

# ORBIFOLD CUP PRODUCTS AND RING STRUCTURES ON HOCHSCHILD COHOMOLOGIES

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ABSTRACT. In this paper we study the Hochschild cohomology ring of convolution algebras associated to orbifolds, as well as their deformation quantizations. In the first case the ring structure is given in terms of a wedge product on twisted polyvectorfields on the inertia orbifold. After deformation quantization, the ring structure defines a product on the cohomology of the inertia orbifold. We study the relation between this product and an  $S^1$ -equivariant version of the Chen–Ruan product. In particular, we give a de Rham model for this equivariant orbifold cohomology.

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## 1. INTRODUCTION

In this paper, we study orbifolds within the language of noncommutative geometry. According to [MO], an orbifold  $X$  can be represented by a proper étale Lie groupoid  $G$ , and different representations of the same orbifold  $X$  are Morita equivalent. A paradigm from noncommutative geometry tells that one should view the groupoid algebra  $\mathcal{A} \rtimes G$  of a proper étale groupoid  $G$  representing the orbifold  $X$  as the “algebra of functions” on  $X$ , where  $\mathcal{A}$  is the sheaf of functions on the unit space  $G_0$  of  $G$ . Though it is noncommutative, the algebra  $\mathcal{A} \rtimes G$  contains much important information of  $X$ .

We provide in this paper a complete description of the ring structures on the Hochschild cohomology of the groupoid algebra  $\mathcal{A} \rtimes G$  and its deformation quantization  $\mathcal{A}^{\hbar} \rtimes G$  when  $X$  is a symplectic orbifold. We thus complete projects initiated in [TA] and [NEPFPOTA]. By our results one obtains a cup product on the space of multivector fields on the inertia orbifold  $\tilde{X}$  associated to  $X$ , and a Frobenius structure on the de Rham cohomology of the inertia orbifold  $\tilde{X}$ . This Frobenius structure is closely related to the Chen-Ruan orbifold cohomology [CHR04], and inspires us to introduce a de Rham model for some  $S^1$ -equivariant Chen-Ruan orbifold cohomology. We prove that the algebra of the Hochschild cohomology of the deformation quantization  $\mathcal{A}^{\hbar} \rtimes G$  is isomorphic to the graded algebra of the Chen-Ruan orbifold cohomology with respect to a natural filtration.

In this paper, we view the algebra  $\mathcal{A} \rtimes G$  as a bornological algebra with the canonical bornology inherited from the Frechet topology and compute its Hochschild cohomology respect to this bornology. In [NEPFPOTA], we constructed a vector space isomorphism

$$H^*(\mathcal{A} \rtimes G, \mathcal{A} \rtimes G) = \Gamma^\infty \left( \wedge^{\bullet-\ell} TB_0 \otimes \wedge^\ell N \right)^G,$$

where  $B_0$  is the space of loops of  $G$  defined as  $\{g \in G \mid s(g) = t(g)\}$ ,  $\ell$  is a locally constant function on  $B_0$ , namely the codimension of the germ of  $B_0$  inside  $G_1$ , and where  $N$  is the normal bundle of  $B_0$  in  $G_1$ . In this article, we determine the cup product on the Hochschild cochain  $H^*(\mathcal{A} \rtimes G, \mathcal{A} \rtimes G)$ . To do so, we need to understand the maps realizing the above isomorphisms of vector spaces. In [NEPFPOTA], the ring structures got lost at the end of the final equality, since there we were dealing with a clumsy chain of quasi-isomorphisms. The first goal of this work is to present a sequence of explicit quasi-isomorphisms of differential graded algebras preserving cup products. Some parts of these quasi-isomorphisms have already appeared in [NEPFPOTA] and [HATA], but in this work we succeeded to put all ingredients together in the right way and thus determine the cup products we were looking for. The new input consists in the following. Firstly, we introduce a complex of fine presheaves  $\mathcal{H}^\bullet$  on  $X$  which has a natural cup product and the global sections of which form a complex quasi-isomorphic to the Hochschild cohomology complex. Secondly, we use Čech cohomology methods to localize the computation of the cohomology ring of  $\mathcal{H}^\bullet(X)$ . Thirdly, we use the twisted cocycle construction and the local quasi-inverse map  $T$  from [HATA] to compute the cup product locally. By gluing together the local cup products to a global one we finally arrive at a transparent computation of the cup product on  $H^*(\mathcal{A} \rtimes G, \mathcal{A} \rtimes G)$ . We would like to mention that in [AN], some similar but incomplete results in the local situation were obtained.

The above sequence of explicit quasi-isomorphisms opens a way to compute the Hochschild cohomology of the deformation quantization  $\mathcal{A}^{\hbar} \rtimes \mathbb{G}$ , which originally has been constructed in [TA]. In the case of a global quotient, the Hochschild cohomology of this algebra has been computed by Dolgushev and Etingof [DOET] as a vector space using van den Berg duality. Our method is completely different from [DOET] and allows even to determine the ring structure on  $H^\bullet(\mathcal{A}^{\hbar} \rtimes \mathbb{G}, \mathcal{A}^{\hbar} \rtimes \mathbb{G})$  in full generality. The crucial step in our approach is that we generalize the above complex of presheaves  $\mathcal{H}^\bullet$  on  $X$  and the associated Čech double complex to the deformed case. With the quasi-isomorphisms for the undeformed algebra, one can check that there are natural morphisms of differential graded algebras from the Hochschild cochain complex of  $\mathcal{A}^{\hbar} \rtimes \mathbb{G}$  to the presheaf complex  $\mathcal{H}_{loc, \hbar}^\bullet$  and the associated Čech double complex. We prove these maps to be quasi-isomorphisms by looking at the  $E_1$  terms of the spectral sequence associated to the  $\hbar$ -filtration, which agrees with the undeformed complexes. To perform the local computations, we generalize the Fedosov–Weinstein–Xu resolution in [Do] for the computation of the Hochschild cohomology of a deformation quantization to the  $\mathbb{G}$ -twisted situation using ideas of Fedosov [FE]. We use essentially an explicit map from the Koszul resolution of the Weyl algebra to the corresponding Bar resolution by Pinczon [PI]. Our main theorem is that we have a natural isomorphism of algebras over  $\mathbb{C}(\!(\hbar)\!)$

$$H^\bullet(\mathcal{A}^{\hbar} \rtimes \mathbb{G}, \mathcal{A}^{\hbar} \rtimes \mathbb{G}) \cong H^{\bullet-\ell}(\tilde{X}, \mathbb{C}(\!(\hbar)\!)),$$

where the product structure on the right hand side is defined (cf. Section 4) by

$$[\alpha] \cup [\beta] = \int_{m_\ell} \text{pr}_1^* \alpha \wedge \text{pr}_2^* \beta. \quad (1.1)$$

This generalizes Alvarez’s result [AL] on the Hochschild cohomology ring of the crossed product algebra of a finite group with the Weyl algebra.

The cup product (1.1) together with integration with respect to the symplectic volume form defines a Frobenius structure on the de Rham cohomology of the inertia orbifold  $\tilde{X}$ . One notices that there is similarity between (1.1) and the de Rham model defined by Chen and Hu [CHHU]. However, Chen and Hu’s model was only defined for abelian orbifolds and works in a formal level. To connect the Hochschild cohomology of  $\mathcal{A}^{\hbar} \rtimes \mathbb{G}$  to the Chen-Ruan orbifold cohomology, we extend the de Rham model to an arbitrary almost complex orbifold using methods from equivariant cohomology theory and [JAKAKI]’s result on obstruction bundles. More precisely, we prove that the algebra  $(H^\bullet(\mathcal{A}^{\hbar} \rtimes \mathbb{G}, \mathcal{A}^{\hbar} \rtimes \mathbb{G}), \cup)$  is isomorphic to the graded algebra of the  $S^1$ -equivariant Chen-Ruan orbifold cohomology with respect to a natural filtration. In general, the Hochschild cohomology and the Chen-Ruan orbifold cohomology are not isomorphic as algebras. By construction, the Chen-Ruan orbifold cohomology depends on the choice of an almost complex structure, but the Hochschild cohomology is independent of the choice of an almost complex or symplectic structure. Therefore, one naturally expects that information on the almost complex structure should be contained in the filtration on the de Rham model. It is a very interesting question whether one can detect different almost complex structures through the filtration on the de Rham model. Our de Rham model and the computation of the Hochschild cohomology ring of the deformed convolution algebra give more insight to the Ginzburg-Kaledin conjecture [GIKA] for hyper-Kähler orbifolds. Our computations within the differential category suggest

that it is crucial to work in the holomorphic category of deformation quantization, otherwise conjecture from [GIKA] that there is an isomorphism between the Hochschild cohomology ring of a deformation quantization and the Chen-Ruan orbifold cohomology will in general not be true. Concerning the de Rham model for orbifold cohomology let us also mention that recently, a similar model has been obtained independantly by R. Kaufmann [KA].

