Proof systems for Dynamic Predicate Logic

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1 Introduction

Goal
To find a sound and complete Fitch style Natural Deduction System $\mathcal{N}$ for DPL.

Strategy
Develop complete Semantic Tableau system $\mathcal{S}$.
Use $\mathcal{S}$ to find complete Deductive Tableau system $\mathcal{D}$
Use $\mathcal{D}$ to find $\mathcal{N}$.

What the reader will find is a motivated description of $\mathcal{S}$, $\mathcal{D}$ and $\mathcal{N}$ plus proofs which show that

$$\pi \models \sigma \Rightarrow \pi \vdash_{\mathcal{S}} \sigma \Rightarrow \pi \vdash_{\mathcal{D}} \sigma \Rightarrow \pi \vdash_{\mathcal{N}} \sigma \Rightarrow \pi \models \sigma$$

2 Preliminaries

Definition 1 (The languages of DPL)

(a) The languages of dynamic predicate logic share the following *logical vocabulary*: $\bot$ (falsum), $\neg$ (negation), $\wedge$ (conjunction); the pair of brackets (,); a denumerable set of syntactic variables $V = \{x_0, x_1, x_2, \ldots\}$; existential quantifiers $\exists x$, for $x \in V$.

(b) A specific language L is identified with its *non-logical vocabulary*: a set of individual constants, and a for each n a set of n-place predicates.

The union of the set of variables and the set of individual constants of L is called the set of *terms* of L. The set of *formulas* of L is defined in the usual way.

The core language can be extended by defining additional logical constants. E.g., we can add '$\rightarrow$' (implication), '$\vee$' (disjunction), and '$\forall x$' (universal quantifiers). The choice of logical primitives is not as optional in DPL as it is in standard predicate logic.

Definition 2 (Extension of core syntax)

(a) $(\phi \to \psi) = \neg(\phi \wedge \neg\psi)$.
(b) $(\phi \vee \psi) = \neg(\neg\phi \wedge \neg\psi)$.
(c) $\forall x\phi = \neg\exists x\neg\phi$.

Below, the letters π, σ (with or without subscripts) vary over finite, possibly empty sequences of formulas, which are also called texts. We will often write $\varphi_1 \,.\, \cdots \,.\, \varphi_n$ instead of $\langle\varphi_1, \ldots, \varphi_n\rangle$. If $\pi = \varphi_1 \,.\, \cdots \,.\, \varphi_n$ and $\sigma = \psi_1 \,.\, \cdots \,.\, \psi_m$, then $\pi \,.\, \sigma$ is short for $\langle\varphi_1, \ldots, \varphi_n, \psi_1 \ \ldots, \psi_m\rangle$.

Definition 3 (States) Let $\mathcal{M} = \langle\mathcal{D}, \mathcal{I}\rangle$ be some model for a language of first order predicate logic.

(a) $\mathbf{1}_{\mathcal{M}}$, the set of *assignments* pertaining to $\mathcal{M}$, is the set of functions from V into $\mathcal{D}$. Whenever it is clear about which model $\mathcal{M}$ we are talking, we will write $\mathbf{1}$ rather than $\mathbf{1}_{\mathcal{M}}$.
(b) Let v be an assignment, $x \in V$, $d \in \mathcal{D}$. $v[x/d]$ is the assignment u such that $u(x) = d$, while for every $y \in V$: if $y \neq x$ then $u(y) = v(y)$.
(c) A *state* based on $\mathcal{M}$ is a set s of assignments. Here $\mathbf{1}$ can be thought of as the *mininimal state*, and $\emptyset$ as the maximal, or *absurd* state. In this connection we will often write '$\mathbf{0}$' rather than '$\emptyset$'.
(d) Let s be a state, $x \in V$, $d \in \mathcal{D}$.
$s[x/d] = \{u \in \mathbf{1} \mid u = v[x/d] \text{ for some } v \in s\}$.

Definition 4 (Dynamic interpretation) Let s be a state based on $\mathcal{M}$. For every formula φ the update $s[\varphi]$ of s with φ is recursively defined as follows:

(i) $s[Rt_1 \ldots t_n] = \{v \in s \mid \langle v(t_1), \ldots, v(t_n)\rangle \in \mathcal{I}(R)\}$.
(ii) $s[\neg\varphi] = \{v \in s \mid \{v\}[\varphi] = \mathbf{0}\}$
(iii) $s[\varphi \wedge \psi] = s[\phi][\psi]$.
(iv) $s[\exists x\varphi] = \cup_{d\in\mathcal{D}}(s[x/d][\varphi])$.

For a text $\pi = \varphi_1 \,.\, \cdots \,.\, \varphi_n$ we write $s[\pi]$ to abbreviate $s[\varphi_1] \ldots [\varphi_n]$.

Examples

- John has a new girlfriend. She is blond
  $\exists xGjx \,.\, Bx$
  (Consider $\mathbf{1}[\exists xGjx][Bx]$).
- John has no new girlfriend. She is blond (??)
  $\neg\exists xGjx \,.\, Bx$
  (Consider $\mathbf{1}[\neg\exists xGjx][Bx]$)
- If John has a new girlfriend, she is blond
  $\exists xGjx \to Bx$
  $\forall x(Gjx \to Bx)$

In the sequel we need a slightly more general notion of entailment than the one you may be prepared for.

Definition 5 (Support) $s \models \sigma$ iff for every $v \in s$, $\{v\}[\pi] \neq \mathbf{0}$.

Definition 6 (Entailment) $\pi \models \sigma$ iff for any model $\mathcal{M}$, and any s based on $\mathcal{M}$, $s[\pi] \models \sigma$.

2.1 Examples

- $\exists x Px \models Px$
- $\exists x Px \,.\, \exists x \neg Px \models \neg Px$
- $\exists x Px \,.\, \neg\neg\exists x \neg Px \models Px$
- $\exists x(Px \wedge Qx) \models \exists x Px \,.\, Qx$
- $\exists x Px \rightarrow Qx \models \forall x(Px \rightarrow Qx)$

Loss of structural properties:

- No Repetition: $\exists x Px \,.\, Px \wedge \exists x \neg Px \not\models Px$.
- No Monotony: $\exists x Px \models Px$, but $\exists x Px \,.\, \exists x \neg Px \not\models Px$.
- No Permutation: $\exists x \neg Px \,.\, \exists x Px \models Px$, but $\exists x Px \,.\, \exists x \neg Px \not\models Px$.
- No Cut: $\exists x Px \,.\, \exists x \neg Px \models \exists x Px$, and $\exists x Px \,.\, \exists x \neg Px \,.\, \exists x Px \models Px$, but $\exists x Px \,.\, \exists x \neg Px \not\models Px$

DPL is not an extension of standard predicate logic.
$\exists x((Ax \wedge \exists x Bx) \wedge Cx) \models_{cl} \exists x(Cx \wedge (Ax \wedge \exists x Bx))$, but
$\exists x((Ax \wedge \exists x Bx) \wedge Cx) \not\models \exists x(Cx \wedge (Ax \wedge \exists x Bx))$.

2.2 Scope and Binding

Definition 7 (Scope island) A formula φ is a *scope island* iff φ is of the form $\neg\psi$.

Definition 8 (Free and bound variables)

(a) A specific occurrence of x in π is *bound by* a specific occurrence of $\exists x$ in π iff

- (i) The occurrence of x is to the right of the occurrence of $\exists x$;
- (ii) There is no occurrence of a scope island φ in π such that the occurrence of $\exists x$ is inside the occurrence of φ and the occurrence of x is not;
- (iii) The occurrence of $\exists x$ is not to the left of another occurrence of $\exists x$ in π for which (i) and (ii) hold.

(b) A given occurrence of a quantifier $\exists x$ in a text π is *active after* π iff

- (i) There is no occurrence of a scope island φ in π such that the occurrence of $\exists x$ is inside the occurrence of φ;
- (ii) The occurrence of $\exists x$ is not to the left of another occurrence of $\exists x$ in π for which (i) holds.

(c) An occurrence of x is *free in* π iff the occurrence of x is not bound by an occurrence of $\exists x$ in π.

(d) An occurrence of x in σ is *free in* σ *after* π iff the occurrence of x in σ is free in $\pi \,.\, \sigma$.

(e) σ *depends on* π iff there is a free occurrence of a variable in σ which is not free in σ after π.

Definition 9 (Substitution)

(a) $[t/x, \pi](\sigma)$ is the result of substituting t for every occurrence of x in σ which is free in σ after π.

