm{i}} \rangle$.

Now we turn to Schrödinger quantum mechanics. As is well-known, the energy levels are given by

$$E_k = (k + \frac{1}{2})\hbar\omega \quad \text{with} \quad k = 0, 1, 2, \ldots, \tag{145}$$

with corresponding wave functions

$$\psi_k(q) = C_k H_k \left(\sqrt{\frac{m\omega}{\hbar}} q \right) \exp\left(-\frac{m\omega}{2\hbar} q^2 \right) \quad \text{for} \quad k = 0, 1, 2, \ldots. \tag{146}$$

Here H_k are Hermite polynomials and C_k are normalisation constants.

(e) Use Eqs. (145) and (146) to calculate the transition amplitude $\langle q_{\mathrm{f}}, t_{\mathrm{f}} | q_{\mathrm{i}}, t_{\mathrm{i}} \rangle$. Compare with the result from question (d).

Problem 18 **The semi-classical approximation**

As can be seen from Eq. (127) the largest contribution to the path integral for small $\hbar$ comes from the paths around a classical path, $q_{\mathrm{cl}}(t)$, which minimises the Euclidean action $S_{\mathrm{E}}[q]$. The semi-classical approximation consists of an expansion around an extremum of the action, and only taking into account the fluctuations around the classical path that are first order in $\hbar$. Assume there is only one minimum of the action. Like in Eq. (137) we expand around the classical solution. Substitution of this expansion into the action yields

$$S_{\mathrm{E}}[q] = S_{\mathrm{E}}[q_{\mathrm{cl}}] + \frac{1}{2}\int_{-\frac{T}{2}}^{\frac{T}{2}} \mathrm{d}t\, \eta \left(-m\frac{\mathrm{d}^2}{\mathrm{d}t^2} + V''(q_{\mathrm{cl}}) \right) \eta + \mathcal{O}(\eta^3). \tag{147}$$

We approximate this by neglecting the terms that are of higher order than η^2. Now assume that the q_n are eigenfunctions of the *fluctuation operator*:

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \frac{1}{m}V''(q_{\mathrm{cl}}) \right) q_n = \lambda_n q_n, \tag{148}$$

with λ_n real, and that the q_n form a complete orthonormal set, in which η can be expanded (this puts some restrictions on V which we will assume to be satisfied):

$$\eta(t) = \sum_{n=0}^{\infty} c_n q_n(t). \tag{149}$$

Of course the q_n satisfy the same boundary conditions as η. The symbol $\int \mathcal{D}\eta$ in the expression for the path integral denotes an integration over all possible paths from η_{i} to η_{f}. These paths can be parametrised by the coefficients c_n:

$$\int \mathcal{D}\eta = N \prod_{n=0}^{\infty} \int \frac{\mathrm{d}c_n}{\sqrt{2\pi\hbar/m}}, \tag{150}$$

where N is defined by this equivalence.

(a) Derive Eq. (147). Substitute the expansion of $\eta(t)$ and express the path integral in $S_{\mathrm{E}}[q_{\mathrm{cl}}]$ and

$$\det\Omega \equiv \prod_n \lambda_n. \tag{151}$$

The eigenvalues of the fluctuation operator, Eq. (148) diverge for $n \to \infty$, but the pre-exponential factor that you have found in the expression for the path integral, is well-defined, though. The proof is based on a theorem that we will not prove here. It states that

$$\frac{\det(-\frac{\mathrm{d}^2}{\mathrm{d}t^2} + W(t))}{\psi_0(\frac{1}{2}T)} = C, \tag{152}$$

where C is a constant that is independent of W and T. $\psi_0(t)$ satisfies $\psi_0(-\frac{1}{2}T) = 0$ and $\partial_t\psi_0(-\frac{1}{2}T) = 1$ and

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2} + W(t) \right) \psi_0(t) = 0. \tag{153}$$

Furthermore it is assumed that W is bounded on $[-\frac{1}{2}T, \frac{1}{2}T]$.

(b) Express the pre-exponential factor in N, C and ψ_0. Calculate the system-independent numerator by computing the path-integral for the harmonic oscillator with $q_{\mathrm{i}} = q_{\mathrm{f}} = 0$. Consider the large-$T$ limit, so that you only have to take the lowest energy eigenvalue into account. Now give the full expression for the semi-classical approximation for the Euclidean path integral.