Our paper is organized as follows. In Section 2, we outline the strategy and the main results of this paper, in Section 3, we provide a detailed computation of the Hochschild cohomology and its ring structure of the algebra  $\mathcal{A} \rtimes \mathbb{G}$ . Next, in Section 4, we compute the Hochschild cohomology and its ring structure of the deformed algebra  $\mathcal{A}^{\hbar} \rtimes \mathbb{G}$ . Then we switch in Section 5 to orbifold cohomology theory. We introduce a de Rham model for some  $S^1$ -equivariant Chen-Ruan orbifold cohomology and connect this model to the ring structure of the Hochschild cohomology of the deformed convolution algebra. In the Appendix, we provide a full introduction to bornological algebras, their modules and their Morita theory. We want to emphasize that the Appendix contains some original results on Morita equivalence of bornological algebras, which to our knowledge has not been covered in the literature before. We have chosen to keep these results in the Appendix to avoid too technical arguments in the main part of our paper.

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## 2. OUTLINE

As is mentioned above, the main goal of this article is to determine the ring structure of the Hochschild cohomology of a deformation quantization on a proper étale Lie groupoid. In this section, we outline the strategy to achieve that goal and begin with a brief overview over the basic notation and results needed from the theory of groupoids. For further details on the latter we refer the interested reader to the monograph [MOMR] and also to our previous article [NEPFPOTA].

Recall that a *groupoid* is a small category  $\mathbb{G}$  with set of objects denoted by  $\mathbb{G}_0$  and set of morphisms by  $\mathbb{G}_1$  such that all morphisms are invertible. The structure maps of a groupoid are depicted in the diagram

$$\mathbb{G}_1 \times_{\mathbb{G}_0} \mathbb{G}_1 \xrightarrow{m} \mathbb{G}_1 \xrightarrow{i} \mathbb{G}_1 \begin{array}{c} \xrightarrow{s} \\ \xrightarrow{t} \end{array} \mathbb{G}_0 \xrightarrow{u} \mathbb{G}_1,$$

where  $s$  and  $t$  are the source and target map,  $m$  is the multiplication resp. composition,  $i$  denotes the inverse and finally  $u$  is the inclusion of objects by identity morphisms. In the most interesting cases, the groupoid carries an additional structure, like a topological or differentiable structure. If  $\mathbb{G}_1$  and  $\mathbb{G}_0$  are both topological spaces and if all structure maps are continuous, then  $\mathbb{G}$  is called a topological groupoid. Such a topological groupoid is called *proper*, if the map  $(s, t) : \mathbb{G}_1 \rightarrow \mathbb{G}_0 \times \mathbb{G}_0$  is a proper map, and *étale*, if  $s$  and  $t$  are both local homeomorphisms. In case  $\mathbb{G}_1$  and

$G_0$  carry the structure of a  $C^\infty$ -manifold such that  $s, t, m, i$  and  $u$  are smooth and  $s, t$  submersions, then  $G$  is said to be a *Lie groupoid*.

The situation studied in this article consists of an orbifold represented by a proper étale Lie groupoid  $G$ . As a topological space, the orbifold coincides with the orbit space  $X = G_0/G$ . In the following we introduce several sheaves on  $G$  and  $X$ . By  $\mathcal{A}$  we always denote the sheaf of smooth functions on  $G_0$ , and by  $\mathcal{A} \rtimes G$  the convolution algebra, i.e. the space  $C_{\text{cpt}}^\infty(G_1)$  together with the convolution product  $*$  which is defined by the formula

$$a_1 * a_2(g) = \sum_{g_1 \cdot g_2 = g} a_1(g_1) a_2(g_2) \quad \text{for all } a_1, a_2 \in C_{\text{cpt}}^\infty(G_1) \text{ and } g \in G_1. \quad (2.1)$$

Next, let  $\omega$  be a  $G$ -invariant symplectic form on  $G_0$  and choose a  $G$ -invariant (local) star product  $\star$  on  $G_0$ . The resulting sheaf of deformed algebras of smooth functions will be denoted by  $\mathcal{A}^{\hbar}$ . The crossed product algebra  $\mathcal{A}^{\hbar} \rtimes G$  has the underlying vector space  $C_{\text{cpt}}^\infty(G_1)[[\hbar]] = \Gamma_{\text{cpt}}^\infty(G_1, s^* \mathcal{A}^{\hbar})$  and carries the product  $\star_c$  given by

$$[a_1 \star_c a_2]_g = \sum_{g_1 \cdot g_2 = g} [a_1]_{g_1} \star [a_2]_{g_2} \quad \text{for all } a_1, a_2 \in C_{\text{cpt}}^\infty(G_1) \text{ and } g \in G_1. \quad (2.2)$$

Hereby,  $[a]_g$  denotes an element of the stalk  $(s^* \mathcal{A}^{\hbar})_g \cong \mathcal{A}_{s(g)}^{\hbar}$ , and it has been used that  $G$  acts from the right on the sheaf  $\mathcal{A}^{\hbar}$  (see [NEPFPOTA, Sec. 2] for details).

For every open subset  $U \subset X$  define the space  $\tilde{\mathcal{A}}(U)$  by

$$\tilde{\mathcal{A}}(U) := (\pi s)_* s^* \mathcal{A}(U) = C^\infty((\pi s)^{-1}(U)),$$

where  $\pi : G_0 \rightarrow X$  is the canonical projection. Denote by  $\tilde{\mathcal{A}}_{\text{fc}}(U)$  the subspace

$$\{f \in C^\infty((\pi s)^{-1}(U)) \mid \text{supp } f \cap (\pi s)^{-1}(K) \text{ is compact for all compact } K \subset U\}$$

of all smooth functions on  $(\pi s)^{-1}(U)$  with fiberwise compact support. Observe that the convolution product  $*$  can be extended naturally by Eq. (2.1) to each of the spaces  $\tilde{\mathcal{A}}_{\text{fc}}(U)$ . Indeed, since for  $K_i := \text{supp } a_i$  with  $a_i \in \tilde{\mathcal{A}}_{\text{fc}}(U)$ ,  $i = 1, 2$  the set

$$\begin{aligned} & m((K_1 \times K_2) \cap (G_1 \times_{G_0} G_1) \cap (\pi s)^{-1}(K)) \\ &= m(((K_1 \cap (\pi s)^{-1}(K)) \times (K_2 \cap (\pi s)^{-1}(K))) \cap (G_1 \times_{G_0} G_1)) \end{aligned}$$

is compact by assumption on the  $a_i$ , the product  $a_1 * a_2$  is well-defined and lies again in  $\tilde{\mathcal{A}}_{\text{fc}}(U)$ . Hence, the spaces  $\tilde{\mathcal{A}}_{\text{fc}}(U)$  all carry the structure of an algebra and form the sectional spaces of a sheaf  $\tilde{\mathcal{A}}_{\text{fc}}$  on  $X$ . Likewise, one constructs the sheaf  $\tilde{\mathcal{A}}_{\text{fc}}^{\hbar}$ . Finally note that the natural maps  $\mathcal{A} \rtimes G \hookrightarrow \tilde{\mathcal{A}}_{\text{fc}}(X)$  and  $\mathcal{A}^{\hbar} \rtimes G \hookrightarrow \tilde{\mathcal{A}}_{\text{fc}}^{\hbar}(X)$  are both algebra homomorphisms.

