1. MANIFOLDS: A REFRESHER

We start by briefly recalling the definition of a manifold. Informally, a manifold is a space that "locally looks like $\mathbb{R}^n". This vague statement is formalized by the notion of an atlas: Let $M$ be set.

**Definition 1.1.** A **smooth atlas** on $M$ is given by a collection of pairs $\{(U_\alpha, x_\alpha)\}_{\alpha \in I}$, where $I$ is some indexing set, with:

1. $U_\alpha \subset M, \alpha \in I$ are subsets that cover $M$: $M = \bigcup \alpha U_\alpha$,
2. $x_\alpha : U_\alpha \to \mathbb{R}^n, \alpha \in I$ are bijections $U_\alpha \cong x_\alpha(U_\alpha) \subset \mathbb{R}^n$ (called "charts"),
3. For all $\alpha, \beta \in I$, the maps

   \[ x_\beta \circ x_\alpha^{-1} : x_\alpha(U_\alpha \cap U_\beta) \cong x_\beta(U_\alpha \cap U_\beta), \]

   are smooth (i.e., $C^\infty$).

**Remark 1.2.** We can define the notion of a complex or holomorphic atlas in a similar way: this time the charts $(U_\alpha, z_\alpha)$ map $z_\alpha : U_\alpha \to \mathbb{C}^n$ and are such that the transition maps $z_\beta \circ z_\alpha^{-1} : q_\alpha(U_\alpha \cap U_\beta) \to \mathbb{C}^n$ are holomorphic. (We say that $f : U \to \mathbb{C}, U \subset \mathbb{C}^n$ is holomorphic if, in standard coordinates $(z^1, \ldots, z^n) \in U \subset \mathbb{C}^n$, the functions given by $z^i \mapsto f(z^1, \ldots, z^i, \ldots, z^n)$, keeping the other $z^j, j \neq i$ fixed, are holomorphic functions of one variable, i.e., satisfy the Cauchy–Riemann equations:

\[
\frac{\partial f_1}{\partial x^i} = \frac{\partial f_2}{\partial y^i}, \quad \frac{\partial f_2}{\partial x^i} = -\frac{\partial f_1}{\partial y^i}, \quad f = f_1 + \sqrt{-1} f_2, \quad z^i = x^i + \sqrt{-1} y^i
\]

**Definition 1.3.** A **smooth** manifold is a set equipped with a **smooth** atlas. A **complex** manifold is a set equipped with a **complex** atlas.

**Mathematical Remark 1.4.** This definition is not quite precise. There are two mathematical objections to this definition, in the sense that the definition above is not quite what we want.

1. An atlas $\{(U_\alpha, x_\alpha)\}_{\alpha \in I}$ on $M$ induces a **topology** by declaring a set $U$ to be open if and only if $x_\alpha(U \cap U_\alpha) \subset \mathbb{R}^n$ is open for all $\alpha \in I$. To avoid pathological behavior, we have to assume this topology to be Hausdorff and second countable. This excludes for example the possibility to turn $\mathbb{R}^n$ itself into a $k$-dimensional manifold for $k < n$. In the usual mathematical definition, one starts with a topological space (Hausdorff and second countable) and defines an atlas as above,
assuming in addition that $U_\alpha \subset M$ are open. It can be checked that the induced atlas-topology agrees with the original one.

ii) The mathematical definition uses the notion of a maximal atlas: We say that two atlases $\{(U_\alpha, x_\alpha)\}_{\alpha \in I}$ and $\{(U_{\alpha'}, x_{\alpha'})\}_{\alpha' \in J}$ are compatible if the collection $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I} \coprod \{(U_{\alpha'}, \varphi_{\alpha'})\}_{\alpha' \in J}$ is still an atlas. (For this one needs to check condition iii) above.) Being compatible defines an equivalence relation, and a maximal atlas is, by definition, an equivalence class of charts.

Remark 1.5. A complex manifold is in particular a smooth manifold, because of the fact that a holomorphic function is smooth. For a local complex chart $(z_1^\alpha, \ldots, z_n^\alpha)$ the underlying smooth chart is given as $(x_1^\alpha(x), \ldots, x_n^\alpha(x))$, or $(x_1^\alpha, \ldots, x_n^\alpha, y_1^\alpha, \ldots, y_n^\alpha)$ by writing $z_i^\alpha = x_i^\alpha + \sqrt{-1} y_i^\alpha$ (the order of coordinates may matter when one assigns an orientation). Our focus in this course is on smooth manifolds, but it is convenient to have the concept of a complex manifold at hand in some cases.

