Logic and Knowledge Representation

Predicate Logic

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Predicate logic

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Socrates is a man.
Therefore… ?

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Additional problems...

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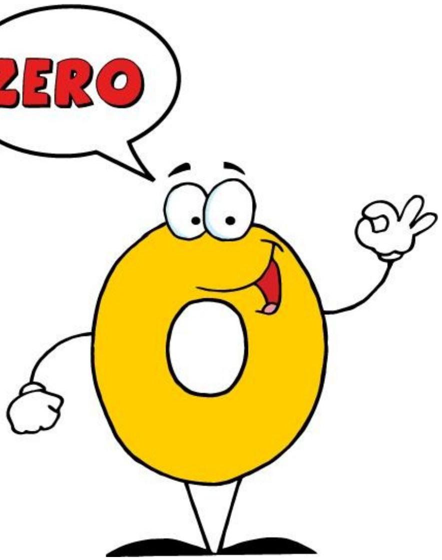
- **No man** is immortal ???



Predicate logic: Quantifiers

Gottlob Frege

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Nota bene: the *x* is a specific individual within *[...]*, and it cannot be outside of *[...]*

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- It is a *"open" statement*: there is no anchoring to actual elements. ex. *Red(car)*: "car is red" does not make complete sense: which car?

Predicate logic

- *A(x)* is a **predicate** with **argument** *x*
- when *t* has a defined **value** (i.e. it refers to a specific entity), *A(t)* is a **proposition** and may be true or false.

- Example: That car is red: *Red(thatcar)*

(assuming *thatcar* to be a shared constant)

`thatcar` →

shared symbol — *correspondence semantics*

Predicate logic

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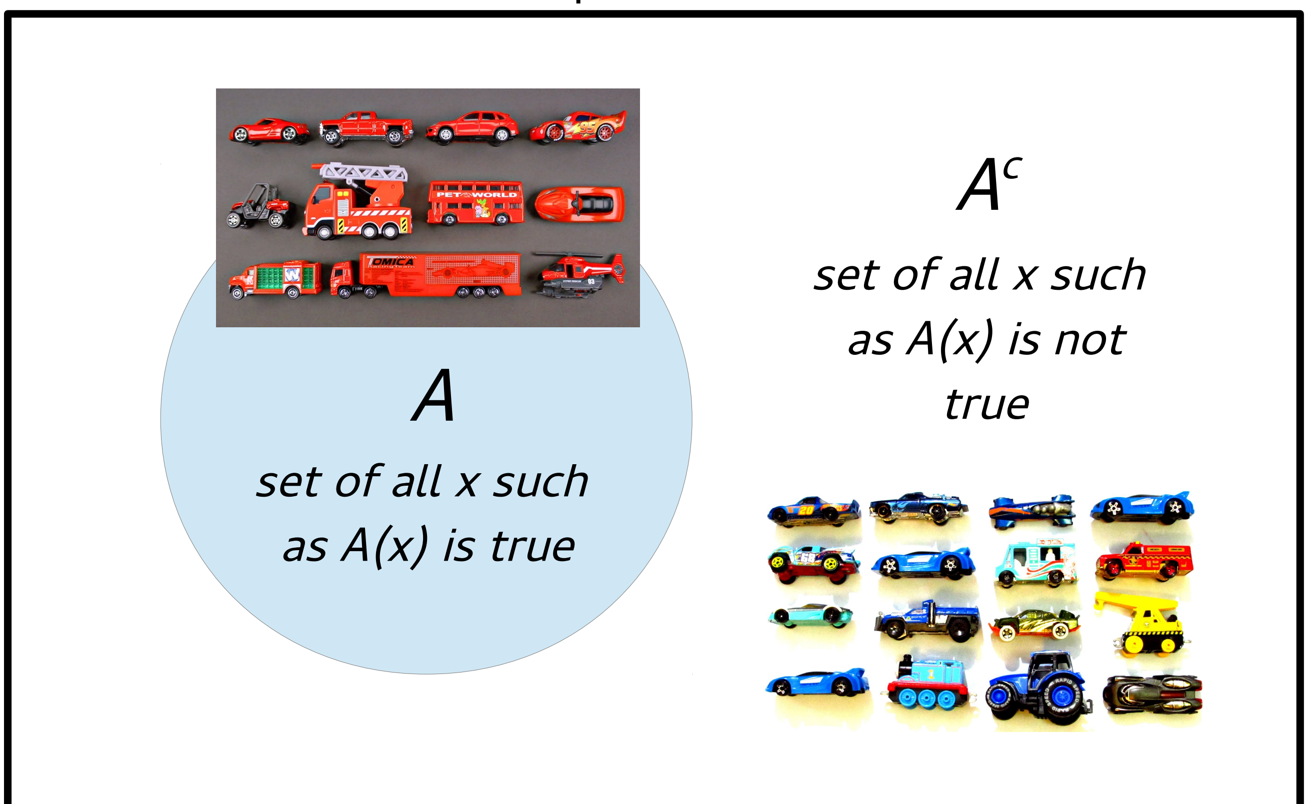
- when *t* has a defined **value** (i.e. it refers to a specific entity), *A(t)* is a **proposition** and may be true or false.