From Appendix A.6 one can derive the following result.

**Theorem O.** *The algebras  $\mathcal{A} \rtimes G$  and  $\mathcal{A}^{\hbar} \rtimes G$  carry in a natural way the structure of a bornological algebra and are both quasi-unital. Likewise, the sheaves  $\tilde{\mathcal{A}}_{\text{fc}}$  and  $\tilde{\mathcal{A}}_{\text{fc}}^{\hbar}$  are sheaves of quasi-unital bornological algebras. Moreover, the natural homomorphisms  $\mathcal{A} \rtimes G \hookrightarrow \tilde{\mathcal{A}}_{\text{fc}}(X)$  and  $\mathcal{A}^{\hbar} \rtimes G \hookrightarrow \tilde{\mathcal{A}}_{\text{fc}}^{\hbar}(X)$  are bounded.*

*Proof.* Prop. A.6 and Prop. A.8 in the appendix show that  $\mathcal{A} \rtimes G$  and  $\mathcal{A}^{\hbar} \rtimes G$  are quasi-unital bornological algebras. By exactly the same methods as in there one shows that  $\tilde{\mathcal{A}}_{\text{fc}}$  and  $\tilde{\mathcal{A}}_{\text{fc}}^{\hbar}$  are sheaves of quasi-unital bornological algebras. That the homomorphisms in Theorem O. are bounded is straightforward.  $\square$

According to Appendix A.4, Theorem O implies in particular that each one of the algebras in the claim is  $\mathbb{H}$ -unital and that the Bar complex provides a projective resolution. This will be the starting point for proving that several of the chain maps constructed in the following steps are indeed quasi-isomorphisms.

To formulate the next step, consider the Hochschild cochain complex (see A.4)

$$C^\bullet(\mathcal{A} \rtimes \mathbb{G}, \mathcal{A} \rtimes \mathbb{G}) := \text{Hom}_{(\mathcal{A} \rtimes \mathbb{G})^\circ}(\text{Bar}_\bullet(\mathcal{A} \rtimes \mathbb{G}), \mathcal{A} \rtimes \mathbb{G}),$$

where  $(\mathcal{A} \rtimes \mathbb{G})^\circ$  is the enveloping algebra (see Sec. A.4), and define for each open  $U \subset X$  the bornological space  $\mathcal{H}_\mathbb{G}^k(U)$  by

$$\mathcal{H}_\mathbb{G}^k(U) := \text{Hom}((\mathcal{A}|_U \rtimes \mathbb{G}|_U)^{\hat{\otimes}^k}, \tilde{\mathcal{A}}_{\text{fc}}(U)),$$

where  $\mathbb{G}|_U$  is the groupoid with object set  $\mathbb{G}|_{U_0} = \pi^{-1}(U)$  and morphism set  $\mathbb{G}|_{U_1} = (\pi s)^{-1}(U)$  and where  $\mathcal{A}|_U$  is the sheaf of smooth functions on  $\pi^{-1}(U)$ . Obviously, the spaces  $\mathcal{H}_\mathbb{G}^k(U)$  form the sectional spaces of a presheaf  $\mathcal{H}_\mathbb{G}^k$  on  $X$  which we denote by  $\mathcal{H}^k$  if no confusion can arise. The Hochschild coboundary map  $\beta := b^*$  on  $C^\bullet(\mathcal{A}|_U \rtimes \mathbb{G}|_U, \mathcal{A}|_U \rtimes \mathbb{G}|_U)$  extends to a coboundary map  $\beta$  on  $\mathcal{H}^\bullet(U)$  by the following definition:

$$\begin{aligned} \beta F(a_1 \otimes \dots \otimes a_k) &:= a_1 F(a_2 \otimes \dots \otimes a_k) + \\ &+ \sum_{i=2}^{k-1} (-1)^{i+1} F(a_1 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_k) \\ &+ F(a_1 \otimes \dots \otimes a_{k-1}) a_k, \text{ for } F \in \mathcal{H}^k(U) \text{ and } a_1, \dots, a_k \in \mathcal{C}_{\text{cpt}}^\infty((\pi s)^{-1}(U)). \end{aligned}$$

Moreover, there is a product  $\cup : \mathcal{H}^\bullet(U) \times \mathcal{H}^\bullet(U) \rightarrow \mathcal{H}^\bullet(U)$ , which is called the *cup product* on  $\mathcal{H}^\bullet(U)$  and which is given as follows:

$$\begin{aligned} \cup : \mathcal{H}^k(U) \times \mathcal{H}^l(U) &\rightarrow \mathcal{H}^{k+l}(U), \quad (F, G) \mapsto F \cup G, \\ F \cup G(a_1 \otimes \dots \otimes a_{k+l}) &:= F(a_1 \otimes \dots \otimes a_k) G(a_{k+1} \otimes \dots \otimes a_{k+l}) \\ &\text{for } a_1, \dots, a_{k+l} \in \mathcal{C}_{\text{cpt}}^\infty((\pi s)^{-1}(U)). \end{aligned}$$

It is straightforward to check that the cup product is associative and passes down to the cohomology of  $\mathcal{H}^\bullet(U)$ .

The compatibility between the Hochschild cohomology ring of  $\mathcal{A} \rtimes \mathbb{G}$  and the ring structure on the cohomology of  $\mathcal{H}^\bullet$  is expressed by the following step and will be proved in Section 3.1.

**Theorem I.** *The canonical embedding*

$$\iota : C^\bullet(\mathcal{A} \rtimes \mathbb{G}, \mathcal{A} \rtimes \mathbb{G}) \rightarrow \mathcal{H}^\bullet(X)$$

*is a quasi-isomorphism which preserves cup products.*

Since  $\mathcal{H}^\bullet$  is a complex of presheaves on  $X$ , one can use localization techniques for the computation of its cohomology ring. Thus, methods from Čech cohomology theory come into play. To make these ideas precise let  $\mathcal{U}$  be an open cover of the orbit space  $X$ , and denote by  $\check{\mathcal{H}}_{\mathcal{U}}^{\bullet, \bullet} := \check{\mathcal{H}}_{\mathbb{G}, \mathcal{U}}^{\bullet, \bullet} := \check{C}_{\mathcal{U}}^{\bullet}(\mathcal{H}_{\mathbb{G}}^{\bullet})$  the Čech double complex associated to the presheaf complex  $\mathcal{H}_{\mathbb{G}}^{\bullet}$ . This means that

$$\check{\mathcal{H}}_{\mathcal{U}}^{p, q} := \check{C}_{\mathcal{U}}^q(\mathcal{H}^p) := \prod_{(U_0, \dots, U_q) \in \mathcal{U}^{q+1}} \mathcal{H}^p(U_0 \cap \dots \cap U_q).$$

The coboundaries on  $\check{\mathcal{H}}_{\mathcal{U}}^{\bullet, \bullet}$  are given in  $p$ -direction by the Hochschild coboundary

$$\beta : \check{\mathcal{H}}_{\mathcal{U}}^{p,q} \rightarrow \check{\mathcal{H}}_{\mathcal{U}}^{p+1,q}$$

and, in  $q$ -direction, by the Čech coboundary

$$\delta : \check{\mathcal{H}}_{\mathcal{U}}^{p,q} \rightarrow \check{\mathcal{H}}_{\mathcal{U}}^{p,q+1}, (H_{(U_0, \dots, U_q)})_{(U_0, \dots, U_q) \in \mathcal{U}^{q+1}} \mapsto \left( \sum_{0 \leq i \leq q+1} (-1)^i H_{(U_0, \dots, \widehat{U}_i, \dots, U_{q+1})|_{U_0 \cap \dots \cap U_{q+1}}} \right)_{(U_0, \dots, U_{q+1}) \in \mathcal{U}^{q+2}}.$$

The cohomology of the double complex  $\check{\mathcal{H}}_{\mathcal{U}}^{\bullet, \bullet}$ , i.e. the cohomology of the total complex  $\text{Tot}_{\oplus}^{\bullet}(\check{\mathcal{H}}_{\mathcal{U}}^{\bullet, \bullet})$ , will be denoted by  $\check{H}_{\mathcal{U}}^{\bullet}(\mathcal{H}^{\bullet})$ . The inductive limit

$$\check{H}^{\bullet}(\mathcal{H}^{\bullet}) := \varinjlim_{\mathcal{U}} \check{H}_{\mathcal{U}}^{\bullet}(\mathcal{H}^{\bullet}),$$

where  $\mathcal{U}$  runs through the set of open covers of  $X$ , then is the Čech cohomology of the presheaf complex  $\mathcal{H}^{\bullet}$ . The crucial claim, which will be proved in Section 3.1 as well, now is the following.