(c) Compute $\langle q_\mathrm{f}, \frac{1}{2}T | q_\mathrm{i}, -\frac{1}{2}T\rangle$ for the harmonic oscillator, using the semi-classical approximation that you obtain with the result of (b). Explain why this is *exactly* the Green's function for the harmonic oscillator as you have found in Problem 17.

Problem 19 **The double well: an introduction to instantons**

It becomes much more interesting to apply the semi-classical approximation to cases where classically the propagator would be zero, but quantum mechanically the transition amplitude does not vanish. We will show that the solution to the Euclidean equation of motion is directly related to the quantum phenomenon of tunnelling. Especially in the case of quantum field theory this Euclidean formalism is essential, and only in this way a rigorous theory can be constructed. However, in this problem we will concentrate on a simpler, quantum mechanical application of the Euclidean formalism, which in principle also could have been solved with quantum mechanics in the well-known Schrödinger description.
Consider a non-relativistic system with the following potential:

$$V(x) = \frac{1}{2}\lambda^4(x^2 - a^2)^2, \quad \lambda, a > 0. \tag{154}$$

The boundary conditions for x are

$$x(-\frac{1}{2}T) = -a \quad \text{and} \quad x(\frac{1}{2}T) = +a. \tag{155}$$

(a) Sketch the potential. What is the classical ground state? Explain why the degeneracy of the classical vacuum will be lifted quantum mechanically, so why the classical ground state energy will split into two energy levels.

We will use the semi-classically approximated Euclidean path-integral for this system. In Problem 18 we have seen that it is dominated by the stationary points of the (Euclidean) action.

(b) Perform a Wick rotation on the Minkowski action. Use the Bogomolny argument to determine the lowest non-trivial solution to the Euclidean field equations, which is stable. For that you should take the approximation of $T \to \infty$, where

$$\dot{x}(\pm\frac{1}{2}T) = 0 \quad \text{and} \quad V\left(x(\pm\frac{1}{2}T)\right) = 0 \quad (\text{for} \quad T \to \infty). \tag{156}$$

(c) Describe the corresponding mechanics problem and explain what the solution from (b) describes in that problem.

(d) Calculate the behaviour of the solution, which you have found under (b), for $t \to \pm\infty$, and explain why the solution is called an *instanton*. What can you now say about the $T \to \infty$ approximation which you have used to find the stationary points of the Euclidean action with finite T?

The energy difference between the two lowest eigenstates is proportional to the tunnelling amplitude, which is proportional to $\exp\left(-\frac{1}{\hbar}S_0\right)$, where S_0 is the action of the instanton solution.

(e) Verify the above statement for the square double well, Figure 4. Explain the procedure you follow.

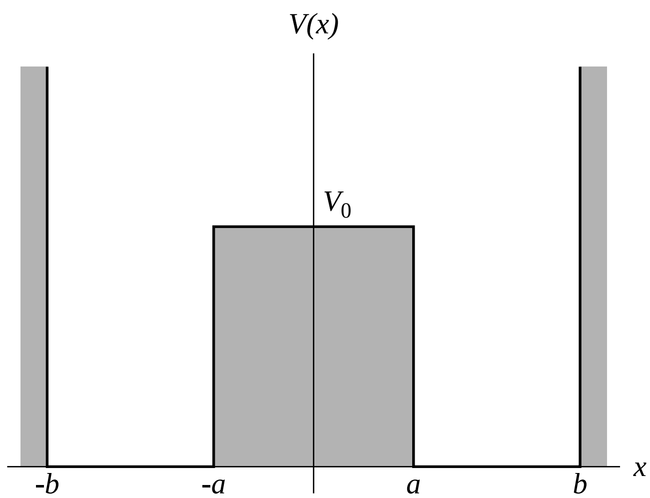

Figure 4: The square double well.

VII. Instantons

Abstract

This time we will consider the quantum aspects of systems with degenerate ground states in more detail. We will completely carry out the instanton calculation of the lowest energy levels, thereby also taking the fluctuations around the classical solutions into account. The same principle will be used to obtain a qualititative expression for the life time of a neutron, as an example of a "false vacuum".

References:
B. Felsager, "Geometry, Particles and Fields", Odense University Press, 1981: Chapter 5.
S. Coleman, "Aspects of symmetry", Cambridge University Press, Cambridge, 1985: Chapter 7 (especially Appendices 1 and 2).