(b) y is *free for* x *in* σ *after* π iff y occurs free in $[y/x, \pi](\sigma)$ at all places where x occurs free in σ after π.

We will write '$[c/x](\sigma)$' rather than '$[c/x, \emptyset](\sigma)$'

Lemma 1
Let u and v be two assignments such that $u(x) = v(x)$ for all variables x occurring free in π. Then $\{u\}[\pi] \neq \emptyset$ iff $\{v\}[\pi] \neq \emptyset$

Moreover ...

Lemma 2

(a) Let x be free for y in π. Let s and s' be two states such that $s' = [y/s(x)]s$. Then $v \in s[[x/y]\pi]$ iff $v[y/s(x)] \in s'[\pi]]$

(b) Let x be free for y in π, and suppose that x does not occur free in π. Let s and s' be two states such that $s' = [x/s(y)]s$. Then $v \in s[\pi]$ iff $v[x/s(y)] \in s'[[x/y]\pi]]$

Definition 10 (Strong entailment)

(a) $i \approx_\pi j$ iff $i(x) = j(x)$ for all x such that $\exists x$ is active after π.

(b) $s \models_\pi \sigma$ iff $s \models \sigma$, and for all $i \in s$ and $j \in \{i\}[\sigma]$ it holds that $i \approx_\pi j$.

(c) $\pi \models\!\!\!\approx \sigma$ iff for any s $s[\pi] \models_\pi \sigma$.

Notice:

(a) If $\pi \models \sigma . \sigma'$, then $\pi \models \sigma$.

(b) It is not generally so that if $\pi \models \sigma . \sigma'$, then $\pi \models \sigma'$

(c) If $\pi \models \sigma . \sigma'$, and $\pi \models\!\!\!\approx \sigma$, then $\pi \models \sigma'$

3 Deductive Tableaus

A sequent is a pair $\langle \pi, \sigma \rangle$. We will write $\pi \Rightarrow \sigma$ rather than $\langle \pi, \sigma \rangle$

Call a text π *atomic* iff it consists of just one atomic sentence, and *simple* iff every formula that occurs in it is an atomic sentence. A sequent $\pi \Rightarrow \sigma$ is *simple* if π is simple and σ is atomic.

Definition 11 Let $\sigma = \varphi_1 . \cdots . \varphi_n$.

$\bigwedge \sigma =_{df} (\varphi_1 \wedge (\varphi_2 \wedge (\ldots \wedge (\varphi_{n-1} \wedge \varphi_n) \ldots)).$

Definition 12 (Deductive Tableau)
A deductive tableau for $\pi \Rightarrow \sigma$ is a tree consisting of sequents. This tree $\mathcal{T}$ is constructed as follows.

(i) The root of $\mathcal{T}$ is the sequent $\pi \Rightarrow \sigma$.

(ii) If a node $\pi' \Rightarrow \sigma'$ is simple or an axiom, then this node will have no successors.

(iii) If a node $\pi' \Rightarrow \sigma'$ is neither simple nor an axiom, then this node will have one or two immediate successors. For $\pi'' \Rightarrow \sigma''$ to be the one immediate successor of $\pi' \Rightarrow \sigma'$, it is neccessary that

$$\frac{\pi'' \Rightarrow \sigma''}{\pi' \Rightarrow \sigma'}$$

is a deductive tableau rule, while for $\pi'' \Rightarrow \sigma''$ and $\pi''' \Rightarrow \sigma'''$ to be the two immediate successors of $\pi' \Rightarrow \sigma'$ it is necessary that

$$\frac{\pi'' \Rightarrow \sigma'' \quad \pi''' \Rightarrow \sigma'''}{\pi' \Rightarrow \sigma'}$$

is a tableau rule.

The deductive tableau rules are given in the table below.

$\pi \,.\, Rt_1 \ldots t_n \,.\, \pi' \Rightarrow Rt_1 \ldots t_n$ (axiom)	Provided $Rt_1 \ldots t_n$ does not depend on π'
$\dfrac{\pi \,.\, \varphi \,.\, \psi \,.\, \pi' \Rightarrow \sigma}{\pi \,.\, \varphi \wedge \psi \,.\, \pi' \Rightarrow \sigma}$ $\mathrm{E}_\wedge$	Provided —
$\dfrac{\pi \Rightarrow \sigma \,.\, \varphi \,.\, \psi \,.\, \sigma'}{\pi \Rightarrow \sigma \,.\, \varphi \wedge \psi \,.\, \sigma'}$ $\mathrm{I}_\wedge$	Provided —
$\dfrac{\pi \,.\, [^c/x](\varphi \,.\, \pi') \Rightarrow [^c/x, \varphi \,.\, \pi'](\sigma)}{\pi \,.\, \exists x \varphi \,.\, \pi' \Rightarrow \sigma}$ $\mathrm{E}_\exists$	Provided c does not occur in π, φ, π', or σ
$\dfrac{\pi \Rightarrow \sigma \,.\, [^t/x](\varphi \,.\, \sigma')}{\pi \Rightarrow \sigma \,.\, \exists x \varphi \,.\, \sigma'}$ $\mathrm{I}_\exists$	Provided t is free for x in $\varphi \,.\, \sigma'$
$\dfrac{\pi \,.\, \pi' \Rightarrow \varphi}{\pi \,.\, \neg\varphi \,.\, \pi' \Rightarrow \bot}$ $\mathrm{E}_\neg$	Provided φ does not depend on π'
$\dfrac{\pi \,.\, \varphi \Rightarrow \bot}{\pi \Rightarrow \neg\varphi}$ $\mathrm{I}_\neg$	Provided —
$\dfrac{\pi \Rightarrow \neg\neg\varphi}{\pi \Rightarrow \varphi}$ $\neg\neg$	Provided —
$\dfrac{\pi \Rightarrow \sigma \,.\, \varphi \wedge \psi \,.\, \sigma'}{\pi \Rightarrow \sigma \,.\, \varphi \,.\, \psi \,.\, \sigma'}$ sen	Provided —
$\dfrac{\pi \Rightarrow \sigma \quad \pi \,.\, \sigma \Rightarrow \sigma'}{\pi \Rightarrow \sigma \,.\, \sigma'}$ seq	Provided —

Definition 13 $\pi \vdash_{\mathcal{D}} \sigma$ iff there exists some finite deductive tableau with root $\pi \Rightarrow \sigma$ such that all sequents at the top are axioms.

Examples

(a) The example below shows why we cannot restrict our attention to sequents $\pi \Rightarrow \varphi$.

$$
\begin{array}{cl}
Ac\,.\,Bc \Rightarrow Ac \quad Ac\,.\,Bc\,.\,Ac \Rightarrow Bc & \\ \cline{1-1}
Ac\,.\,Bc \Rightarrow Ac\,.\,Bc & \text{seq} \\ \cline{1-1}
Ac\,.\,Bc \Rightarrow \exists x Ax\,.\,Bx & \mathrm{I}_{\exists} \\ \cline{1-1}
Ac\,.\,Bc \Rightarrow \exists x Ax \wedge Bx & \mathrm{I}_{\wedge} \\ \cline{1-1}
Ac \wedge Bc \Rightarrow \exists x Ax \wedge Bx & \mathrm{E}_{\wedge} \\ \cline{1-1}
\exists x(Ax \wedge Bx) \Rightarrow \exists x Ax \wedge Bx & \mathrm{E}_{\exists}
\end{array}
$$

(b) The rules $\neg\neg$ and sen, which differ from the other rules in that successor sequent is more complex than the predecessor, could be replaced by one rule that would allow for a strong form of reductio ad absurdum.