**Theorem II.** *The presheaves  $\mathcal{H}^p$  are all fine, hence the Čech cohomology of the presheaf complex  $\mathcal{H}^{\bullet}$  is concentrated in degree  $q = 0$ , i.e.  $\check{H}_{\mathcal{U}}^{\bullet}(\mathcal{H}^{\bullet})$  is canonically isomorphic to the cohomology of the cochain complex  $\check{\mathcal{H}}_{\mathcal{U}}^{\bullet, 0}$ . Moreover, the Čech cohomology  $\check{H}^{\bullet}(\mathcal{H}^{\bullet})$  is given by the global sections of a cohomology sheaf on  $X$ . Finally, for each sufficiently fine and locally finite open covering  $\mathcal{U}$  of  $X$  the canonical chain map*

$$\mathcal{H}^p(X) \rightarrow \check{Z}_{\mathcal{U}}^{p,0}(\mathcal{H}^{\bullet}), \quad H \mapsto (H|_U)_{U \in \mathcal{U}}$$

is a quasi-isomorphism, where  $\check{Z}_{\mathcal{U}}^{p,0}(\mathcal{H}^{\bullet}) := \{H = (H_U)_{U \in \mathcal{U}} \in \check{\mathcal{H}}_{\mathcal{U}}^{p,0} \mid \delta H = 0\}$ .

By Theorem II one only needs to compute the cohomology of the complexes  $\mathcal{H}^{\bullet}(U)$  for all elements  $U$  of a sufficiently fine open covering of  $X$ . This is the purpose of the following steps.

Let us consider now a weak equivalence of proper étale groupoids  $\varphi : \mathbf{H} \hookrightarrow \mathbf{G}$ . Assume further that  $\varphi$  is an open embedding and denote by  $\mathcal{H}_{\mathbf{H}}^{\bullet}$  and  $\mathcal{H}_{\mathbf{G}}^{\bullet}$  the complexes of presheaves as defined above. Then  $\varphi$  induces a bounded linear map  $\varphi_* : \mathcal{C}_{\text{cpt}}^{\infty}(\mathbf{H}_1) \rightarrow \mathcal{C}_{\text{cpt}}^{\infty}(\mathbf{G}_1)$  by putting for  $a \in \mathcal{C}_{\text{cpt}}^{\infty}(\mathbf{H}_1)$ ,  $g \in \mathbf{G}_1$

$$\varphi_*(a)(g) = \begin{cases} a \circ \varphi^{-1}(g), & \text{if } g \in \text{im } \varphi, \\ 0, & \text{else.} \end{cases}$$

Moreover, one obtains bounded chain maps

$$\begin{aligned} \varphi^* : C^{\bullet}(\mathcal{A} \rtimes \mathbf{G}, \mathcal{A} \rtimes \mathbf{G}) &\rightarrow C^{\bullet}(\mathcal{A} \rtimes \mathbf{H}, \mathcal{A} \rtimes \mathbf{H}), & F &\mapsto \varphi^*(F) & \text{and} \\ \varphi^* : \mathcal{H}_{\mathbf{G}}^{\bullet} &\rightarrow \mathcal{H}_{\mathbf{H}}^{\bullet}, & F &\mapsto \varphi^*(F), \end{aligned}$$

where in both cases  $\varphi^*(F)$  is defined by

$$\varphi^*(F)(a_1 \otimes \dots \otimes a_k) = F(\varphi_* a_1 \otimes \dots \otimes \varphi_* a_k) \circ \varphi \quad \text{for } a_1, \dots, a_k \in \mathcal{C}_{\text{cpt}}^{\infty}(\mathbf{H}_1).$$

By Theorem I, the chain map  $\iota : C^{\bullet}(\mathcal{A} \rtimes \mathbf{G}, \mathcal{A} \rtimes \mathbf{G}) \rightarrow \mathcal{H}_{\mathbf{G}}^{\bullet}(X)$  is a quasi-isomorphism. Hence by Theorems A.11 and A.14 from the Appendix the following result holds true.

**Theorem III.** Under the assumptions on  $\mathbf{G}$ ,  $\mathbf{H}$  and  $\varphi$  from above, the convolution

algebras  $\mathcal{A} \rtimes \mathbb{G}$  and  $\mathcal{A} \rtimes \mathbb{H}$  are Morita equivalent as bornological algebras. Moreover, there is a commutative diagram consisting of quasi-isomorphisms

$$\begin{array}{ccc} C^\bullet(\mathcal{A} \rtimes \mathbb{G}, \mathcal{A} \rtimes \mathbb{G}) & \xrightarrow{\varphi^*} & C^\bullet(\mathcal{A} \rtimes \mathbb{H}, \mathcal{A} \rtimes \mathbb{H}) \\ \downarrow \iota & & \downarrow \iota \\ \mathcal{H}_{\mathbb{G}}^\bullet(X) & \xrightarrow{\varphi^*} & \mathcal{H}_{\mathbb{H}}^\bullet(X) \end{array}$$

such that the upper horizontal chain map coincides with the natural isomorphism between the Hochschild cohomologies  $H^\bullet(\mathcal{A} \rtimes \mathbb{G}, \mathcal{A} \rtimes \mathbb{G}) \rightarrow H^\bullet(\mathcal{A} \rtimes \mathbb{H}, \mathcal{A} \rtimes \mathbb{H})$  induced by the Morita context between  $\mathcal{A} \rtimes \mathbb{G}$  and  $\mathcal{A} \rtimes \mathbb{H}$ .

As an application of this result, we consider a point  $x \in \mathbb{G}_0$  in the object set of a proper étale Lie groupoid  $\mathbb{G}$ , and denote by  $\mathbb{G}_x$  the isotropy group of  $x$  that means the group of all arrows starting and ending at  $x$ . Choose for each  $g \in \mathbb{G}_x$  an open connected neighborhood  $W_g \subset \mathbb{G}_1$  (which can be chosen to be sufficiently small) such that both  $s|_{W_g} : W_g \rightarrow \mathbb{G}_0$  and  $t|_{W_g} : W_g \rightarrow \mathbb{G}_0$  are diffeomorphisms onto their images. Let  $M_x$  be the connected component of  $x$  in  $\bigcap_{g \in \mathbb{G}_x} s(W_g) \cap t(W_g)$  and put  $M_g := W_g \cap s^{-1}(M_x)$  for all  $g \in \mathbb{G}_x$ . Define an action of  $\mathbb{G}_x$  on  $M_x$  by

$$\mathbb{G}_x \times M_x \rightarrow M_x, \quad (g, y) \mapsto t(s|_{W_g}^{-1}(y)).$$