We can create a map with all the possible values of *x*, where *A(x)* is true, and where is false, i.e. where $\neg A(x)$ is true

Predicate logic and Venn diagrams

- Considering just one predicate *A(x),* we have:

domain or **universe** of the predicate variable x



Predicate logic: new features

- **Individual variables**: x, y, z, . . . may be considered to vary over one (or more) universes.

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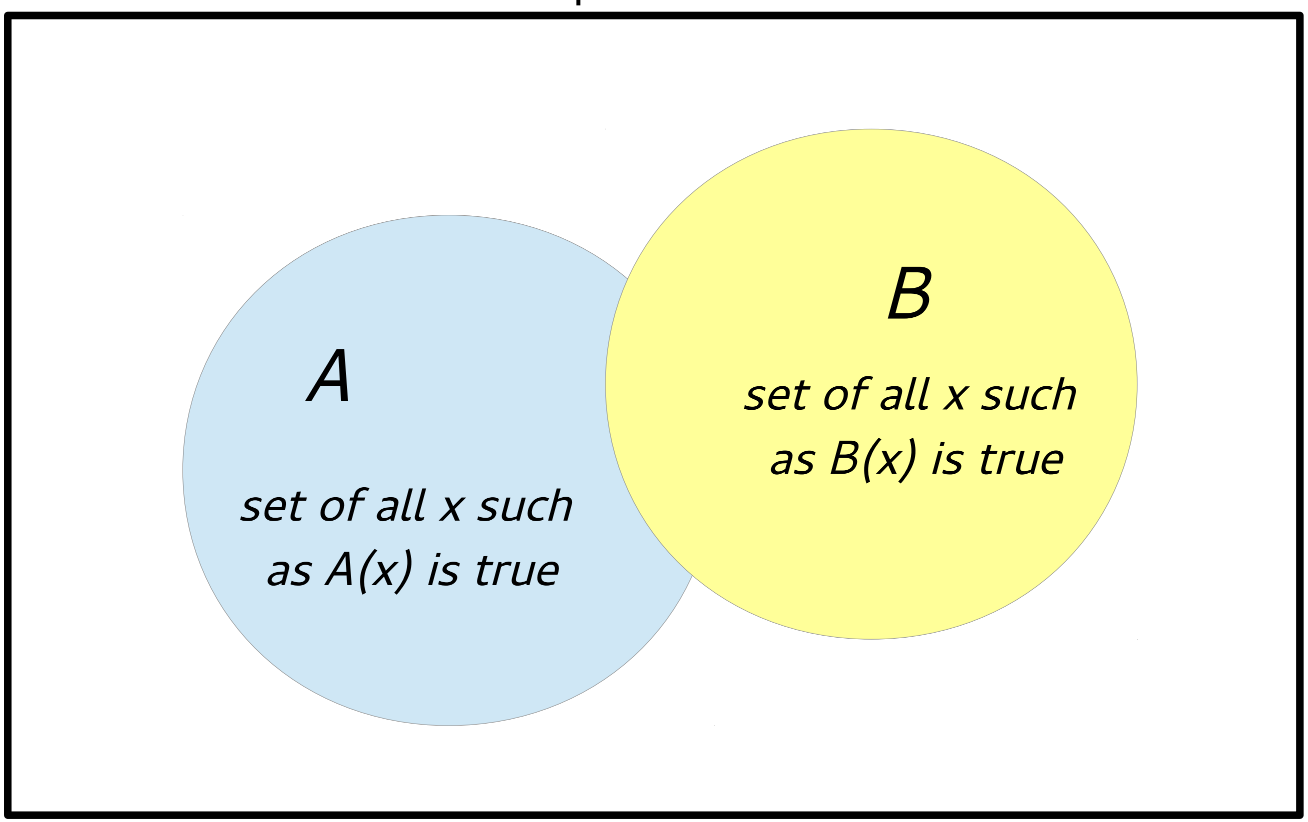
First–order logic (FOL):
no quantifiers over predicates, e.g. ∀P [..]
no predicate applies on predicates. e.g. P(Q, R)