$$
\dfrac{\pi\,.\,\neg \bigwedge \sigma \Rightarrow \bot}{\pi \Rightarrow \sigma}\ \text{reductio}
$$

$$
\begin{array}{cl}
Ac\,.\,Bc \Rightarrow Ac \quad Ac\,.\,Bc\,.\,Ac \Rightarrow Bc & \\ \cline{1-1}
Ac\,.\,Bc \Rightarrow Ac\,.\,Bc & \text{seq} \\ \cline{1-1}
Ac\,.\,Bc \Rightarrow \exists x Ax\,.\,Bx & \mathrm{I}_{\exists} \\ \cline{1-1}
Ac \wedge Bc \Rightarrow \exists x Ax\,.\,Bx & \mathrm{E}_{\wedge} \\ \cline{1-1}
\exists x(Ax \wedge Bx) \Rightarrow \exists x Ax\,.\,Bx & \mathrm{E}_{\exists} \\ \cline{1-1}
\exists x(Ax \wedge Bx) \Rightarrow \exists x Ax \wedge Bx & \mathrm{I}_{\wedge} \\ \cline{1-1}
\exists x(Ax \wedge Bx)\,.\,\neg(\exists x Ax \wedge Bx) \Rightarrow \bot & \mathrm{E}_{\neg} \\ \cline{1-1}
\neg(\exists x Ax \wedge Bx) \Rightarrow \neg\exists x(Ax \wedge Bx) & \mathrm{I}_{\neg} \\ \cline{1-1}
\neg\neg\exists x(Ax \wedge Bx)\,.\,\neg(\exists x Ax \wedge Bx) \Rightarrow \bot & \mathrm{E}_{\neg} \\ \cline{1-1}
\neg\neg\exists x(Ax \wedge Bx) \Rightarrow \neg\neg(\exists x Ax \wedge Bx) & \mathrm{I}_{\neg} \\ \cline{1-1}
\neg\neg\exists x(Ax \wedge Bx) \Rightarrow \exists x Ax \wedge Bx & \neg\neg \\ \cline{1-1}
\neg\neg\exists x(Ax \wedge Bx) \Rightarrow \exists x Ax\,.\,Bx & \text{sen}
\end{array}
$$

For practical purposes, it's pleasant to add the following rule:

$\dfrac{\pi \Rightarrow \bot}{\pi \Rightarrow \sigma}$ ex falso	Provided —

However, this rule is in fact not needed.

Proposition 1

(a) If $\pi \vdash_{\mathcal{D}} \sigma$, then $\pi\,.\,\pi' \vdash_{\mathcal{D}} \sigma$, provided that σ does not depend on π'.
(b) $\pi\,.\,\varphi \vdash_{\mathcal{D}} \varphi$, provided that φ does not depend on φ.

Proof:

(a) ...
(b) ...

Corollaries

(a) If $\pi \vdash_{\mathcal{D}} \bot$, then $\pi \vdash_{\mathcal{D}} \sigma$
(b) $\pi\,.\,\neg\varphi \vdash_{\mathcal{D}} \neg\varphi$

3.1 Intermediate Tableaus

With one exception the rules for intermediate tableaus are the same as the rules for deductive tableaus. The elimation rule for the existential quantifier

$\dfrac{\pi\,.\,[^c/x](\varphi\,.\,\pi') \Rightarrow [^c/x,\varphi\,.\,\pi'](\sigma)}{\pi\,.\,\exists x\varphi\,.\,\pi' \Rightarrow \sigma}\ \mathrm{E}_\exists$	Provided c does not occur in π, φ, π', or σ

is replaced by the following two rules:

$\dfrac{\pi\,.\,\exists y[^y/x](\varphi\,.\,\pi') \Rightarrow [^y/x,\varphi\,.\,\pi'](\sigma)}{\pi\,.\,\exists x\varphi\,.\,\pi' \Rightarrow \sigma}\ \mathrm{alf}$	Provided y is free for x in $\varphi\,.\,\pi'\,.\,\sigma$
$\dfrac{\pi\,.\,\varphi\,.\,\pi' \Rightarrow \sigma}{\pi\,.\,\exists x\varphi\,.\,\pi' \Rightarrow \sigma}\ \mathrm{pron}$	Provided x is free in $\varphi\,.\,\pi'\,.\,\sigma$ after π

Let's call the resulting system $\mathcal{D}'$. We will show now that if $\pi \vdash_{\mathcal{D}} \sigma$, then $\pi \vdash_{\mathcal{N}} \sigma$ by proving the following two lemmata.

Lemma 3
If $\pi \vdash_{\mathcal{D}} \sigma$, then $\pi \vdash_{\mathcal{D}'} \sigma$

Lemma 4
If $\pi \vdash_{\mathcal{D}'} \sigma$, then $\pi \vdash_{\mathcal{N}} \sigma$

4 Semantic Tableaus

A *generalised sequent* (g-sequent) is a pair $\langle \pi, \Delta \rangle$, where π is a text and Δ a set of texts. We will write and $\pi \Rightarrow \Delta$ rather than and $\langle \pi, \Delta \rangle$. A generalised sequent $\pi \Rightarrow \Delta$ is *simple* iff π is simple and each σ in Δ is atomic.

Definition 14 (Generalised Entailment) Let Δ be a set of texts. $\pi \models \Delta$ iff the following holds for any state s: for every $i \in s[\pi]$, there exists some $\sigma \in \Delta$ such that $\{i\}[\sigma] \neq \emptyset$.

The definition says that $\pi \models \{\sigma\}$ iff for any s, $s[\pi]$ supports σ. So, when Δ is a singleton, we are back in the old case. We will always write '$\pi \models \sigma$' rather than $\pi \models \{\sigma\}$.

Semantic tableaus are constructed by applying the following rules.

$\pi\,.\,Rt_1 \ldots t_n\,.\,\pi' \Rightarrow \Delta, Rt_1 \ldots t_n$ (axiom)	Provided $Rt_1 \ldots t_n$ does not depend on π'
$\dfrac{\pi\,.\,\varphi\,.\,\psi\,.\,\pi' \Rightarrow \Delta}{\pi\,.\,\varphi \wedge \psi\,.\,\pi' \Rightarrow \Delta}\ \mathrm{E}_\wedge$	Provided —
$\dfrac{\pi \Rightarrow \Delta, \sigma\,.\,\varphi\,.\,\psi\,.\,\sigma'}{\pi \Rightarrow \Delta, \sigma\,.\,\varphi \wedge \psi\,.\,\sigma'}\ \mathrm{I}_\wedge$	Provided —
$\dfrac{\pi\,.\,[^c/x](\varphi\,.\,\pi') \Rightarrow [^c/x, \varphi\,.\,\pi'](\Delta)}{\pi\,.\,\exists x\varphi\,.\,\pi' \Rightarrow \Delta}\ \mathrm{E}_\exists$	Provided c does not occur in π, φ, π', or Δ
$\dfrac{\pi \Rightarrow \Delta, \sigma\,.\,[^t/x](\varphi\,.\,\sigma')}{\pi \Rightarrow \Delta, \sigma\,.\,\exists x\varphi\,.\,\sigma'}\ \mathrm{I}_\exists$	Provided t is free for x
$\dfrac{\pi\,.\,\pi' \Rightarrow \Delta, \varphi}{\pi\,.\,\neg\varphi\,.\,\pi' \Rightarrow \Delta}\ \mathrm{E}_\neg$	Provided φ does not depend on π'
$\dfrac{\pi\,.\,\varphi \Rightarrow \Delta}{\pi\,.\ \Rightarrow \Delta, \neg\varphi}\ \mathrm{I}_\neg$	Provided no σ in Δ depends on φ
$\dfrac{\pi \Rightarrow \Delta, \sigma' \quad \pi\,.\,\sigma' \Rightarrow \Delta, \sigma''}{\pi \Rightarrow \Delta, \sigma'\,.\,\sigma''}\ \mathrm{seq}$	Provided no σ in Δ depends on σ'

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{Ac\,.\,Bc \Rightarrow Ac \quad Ac\,.\,Bc\,.\,Ac \Rightarrow Bc}{Ac\,.\,Bc \Rightarrow Ac\,.\,Bc}\ \mathrm{seq}}{Ac\,.\,Bc \Rightarrow \exists x Ax\,.\,Bx}\ \mathrm{I}_\exists}{Ac \wedge Bc \Rightarrow \exists x Ax\,.\,Bx}\ \mathrm{E}_\wedge}{\exists x(Ax \wedge Bx) \Rightarrow \exists x Ax\,.\,Bx}\ \mathrm{E}_\exists}{\Rightarrow \exists x Ax\,.\,Bx, \neg\exists x(Ax \wedge Bx)}\ \mathrm{I}_\neg}{\neg\neg\exists x(Ax \wedge Bx) \Rightarrow \exists x Ax\,.\,Bx}\ \mathrm{E}_\neg$$

Proposition 2
If $\pi \vdash_{\mathcal{S}} \sigma$, then $\pi \vdash_{\mathcal{D}} \sigma$
Proof: The proposition follows from the following claim:

$$\text{If } \pi \vdash_{\mathcal{S}} \sigma_1, \ldots, \sigma_n, \text{ then } \pi\,.\,\neg \bigwedge \sigma_1\,.\,\cdots\,.\,\neg \bigwedge \sigma_n \vdash_{\mathcal{D}} \bot$$

The claim is proved with induction on the number of rules employed in in the closed semantic tableau for $\pi \Rightarrow \sigma_1, \ldots, \sigma_n$.