It is now straightforward to check that the canonical embedding

$$\mathbb{G}_x \times M_x \hookrightarrow \mathbb{G}|_{\pi(M_x)}, \quad (y, g) \mapsto s|_{W_g}^{-1}(y),$$

is open and a weak equivalence of Lie groupoids. In this article, we will call a manifold  $M_x$  together with a  $\mathbb{G}_x$ -action on  $M_x$  and a  $\mathbb{G}_x$ -equivariant embedding  $\iota_x : M_x \hookrightarrow \mathbb{G}_0$  a *slice* around  $x$ , if the induced embedding  $\mathbb{G}_x \times M_x \hookrightarrow \mathbb{G}|_{\pi(M_x)}$  is open and a weak equivalence of Lie groupoids. The argument above shows that for every point  $x \in \mathbb{G}_0$  there exists a slice. As a corollary to the above one obtains

**Theorem IIIb.** *Let  $x \in \mathbb{G}_0$  be a point and  $\iota_x : M_x \hookrightarrow \mathbb{G}_0$  a slice around  $x$ . Let  $\varphi_x : \mathbb{G}_x \times M_x \hookrightarrow \mathbb{G}|_{U_x}$  with  $U_x := \pi \iota_x(M_x)$  be the corresponding weak equivalence. Then the convolution algebras  $\mathcal{C}^\infty(M_x) \rtimes \mathbb{G}_x$  and  $\mathcal{A}|_{U_x} \rtimes \mathbb{G}|_{U_x}$  are Morita equivalent. Moreover, the canonical chain map*

$$\varphi_x^* : \mathcal{H}^\bullet(U_x) \rightarrow C^\bullet(\mathcal{C}^\infty(M_x) \rtimes \mathbb{G}_x, \mathcal{C}^\infty(M_x) \rtimes \mathbb{G}_x)$$

*is a quasi-isomorphism which implements the quasi-isomorphism induced in Hochschild cohomology by the Morita context between  $\mathcal{C}^\infty(M_x) \rtimes \mathbb{G}_x$  and  $\mathcal{A}|_{U_x} \rtimes \mathbb{G}|_{U_x}$ .*

Theorems II and III enable us to localize the computation of the Hochschild cohomology rings. Locally, we have the following result, also shown in Section 3.

**Theorem IV.** *Let  $M$  be a smooth manifold, and  $\Gamma$  a finite group acting on  $M$ . Then the Hochschild cohomology ring  $H^\bullet(\mathcal{C}_{\text{cpt}}^\infty(M) \rtimes \Gamma, \mathcal{C}_{\text{cpt}}^\infty(M) \rtimes \Gamma)$  is given as follows. As a vector space, one has*

$$H^\bullet(\mathcal{C}_{\text{cpt}}^\infty(M) \rtimes \Gamma, \mathcal{C}_{\text{cpt}}^\infty(M) \rtimes \Gamma) = \bigoplus_{\gamma \in \Gamma} \Gamma^\infty \left( \Lambda^{\bullet - \ell(\gamma)} T M^\gamma \otimes \Lambda^{\ell(\gamma)} N^\gamma \right)^\Gamma,$$



where  $\ell(\gamma)$  is the codimension of  $M^\gamma$  in  $M$ , and  $N^\gamma$  is the normal bundle to  $M^\gamma$  in  $M$ . For elements

$$\xi = (\xi_\alpha)_{\alpha \in \Gamma}, \eta = (\eta_\beta)_{\beta \in \Gamma} \in \bigoplus_{\gamma \in \Gamma} \Gamma^\infty \left( \Lambda^{\bullet - \ell(\gamma)} TM^\gamma \otimes \Lambda^{\ell(\gamma)} N^\gamma \right)^\Gamma$$

their cup product is given by

$$(\xi \cup \eta)_\gamma = \sum_{\substack{\alpha, \beta = \gamma, \\ \ell(\alpha) + \ell(\beta) = \ell(\gamma)}} \xi_\alpha \wedge \eta_\beta.$$

By globalization of Theorem IV, one obtains the following result.

**Theorem V.** *Let  $\mathbf{G}$  be a proper étale groupoid. Denote by*

$$\mathbf{B}_0 := \{g \in \mathbf{G}_1 \mid s(g) = t(g)\}$$

*its space of loops, and by  $N \rightarrow \mathbf{B}_0$  the normal bundle to  $T\mathbf{B}_0$  in  $T|_{\mathbf{B}_0}\mathbf{G}_1$ . Then the Hochschild cohomology ring of the convolution algebra is given by*

$$H^\bullet(\mathcal{A} \rtimes \mathbf{G}, \mathcal{A} \rtimes \mathbf{G}) = \Gamma^\infty \left( \Lambda^{\bullet - \ell} T\mathbf{B}_0 \otimes \Lambda^\ell N \right)^\mathbf{G},$$

*where  $\ell$  is the locally constant function on  $\mathbf{B}_0$  the value of which at  $g \in \mathbf{B}_0$  coincides with the codimension of the germ of  $\mathbf{B}_0$  at  $g$  within  $\mathbf{G}_1$ . The cup-product is given by the formula*

$$\xi \cup \eta = \int_m pr_1^* \xi \wedge pr_2^* \eta,$$

*for “multivectorfields”  $\xi, \eta \in \Gamma^\infty \left( \Lambda^{\bullet - \ell} T\mathbf{B}_0 \otimes \Lambda^\ell N \right)^\mathbf{G}$ . In the formula above, the maps  $m, pr_1, pr_2 : \mathbf{S} \rightarrow \mathbf{B}_0$  are the multiplication and the projection onto the first and second component, where*

$$\mathbf{S} := \{(g_1, g_2) \in \mathbf{B}_0 \times \mathbf{B}_0 \mid s(g_1) = t(g_2)\}.$$

*Finally, the integral over  $m$  simply means summation over the discrete fiber of  $m$ .*

**Remark 2.1.** We remark that one can also compute the Gerstenhaber bracket on  $H^\bullet(\mathcal{A} \rtimes \mathbf{G}, \mathcal{A} \rtimes \mathbf{G})$  by tracing down the quasi-isomorphisms constructed in Theorem I-III. Though, there is no natural Gerstenhaber bracket defined on the complex  $\mathcal{H}^\bullet$ , the bracket is well defined on the subcomplex of local cochains which take values in compactly supported functions. And this subcomplex is quasi-isomorphic to the whole complex by Teleman’s localization as is explained in Section 3. Therefore, the Gerstenhaber bracket is well defined on the Hochschild cohomology. Using the presheaf  $\mathcal{H}^\bullet$  and the local computation in [HATA], one can generalize the computation of the Gerstenhaber bracket from [HATA] to general orbifolds.

Let us now consider the deformed case. The strategy in computing the Hochschild cohomology of the deformed algebra  $\mathcal{A}^\hbar \rtimes \mathbf{G}$  is basically the same as in the undeformed algebra. We define the deformed analogue of the complex of presheaves  $\mathcal{H}_\mathbf{G}^\bullet$  in the obvious way and denote it by  $\mathcal{H}_{\mathbf{G}, \hbar}^\bullet$ . The associated Čech complex is denoted by  $\check{\mathcal{H}}_{\mathcal{U}, \hbar}^\bullet$ . With this, the deformed versions of Theorems I-III are straightforward to prove: the maps in these theorems generalize trivially to the sheaf  $\mathcal{A}^\hbar$ . Using the  $\hbar$ -adic filtrations on the complexes, Theorems I-III imply that in the zero’th order approximation these maps are quasi-isomorphisms. By an easy spectral sequence argument, cf. Section 4 one then shows this must be quasi-isomorphisms in general.

Again, this enables us to localize the computation of the Hochschild cohomology rings. For a global quotient orbifold, we have the following result, proved in Section 4.4.