Predicate logic and Venn diagrams

- When a statement contains more predicates, we can map the different sets and translate the logical operators in set operators. The resulting set is where our original statement is true.

domain or **universe** of the predicate variable x



Predicate logic and syllogisms

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but the universe may be empty!

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Syllogisms assume that in **A** there is always some student:

$\ldots \land \exists x\ \mathit{Student}(x)$

explicit universe:

$\forall x\ [\mathit{Student}(x) \rightarrow \mathit{WearsUniform}(x)]$

Example

any two individuals communicate, if there is an interpreter

$$(\forall x)\,(\forall y)\,(\forall l_1)\,(\forall l_2)\,((s(x, l_1) \wedge (s(y, l_2) \wedge (\exists z)\,(s(z, l_1) \wedge s(z, l_2)))) \supset c(x, y))$$

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negate

$$(\exists x)\,(\exists y)\,(\exists l_1)\,(\exists l_2)\,((s(x, l_1) \wedge (s(y, l_2) \wedge (\exists z)\,(s(z, l_1) \wedge s(s, l_2)))) \wedge \neg c(x, y))$$

there are two individuals who do not communicate despite the presence of an interpreter

A language consists of **symbols**, ...

- **Alphabet**
  - *terms:*
    - *constants:* c1, c2, ...
    - *variables:* v1, v2, ...
    - *functors:* f(t1, .., tn), where t1, .. tn are terms

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A language consists of a **syntax** (rules to aggregate symbols), ...

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- **A formula is**
  - an atomic formula
  - if F is a formula, then ¬F is a formula
  - if F and G are formulas, then (F o G) is a formula, where o is a binary connective.
  - if F is a formula, (∀x) F and (∃x) F are formulas, where x is a variable.

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  - unbound variables are **free**.
- A **closed formula** or ***statement*** is a formula with no free variables.

Another example: Peano arithmetics

- **Alphabet**
  - *terms:*
    - *constants:* 1, 2, 3, 4, 5, 6, 7, 8, 9
    - *variables:* x, y, z, ...
    - *functors (arity):* 0/0, +/2, */2
  - *predicates (arity):* </2, ≤/2, ≈/2, ...
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  (∀x) (∀y) (x ≤ y ≡ (∃z) (x+z≈y))

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  - *predicates (arity):* $</2$, $\leq/2$, $\approx/2$, ...
  - *quantifiers:* $\forall$, $\exists$
  - *propositional logic connectives:* $\top$, $\bot$, $\neg$, $\wedge$, $\vee$, $\supset$, $\equiv$, ...

trade-off between complexity of signature and of quantification

- Examples:

  $(\forall x)\ (\forall y)\ (x \leq y \equiv (\exists z)\ (x+z \approx y))$ ← Definition of $\leq$

  $(\exists x)\ (\forall y)\ (x+y \approx y)$ ← Definition of 0

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- An **assignment** A instantiates each variable v by giving it a value v^A taken from D.