5 Rigid Semantic Tableaus

A rigid sequent (r-sequent) is a pair $\langle \pi, \Sigma \rangle$, where π is a text and Σ a sequence of texts. We will write $\pi \Rightarrow \Sigma$ rather than $\langle \pi, \Sigma \rangle$.

Rules for r-sequents are given in the table below.

$\pi \,.\, Rc_1 \ldots c_n \,.\, \pi' \Rightarrow \Sigma; Rc_1 \ldots c_n; \Sigma'$ (axiom)	
$\dfrac{\pi \,.\, \varphi \,.\, \psi \,.\, \pi' \Rightarrow \Sigma}{\pi \,.\, \varphi \wedge \psi \,.\, \pi' \Rightarrow \Sigma}\ \mathrm{E}_\wedge$	Provided π is simple
$\dfrac{\pi \Rightarrow \Sigma; \varphi \,.\, \psi \,.\, \sigma; \Sigma'}{\pi \Rightarrow \Sigma; \varphi \wedge \psi \,.\, \sigma; \Sigma'}\ \mathrm{I}_\wedge$	Provided $\pi \Rightarrow \Sigma$ is simple
$\dfrac{\pi \,.\, [^c/x](\varphi \,.\, \pi') \Rightarrow [^c/x, \varphi \,.\, \pi'](\Sigma)}{\pi \,.\, \exists x\varphi \,.\, \pi' \Rightarrow \Sigma}\ \mathrm{E}_\exists$	Provided π is simple, and c does not occur in π, φ, π', or Σ
$\dfrac{\pi \Rightarrow \Sigma; [^c/x](\varphi \,.\, \sigma); \Sigma'; \exists x\varphi \,.\, \sigma}{\pi \Rightarrow \Sigma; \exists x\varphi \,.\, \sigma; \Sigma'}\ \mathrm{I}_\exists$	Provided $\pi \Rightarrow \Sigma$ is simple
$\dfrac{\pi \,.\, \pi' \Rightarrow \Sigma; \varphi}{\pi \,.\, \neg\varphi \,.\, \pi' \Rightarrow \Sigma}\ \mathrm{E}_\neg$	Provided π is simple
$\dfrac{\pi \,.\, \varphi \Rightarrow \Sigma; \Sigma'}{\pi \Rightarrow \Sigma; \neg\varphi; \Sigma'}\ \mathrm{I}_\neg$	Provided $\pi \Rightarrow \Sigma$ is simple
$\dfrac{\pi \Rightarrow \Sigma; \varphi; \Sigma' \quad \pi \,.\, \varphi \Rightarrow \Sigma; \sigma; \Sigma'}{\pi \Rightarrow \Sigma; \varphi \,.\, \sigma; \Sigma'}\ \mathrm{seq}$	Provided $\pi \Rightarrow \Sigma$ is simple, and φ is a negation or atomic

Note that these rules are special cases of the semantic tableau rules.

Definition 15 (Rigid Tableau)
Let $c_1, c_2, \ldots$ be an enumeration of a countable set of individual constants. Given this enumeration, a *rigid tableau* for $\pi \Rightarrow \Sigma$ is a tableau constructed according to the rules given above provided that:

(i) Whenever the rigid rule $\mathrm{E}_\exists$ is applied, i.e. whenever a node

$$\pi \,.\, \exists x\varphi \,.\, \pi' \Rightarrow \Sigma$$

is succeeded by a node

$$\pi \,.\, [^c/x](\varphi \,.\, \pi') \Rightarrow [^c/x, \varphi \,.\, \pi'](\Sigma)$$

then the individual constant c is the first of $c_1, c_2, \ldots$ that does not occur in π, φ, π', or Σ.

(ii) Whenever the rigid rule $\mathrm{I}_\exists$ is applied, i.e., whenever a node

$$\pi \Rightarrow \Sigma; \exists x\varphi \,.\, \sigma; \Sigma'$$

is succeeded by a node

$$\pi \Rightarrow \Sigma; [^c/x](\varphi \,.\, \sigma); \Sigma'; \exists x\varphi \,.\, \sigma$$

then the individual c is the first of $c_1, c_2, \ldots$ such that $[^c/x](\varphi\sigma)$ has not already been introduced by an application of $\mathrm{I}_\exists$ as a text in the right-hand side of any of the predecessors of the sequent $\pi \Rightarrow \Sigma; \exists x\varphi \,.\, \sigma; \Sigma'$.

Proposition 3
If $\pi \vdash_{\mathcal{R}} \Sigma$, then $\pi \vdash_{\mathcal{S}} \{\sigma \mid \sigma \text{ occurs in the sequence } \Sigma\}$

Lemma 5

(i) Suppose there is some c not occurring in $\varphi \,.\, \psi' \,.\, \sigma$ such that $i \in s[\pi \,.\, [^c/x](\varphi \,.\, \pi')]$ and $\{i\}[[^c/x, \varphi \,.\, \pi'](\sigma)] = \emptyset$. Then $i \in s[\pi \,.\, \exists x\varphi \,.\, \pi']$ and $\{i\}[\sigma] = \emptyset$.

(ii) Suppose there is some $i \in s[\pi \,.\, \pi']$, such that $\{i\}[\varphi] = \emptyset$. Then $i \in s[\pi \,.\, \neg\varphi \,.\, \pi']$.

Proposition 4
Fix an enumeration $c_1, c_2, \ldots$ of a countable set of individual constants. Let $\pi \Rightarrow \Sigma$ be any sequent. Given the enumeration, there exists exactly one rigid tableau for $\pi \Rightarrow \Sigma$.

Proposition 5
Suppose the rigid tableau $\mathcal{T}$ for $\pi \Rightarrow \Sigma$ contains an open branch $\mathcal{B}$.

Consider the model $\mathcal{M} = \langle \mathcal{D}, \mathcal{I} \rangle$ given by

(a) $\mathcal{D} = \{c \mid c \text{ occurs in any of the sequents constituting } \mathcal{B}\}$

(b) $\mathcal{I}(c_i) = c_i$

(c) If R is an n-place predicate, then $\langle c_{i_1}, \ldots, c_{i_n} \rangle \in \mathcal{I}(R)$ iff there is some node $\pi \Rightarrow \Sigma$ on $\mathcal{B}$ such that $\pi = \pi' \,.\, Rc_{i_1}, \ldots, c_{i_n} \,.\, \pi''$.

Now consider any sequent $\pi \Rightarrow \sigma$ such that $\pi \Rightarrow \sigma_1; \ldots; \sigma; \ldots; \sigma_n$ occurs on the branch $\mathcal{B}$.

Claim: There is some $i \in \mathbf{1}[\pi]$ such that $\{i\}[\sigma] = \mathbf{0}$.

Proof of the claim:
The proof is by induction on the complexity of $\pi \Rightarrow \sigma$, where the complexity of $\pi \Rightarrow \sigma$ is given by the number $m = 2j + l$ where j and l are determined as follows.

- j is the number of logical constants ($\neg$, $\wedge$, $\exists$) occurring in the sequence $\pi \,.\, \sigma$.
- l is the number of formulas in σ.

(i) $m = 1$. Notice that in this case $\pi \Rightarrow \sigma$ is simple. Given the defintion of $\mathcal{M}$, it holds that $\mathbf{1}[\pi] = \mathbf{1}$, and $\mathbf{1}[\pi \,.\, \sigma] = \mathbf{0}$.

(ii) Now assume that the claim holds for any sequent the complexity of which is not larger than k. Consider a sequent $\pi \Rightarrow \sigma$ with complexity $k + 1$. There are three major subcases:

Case (a): π is not simple. Then $\pi \Rightarrow \sigma_1; \ldots; \sigma; \ldots; \sigma_n$is derived from its immediate predecessor by an application of either of the following rules:

– $\mathbf{E}_\wedge$**:** This case is easy. $\pi \Rightarrow \sigma_1; \ldots; \sigma; \ldots; \sigma_n$ is of the form

$$\pi' \,.\, (\varphi \wedge \psi) \,.\, \pi'' \Rightarrow \sigma_1; \ldots; \sigma; \ldots; \sigma_n$$

while its immediate predecessor is

$$\pi' . \varphi . \psi . \pi'' \Rightarrow \sigma_1; \ldots; \sigma; \ldots; \sigma_n$$

Given the semantic rule for $\wedge$-formulas it holds that if $i \in s[\varphi . \psi]$, then $i \in s[\varphi \wedge \psi]$. Given the induction hypothesis we may assume that there is some $i \in \mathbf{1}[\pi . \varphi . \psi . \pi']$ such that $\{i\}[\sigma] = \mathbf{0}$. Hence, there is some $i \in \mathbf{1}[\pi . (\varphi \wedge \psi) . \pi']$ such that $\{i\}[\sigma] = \mathbf{0}$.