**Theorem VI.** *Let  $M$  be a smooth symplectic manifold, and  $\Gamma$  a finite group acting on  $M$  preserving the symplectic structure. Then the Hochschild cohomology ring  $H^\bullet(\mathcal{A}_{\text{cpt}}^{((\hbar))}(M) \rtimes \Gamma, \mathcal{A}_{\text{cpt}}^{((\hbar))} \rtimes \Gamma)$  is given as follows. As a vector space, one has*

$$H^\bullet(\mathcal{A}_{\text{cpt}}^{((\hbar))}(M) \rtimes \Gamma, \mathcal{A}_{\text{cpt}}^{((\hbar))} \rtimes \Gamma) \cong \bigoplus_{(\gamma) \subseteq \Gamma} H_{Z(\gamma)}^{\bullet-\ell(\gamma)}(M^\gamma, \mathbb{C}((\hbar))),$$

where  $Z(\gamma)$  is the centralizer of  $\gamma$  in  $\Gamma$ , and  $(\gamma)$  stands for the conjugacy class of  $\gamma$  inside  $\Gamma$ . For elements

$$\alpha = (\xi_\gamma)_{\gamma \in \Gamma}, \beta = (\beta_\gamma)_{\gamma \in \Gamma} \in \bigoplus_{\gamma \in \Gamma} \left( H^{\bullet-\ell(\gamma)}(M^\gamma, \mathbb{C}((\hbar))) \right)^\Gamma$$

their cup product is given by

$$\alpha \cup \beta = \sum_{\substack{\gamma_1 \gamma_2 = \gamma, \\ \ell(\gamma_1) + \ell(\gamma_2) = \ell(\gamma_1 \gamma_2)}} i_{\gamma_1}^* \alpha_{\gamma_1} \wedge i_{\gamma_2}^* \alpha_{\gamma_2}.$$

Given this result, one might hope for a quasi-isomorphism  $\mathcal{H}_{\mathbf{G}, \hbar}^\bullet \rightarrow \Omega_{\tilde{X}}^{\bullet-\ell}$  to exist, which implements the isomorphism of the theorem above. The situation however is more complicated than that, and this is where the deformed case notably differs from the undeformed case.

First of all, it turns out one has to consider a sub-complex of presheaves  $\mathcal{H}_{\mathbf{G}, \text{loc}, \hbar}^\bullet \subset \mathcal{H}_{\mathbf{G}, \hbar}^\bullet$ , of cochains that are local in a sense explained in the beginning of Section 4. Second, instead of one quasi-isomorphism, there is a chain

$$\mathcal{H}_{\mathbf{G}, \text{loc}, \hbar}^\bullet \hookrightarrow \mathcal{C}_{\tilde{X}}^{\bullet, \bullet} \hookrightarrow \Omega_{\tilde{X}}^{\bullet-\ell},$$

where the intermediate double complex of sheaves  $\mathcal{C}_{\tilde{X}}^{\bullet, \bullet}$  is a twisted version of the Fedosov–Weinstein–Xu resolution used in [DO]. With this, we finally obtain the following result:

**Theorem VII.** *Let  $\mathbf{G}$  be a proper étale groupoid with an invariant symplectic structure, modeling a symplectic orbifold  $X$ . For any invariant deformation quantization  $\mathcal{A}^\hbar$  of  $\mathbf{G}$ , we have a natural isomorphism*

$$H^\bullet(\mathcal{A}^{((\hbar))} \rtimes \mathbf{G}, \mathcal{A}^{((\hbar))} \rtimes \mathbf{G}) \cong H^{\bullet-\ell}(\tilde{X}, \mathbb{C}((\hbar))).$$

With this isomorphism, the cup product is given by

$$\alpha \cup \beta = \int_{m_\ell} pr_1^* \alpha \wedge pr_2^* \beta,$$

for  $\alpha, \beta \in H^{\bullet-\ell}(\tilde{X}, \mathbb{C}((\hbar)))$  and where  $m_\ell$  is the restriction of  $m$ , cf. Theorem V, to those connected components of  $\mathbf{S}$  that satisfy

$$\ell(g_1 g_2) = \ell(g_1) + \ell(g_2), \quad (g_1, g_2) \in \mathbf{S}.$$

Moreover, this cup product and symplectic volume form together define a Frobenius algebra structure on  $H^{\bullet-\ell}(\tilde{X}, \mathbb{C}((\hbar)))$ .

On the other hand, on  $H^{\bullet-\ell}(\tilde{X}, \mathbb{C}((t)))$ , there is the famous Chen–Ruan orbifold product [CHRU04]. In Section 5, we study the connection between the cup product

defined in Theorem VI and the Chen-Ruan orbifold product. We introduce a de Rham model for some particular  $S^1$ -equivariant Chen-Ruan orbifold cohomology and relate this de Rham model to the above computation of Hochschild cohomology of  $\mathcal{A}^{\hbar} \rtimes \mathbb{G}$ .

Given an arbitrary almost complex orbifold  $X$ , we introduce a trivial  $S^1$ -action on  $X$ , but a nontrivial  $S^1$ -action on the bundle  $TX \rightarrow X$  by rotating each fiber. This  $S^1$ -action is compatible with all the orbifold structures on  $X$  and the inertia orbifold  $\tilde{X}$ . Therefore, we have the  $S^1$ -equivariant Chen-Ruan orbifold cohomology  $(H_{CR}^\bullet(X)((t)), \star_t)$  as introduced in Section 5.1 with  $\star_t$  the equivariant Chen-Ruan orbifold product.

The de Rham model  $(HT^\bullet(X)((t)), \wedge_t)$  for the above  $S^1$ -equivariant Chen-Ruan orbifold cohomology is defined as a vector space equal to  $H^\bullet(\tilde{X})((t))[\ell]$  with the product defined by putting

$$(\xi \wedge_t \eta)_\gamma := \sum_{\gamma = \gamma_1 \gamma_2} \iota_\gamma^* (\iota_{\gamma_1^*}(\xi_{\gamma_1}) \wedge \iota_{\gamma_2^*}(\eta_{\gamma_2})), \quad \xi, \eta \in H^\bullet(\tilde{X})((t)),$$

where  $\iota_{\gamma_i}$  is the embedding of  $X^{\gamma_i}$  into  $X$ . The following theorem is proved in Section 5.

**Theorem VIII.** *The two algebras  $(H_{CR}^\bullet(X)((t)), \star_t)$  and  $(HT^\bullet(X)((t)), \wedge_t)$  are isomorphic.*

To connect  $(HT^\bullet(X)((t)), \wedge_t)$  to the above Hochschild cohomology ring, we define a decreasing filtration  $\mathcal{F}^*$  on  $HT^\bullet(X)((t))$  by

$$\mathcal{F}^* = \{\alpha \in H^\bullet(X^\gamma)((t)) \mid \deg(\alpha) - \ell(\gamma) \geq *\}.$$

We prove in Section 5 the following result and thus finish our article.

**Theorem IX** *The graded algebra  $gr(HT^\bullet(X)((t)))$  associated to  $(HT^\bullet(X)((t)), \wedge_t)$  with respect to the filtration  $\mathcal{F}^*$  is isomorphic to the Hochschild cohomology algebra  $(H^\bullet(\mathcal{A}^{\hbar} \rtimes \mathbb{G}, \mathcal{A}^{\hbar} \rtimes \mathbb{G}), \cup)$  by identifying  $t$  with  $\hbar$ .*

### 3. CUP PRODUCT ON THE HOCHSCHILD COHOMOLOGY OF THE CONVOLUTION ALGEBRA

**3.1. Localization methods.** We start with the proof of Theorem I by using a localization method going back to Teleman [TE]. Recall that the orbifold  $X = \mathbb{G}_0/\mathbb{G}$  represented by a proper étale Lie groupoid  $\mathbb{G}$  carries in a natural way a sheaf  $\mathcal{C}_X^\infty$  of smooth functions. More precisely, for every open  $U \subset X$  the algebra  $\mathcal{C}^\infty(U)$  coincides naturally with the algebra  $(\mathcal{C}^\infty(\pi^{-1}(U)))^{\mathbb{G}}$  of smooth functions on  $\mathbb{G}_0$  invariant under the action of  $\mathbb{G}$ . Clearly,  $C^k(\mathcal{A} \rtimes \mathbb{G}, \mathcal{A} \rtimes \mathbb{G})$  is a module over  $\mathcal{C}^\infty(X)$ , and  $\mathcal{H}^k$  is a module presheaf over the  $\mathcal{C}_X^\infty$  for every  $k \in \mathbb{N}$ . Since  $\mathcal{C}_X^\infty$  is a fine sheaf, this implies in particular that  $\mathcal{H}^k$  has to be a fine presheaf.