Problem 20 **The lowest energy levels**

We will calculate, in the semi-classical approximation, the Euclidean transition amplitudes

$$\langle -a|e^{-\frac{1}{\hbar}HT}|-a\rangle = \langle a|e^{-\frac{1}{\hbar}HT}|a\rangle \tag{157}$$

and

$$\langle a|e^{-\frac{1}{\hbar}HT}|-a\rangle = \langle -a|e^{-\frac{1}{\hbar}HT}|a\rangle \tag{158}$$

for a double well potential which around the minima resembles the potential of a harmonic oscillator of frequency ω. The first step—the calculation of the stationary points of the Euclidean action that obey the boundary conditions—you have performed in Problem 19. The next step is to calculate the fluctuations around the classical solution. The full tunnelling process also has contributions from multi-instanton solutions that interpolate between two classical vacua, and this will be calculated in the so-called *dilute gas approximation.* The resulting (Euclidean) path-integral will be compared to its decomposition in terms of the eigenfunctions of the Hamilton operator for large T, see Problem 16(c), to obtain the lowest-lying energy levels.

For $T \to \infty$ there is an indeterminacy in the position of the instanton, due to the presence of an integration constant which is not fixed by the boundary conditions. Phrased differently, a particular instanton solution does not possess the time translation symmetry of the Euclidean field equations, so applying this symmetry operation to this particular solution yields a family of solutions. Thus, a perturbation of a particular solution which shifts the whole solution in time will also be a solution. This means that $\frac{\mathrm{d}\bar{x}}{\mathrm{d}t}$ is a *zero-mode* of the fluctuation operator, *i.e.*, an eigenfunction with eigenvalue zero. Here $\bar{x}(t)$ is the instanton solution from Problem 19.

(a) Prove that $\frac{\mathrm{d}\bar{x}}{\mathrm{d}t}$ is a zero-mode of the fluctuation operator, for $T \to \infty$. Normalise this zero-mode. (For this you don't need the explicit expression for $\bar{x}(t)$ from Problem 16(b), but only the Bogomolny equation.) Explain why $\bar{x}$ corresponds to the lowest eigenvalue, so the spectrum of the fluctuation operator is bounded from below by zero.

It follows that the integration over c_0 blows up the path integral for large T. In order to handle this, the integral over c_0 is rewritten as an integration over the position t_0 of the instanton in time.

(b) Split off the integration over c_0, so express the pre-exponential factor in an integration over c_0 and $\det' \Omega$, where the latter can be formally written as

$$\det{}' \Omega = \lim_{T \to \infty} \frac{\det \Omega}{\lambda_0(T)}. \tag{159}$$

In order to change from an integration over c_0 to an integration over t_0, one needs to know how $\mathrm{d}c_0$ compares to $\mathrm{d}t_0$. To this end, compare a (small) shift in time for a fluctuation around the instanton solution with an arbitrary (small) fluctuation, so compare $\bar{x}(t + \mathrm{d}t_0)$ with $\bar{x}(t) + \eta(t)$. Now transform the integration over c_0 into an integration over t_0. What are the integration limits for the latter integration? Assume that $\lambda_0(T)$ drops off exponentially with T for large T. Perform the integration over t_0. Give the transition amplitude in terms of $\det' \Omega$ and S_0, in the limit of large T.

Now one has to regularise $\det' \Omega$. The trick is to write

$$\det{}' \Omega = \frac{\det\left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \omega^2\right)}{K^2}, \tag{160}$$

where $\det(-\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \omega^2)$ is the regularised determinant for the harmonic oscillator, see Problem 18, and K is defined by

$$K = \sqrt{\frac{\det\left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \omega^2\right)}{\det'\left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \frac{1}{m}V''(\bar{x}(t))\right)}}, \tag{161}$$

where the prime on the determinant indicates that the lowest eigenvalue is to be omitted. The calculation of K is postponed to Problem 21. For the moment it is only important that it is a finite quantity, as will be seen in Problem 21.

(c) Express the tunnelling amplitude due to a single instanton, in the large-T limit, in terms of K.