– **E$_\exists$:** In this case $\pi \Rightarrow \sigma_1; \ldots; \sigma; \ldots; \sigma_n$ is of the form

$$\pi' . \exists x\varphi . \pi'' \Rightarrow \sigma_1; \ldots; \sigma; \ldots; \sigma_n$$

while its immediate predecessor is

$$\pi' . [^c/x](\varphi . \pi'') \Rightarrow [^c/x, \varphi . \pi'](\sigma_1); \ldots; [^c/x; \varphi . \pi'](\sigma); \ldots; [^c/x, \varphi . \pi'](\sigma_n)$$

– **E$_\neg$:** In this case we have that $\pi \Rightarrow \sigma_1; \ldots; \sigma; \ldots; \sigma_n$ is of the form

$$\pi' . \neg\varphi . \pi'' \Rightarrow \sigma_1; \ldots; \sigma; \ldots; \sigma_n$$

whereas its immediate predecesoor is $\pi \Rightarrow \sigma_1; \ldots; \sigma; \ldots; \sigma_n$ is of the form

$$\pi' . \pi'' \Rightarrow \sigma_1; \ldots; \sigma; \ldots; \sigma_n; \varphi$$

Case (b): π is simple, and σ is not active. In this case there must be a predecessor of the form $\pi' \Rightarrow \sigma$ on the branch $\mathcal{B}$ where π' is simple and σ is active.

Case (c): π is simple, and σ is active. Then $\pi \Rightarrow \sigma_1; \ldots; \sigma; \ldots; \sigma_n$is derived from its immediate predecessor by an application of either of the following rules:

– **I$_\wedge$:** In this case $\pi \Rightarrow \sigma_1; \ldots; \sigma; \ldots; \sigma_n$ is of the form

$$\pi \Rightarrow \sigma_1; \ldots; (\varphi \wedge \psi) . \sigma'; \ldots; \sigma_n$$

while its immediate predecessor is

$$\pi \Rightarrow \sigma_1; \ldots; \varphi . \psi . \sigma'; \ldots; \sigma_n$$

– **I$_\exists$:** In this case $\pi \Rightarrow \sigma_1; \ldots; \sigma; \ldots; \sigma_n$ is of the form

$$\pi \Rightarrow \sigma_1; \ldots; \exists x\varphi . \sigma; \ldots; \sigma_n$$

while its immediate predecessor is

$$\pi \Rightarrow \sigma_1; \ldots; [^c/x](\varphi . \sigma'); \ldots; \sigma_n; \exists x\varphi . \sigma'$$

– **I$_\neg$:** In this case $\pi \Rightarrow \sigma_1; \ldots; \sigma; \ldots; \sigma_n$ is of the form

$$\pi \Rightarrow \sigma_1; \ldots; \neg\varphi; \ldots; \sigma_n$$

while its immediate predecessor is

$$\pi . \varphi \Rightarrow \sigma_1; \ldots\ldots; \sigma_n$$

– seq: This case has two subsubcases:

(a) $\pi \Rightarrow \sigma_1; \ldots; \sigma; \ldots; \sigma_n$ is either of the form

$$\pi \Rightarrow \sigma_1; \ldots; \neg\varphi \,.\, \sigma'; \ldots; \sigma_n$$

while its two immediate predecessors are

$$\pi \Rightarrow \sigma_1; \ldots; \neg\varphi; \ldots; \sigma_n$$

and

$$\pi \,.\, \neg\varphi \Rightarrow \sigma_1; \ldots; \sigma'; \ldots; \sigma_n$$

(b) $\pi \Rightarrow \sigma_1; \ldots; \sigma; \ldots; \sigma_n$ is of the form

$$\pi \Rightarrow \sigma_1; \ldots; Ra_1 \ldots a_n \,.\, \sigma'; \ldots; \sigma_n$$

while its two immediate predecessors are

$$\pi \Rightarrow \sigma_1; \ldots; Ra_1 \ldots a_n; \ldots; \sigma_n$$

and

$$\pi \,.\, Ra_1 \ldots a_n \Rightarrow \sigma_1; \ldots; \sigma'; \ldots; \sigma_n$$

Theorem 1 (Completeness)
If $\pi \models \Sigma$, then $\pi \vdash_{\mathcal{R}} \Sigma$

6 Natural Deduction

Examples.

(a) Note that $Rxy\,.\,\exists xAx\,.\,\exists yBy \models \exists x\exists yRxy$.
But how to derive

1	Rxy	premise
2	$\exists xAx$	premise
3	$\exists yBy$	premise
4	$\exists yRxy$	???

(b) Note that $\exists xAx\,.\,Bx\,.\,\exists xAx \not\models Bx$
However, $\exists xAx\,.\,Bx \models \exists xAx\,.\,Bx$
and $\exists xAx\,.\,Bx \models Bx$

(c) How to derive

1	Ac	premise
2	Bc	premise
3	$\exists xAx$	
4	Bx	

Let $\varphi_1\,.\,\ldots\,.\,\varphi_k$ be a text. A *natural deduction* from $\varphi_1\,.\,\ldots\,.\,\varphi_k$ is a sequence $S_1, S_2, \ldots$ of *steps* $S_n = \langle F_n, A_n, Q_n\rangle$ such that

(i) F_n is a formula;

(ii) $A_n \subseteq \{k \mid k \leq n\}$; if $k \in A_n$ then the formula F_k is one of the *assumptions at* n;

(iii) Q_n is a partial function assigning a natural number, called the *(quantifier) index* $Q_n(\exists x, i)$, to the i-th occurrence of the quantifier $\exists x$ in the formula F_n.

In a making a natural deduction we have to keep track at each step S_{n+1} of the index of the quantifier by which a free occurrence of the variable x in F_{n+1} is going to be bound. $C_n(x)$ is supposed to tell this. More precisely:

C_n is a partial function assigning a natural number to some of the variables as follows.

(a) If no occurrence of $\exists x$ in F_n is active after F_n, then $C_n(x) = C_m(x)$, where m is the largest number smaller than n such that $C_m(x)$ is defined and $A_m \subseteq A_n$;

(b) If the i-th occurrence of $\exists x$ in F_n is active after F_n,
then $C_n(x) = Q_n(\exists x, i)$.

Usually, the largest number smaller than n such that $C_m(x)$ is defined and $A_m \subseteq A_n$ happens to be $n-1$. The only exception is the case where in S_n an assumption is withdrawn.

Another useful abbreviation is given by $M_n(\exists x)$, where $M_n(\exists x)$ denotes the maximal number in $\{l \mid l = Q_m(\exists x, i)$ for some i and some $m < n\}$

To qualify as a natural deduction the following conditions should be fulfilled:
Adding a premise For $1 \leq n \leq k$ the following must hold:

- $F_n = \varphi_n$.
- $A_n = \emptyset$.

- For every x and i such that $\exists x$ occurs at least i times in F_n, $Q_n(\exists x, i) = M_{n-1}(\exists x) + i$.

Each F_n is called a premise.

For $n > k$ either of the following must hold:

Making an assumption

- $A_n = A_{n-1} \cup \{n\}$.
- For every x and i such that $\exists x$ occurs at least i times in F_n, $Q_n(\exists x, i) = M_{n-1}(\exists x) + i$.

In this case we say that F_n is an assumption *made* at step n.

Repetition There is some $m < n$ such that

- $F_m = F_n$
- $A_m \subseteq A_{n-1} = A_n$.
- For every x such that x occurs free in F_n, $C_{n-1}(x) = C_{m-1}(x)$;
- For every x and i such that $\exists x$ occurs at least i times in F_n, $Q_n(\exists x, i) = Q_m(\exists x, i)$.

In this case we say that $F_n . \cdots . F_{n+l}$ is obtained by repetition.