Given an atlas $\{(U_\alpha, x_\alpha)\}_{\alpha \in I}$ for a manifold $M$, we can write out $x_\alpha: U_\alpha \to \mathbb{R}^n$ in coordinates:

$$x_\alpha(x) := (x_1^\alpha(x), \ldots, x_n^\alpha(x)), \quad x \in U_\alpha.$$ If $x \in U_\alpha \cap U_\beta$ we have two charts around $x$ and the local coordinates are related by

(2) $$(x_\beta \circ x_\alpha^{-1})(x_1^\alpha(x), \ldots, x_n^\alpha(x)) = (x_1^\beta(x), \ldots, x_n^\beta(x)).$$

Remark 1.6 (“clutching and pasting”). A slightly different point of view on manifolds is given by focusing on the transition functions $\varphi_{\alpha\beta} := x_\alpha \circ x_\beta^{-1}$, which are by definition local diffeomorphisms on $\mathbb{R}^n$. We now think of $M$ as consisting of pieces $\tilde{U}_\alpha := x_\alpha(U_\alpha) \subset \mathbb{R}^n$ which are glued together using the transition functions $\varphi_{\alpha\beta}$:

(3) $$M \cong \bigsqcup_{\alpha \in I} \tilde{U}_\alpha / \sim,$$

where $x \sim y$ means that $\varphi_{\alpha\beta}(x) = y$, for $x \in \tilde{U}_\alpha$ and $y \in \tilde{U}_\beta$. The quotient makes sense because the transition functions $\{\varphi_{\alpha\beta}\}_{\alpha, \beta \in I}$ satisfy the following properties ensuring that $\sim$ defines an equivalence relation:

(4a) $\varphi_{\alpha\alpha} = \text{id}$ (Reflexivity)

(4b) $\varphi_{\beta\alpha} = \varphi_{\alpha\beta}^{-1}$ (Symmetry)

(4c) $\varphi_{\alpha\beta} \circ \varphi_{\beta\gamma} = \varphi_{\alpha\gamma}$ (Transitivity)

Conversely, given a bunch of open subsets $U_\alpha \subset \mathbb{R}^n$, together with local diffeomorphisms $\varphi_{\alpha\beta}: U \to U'$ with $U \subset U_\alpha$ and $U' \subset U_\beta$, satisfying the three properties above, we can define a smooth manifold structure on $M$ defined by (3).
Example 1.7 (Projective spaces). Consider $\mathbb{P}^n = \mathbb{CP}^n$, the space of all one-dimensional lines in $\mathbb{C}^{n+1}$. We denote the usual homogeneous coordinates by $[z^0, \ldots, z^n] \in \mathbb{CP}^n$. The standard manifold charts are given by

$$U_i = \{[z^0, \ldots, z^n] \in \mathbb{CP}^n, z^i \neq 0\},$$

with coordinate charts $\varphi_i : U_i \to \mathbb{C}^n$, given by

$$\varphi_i([z^0, \ldots, z^n]) = \left(\frac{z^0}{z^i}, \ldots, \frac{z^{i-1}}{z^i}, \frac{z^{i+1}}{z^i}, \ldots, \frac{z^n}{z^i}\right).$$

The transition maps $\varphi_{ij} = \varphi_i \circ \varphi_j^{-1}$ are given by $\varphi_{ij}(z^0, \ldots, z^n) = (z^i)^{-1} \cdot (z^0, \ldots, z^n)$, where we have identified $\varphi_j(U_j)$ with the affine hyperplane $\{(z^0, \ldots, z^n), z^i = 1\} \subset \mathbb{C}^{n+1}$.

The main idea behind the definition of a manifold is that we can use the local structure on $M$, as being equal to $\mathbb{R}^n$, to introduce key concepts from analysis such as smooth functions, mappings and vector fields on $M$. All we have to do is phrase the definition in terms of local charts, and check that the definition is invariant under change of coordinates (2).