Next we have to consider the contribution of multi-instanton solutions, so we also take solutions into account that move back and forth a certain number of times between two classical vacua. Assume a large separation between the instantons (compared to $\frac{1}{\omega}$), so approximate multi-instanton solutions by a linear superposition of single instanton solutions. This is the *dilute gas approximation.*

(d) What is the action of the n-instanton solution $\bar{x}_n$?

Now the pre-exponential factor for the multi-instanton solution has to be determined. As in the one-instanton case, it arises from the fluctuations around the classical solution, so it is closely related to the spectrum of the fluctuation operator. Again one has to distinguish between the zero-modes and the other modes.

(e) In the large-T limit, the zero-modes correspond to shifting the positions of the instantons in time. Calculate the factor due to integration over these positions.

The other modes give rise to a determinant which can be regularised analagously to

Eq. (160). This time one is led to consider

$$K_n = \sqrt{\frac{\det\left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2}+\omega^2\right)}{\det'\left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2}+\frac{1}{m}V''(\bar{x}_n(t))\right)}}. \tag{162}$$

(f) Explain why $V''(\bar{x}_n(t))$ may be approximated by

$$V''(\bar{x}_n(t)) = m\omega^2 + \sum_{i=1}^{n} \Delta(t-t_i), \tag{163}$$

where $\Delta(t-t_i)$ differs appreciably from zero only around the position t_i (in time) of the ith instanton. Prove that

$$K_n = K^n, \tag{164}$$

where K is given in Eq. (161).

(g) Write down the contribution to the transition amplitude from the n-instanton solution and calculate the resulting transition amplitudes, Eqs. (157) and (158). Which n give the largest contribution? Is the dilute gas approximation a good approximation?

(h) The transition amplitudes can also be expressed in terms of the energy spectrum and the eigenfunctions of the Hamiltonian, see Problem 16(c). Compare this expression to the amplitudes calculated under (g), in the large-T limit. Determine the energy of the ground state and of the first excited state. Discuss the parity of the wave functions of these states.

<u>Problem 21</u> **The final calculations of the energy split**

In the previous problem, the energy split has been calculated between the two lowest levels for the double well potential as given in Eq. (154). The result still contains the yet unspecified expression for K. In order to complete the calculation of the energy split, this quantity will now be determined.
From the definitions one has

$$K^2 = \frac{\det\left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2}+\omega^2\right)}{\det\left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2}+\frac{1}{m}V''(\bar{x}(t))\right)}\lambda_0. \tag{165}$$

We will first calculate the quotient of the determinants. According to Eq. (152),

$$\frac{\det\left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2}+\omega^2\right)}{\det\left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2}+\frac{1}{m}V''(\bar{x}(t))\right)} = \frac{\psi^{(0)}(\frac{1}{2}T)}{\psi_0(\frac{1}{2}T)}. \tag{166}$$

Here $\psi^{(0)}(t)$ satisfies

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2}+\omega^2\right)\psi^{(0)}(t) = 0 \tag{167}$$

with boundary conditions

$$\psi^{(0)}(-\frac{1}{2}T) = 0 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}\psi^{(0)}(-\frac{1}{2}T) = 1, \tag{168}$$

and $\psi_0(t)$ satisfies

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2}+\frac{1}{m}V''(\bar{x}(t))\right)\psi_0(t)=0 \tag{169}$$

with boundary conditions

$$\psi_0(-\frac{1}{2}T)=0 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}\psi_0(-\frac{1}{2}T)=1. \tag{170}$$

An exact expression for $\psi^{(0)}$ is easily obtained, but we still have to find $\psi_0(\frac{1}{2}T)$. Eq. (169) has two linearly independent solutions. In the limit of large T, one is the zero-mode that has already been calculated; denote it by x_0. In order to find the other one, which will be denoted by y_0, we consider the Wronskian

$$W(x_0,y_0)=\begin{vmatrix} x_0 & y_0 \\ \partial_t x_0 & \partial_t y_0 \end{vmatrix}. \tag{171}$$

This has the property that it is constant (verify this!). Asymptotically

$$x_0(t)\to Ae^{\omega t} \qquad \text{for} \qquad t\to-\infty, \tag{172}$$
$$x_0(t)\to Ae^{-\omega t} \qquad \text{for} \qquad t\to+\infty, \tag{173}$$

where A is some constant. We multiply y_0 by a scalar, such that

$$W(x_0,y_0)=-2A^2\omega. \tag{174}$$

(a) Calculate A. Determine the asymptotic behaviour of $y_0(t)$ for $t\to-\infty$ and for $t\to+\infty$. Express $\psi_0(t)$ in terms of $x_0(t)$ and $y_0(t)$. Calculate $\psi_0(\frac{1}{2}T)$ for large T.