Next recall from [NEPFPOTA, Sec. 3, Step I] that there is a canonical isomorphism

$$\wedge : C^k(\mathcal{A} \rtimes \mathbb{G}, \mathcal{A} \rtimes \mathbb{G}) \rightarrow \text{Hom}(\mathcal{A} \rtimes \mathbb{G}^{\hat{\otimes} k}, \mathcal{C}^\infty(\mathbb{G}_1)) = C_{\text{red}}^k(\mathcal{A} \rtimes \mathbb{G}, \mathcal{C}^\infty(\mathbb{G}_1)), \quad F \mapsto \hat{F}. \quad (3.1)$$

Hereby, the map  $\hat{F} : \mathcal{A} \rtimes \mathbb{G}^{\hat{\otimes} k} \rightarrow \mathcal{C}^\infty(\mathbb{G}_1)$  is uniquely determined by the requirement that for every compact  $K \subset \mathbb{G}_1$  and all  $a_1, \dots, a_k \in \mathcal{A} \rtimes \mathbb{G}$  the relation

$$\hat{F}(a_1 \otimes \dots \otimes a_k)|_K = F(\varphi_K \delta_u \otimes a_1 \otimes \dots \otimes a_k \otimes \varphi_K \delta_u)$$

holds true, where  $\varphi_K : \mathbb{G}_0 \rightarrow [0, 1]$  is a smooth function with compact support such that  $\varphi_K(x) = 1$  for all  $x$  in a neighborhood of  $s(K) \cup t(K)$ , and where  $\delta_u : \mathbb{G}_1 \rightarrow \mathbb{R}$  is the locally constant function which coincides with 1 on  $\mathbb{G}_0$  and which vanishes elsewhere.

Now let us fix a smooth function  $\varrho : \mathbb{R} \rightarrow [0, 1]$  which has support in  $(-\infty, \frac{3}{4}]$  and which satisfies  $\varrho(r) = 1$  for  $r \leq \frac{1}{2}$ . For  $\varepsilon > 0$  we denote by  $\varrho_\varepsilon$  the rescaled function  $r \mapsto \varrho(\frac{r}{\varepsilon})$ . Next choose a  $\mathbb{G}$ -invariant complete riemannian metric on  $\mathbb{G}_0$ , and denote by  $d$  the corresponding geodesic distance on  $\mathbb{G}_0$  (where we put  $d(x, y) = \infty$ , if  $x$  and  $y$  are not in the same connected component of  $\mathbb{G}_0$ ). Then  $d^2$  is smooth on the set of pairs of points of  $\mathbb{G}_0$  having finite distance. Put for every  $k \in \mathbb{N} \cup \{-1\}$ ,  $i = 1, \dots, k+1$  and  $\varepsilon > 0$ :

$$\Psi_{k,i,\varepsilon}(g_0, g_1, \dots, g_k) = \prod_{j=0}^{i-1} \varrho_\varepsilon(d^2(s(g_j), t(g_{j+1}))), \quad \text{where } g_j \in \mathbb{G}_1 \text{ and } g_{k+1} := g_0.$$

Moreover, put  $\Psi_{k,\varepsilon} := \Psi_{k,k+1,\varepsilon}$ . Using the above identification (3.1) we then define for  $F \in C^k := C^k(\mathcal{A} \rtimes \mathbb{G}, \mathcal{A} \rtimes \mathbb{G})$  a Hochschild cochain  $\Psi^{k,\varepsilon} F$  as follows:

$$\begin{aligned} \Psi^{k,\varepsilon} F(a_1 \otimes \dots \otimes a_k)(g_0) &:= F(\Psi_{k,\varepsilon}(g_0^{-1}, -, \dots, -) \cdot (a_1 \otimes \dots \otimes a_k))(g_0), \\ &\text{for } g_0 \in \mathbb{G}_1 \text{ and } a_1, \dots, a_k \in \mathcal{C}_{\text{cpt}}^\infty(\mathbb{G}_1). \end{aligned}$$

One immediately checks that  $\Psi^{\bullet,\varepsilon}$  forms a chain map on the Hochschild cochain complex. Likewise, one defines a chain map  $\Psi^{\bullet,\varepsilon}$  acting on the sheaf of cochain complexes  $\mathcal{H}^\bullet$ . In [NEPFPOTA, Sec. 3, Step 2] it has been shown that there exist homotopy operators  $H^{k,\varepsilon} : C^k \rightarrow C^{k-1}$  such that

$$(\beta H^{k,\varepsilon} + H^{k+1,\varepsilon} \beta) F = F - \Psi^{k,\varepsilon} F \quad (3.2)$$

for all  $F \in C^k$ . By a similar argument like in [NEPFPOTA] one shows that this algebraic homotopy holds also for  $F \in \mathcal{H}^k(X)$ . By completeness of the metric  $d$ , the cochain  $\Psi^{\bullet,\varepsilon} F$  is an element of  $C_{\text{red}}^k(\mathcal{A} \rtimes \mathbb{G}, \mathcal{A} \rtimes \mathbb{G})$  for  $F \in C^k$  or  $F \in \mathcal{H}^k(X)$ . Hence  $\Psi^{\bullet,\varepsilon}$  is a quasi-inverse to the canonical embedding  $C_{\text{red}}^\bullet(\mathcal{A} \rtimes \mathbb{G}, \mathcal{A} \rtimes \mathbb{G}) \hookrightarrow C^\bullet(\mathcal{A} \rtimes \mathbb{G}, \mathcal{A} \rtimes \mathbb{G})$  resp. to  $\iota : C_{\text{red}}^\bullet(\mathcal{A} \rtimes \mathbb{G}, \mathcal{A} \rtimes \mathbb{G}) \rightarrow \mathcal{H}^\bullet(X)$ . This proves Theorem I.

Next, we study the properties of the Čech double complex  $\check{\mathcal{H}}_{\mathcal{U}}^{\bullet,\bullet}$  associated to an open covering  $\mathcal{U}$  of  $X$  and prove Theorem II. We already have shown above that each presheaf  $\mathcal{H}^k$  is fine. Denote by  $\hat{\mathcal{H}}^k$  the sheaf associated to the presheaf  $\mathcal{H}^k$ . Then the Čech cohomology of  $\mathcal{H}^\bullet$  coincides with the Čech cohomology of  $\hat{\mathcal{H}}^\bullet$ , and the latter is given by the global sections of the cohomology sheaf of  $\hat{\mathcal{H}}^\bullet$  (see for example [SP, Sec. 6.8]). To prove the last part of Theorem II choose a locally finite open covering  $\mathcal{U}$  of  $X$  such that each element  $U \in \mathcal{U}$  is relatively compact and let  $(\varphi_U)_{U \in \mathcal{U}}$  be a subordinate partition of unity by smooth functions on  $X$ . Then the Čech double complex  $\check{C}_{\mathcal{U}}^\bullet(\mathcal{H}^\bullet)$  collapses at the  $E_1$  term, hence its cohomology can be computed by the cohomology of  $\check{Z}_{\mathcal{U}}^{\bullet,0}(\mathcal{H}^\bullet)$ . By assumption on  $\mathcal{U}$  there exists for every  $U \in \mathcal{U}$  a  $\varepsilon_U > 0$  such that for every  $H_U \in \mathcal{H}^p(U)$  the cochain  $\varphi_U \Psi^{p,\varepsilon_U} H_U \in \mathcal{H}^p(X)$  can be extended by zero to an element of  $\mathcal{H}^p(X)$  which we

also denote by  $\varphi_U \Psi^{p,\varepsilon_U} H_U$ . Then it is easily checked that the restriction map

$$\mathcal{H}^p(X) \rightarrow \check{Z}_{\mathcal{U}}^{p,0}(\mathcal{H}^\bullet) \subset \prod_{U \in \mathcal{U}} \mathcal{H}^p(U), \quad H \mapsto (H|_U)_{U \in \mathcal{U}}$$

is a quasi-isomorphism with quasi-inverse given by

$$\check{Z}_{\mathcal{U}}^{p,0}(\mathcal{H}^\bullet) \ni (H_U)_{U \in \mathcal{U}} \mapsto \sum_{U \in \mathcal{U}} \varphi_U \Psi^{p,\varepsilon_U} H_U \in \mathcal{H}^p(X). \quad (3.3)$$

This finishes the proof of Theorem II.