A language consists of a **semantic** (rules to interpret it)

- Terms are interpreted recursively from the interpretation of their elements: for each term t, $t^{I,A}$ is defined as:
  - c^I for a constant c,
  - v^A for a variable v,
  - $f^I(t1^{I,A}, t2^{I,A},... tn^{I,A})$ for a functional term f(t1, , ...tn).

- A **model** M(D, I) is defined by the domain D and the interpretation I.

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- $T^{I,A} = \mathbf{V}$; $\bot^{I,A} = \mathbf{F}$
- $(\neg X^{I,A}) = \neg X^{I,A}$
- $(X \text{ o } Y)^{I,A} = X^{I,A} \bullet Y^{I,A}$ for coupled operators o and •
- $((\forall x) F)^{I,A} = \mathbf{V}$ *if and only if* $F^{I,B} = \mathbf{V}$ for all assignation B equal to A save for x.
- $((\exists x) F)^{I,A} = \mathbf{V}$ *if and only if* $F^{I,B} = \mathbf{V}$ for (at least) one assignation B equal to A save for x.

Semantics of formulas

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  if $F^{I,A} = \mathbf{V}$ for all assignments A.
- A formula F is ***valid***
  if F is true in any M(D,I).
- A set S of formulas is ***satisfiable*** in M(D,I)
  if there exists (at least) an assignement A such that $F^{I,A} = \mathbf{V}$ for all F belonging to S.

NOTA BENE: A formula F is ***valid***
if and only if $\{\neg F\}$ is not satisfiable.

Example

- Consider
  - the domain D = {"France", "Vatican", "Japan", "to have diplomatic relations with"}
  - in the interpretation I which matches

    f to f^I as "France", ...

    D to the only relation of D,

- We can then evaluate the truth and validity of formula as *D*(x, y), *D*(france, vatican), and all their combinations

Herbrand model

- A model M(D,I) for a first-order language L is an **Herbrand model**, if and only if:
  - D contains only closed terms of L
  - For each closed term t, $t^I = t$

Noting with F{v/d} the outcome of a substitution of v by d in F (note that F{v/d} is still a formula!)...

In an Herbrand model:
  - For any formula F, **(∀v) F is true** if and only if
    F{v/d} is true for any d ∈ D
  - For any formula F, **(∃v) F is true** if and only if
    F{v/d} it true for at least a d ∈ D

Replacing quantifiers

- M(D,I) is an Herbrand Model for a first order language L:

- If γ is a formula of L, γ is true in M if and only if
  γ(d) is true for any d $\in$ D ;
- If δ is a formula of L, δ is true in M if and only if
  δ(d) is true for (at least a) d $\in$ D

γ-rule	γ(d)	δ-rule	δ(d)
$(\forall x)$ F	F{x/d}	$(\exists x)$ F	F{x/d}
$\neg(\exists x)$ F	¬F{x/d}	$\neg(\forall x)$ F	¬F{x/d}

Example of resolution

$((\forall x)\ (P(x) \lor Q(x)) \supset ((\exists x)\ P(x) \lor (\forall x)\ Q(x)))$

start from the negated formula (refutation):

1. $[\neg((\forall x)\ (P(x) \lor Q(x)) \supset ((\exists x)\ P(x) \lor (\forall x)\ Q(x)))]$

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3. $[\neg((\exists x)\ P(x) \vee (\forall x)\ Q(x))]$ development of 1. (α-rule)

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7. [¬P(c)] a version of 4. (γ-rule, *taking the same c*)

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11. [] resolving clause of 6. and 10.