Examples

$\exists x Ax . Bx \models Bx$;
$\exists x Ax . Bx \models \exists x Ax \wedge Bx$;
$\exists x Ax . Bx . \exists x Ax \not\models Bx$;
$\exists x Ax . Bx . \exists x Ax \models \exists x Ax \wedge Bx$.

1	$\exists^1 x Ax$	premise
2	Bx	premise
3	Bx	repetition, 2

1	$\exists^1 x Ax$	premise
2	Bx	premise
3	$\exists^1 x Ax \wedge Bx$	$\text{I}_\wedge$, 1,2

1	$\exists^1 x Ax$	premise
2	Bx	premise
3	$\exists^2 x Ax$	premise
4	Bx	???

1	$\exists^1 x Ax$	premise
2	Bx	premise
3	$\exists^2 x Ax$	premise
4	$\exists^1 x Ax \wedge Bx$	$\text{I}_\wedge$, 1,2

Elimination of $\exists x$ There is some $m < n$ such that

- $F_m = \exists x F_n$.
- $A_m \subseteq A_{n-1} = A_n$.
- $C_{n-1}(x) = Q_m(\exists x, 1)$;
  for every y such that $y \neq x$, and y occurs free in F_n, $C_{n-1}(y) = C_{m-1}(y)$.
- If $\exists x$ occurs at least i times in F_n, $Q_n(\exists x, i) = Q_m(\exists x, i+1)$.
  For every $y \neq x$ and i such that $\exists y$ occurs at least i times in F_n, $Q_n(\exists y, i) = Q_m(\exists y, i)$.

In this case we say that F_n is obtained from F_m by $\mathrm{E}_\exists$.

Introduction of $\exists x$ There exist t, $\chi_0, \chi_1 \ldots \chi_l$, and $m < n - l$ such that

- $F_m \,.\, F_{m+1} \,.\, \cdots \,.\, F_{m+l} = [^t/x](\chi_0 \,.\, \chi_1 \,.\, \cdots \,.\, \chi_l)$
  $F_n \,.\, F_{n+1} \,.\, \cdots \,.\, F_{n+l} = \exists x \chi_0 \,.\, \chi_1 \,.\, \cdots \,.\, \chi_l$
- $A_m \subseteq A_{m+1} \subseteq \ldots \subseteq A_{m+l} \subseteq A_{n-1} = A_n = \ldots = A_{n+l}$.
- For every y such that y occurs free in $\exists x \chi_0 \,.\, \cdots \,.\, \chi_l$, $C_{n-1}(y) = C_{m-1}(y)$.
- $Q_n(\exists x, 1) = M_{n-1}(\exists x) + 1$;
  for $i > 1$, $Q_n(\exists x, i) = Q_m(\exists x, i - 1)$;
  for every y such that $y \neq x$, $Q_n(\exists y, i) = Q_m(\exists y, i)$.
  For every $1 \leq j \leq l$, and every y, $Q_{n+j}(\exists y, i) = Q_{m+j}(\exists y, i)$.

In this case we say that $F_n \,.\, \cdots \,.\, F_{n+l}$ is obtained from $F_m \,.\, \cdots \,.\, F_{m+l}$ by $\mathrm{I}_\exists$

Example

$\exists x Rxx \models \exists x \exists y Rxy \wedge Ryx$

1	$\exists^1 x Rxx$	premise
2	Rxx	$\mathrm{E}_\exists$, 1
3	Rxx	repetition, 2
4	$\exists^1 y Rxy$	
5	Ryx	$\mathrm{I}_\exists$, 2–3
6	$\exists^2 x \exists^1 y Rxy$	
7	Ryx	$\mathrm{I}_\exists$, 4–5
8	$\exists^2 x \exists^1 y Rxy \wedge Ryx$	$\mathrm{I}_\wedge$, 6,7

Variable switch There exist y, x, χ, and m, l with $m < n - l$ such that

- $F_m = \exists x \chi$
  $F_n = \exists y [^y/x](\chi)$
  $F_{n+1} \,.\, \cdots \,.\, F_{n+l} = [^y/x, \chi](F_{m+1} \,.\, \cdots \,.\, F_{m+l})$
- $A_m \subseteq A_{m+1} \subseteq \ldots \subseteq A_{m+l} \subseteq A_{n-1} = A_n = \ldots = A_{n+l}$.
- For every z such that z occurs free in $F_n \,.\, \cdots \,.\, F_{n+l}$, $C_{n-1}(z) = C_{m-1}(z)$.
- $Q_n(\exists y, 1) = M_{n-1}(\exists y) + 1$; for $i > 1$, $Q_n(\exists y, i) = Q_m(\exists y, i - 1)$;
  $Q_n(\exists x, i) = Q_m(\exists x, i + 1)$;
  for every z such that $z \neq x$ and $z \neq y$, $Q_n(\exists z, i) = Q_m(\exists y, i)$.
  For every $1 \leq j \leq l$, and every z, $Q_{n+j}(\exists z, i) = Q_{m+j}(\exists z, i)$.

In this case we say that $F_n . \cdots . F_{n+l}$ is obtained from $F_m . \cdots . F_{m+l}$ by *variable switch*.

Example

$\exists x(Ax \wedge \exists x \neg Ax) \models \neg Ax$

1	$\exists^1 x(Ax \wedge \exists^2 x \neg Ax)$	premise
2	$\exists^1 y(Ay \wedge \exists^2 x \neg Ax)$	variable switch, 1
3	$Ay \wedge \exists^2 x \neg Ax$	$\mathrm{E}_\exists$, 2
4	$\exists^2 x \neg Ax$	$\mathrm{E}_\wedge$, 3
5	$\neg Ax$	$\mathrm{E}_\exists$, 4

Elimination of $\wedge$ There exist and $m < n$ such that

- $F_m = F_n \wedge F_{n+1}$.
- $A_m \subseteq A_{n-1} = A_n = A_{n+1}$.
- For every x such that x occurs free in F_m, $C_{n-1}(x) = C_{m-1}(x)$;
- For every x and i such that $\exists x$ occurs at least i times in F_n, $Q_n(\exists x, i) = Q_m(\exists x, i)$.
  For every x and i such that $\exists x$ occurs at least i times in F_{n+1}, $Q_{n+1}(\exists x, i) = Q_m(\exists x, i + j)$, where j is the number of times $\exists x$ occurs in F_n.

In this case we say that F_n is obtained from F_m by $\mathrm{E}_\wedge$.

Introduction of $\wedge$ There is some m with $m < n - 1$ such that

- $F_n = F_m \wedge F_{m+1}$.
- $A_m \subseteq A_{m+1} \subseteq A_{n-1} = A_n$.
- For every x such that x occurs free in F_n, $C_{n-1}(x) = C_{m-1}(x)$.
- Q_n is determined as follows:
  Suppose $\exists x$ occurs j times in F_m. Then for every $i \leq j$, $Q_n(\exists x, i) = Q_m(\exists x, i)$, and for every $i > j$, if $\exists x$ occurs at least i times in F_n, $Q_n(\exists x, i) = Q_{m+1}(\exists x, i - j)$.

In this case we say that F_n is obtained from F_m by $\mathrm{I}_\wedge$.

Elimination of $\perp$ There is some $m \leq n$

- $F_m = \perp$.
- $A_m \subseteq A_{n-1} = A_n$.
- For every x and i such that $\exists x$ occurs at least i times in F_n, $Q_n(\exists x, i) \leq M_{n-1}(\exists x) + i$.

In this case we say that F_n is obtained from F_m by $\mathrm{E}_\perp$.

Elimination of $\neg$ There exist χ and l, m with $l \leq m \leq n$ such that

- $F_l = \neg\chi$, $F_m = \chi$, and $F_n = \perp$.
- $A_l \subseteq A_m \subseteq A_{n-1} = A_n$.
- For every x such that x occurs free in χ, $C_{l-1}(x) = C_{m-1}(x)$

In this case we say that F_n is obtained from F_m by $\mathrm{I}_\perp$.

Introduction of $\neg$ There is some $m < n$ such that

- F_m is an assumption made at step m, $F_{n-1} = \perp$, and $F_n = \neg F_m$.
- $A_m = A_{n-1}$, and $A_n = A_{n-1} \backslash \{m\}$
- For every x and i such that $\exists x$ occurs at least i times in F_n, $Q_n(\exists x, i) = Q_m(\exists x, i)$;

In this case we say that F_n is obtained from F_m by $\mathrm{I}_\neg$.

The assumption χ made at step m is *withdrawn* at step n.