Next we want to determine the lowest eigenvalue, λ_0, which is a function of T. We have the eigenvalue equation

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2}+\frac{1}{m}V''(\bar{x}(t))\right)\psi_\lambda(t)=\lambda\psi_\lambda(t), \tag{175}$$

with boundary conditions

$$\psi_\lambda(-\frac{1}{2}T)=0 \quad \text{and} \quad \psi_\lambda(\frac{1}{2}T)=0 \tag{176}$$

Instead of this boundary value problem, we rather consider the initial value problem

$$\psi_\lambda(-\frac{1}{2}T)=0 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}\psi_\lambda(-\frac{1}{2}T)=1. \tag{177}$$

and then impose the condition

$$\psi_\lambda(\frac{1}{2}T)=0. \tag{178}$$

Note that above we have solved this initial value problem for $\lambda=0$.

(b) Transform the eigenvalue equation Eq. (175) with initial values Eq. (177) into the integral equation

$$\psi_\lambda(t)=\psi_0(t)-\lambda\int_{-\frac{1}{2}T}^{t}G(t,t')\psi_\lambda(t')\mathrm{d}t', \tag{179}$$

where the Green function satisfies

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2}+\frac{1}{m}V''(\bar{x}(t))\right)G(t,t')=0 \quad \text{for each } t' \tag{180}$$

and

$$G(t,t')\big|_{t'=t}=0 \quad \text{and} \quad \frac{\partial}{\partial t}G(t,t')\bigg|_{t'=t}=1. \tag{181}$$

Express the Green functions in terms of x_0 and y_0. Iterate the integral equation Eq. (179), *i.e.*, substitute the right-hand side in the integrand, discarding the λ^2 terms. Calculate $\lambda=\lambda_0$ from Eq. (178), in the limit of large T.

(c) Calculate the two lowest energies.

Problem 22 **Bounces, and the life-time of the neutron**

In this problem, the life-time of the neutron will be calculated qualitatively. The electron is trapped close to the proton; assume the potential has the shape shown in Figure 5. In the well around $x=0$ the electron is bound in the neutron, but it could tunnel through the barrier and become free.

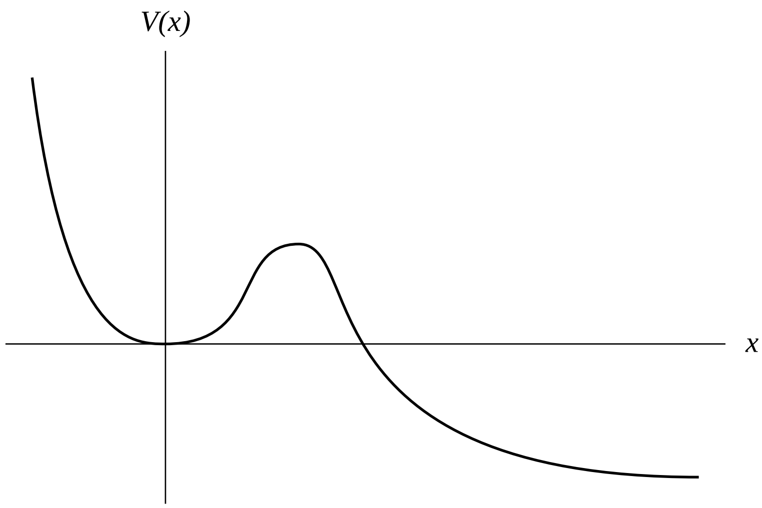

Figure 5: The potential for an electron within a neutron.

We will calculate the transition amplitude from $x_\mathrm{i}=0$ to $x_\mathrm{f}=0$, in the semi-classical approximation, as a Euclidean path integral. The action has a trivial stationary point with $x_\mathrm{i}=0$ and $x_\mathrm{f}=0$, namely the electron remaining at $x=0$ throughout. The simplest non-trivial stationary point with $x_\mathrm{i}=0$ and $x_\mathrm{f}=0$, in the limit of large T, is called a *bounce*. Its action will be denoted by B.