Prenex form

- A formula is in **prenex** form if it is written as a sequence of quantifiers (prefix) followed by a quantifier-free part (matrix).

prefix matrix

$(Q_1\ x_1) \ldots (Q_n\ x_n)\ M$

$Q_i \in \{\forall, \exists\}$

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helpers

$\neg(\exists x)\ A(x) \equiv (\forall x)\ \neg A(x)$

$\neg(\forall x)\ A(x) \equiv (\exists x)\ \neg A(x)$

$((\forall x)\ A(x) \land B) \equiv (\forall x)\ (A(x) \land B)$

$(A \land (\forall x)\ B(x)) \equiv (\forall x)\ (A \land B(x))$

$((\exists x)\ A(x) \land B) \equiv (\exists x)\ (A(x) \land B)$

$(A \land (\exists x)\ B(x)) \equiv (\exists x)\ (A \land B(x))$

$((\forall x)\ A(x) \supset B) \equiv (\exists x)\ (A(x) \supset B)$

$(A \supset (\forall x)\ B(x)) \equiv (\forall x)\ (A \supset B(x))$

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$Qi \in \{\forall, \exists\}$

non-deterministic process

a good practice: existential quantifiers in the *leftmost* possible positions.

helpers

$\neg(\exists x)\ A(x) \equiv (\forall x)\ \neg A(x)$

$\neg(\forall x)\ A(x) \equiv (\exists x)\ \neg A(x)$

$((\forall x)\ A(x)) \wedge B \equiv (\forall x)\ (A(x) \wedge B)$

$A \wedge ((\forall x)\ B(x)) \equiv (\forall x)\ (A \wedge B(x))$

$((\exists x)\ A(x)) \wedge B \equiv (\exists x)\ (A(x) \wedge B)$

$A \wedge ((\exists x)\ B(x)) \equiv (\exists x)\ (A \wedge B(x))$

$((\forall x)\ A(x)) \supset B \equiv (\exists x)\ (A(x) \supset B)$

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Example

$(\forall x)\ (\exists y)\ \neg(A(x) \supset A(y))$

$\neg(\exists x)\ A(x) \equiv (\forall x)\ \neg A(x)$

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1. $(\forall x)\ \neg(\forall y)\ (A(x) \supset A(y))$

Example

$(\forall x)\ (\exists y)\ \neg(A(x) \supset A(y))$

1. $(\forall x)\ \neg(\forall y)\ (A(x) \supset A(y))$
2. $(\forall x)\ \neg(A(x) \supset (\forall y)\ A(y))$

$\neg(\exists x)\ A(x) \equiv (\forall x)\ \neg A(x)$

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$(\forall x)\ (\exists y)\ \neg(A(x) \supset A(y))$

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5. $\neg(\forall y)\ ((\exists x)\ A(x) \supset A(y))$

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6. $\neg(\forall y)\ (\exists x)\ (A(x) \supset A(y))$
7. $(\exists y)\ (\forall x)\ \neg(A(x) \supset A(y))$

Skolemization

- Skolemization is a transformation in which a formula in prenex form, as

  $$(Q1x1)\ (Q2x2)\ \ldots(\exists xk)\ldots(Qnxn)\ F$$

  is transformed in

  $$(Q1x1)\ (Q2x2)\ \ldots(Qnxn)\ F\{xk/f(x1,\ x2\ \ldots xk\text{-}1)\}$$

  where f is a new functor (called ***Skolem function***) that does not belong to the language.

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- The two formulas are not equivalent, but they have the same satisfiability.

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  non-deterministic process

  a good practice: start from external quantifiers

Herbrand model lemma

- Let S be a set of statements in Skolem form

S has a model
if and only if
S has a Herbrand model

Authomatic proving in practice

To prove the validity of a formula F:

- rename variables if necessary,
  e.g. $((\forall x)\ p(x) \supset (\forall x)\ r(x))$ in $((\forall x)\ p(x) \supset (\forall y)\ r(y))$
- transform $\neg F$ in prenex form
- skolemize
- remove the quantifiers
- transform in CNF
- use the resolution method
- apply unification


Si..., ALORS...
QUOI, « SI ALORS » ?...
LOGIQUE FORMELLE... T'AS PLUS QU'À REMPLIR.