As a simple example, define a function $f$ on $M$ to be smooth if for each chart $(U_\alpha, \varphi_\alpha)$ the function $f_\alpha := f \circ \varphi_\alpha^{-1}$ is a smooth function on $\mathbb{R}^n$. By the chain rule for derivatives, this definition is independent of the choice of local chart, and therefore makes sense on the manifold $M$. The space of smooth functions on $M$ is denoted by $C^\infty(M)$. From the point of view of Remark 1.6, a smooth function $f \in C^\infty(M)$ is given by a collection of smooth functions $\{f_\alpha \in C^\infty(U_\alpha)\}_{\alpha \in I}$ on pieces of $\mathbb{R}^n$, that agree on overlaps:

$$f_\alpha(x) = f_\beta(\varphi_{\alpha\beta}(x)), \text{ for all } x \in U_\alpha \cap U_\beta.$$  

Another, more general example is given by the notion of a smooth map between manifolds: A map $f : M \to N$ is said to be smooth if, for atlases $(U_\alpha, \varphi_\alpha)$ for $M$ and $(V_\beta, y_\beta)$ of $N$, the composition

$$y_\beta \circ f \circ \varphi_\alpha^{-1} : U_\alpha \to V_\beta,$$

is smooth. By the chain rule, this notion of smoothness is independent of local coordinates.

2. Tangent bundle and vector fields

Recall that for an open subset $U \subset \mathbb{R}^n$, its tangent bundle is defined to be $TU := U \times \mathbb{R}^n$. If we write $x \in U$ in coordinates $x = (x^1, \ldots, x^n)$, the tangent space $T_xU = \mathbb{R}^n$ to $U$ at $x$ has the basis $\{\partial/\partial x^1, \ldots, \partial/\partial x^n\}$. Let $\varphi : U \to V$ with $V \subset \mathbb{R}^n$ be a diffeomorphism sending $x \in U$ to $\varphi(x) = \varphi(x) = (y^1, \ldots, y^n) \in V$. Its tangent map acts on the tangent
space by the Jacobi matrix:

\[ T_x \varphi \left( \frac{\partial}{\partial x^i} \right) = \sum_j \frac{\partial y^j}{\partial x^i}(x) \frac{\partial}{\partial y^j} \in T_{\varphi(x)} V. \]

Varying the basepoint \( x \in U \), these matrices together form the tangent mapping \( T\varphi : TU \to TV \), and by the chain rule for Jacobi matrices we see that

\[ T(\psi \circ \varphi) = T\psi \circ T\varphi, \]

where \( \psi : V \to W \) is another diffeomorphism.

Suppose now that we are given a manifold structure on \( M \) provided by an atlas \( \{(U_\alpha, x_\alpha)\}_{\alpha \in I} \), with associated “gluing data” \( \{\tilde{U}_\alpha := x_\alpha(U_\alpha) \subset \mathbb{R}^n\} \) with transition functions \( \varphi_{\alpha\beta} := x_\beta \circ x_\alpha^{-1} \) satisfying the conditions (4a)–(4c). Then the subsets

\[ \{T\tilde{U}_\alpha = \tilde{U}_\alpha \times \mathbb{R}^n \subset \mathbb{R}^{2n}\}, \]

together with the local diffeomorphisms \( \psi_{\alpha\beta} := T\varphi_{\alpha\beta} : T\tilde{U}_\alpha \to T\tilde{U}_\beta \), satisfy the same “cocycle conditions” (4a)–(4c) by the chain rule (7). It therefore defines another \( 2n \)-dimensional manifold, called the tangent bundle \( TM \) of \( M \). It comes equipped with a canonical projection \( \pi : TM \to M \), and the fiber \( \pi^{-1}(x) = T_x M \) is called the tangent space of \( M \) at \( x \). In the “gluing picture” of Remark 1.6 we have

\[ TM := \bigsqcup_\alpha (\tilde{U}_\alpha \times \mathbb{R}^n) / \sim, \]

and we therefore see that, given a smooth map \( f : M \to N \), the collection

\[ \bigsqcup_{\alpha, \beta} T(y_\beta \circ f \circ x_\alpha^{-1}) : \bigsqcup_\alpha T(x_\alpha(U_\alpha)) \to \bigsqcup_\beta T(y_\beta(V_\beta)), \]

where \( \{(V_\beta, y_\beta)\} \) is an atlas for \( N \), descends to the quotient to define a smooth map \( Tf : TM \to TN \).