(a) Describe the bounce. Sketch a graph of $\bar{x}_{\mathrm{bounce}}(t)$.

In the large-T limit, there are approximate multi-bounce solutions, consisting of a number of bounces widely separated in time. All these solutions contribute to the transition amplitude. Thus the situation is very similar to the double well, see Problem 20, where multi-instanton solutions contribute to the tunnelling amplitude. The contribution from the n-bounce solutions is given by

$$\left(\int_{x_{\rm i}=0}^{x_{\rm f}=0} \mathcal{D}x \exp\left(-\frac{1}{\hbar} S_{\rm E}[x]\right)\right)_{n \text{ bounces}} = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} e^{-\frac{1}{2}\omega T} \left(\frac{B}{2\pi\hbar}\right)^{\frac{n}{2}} \frac{T^n}{n!} K^n e^{-nB/\hbar}. \quad (182)$$

(b) Explain the meaning and origin of each of the factors in the right-hand side of Eq. (182). Perform the summation over all multi-bounce solutions. Calculate the "ground state" energy. Discuss the significance of the terms you have found.

Naively one would expect that

$$K = \sqrt{\frac{\det\left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \omega^2\right)}{\det'\left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \frac{1}{m} V''(\bar{x}_{\rm bounce}(t))\right)}}, \quad (183)$$

where the prime on the determinant indicated that the zero eigenvalue has to be omitted. This however turns out to be not the case.

(c) In the large-T limit, $\frac{\mathrm{d}}{\mathrm{d}t}\bar{x}_{\rm bounce}(t)$ is a zero mode of the fluctuation operator, due to the time translation invariance of the equations of motion. Sketch a graph of $\frac{\mathrm{d}}{\mathrm{d}t}\bar{x}_{\rm bounce}(t)$. Show that the fluctuation operator has a negative eigenvalue.

This problem arises because the state that we consider—the particle in the bottom of the well—is not an eigenstate of the Hamiltonian: it is unstable. The energy of an unstable state can only be defined by analytic continuation. We will not go through the arguments of this analytic continuation, but state that the imaginary part of the one-bounce contribution to the functional integral is given by

$$\operatorname{Im}\left(\int_{x_{\rm i}=0}^{x_{\rm f}=0} \mathcal{D}x \exp\left(-\frac{1}{\hbar} S_{\rm E}[x]\right)\right)_{\text{one bounce}} = \quad (184)$$

$$\frac{1}{2} N \left(\frac{B}{2\pi\hbar}\right)^{\frac{1}{2}} T \left|\det'\left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \frac{1}{m} V''(\bar{x}_{\rm bounce}(t))\right)\right|^{-\frac{1}{2}} e^{-B/\hbar}. \quad (185)$$

(d) Compute $\operatorname{Im} K$. Determine the life-time of the unstable state.

VIII. Quantisation of Solitons

Abstract

This session is devoted to the quantisation of the soliton of the sine–Gordon model. The quantum fluctuations around the soliton and around the vacuum are studied in the semi-classical approximation. After some generalities, the relevant fluctuation operators are diagonalised. The quantum corrections to the soliton mass are then calculated. They turn out to have have a divergent term, which can be interpreted as a mass renormalisation.

References:
B. Felsager, "Geometry, Particles and Fields", Odense University Press, 1981: Chapter 4.
S. Coleman, "Aspects of symmetry", Cambridge University Press, Cambridge, 1985: Chapter 4.
R. Rajaraman, "Solitons and Instantons", North-Holland, Amsterdam, 1987: Chapters 5 and 7.

Problem 23 **Semi-classical approximation for the soliton**

We consider the sine–Gordon model defined by the Lagrangian density

$$\mathcal{L}(\phi, \partial_\mu \phi) = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{m^4}{\lambda}\left(1 - \cos\frac{\sqrt{\lambda}}{m}\phi\right). \tag{186}$$

In Problem 10 we have calculated the stationary classical solutions, now we will study the quantum fluctuations around these solutions, in the semi-classical approximation.

(a) As an introduction to quantum corrections on classical solutions, read the photocopies from the book by Rajaraman. Consider a (time-dependent) fluctuation $\delta\phi$ around a time-independent classical solution $\bar{\phi}$. Calculate the action and determine the fluctuation operator $\Omega_{\bar{\phi}}$.