Elimination of $\neg\neg$ There is some $m < n$ such that

- $F_m = \neg\neg F_n$.
- $A_m \subseteq A_{n-1} = A_n$.
- For every x such that x occurs free in F_n, $C_{n-1}(x) = C_{m-1}(x)$;
- For every x and i such that $\exists x$ occurs at least i times in F_n, $Q_n(\exists x, i) = Q_m(\exists x, i)$.

In this case we say that F_n is obtained from F_m by $\mathrm{E}_{\neg\neg}$.

Example

$\neg(\exists x Ax \wedge \neg Bx) \,.\, \exists x Ax \models Bx$

1	$\neg(\exists^1 x Ax \wedge \neg Bx)$	premise
2	$\exists^2 x Ax$	premise
3	$\neg Bx$	assumption
4	$\exists^2 x Ax \wedge \neg Bx$	$\mathrm{I}_\wedge$
5	$\bot$	$\mathrm{E}_\neg$, 1, 4
6	$\neg\neg Bx$	$\mathrm{I}_\neg$
7	Bx	$\neg\neg$, 6

Definition 16

Consider a natural deduction. We define T_n, also called the *text at* the n-th step, to be the sequence of formulas $\chi_1 \,.\, \chi_2 \,.\, \cdots$ determined as follows:

(i) $\chi_1 = F_i$ for the smallest $i \leq n$ such that $A_i \subseteq A_n$;

(ii) If $\chi_k = F_j$, then $\chi_{k+1} = F_i$ for the smallest i such that $j \leq i \leq n$ and $A_i \subseteq A_n$.

Definition 17

$\varphi_1 \,.\, \cdots \,.\, \varphi_k \vdash_{\mathcal{N}} \psi_1 \,.\, \cdots \,.\, \psi_l$ iff there exists a natural deduction from $\varphi_1 \,.\, \cdots \,.\, \varphi_k$ such that for some n the following holds:

(i) $F_n \,.\, \cdots \,.\, F_{n+l} = \psi_1 \,.\, \cdots \,.\, \psi_l$;

(ii) for each i such that $n \leq i \leq n+l$, $A_i = \emptyset$;

(iii) for each x occurring free in $F_n \,.\, \cdots \,.\, F_{n+l}$, $C_n(x) = C_k(x)$.

Definition 18

Consider a natural deduction from $\varphi_1 \,.\, \cdots \varphi_k$.

Consider $T_n = \varphi_1 \,.\, \cdots \varphi_k \,.\, \chi_1 \,.\, \cdots \,.\, \chi_m$ for some n such that $k \leq n$.

Let $\mathcal{M} = \langle \mathcal{D}, \mathcal{I} \rangle$ be some model and suppose there are assignments $v_0, \ldots, v_m$ pertaining to $\mathcal{M}$ such that

(i) $v_{i+1} \in \{v_i\}[\chi_{i+1}]$ for every $0 \leq i < m$;

(ii) for all x and all $0 \leq i, j \leq m$ such that $C_i(x) = C_j(x)$ it holds that $v_i(x) = v_j(x)$.

In this case we say that the sequence $v_0, \ldots, v_m$ *follows* the deduction up to n.

Lemma 6
Consider a natural deduction from $\varphi_1 \ldots \cdot \varphi_k$.
Consider $T_k = \varphi_1 \ldots \cdot \varphi_k$.
Let $\mathcal{M}$ be any model $\langle \mathcal{D}, \mathcal{I} \rangle$.
Any sequence of assignments $v_0, \ldots, v_k$ pertaining to $\mathcal{M}$ which has the property mentioned under (i) above follows the deduction up to k.

Lemma 7
Consider a natural deduction from $\varphi_1 \ldots \cdot \varphi_k$.
Consider $T_n = \varphi_1 \ldots \cdot \varphi_m$ for some n such that $k \leq n$, and assume F_{n+1} is not an assumption.
Let $\mathcal{M}$ be any model.
Then any sequence $v_0, \ldots, v_m$ pertaining to $\mathcal{M}$ which follows the deduction up to n can be extended to a sequence $v_0, \ldots, v_m, v_{m+1}$ which follows the deduction op to $n + 1$.

Theorem 2 (Soundness)
If $\pi \vdash_{\mathcal{N}} \sigma$, then $\pi \models \sigma$

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$\dfrac{\pi; [^c/x]\varphi; [^c/x]\pi' \Rightarrow [^c/x]\sigma}{\pi; \exists x; \varphi; \pi' \Rightarrow \sigma}\ \mathrm{E}_\exists$	Provided c does not occur in π, φ, π', or σ
$\dfrac{\pi \Rightarrow \sigma; [^t/x]\varphi; [^t/x]\sigma'}{\pi \Rightarrow \sigma; \exists x; \varphi; \sigma'}\ \mathrm{I}_\exists$	Provided t is free for x

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg\varphi; \varphi; \psi \Rightarrow}{\neg\varphi; \varphi \wedge \psi \Rightarrow}\ \mathrm{seq}}{\neg\varphi \Rightarrow \neg(\varphi \wedge \psi)}\ \mathrm{I}_\neg}{\neg\neg(\varphi \wedge \psi); \neg\varphi \Rightarrow}\ \mathrm{E}_\neg}{\neg\neg(\varphi \wedge \psi) \Rightarrow \neg\neg\varphi}\ \mathrm{I}_\neg \qquad \neg\neg(\varphi \wedge \psi); \neg\neg\varphi \Rightarrow \varphi}{\neg\neg(\varphi \wedge \psi) \Rightarrow \varphi}\ \text{c-cut}$$

However,

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg\psi; \varphi; \psi \Rightarrow (??)}{\neg\psi; \varphi \wedge \psi \Rightarrow}\ \mathrm{seq}}{\neg\psi \Rightarrow \neg(\varphi \wedge \psi)}\ \mathrm{I}_\neg}{\neg\neg(\varphi \wedge \psi); \neg\psi \Rightarrow}\ \mathrm{E}_\neg}{\neg\neg(\varphi \wedge \psi) \Rightarrow \neg\neg\psi}\ \mathrm{I}_\neg \qquad \neg\neg(\varphi \wedge \psi); \neg\neg\psi \Rightarrow \psi}{\neg\neg(\varphi \wedge \psi) \Rightarrow \psi}\ \text{c-cut}$$

Meaning To know the meaning of a sentence is to know the change it brings about in the information state of anyone who accepts the news conveyed by it. In other words, the meaning of a sentence is a function from information states into information states.

Notation Let s be an information state and φ a sentence with meaning $[\varphi]$. We write $s[\varphi]$ for the information state that results when s is updated with φ.

Acceptance Sometimes the information conveyed by φ will already be subsumed by s. In such a case, we say that φ is *accepted* in s, or that s *supports* φ, and we write this as:

$$s \models \varphi$$

Logical validity Updating any state s with the premises $[\varphi_1], \ldots, [\varphi_n]$ in that order, yields a state in which the conclusion ψ is to be accepted.

$$\varphi_1, \ldots, \varphi_n \models \psi \text{ iff for any } s, s[\varphi_1], \ldots, [\varphi_n] \models \psi$$

More examples

- $\exists xPx.$ `might` $\neg Px$ is inconsistent.
- $\exists xPx.$ `might` $\forall y\neg Py$ is inconsistent.
- $\exists xPx.\forall y$ `might` $\neg Py$ is consistent.
- $\exists xPx.\forall y$ `might` $(y \neq x)$ is consistent.
- $\exists x(Px \wedge \forall y$ `might` $(y \neq x))$ is inconsistent.

Definition 19 (Support) A state s supports a text π iff $\{i\}[\pi] \neq \emptyset$ for every $i \in s$.