Smooth sections of the projection \( \pi \), i.e. smooth maps \( X : M \to TM \) satisfying \( \pi \circ X = \text{id}_M \), are called vector fields. In local coordinates \( (x_1^\alpha, \ldots, x_n^\alpha) \) on \( U_\alpha \) a vector field can be written as

\[ X = \sum_i X^i_\alpha(x) \frac{\partial}{\partial x^i_\alpha}, \]

where the “coefficients” \( X^i_\alpha(x) \) are smooth functions of \( x \in U_\alpha \). When \( x \in U_\alpha \cap U_\beta \) and we change to coordinates \( (x_1^\beta, \ldots, x_n^\beta) \), we see from (6) that

\[ X = \sum_i X^i_\beta(x) \frac{\partial}{\partial x^i_\beta}, \quad \text{with} \quad X^i_\beta = \sum_j X^j_\alpha \frac{\partial x^i_\beta}{\partial x^j_\alpha}. \]

This explains the physicists’ point of view on vector fields: for them, a vector field is given by a vector of functions \( X^i_\alpha(x) \) in local coordinates, which transforms as above under coordinate changes. We write \( \mathfrak{X}(M) \) for the vector space of all vector fields on \( M \).

The following properties are easy to check:
• A diffeomorphism \( f: M \to N \) induces a push forward map \( f_*: \mathcal{X}(M) \to \mathcal{X}(N) \) by the formula
  \[
f_*(X)(y) := T_{f^{-1}(y)}(X)
  \]
• A vector field \( X \in \mathcal{X}(M) \) acts on \( C^\infty(M) \) by taking “directional derivatives” \( f \mapsto X(f) \). Once again in local coordinates
  \[
  X(f)(x) := \sum_i X^i(x) \frac{\partial f}{\partial x^i}.
  \]
• There is a “Lie bracket” of vector fields given in local coordinates
  \[
  [X, Y] = \sum_{i,j=1}^n \left( X^i_a(x) \frac{\partial Y^j}{\partial x^i_a} - Y^i_a(x) \frac{\partial X^j}{\partial x^i_a} \right) \frac{\partial}{\partial x^j_a}.
  \]

3. COTANGENT BUNDLE AND DIFFERENTIAL FORMS

The cotangent bundle is dual to the tangent bundle. For \( U \subset \mathbb{R}^n \) we define the cotangent space \( T^*_x U \), \( x \in U \) as the vector space with basis \( \{dx^i\}_{i=1}^n \) dual to the basis \( \{\partial/\partial x^i\}_{i=1}^n \) of \( T_x U \):
  \[
dx^i \left( \frac{\partial}{\partial x^j} \right) = \delta^i_j.
  \]
This duality implies, by the rule (6), that a diffeomorphism \( \varphi: U \to V \) with \( V \subset \mathbb{R}^n \) sending \( x \in U \) to \( \varphi(x) = y(x) = (y^1, \ldots, y^n) \in V \) sends
  \[
  T_y \varphi(dy^j) = \sum_{j=1}^n \frac{\partial y^j}{\partial x^i} dx^i,
  \]
i.e. the covectors \( dx^i \) transform according to the inverse of the Jacobi matrix. Again, the pieces \( T^* U \) together with the inverse of the Jacobi matrices of the transition functions \( T^* \varphi_a \) satisfy the conditions (4a)–(4c) and define a manifold called the cotangent bundle \( T^* M \). Again there is an obvious smooth projection \( \pi: T^* M \to M \) and sections are called differential 1-forms (these are sometimes called covector fields). By definition a differential 1-form \( \theta \) maps a point \( x \in M \) to a linear map \( T_x M \to \mathbb{R} \). In local coordinates, \( \theta \) can be written as
  \[
  \theta = \sum_a \theta^a_i(x) dx^i.
  \]
Changing to another chart \( \varphi_\beta \), equation (9) implies the transformation rule
  \[
  \theta^\beta_i = \sum_j \frac{\partial x^i_\beta}{\partial x^j_\alpha} \theta^a_j.
  \]
We shall write \( \Omega^1(M) \) for the vector space of all differential 1-forms on \( M \). In higher degrees, a \( k \)-form \( \omega \) maps a point \( x \in M \) to an antisymmetric linear \( k \)-form on \( T_x M \), i.e. an element in \( \bigwedge^k T^*_x M \). To define what it means for \( \omega \) to be smooth, we have to define
a manifold structure on $\bigwedge^* T^* M$. In brief: locally for $U \subset \mathbb{R}^n$, a basis of $\bigwedge^k T^*_x U$ is given by \{\(dx^i \wedge \ldots \wedge dx^k\)\} so that $\omega$ can be written in local coordinates as

$$\omega = \sum_{i_1, \ldots, i_k} \omega_{i_1, \ldots, i_k}^a dx^{i_1} \wedge \ldots \wedge dx^{i_k},$$

where $\omega_{i_1, \ldots, i_k}^a$ is a collection of smooth functions on $U_a$, which are antisymmetric under permutations of the $k$-indices $i_1, \ldots, i_k$. The transformation rules under changes of local coordinates are given by

$$\omega_{i_1, \ldots, i_k}^\beta = \sum_{j_1, \ldots, j_k} \frac{\partial x^{j_1}_a}{\partial x^{i_1}_b} \ldots \frac{\partial x^{j_k}_a}{\partial x^{i_k}_b} \omega_{j_1, \ldots, j_k}^a. \tag{10}$$