Expand the fluctuations around a classical solution as

$$\delta\phi(x,t) = \sum_q c_q(t)\eta_q(x). \tag{187}$$

Choose $\eta_q(x)$ to be the complete orthonormal set of bounded eigenfunctions of the following operator:

$$-\partial_x^2 + m^2 \cos\frac{\sqrt{\lambda}}{m}\bar{\phi} \tag{188}$$

with corresponding eigenvalues ω_q^2.

(b) Show that in the semi-classical approximation the action can be written as the action of the classical solution, plus the action of a set of harmonic oscillators with frequencies ω_q. These frequencies depend on the classical solution around which you expand.

(c) Discuss the energy spectrum of the quantised solution. In particular, what is the ground state energy?

Problem 24 **The quantum correction on the soliton mass**

We wish to determine the quantum corrected mass of the soliton. Naively one might think it is sufficient to determine the ground state energy of the soliton. However, in the previous problem we have seen that this is divergent. In order to calculate the mass one has to subtract the ground state energy of the vacuum, which is also divergent. Therefore we have to determine the spectra of the static fluctuation operators around the soliton and the vacuum, Eq. (188).

(a) Solve the eigenvalue equation of the vacuum fluctuation operator. It is convenient to write

$$\omega_k^2 = k^2 + m^2. \tag{189}$$

Explain that only the bounded solutions are relevant. Next impose boundary conditions

$$\eta_k(-\frac{L}{2}) = \eta_k(\frac{L}{2}) = 0 \tag{190}$$

and give the allowed values of k.

Now the eigenvalue equation of the soliton fluctuation operator has to be solved.

(b) Give the classical soliton solution centered at $x = 0$. Determine the eigenvalue equation for the fluctuation operator around this solution. It is convenient to write

$$\omega_q^2 = q^2 + m^2. \tag{191}$$

Substitute

$$\eta_q(x) = \frac{\theta_q(x)}{\cosh mx} \tag{192}$$

and derive the differential equation for $\theta_q(x)$. Multiply this equation by a suitable function of x and perform a Laplace transformation. Solve the resulting functional equation for the Laplace transform $\Theta(s)$ of $\theta(x)$.[16] Transform back and calculate the eigenfunctions η_q of the soliton fluctuation operator. Recall that only the bounded solutions are of physical interest.

(c) Impose the boundary condition

$$\eta_q(-\frac{L}{2}) = \eta_q(\frac{L}{2}) = 0 \tag{195}$$

on the eigenfunctions corresponding to the continuous spectrum, and show that for large L the restriction on q is given by

$$q_n L + \delta(q_n) = 2\pi n \qquad (n \in \mathbf{Z}) \tag{196}$$

[16] For a functional equation of the form

$$A(s+2c)\Theta(s+c) + A(s-2c)\Theta(s-c) = \ldots \tag{193}$$

where A is a given function and c a constant, one may try a substitution

$$\Theta(s) = \frac{X(s)}{A(s-c)A(s+c)}. \tag{194}$$

with

$$\delta(q) = 2\arctan(\frac{m}{q}). \tag{197}$$

Plot $\delta(q)$ versus q. Take a fixed, large value of L and plot $(2\pi n - qL)$ for various n, in the same figure. Compare the frequencies of the fluctuation operator in the vacuum sector and the soliton sector and interpret $\delta(q)$.

(d) Interpret the lowest eigenfunction of the soliton fluctuation operator. Explain that it does not contribute to the ground state energy of the soliton.

(e) Show that the difference between the ground state energy of the soliton and the ground state energy of the vacuum can be written as

$$\Delta E = E_0 - \frac{1}{2}m - \frac{1}{2}\sum_{n\neq 0}\frac{q_n\delta(q_n)}{\sqrt{m^2+q_n^2}}\frac{1}{L}, \tag{198}$$

where E_0 is the energy of the (static) classical solution. For $L \to \infty$ the sum over n becomes an integral over q. Show that the energy difference ΔE is the sum of three terms: the classical energy E_0, another finite term, and a divergent term proportional to

$$\int_0^\infty \frac{\mathrm{d}q}{\sqrt{m^2+q^2}}. \tag{199}$$

Calculate these contributions and give the quantum corrected mass of the soliton.

Next week it will be demonstrated that the divergent term is to be interpreted as a mass renormalisation, and so a finite mass correction is obtained.