Proposition 6 (Soundness Deductive Tableaus) If $\pi \vdash_{\mathcal{D}} \sigma$, then $\pi \models \sigma$
Proof: Check the following:

(i) If $Rt_1 \dots t_1$ does not depend on π', then $\pi \,.\, Rt_1 \dots t_n \,.\, \pi' \models Rt_1 \dots t_n$.
(ii) If $\pi \,.\, \varphi \,.\, \psi \,.\, \pi' \models \sigma$, then $\pi \,.\, \varphi \wedge \psi \,.\, \pi' \models \sigma$.
(iii) If $\pi \models \sigma \,.\, \varphi \,.\, \psi \,.\, \sigma'$, then $\pi \models \sigma \,.\, \varphi \wedge \psi \,.\, \sigma'$.
(iv) If $\pi \,.\, [^c/x](\varphi \,.\, \pi') \models [^c/x, \varphi \,.\, \pi'](\sigma)$, and c does not occur in π, φ, π', or σ, then $\pi \,.\, \exists x \varphi \,.\, \pi' \models \sigma$.
(v) If $\pi \models \sigma \,.\, [^t/x](\varphi \,.\, \sigma')$, and t is free for x in $\varphi \,.\, \sigma'$, then $\pi \models \sigma \,.\, \exists x \varphi \,.\, \sigma'$.
(vi) If $\pi \,.\, \pi' \models \varphi$, and φ does not depend on π', then $\pi \,.\, \neg\varphi \,.\, \pi' \models \sigma$.
(vii) If $\pi \,.\, \varphi \models \emptyset$, then $\pi \models \neg\varphi$.
(viii) If $\pi \,.\, \neg\varphi \models \emptyset$, then $\pi \models \varphi$.
(ix) If $\pi \models \sigma$ and $\pi \,.\, \sigma \models \sigma'$, then $\pi \models \sigma \,.\, \sigma'$.

Elimination rule for $\wedge$

$$\begin{array}{lll} \vdots & \pi' & \\ m & \varphi \wedge \psi & \\ \vdots & \pi & \\ n & \varphi & E_\wedge, m \\ n+1 & \psi & E_\wedge, m \end{array}$$

Conditions:

1. φ does not depend on $\varphi \wedge \psi \,.\, \tau(\pi)$;
2. ψ does not depend on $\psi \,.\, \tau(\pi)$.

Introduction rule for $\wedge$

$$\begin{array}{lll} \vdots & \pi' & \\ m & \varphi & \\ m+1 & \psi & \\ \vdots & \pi & \\ n & \varphi \wedge \psi & I_\wedge, m, m+1 \end{array}$$

Conditions:

1. φ does not depend on $\psi \,.\, \tau(\pi)$;
2. ψ does not depend on $\tau(\pi)$.

Elimination rule for $\neg$

$$\begin{array}{lll} \vdots & \pi'' & \\ k & \varphi & \\ \vdots & \pi' & \\ m & \neg\varphi & \\ \vdots & \pi & \\ n & \bot & E_\neg, k, m \end{array}$$

Conditions:

1. $\neg\varphi$ does not depend on $\varphi \centerdot \tau(\pi')$

Introduction rule for $\neg$

$$\begin{array}{lll} \vdots & \pi & \\ m & \varphi & \text{assumption} \\ \vdots & \pi' & \\ n-1 & \bot & \\ \hline n & \neg\varphi & I_\neg \end{array}$$

Conditions:

1. —

Introduction rule for $\exists$

$$\begin{array}{lll} \vdots & \pi' & \\ m & [^t/x]\varphi & \\ \vdots & \pi & \\ n & \exists x\varphi & I_\exists, m \end{array}$$

Conditions:

1. t is free for x in φ;
2. $\exists x\varphi$ does not depend on $[t/x]\varphi \centerdot \tau(\psi)$.

Elimination rule for $\exists$

$$\begin{array}{lll} \vdots & \pi' & \\ m & \exists x\varphi & \\ \vdots & \pi & \\ n & \varphi & E_\exists, m \end{array}$$

Conditions:

1. φ does not depend on $\varphi \centerdot \tau(\pi)$.

$\neg\neg$-rule

$$\begin{array}{lll} \vdots & \pi' & \\ m & \neg\neg\varphi & \\ \vdots & \pi & \\ n & \varphi & \neg\neg, m \end{array}$$

Conditions:

1. φ does not depend on $\tau(\pi)$.

Recapitulation rule

$$\begin{array}{lll} \vdots & \pi' & \\ m & \exists x\varphi & \\ \vdots & \pi & \\ n & \exists y[y/x]\varphi & R_\exists, m \end{array}$$

Conditions:

1. y is free for x in φ;
2. $\exists y[y/x]\varphi$ does not depend on $\varphi \,.\, \tau(\pi)$.

Proof:

(a) and (b) are left to the reader.

(c)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\pi\,.\,\neg\varphi \Rightarrow \chi}{\pi\,.\,\neg\varphi\,.\,\neg\chi \Rightarrow \bot}\,\mathrm{E}_{\neg}}{\pi\,.\,\neg\chi \Rightarrow \neg\neg\varphi}\,\mathrm{I}_{\neg} \quad \pi\,.\,\neg\chi\,.\,\neg\neg\varphi \Rightarrow \varphi}{\pi\,.\,\neg\chi \Rightarrow \varphi} \qquad \dfrac{\dfrac{\dfrac{\pi\,.\,\varphi\,.\,\neg\psi \Rightarrow \chi}{\pi\,.\,\varphi\,.\,\neg\psi\,.\,\neg\chi \Rightarrow \bot}\,\mathrm{E}_{\neg}}{\pi\,.\,\varphi\,.\,\neg\chi \Rightarrow \neg\neg\psi}\,\mathrm{I}_{\neg} \quad \pi\,.\,\varphi\,.\,\neg\chi\,.\,\neg\neg\psi \Rightarrow \psi}{\pi\,.\,\varphi\,.\,\neg\chi \Rightarrow \psi}}{\pi\,.\,\neg\chi \Rightarrow \varphi\,.\,\psi}\,\mathrm{seq}}{\pi\,.\,\neg\chi \Rightarrow \varphi \wedge \psi}\,\mathrm{I}_{\wedge}}{\pi\,.\,\neg\chi\,.\,\neg(\varphi \wedge \psi) \Rightarrow \bot}\,\mathrm{E}_{\neg}}{\pi\,.\,\neg(\varphi \wedge \psi) \Rightarrow \neg\neg\chi}\,\mathrm{I}_{\neg} \qquad \pi\,.\,\neg(\varphi \wedge \psi)\,.\,\neg\neg\chi}{\pi\,.\,\neg(\varphi \wedge \psi) \Rightarrow \chi}$$

(d)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\pi\,.\,\neg[^{c}/x]\varphi \Rightarrow \psi}{\pi\,.\,\neg\psi\,.\,\neg[^{c}/x]\varphi \Rightarrow \bot}\,\mathrm{E}_{\neg}}{\pi\,.\,\neg\psi \Rightarrow \neg\neg[^{c}/x]\varphi}\,\mathrm{I}_{\neg} \quad \pi\,.\,\neg\psi\,.\,\neg\neg[^{c}/x]\varphi \Rightarrow [^{c}/x]\varphi}{\pi\,.\,\neg\psi \Rightarrow [^{c}/x]\varphi}}{\pi\,.\,\neg\psi \Rightarrow \exists x\varphi}\,\mathrm{I}_{\exists}}{\pi\,.\,\neg\psi\,.\,\neg\exists x\varphi \Rightarrow \bot}\,\mathrm{E}_{\neg}}{\pi\,.\,\neg\exists x\varphi \Rightarrow \neg\neg\psi}\,\mathrm{I}_{\neg} \qquad \pi\,.\,\neg\exists x\varphi\,.\,\neg\neg\psi \Rightarrow \psi}{\pi\,.\,\neg\exists x\varphi \Rightarrow \psi}$$

Example

1	$\neg\neg\exists x(Ax \wedge Bx)$	premise
2	$\neg(\exists x Ax \wedge Bx)$	assumption
3	$\exists x(Ax \wedge Bx)$	assumption
4	$Ax \wedge Bx$	$\mathrm{E}_{\exists}$, 3
5	Ax	$\mathrm{E}_{\wedge}$, 4
6	Bx	$\mathrm{E}_{\wedge}$, 4
7	$\exists x Ax$	$\mathrm{I}_{\exists}$, 5
8	Bx	$\mathrm{I}_{\exists}$, 6,7
9	$\exists x Ax \wedge Bx$	$\mathrm{I}_{\wedge}$, 7, 8
10	$\bot$	$\mathrm{E}_{\neg}$, 2,9
11	$\neg\exists x(Ax \wedge Bx)$	$\mathrm{I}_{\neg}$
12	$\bot$	$\mathrm{E}_{\neg}$, 1,11
13	$\neg\neg(\exists x Ax \wedge Bx)$	$\mathrm{I}_{\neg}$
14	$\exists x Ax \wedge Bx$	$\neg\neg$, 13
15	$\exists x Ax$	$\mathrm{E}_{\wedge}$, 14
16	Bx	$\mathrm{E}_{\wedge}$, 14,15