Again, the following properties are easy to derive:

- A smooth map $f: M \to N$ induces a pull-back map $f^*: \Omega^k(N) \to \Omega^k(M)$ defined as
  $$f^* \omega(x)(V_1, \ldots, V_k) := \omega(f(x))(T_x f(V_1), \ldots, T_x f(V_k)), \quad \text{for } x \in M, V_1, \ldots, V_k \in T_x M.$$

  - By definition, there is a pairing
    $$\Omega^1(M) \times \mathfrak{X}(M) \to \mathbb{R}, \quad (\theta, X) \mapsto \theta(X).$$

  - The total derivative of a function $f \in C^\infty(M)$, written in local coordinates
    $$df|_U = \sum_i \frac{\partial f}{\partial x^i} dx^i,$$

    defines a 1-form $df \in \Omega^1(M)$. This is consistent with (or can be derived from) the action (8) of vector fields on functions: in other words, we can now write $X(f) := df(X)$.

  - The formula
    $$d\omega|_U := \sum_{i_1, i_2, \ldots, i_k} \frac{\partial \omega_{i_1, \ldots, i_k}^a}{\partial x^i} dx^i \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k},$$

    defines an operator $d: \Omega^k(M) \to \Omega^{k+1}(M)$, called the exterior derivative. There is a coordinate independent formula for this derivative as follows

$$d\omega(X_0, \ldots, X_k) = \sum_{i=0}^k (-1)^i X_i (\omega(X_0, \ldots, \hat{X}_i, \ldots, X_k))$$
$$+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k), \tag{11}$$

where the hat means that we omit that term from the argument. It is not difficult to prove that $dd\omega = 0$ for all $\omega \in \Omega^k(M)$, because we can change the order in which we take partial derivatives.
• For any vector field $X \in \mathfrak{X}(M)$ there is an operator $\iota_X : \Omega^k(M) \to \Omega^{k-1}(M)$ called contraction with $X$ and defined as

$$(\iota_X \omega)(x_0) := \omega(x)(X,-,\ldots,-) : \bigwedge^k T_x M \to \mathbb{R}.$$

• Cartan’s magic formula

$$L_X := \iota_X \circ d + d \circ \iota_X : \Omega^k(M) \to \Omega^k(M)$$

defines an action of vector fields on $k$-forms. This extends the action of $\mathfrak{X}(M)$ on $C^\infty(M) = \Omega^0(M)$.

• Using a partition of unity, the integral $\int_M \alpha$ of an $n$-form over an $n$-dimensional manifold is well defined, i.e., independent of the choice of local coordinates. This is because the transformation rule (10) for an $n$-form is exactly given by multiplication with the Jacobian (i.e., the determinant of the Jacobi matrix) which appears in the change of coordinates of multidimensional integrals. When $M$ is a manifold with boundary $\partial M$, Stokes’ theorem asserts that

$$\int_M \omega = \int_{\partial M} \beta,$$

for $\beta \in \Omega^{k-1}(M)$.

4. CALCULUS ON COMPLEX MANIFOLDS

When the manifold $M$ is complex, the differential calculus on $M$ is a bit richer when we complexify the tangent bundle. Let us first again consider the local situation $U \subset \mathbb{C}^n$. Using the coordinates $z^i = x^i + \sqrt{-1} y^i$ with $i = 1, \ldots, n$, a real basis for the tangent space $T_z U$ is given by $\{ \partial/\partial x^i, \partial/\partial y^i \}_{i=1}^n$. On $\mathbb{C}^n$, we can also use the complex coordinates $(z^i, \bar{z}^i)$, so it is convenient to introduce the complex basis

$$\frac{\partial}{\partial z^i} = \frac{1}{2} \left( \frac{\partial}{\partial x^i} - \sqrt{-1} \frac{\partial}{\partial y^i} \right), \quad \frac{\partial}{\partial \bar{z}^i} = \frac{1}{2} \left( \frac{\partial}{\partial x^i} + \sqrt{-1} \frac{\partial}{\partial y^i} \right).$$

The complex tangent space is then defined as $T^C_z U = \text{span}_\mathbb{C}\{ \partial/\partial z^i, \partial/\partial \bar{z}^i, i = 1, \ldots, n \}$. In a similar way we define the complex tangent bundle $T^C M$: this is a complex manifold, just like the tangent bundle to a smooth manifold is a smooth manifold in its own right.

