Exercises for lecture 2

Exercise 1: Maxwell theory in differential form notation

In this exercise we assume a flat Lorentzian metric, $g_{\mu\nu} = \eta_{\mu\nu}$. Recall that the current one-form can be written in components as $J = J_\mu dx^\mu = \rho dx^0 + j_i dx^i$.

a. Show that the inhomogeneous Maxwell equations of motion, $\nabla \cdot E = \rho$ and $\nabla \times B - \partial E/\partial t = j$ can be written as $d \star F = \star J$.

b. What does the equation $d^2 = 0$ imply for $\rho$ and $j$? Explain that the resulting equation can be interpreted as conservation of charge.

c. To gauge fix the Maxwell gauge symmetry, one often chooses the condition $\partial_\mu A^\mu = 0$. Write this condition in differential form notation.

Exercise 2: Equations of motion for Maxwell theory

Show that the Euler-Lagrange equations for the action $S = \int F \wedge \star F + A \wedge \star J$ are indeed Maxwell’s homogeneous equations of motion (2.49).

* Exercise 3: The Dirac monopole (hand-in exercise)

In this exercise, we will work with three-dimensional polar coordinates $(r, \theta, \phi)$, defined by

\[
\begin{align*}
x &= r \sin \theta \cos \phi \\
y &= r \sin \theta \sin \phi \\
z &= r \cos \theta
\end{align*}
\]

in terms of the cartesian coordinates $(x, y, z)$. Of course, this map is not 1-to-1; in particular, $\theta$ and $\phi$ are only defined up to multiples of $2\pi$. The time coordinate $t$ will not play a role in this exercise, so you may assume we are working on $\mathbb{R}^3$.

We will consider the field strength

\[ F = \frac{g}{4\pi} \sin \theta \ d\theta \wedge d\phi \]

with $g \neq 0$.

a. Compute $\int_{S^2} F$, where $F$ is a two-sphere centered at the origin of $\mathbb{R}^3$.

b. Use the previous result and Stokes’ theorem to show that $F$ can not be a closed form.

c. Naively computing $dF$, one still seems to find $dF = 0$. How can this apparent contradiction with the result of (b) be understood?
d. One way to understand the result of (c) is to compute $\star F$. Show that indeed $\star F$ has a singularity at the origin of $\mathbb{R}^3$.

Summarizing, we have found that our field strength $F$ is well-defined and closed only on $(\mathbb{R}^3)^* \equiv \mathbb{R}^3 \setminus \{0,0,0\}$. As this manifold is not topologically trivial, we cannot use Poincaré’s lemma to conclude that $F = dA$ everywhere. However, if we define $D^+$ and $D^-$ as the regions where $\theta \neq \pi$ and $\theta \neq 0$ respectively (that is, $D^+$ is $\mathbb{R}^3$ excluding the negative $z$-axis, and $D^-$ is $\mathbb{R}^3$ excluding the positive $z$-axis), these two regions are topologically trivial. On these regions, we now consider the 1-forms

$$A^+ = \frac{g}{4\pi}(1 - \cos \theta)d\phi, \quad A^- = -\frac{g}{4\pi}(1 + \cos \theta)d\phi$$

e. (Easy:) Show that $F = dA^+$ and $F = dA^-$ in $D^+$ and $D^-$ respectively.

f. Compute $A^+ - A^-$. Where is this 1-form defined? In particular: is that space topologically trivial? Can $A^+$ be obtained from $A^-$ using a gauge transformation?

One can slightly generalize the concept of a gauge transformation as follows. In quantum mechanics, one is interested in the wave function $\psi(x)$ of a particle. Under a transformation of the potential

$$A \rightarrow A + \omega$$

with $\omega$ a closed one-form, the wave function transforms as

$$\psi(x) \rightarrow \psi(x) \exp \left( i \int_\gamma \omega \right)$$

where $\gamma$ is a path from an arbitrarily chosen base point to the point to $x$. A large gauge transformation is a transformation of $A$ by a one-form $\omega$ such that the transformation of $\psi(x)$ is well-defined.

g. In the previous sentence, “well-defined” means that the transformed value of $\psi(x)$ should not depend on the choice of a path $\gamma$. Argue that in a topologically trivial space, this condition is automatically satisfied if $\omega$ is closed.

h. Which additional condition should $\omega$ satisfy if the space is not topologically trivial? (In particular: if it is not simply connected?)

i. In our example, which condition should $g$ satisfy so that $A^+$ and $A^-$ are related by a large gauge transformation?

The upshot of this exercise is therefore that, assuming the condition found in (i) is satisfied, our field strength $F$ describes a “good” electromagnetic field configuration on $\mathbb{R}^3 \setminus \{0,0,0\}$. The interpretation of this configuration is that there is a “defect” at the singular point in the origin – an object which can be interpreted as a particle.

j. Show that this particle does not have an electric charge, but that by replacing the $E$-field with the $B$-field, it can be considered to have a “magnetic charge”. This particle (which has never been observed in nature!) is called the Dirac magnetic monopole.