In the notation above the Cauchy–Riemann equations are given by the simple equation $\partial f / \partial \bar{z} = 0$. The transition functions $\varphi : z^i \mapsto w^i(z^1, \ldots, z^n)$ of the complex manifold are by definition holomorphic, so $\partial w^i / \partial \bar{z}^j = 0$ and therefore

$$T_z \varphi \left( \frac{\partial}{\partial z^j} \right) = \sum_{j=1}^n \frac{\partial w^j}{\partial z^i} \frac{\partial}{\partial w^j}, \quad T_z \varphi \left( \frac{\partial}{\partial \bar{z}^j} \right) = \sum_{j=1}^n \frac{\partial \bar{w}^j}{\partial \bar{z}^i} \frac{\partial}{\partial \bar{w}^j}.$$

It follows that the transition functions for the complex tangent bundle $T^C M$ have the form

$$\left( \begin{array}{cc} \partial w^j / \partial z^i & 0 \\ 0 & \partial \bar{w}^j / \partial \bar{z}^i \end{array} \right).$$
Because of this special shape of the transition matrix, with off-diagonal terms in this $2 \times 2$ matrix equal to zero, the complex tangent bundle splits as

$$T^C M = T^{(1,0)} M \oplus T^{(0,1)} M,$$

with $T^{(1,0)} M$ the subspace spanned by $\partial / \partial z^i$ and $T^{(0,1)} M$ spanned by $\partial / \partial \bar{z}^i$ in a local complex chart $(z^1, \ldots, z^n)$. Dually this leads to a decomposition of the space of complex differential 1-forms (these are sections of the complex cotangent bundle)

$$\Omega^1_C(M) = \Omega^{(1,0)}(M) \oplus \Omega^{(0,1)}(M),$$

where $\alpha \in \Omega^{(1,0)}(M)$ when in local holomorphic coordinates $z = (z^1, \ldots, z^n)$ can be written as $\alpha = \sum_i \alpha_i (z, \bar{z}) dz^i$ (no $d\bar{z}^i$s) and $\beta \in \Omega^{(0,1)}(M)$ when $\beta = \sum_i \beta_i (z, \bar{z}) d\bar{z}^i$ (no $dz^i$s). Going over to higher degree differential forms, we get

$$\Omega^k_C(M) = \bigoplus_{p+q=k} \Omega^{(p,q)}(M),$$

with $\alpha \in \Omega^{(p,q)}(M)$ if locally, in some holomorphic coordinate system $z = (z^1, \ldots, z^n)$ we have

$$\alpha = \sum_{i_1, \ldots, i_p, j_1, \ldots, j_q} \alpha_{i_1, \ldots, i_p, j_1, \ldots, j_q} (z, \bar{z}) dz^{i_1} \wedge \ldots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \ldots \wedge d\bar{z}^{j_q}.$$

The exterior differential

$$d\alpha = \sum_{i_1, \ldots, i_p, j_1, \ldots, j_q} \left( \frac{\partial \alpha_{i_1, \ldots, i_p, j_1, \ldots, j_q}}{\partial z^i} dz^i + \frac{\partial \alpha_{i_1, \ldots, i_p, j_1, \ldots, j_q}}{\partial \bar{z}^j} d\bar{z}^j \right) \wedge dz^{i_1} \wedge \ldots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \ldots \wedge d\bar{z}^{j_q},$$

accordingly splits as $d = \partial + \bar{\partial}$, where $\partial : \Omega^{(p,q)}(M) \to \Omega^{(p+1,q)}(M)$ and $\bar{\partial} : \Omega^{(p,q)}(M) \to \Omega^{(p,q+1)}(M)$. The fundamental property $d \circ d = 0$ of the exterior differential now amounts to

$$\partial \circ \partial = 0, \quad \partial \circ \bar{\partial} + \bar{\partial} \circ \partial = 0, \quad \bar{\partial} \circ \bar{\partial} = 0.$$