**3.2. The global quotient case.** In this part, we provide a complete proof of Theorem IV. Let  $\Gamma$  be a finite group acting on a smooth orientable manifold  $M$ . This defines a transformation groupoid  $\mathbf{G} := (\Gamma \times M \rightrightarrows M)$ . In this case, the groupoid algebra of  $\mathbf{G}$  is equal to the crossed product algebra  $\mathcal{C}_{\text{cpt}}^\infty(M) \rtimes \Gamma$ .

In [NEPFPOTA, Thm. 3], we proved that as a vector space the Hochschild cohomology of the algebra  $\mathcal{C}_{\text{cpt}}^\infty(M) \rtimes \Gamma$  is equal to

$$H^\bullet(\mathcal{C}_{\text{cpt}}^\infty(M) \rtimes \Gamma, \mathcal{C}_{\text{cpt}}^\infty(M) \rtimes \Gamma) = \left( \bigoplus_{\gamma \in \Gamma} \Gamma^\infty(\Lambda^{\bullet-\ell(\gamma)} TM^\gamma \otimes \Lambda^{\ell(\gamma)} N^\gamma) \right)^\Gamma,$$

where  $\ell(\gamma)$  is the codimension of  $M^\gamma$  in  $M$ , and  $N^\gamma$  is the normal bundle to  $M^\gamma$  in  $M$ . The main goal of this section is to compute the cup product between multi-vector fields on the inertia orbifold, that means between elements of the right hand side of the preceding equation, from the cup product on the Hochschild cohomology of the left hand side of that equation. We hereby restrict our considerations to the particular case, where  $M$  carries a  $\Gamma$ -invariant riemannian metric such that  $M$  is geodesically convex. This condition is in particular satisfied for a linear  $\Gamma$ -representation space carrying a  $\Gamma$ -invariant scalar product. Theorem IV can therefore be immediately reduced to the case considered in the following by Theorems II and III and the slice theorem.

In the first part of our construction, we outline how to determine the Hochschild cohomology as a vector space. To this end we construct two cochain maps  $L$  and  $T$  between the Hochschild cochain complex and the space of sections of multi-vector fields on the inertia orbifold. These two cochain maps are actually quasi-isomorphisms. The map  $L$  has already been constructed in [NEPFPOTA], the map  $T$  in [HATA]. In the second part of our construction, we will use the cochain maps  $T$  and  $L$  to compute the cup product.

**3.2.1. The cochain map  $L$ .** Following [NEPFPOTA, Theorem 3.1] we construct

$$L : C^\bullet(\mathcal{C}_{\text{cpt}}^\infty(M) \rtimes \Gamma, \mathcal{C}_{\text{cpt}}^\infty(M) \rtimes \Gamma) \longrightarrow \left( \bigoplus_{\gamma \in \Gamma} \Gamma^\infty(\Lambda^{\bullet-\ell(\gamma)} TM^\gamma \otimes \Lambda^{\ell(\gamma)} N^\gamma) \right)^\Gamma.$$

The map  $L$  is the composition of three cochain maps  $L_1$ ,  $L_2$  and  $L_3$  defined in the following.

To define the first map  $L_1$  recall that  $\Gamma$  acts on  $F \in C^k(\mathcal{C}_{\text{cpt}}^\infty(M), \mathcal{C}_{\text{cpt}}^\infty(M) \rtimes \Gamma)$  by

$$\gamma F := \left( \mathcal{C}_{\text{cpt}}^\infty(M)^{\hat{\otimes} k} \ni f_1 \otimes \cdots \otimes f_k \mapsto \delta_\gamma \cdot F(\gamma^{-1}(f_1) \otimes \cdots \otimes \gamma^{-1}(f_k)) \cdot \delta_{\gamma^{-1}} \right).$$

Given  $f \in \mathcal{C}_{\text{cpt}}^\infty(M)$  and  $\gamma \in \Gamma$  we hereby (and in the following) use the notation  $f\delta_\gamma$  for the function in  $\mathcal{C}_{\text{cpt}}^\infty(M) \rtimes \Gamma$  which maps  $(\sigma, p) \in \Gamma \times M$  to  $f(\gamma p)$ , if  $\sigma = \gamma$ ,

and to 0 else. Now we put

$$\begin{aligned} L_1 : C^k(\mathcal{C}_{\text{cpt}}^\infty(M) \rtimes \Gamma, \mathcal{C}_{\text{cpt}}^\infty(M) \rtimes \Gamma) &\longrightarrow C^k(\mathcal{C}_{\text{cpt}}^\infty(M), \mathcal{C}_{\text{cpt}}^\infty(M) \rtimes \Gamma)^\Gamma, \\ F &\longmapsto L_1 F := \left( \mathcal{C}_{\text{cpt}}^\infty(M)^{\hat{\otimes} k} \ni f_1 \otimes \cdots \otimes f_k \longmapsto F(f_1 \delta_e \otimes \cdots \otimes f_k \delta_e) \right). \end{aligned}$$

Next we explain how the map  $L_2$  is constructed. It has the following form:

$$L_2 : C^\bullet(\mathcal{C}_{\text{cpt}}^\infty(M), \mathcal{C}_{\text{cpt}}^\infty(M) \rtimes \Gamma)^\Gamma \longrightarrow \left( \bigoplus_{\gamma \in \Gamma} \Gamma^\infty(\Lambda^\bullet T|_{M^\gamma} M) \right)^\Gamma,$$

where  $T|_{M^\gamma} M$  is the restriction of the vector bundle  $TM$  to  $M^\gamma$ , and where the differential on the complex  $\bigoplus_{\gamma \in \Gamma} \Gamma^\infty(\Lambda^\bullet T|_{M^\gamma} M)$  is given by the  $\wedge$ -product with a nowhere vanishing vector field  $\kappa$  on  $M$  which we define later. Actually, we will define a  $\Gamma$ -equivariant chain map  $L_2$  slight more general than what is stated above, namely a map

$$L_2 : C^\bullet(\mathcal{C}_{\text{cpt}}^\infty(M), \mathcal{C}_{\text{cpt}}^\infty(M) \rtimes \Gamma) \longrightarrow \bigoplus_{\gamma \in \Gamma} \Gamma^\infty(\Lambda^\bullet T|_{M^\gamma} M).$$

As a  $\mathcal{C}_{\text{cpt}}^\infty(M)$ - $\mathcal{C}_{\text{cpt}}^\infty(M)$  bimodule,  $\mathcal{C}_{\text{cpt}}^\infty(M) \rtimes \Gamma$  has a natural splitting into a direct sum of submodules  $\bigoplus_{\gamma \in \Gamma} \mathcal{C}_{\text{cpt}}^\infty(M)_\gamma$ . Accordingly, the Hochschild cochain complex  $C^\bullet(\mathcal{C}_{\text{cpt}}^\infty(M), \mathcal{C}_{\text{cpt}}^\infty(M) \rtimes \Gamma)$  naturally splits as a direct sum

$$\bigoplus_{\gamma \in \Gamma} C^\bullet(\mathcal{C}_{\text{cpt}}^\infty(M), \mathcal{C}_{\text{cpt}}^\infty(M_\gamma)).$$

Therefore, to define the map  $L_2$ , it is enough to consider each single map

$$L_2^\gamma : C^\bullet(\mathcal{C}_{\text{cpt}}^\infty(M), \mathcal{C}_{\text{cpt}}^\infty(M_\gamma)) \longrightarrow \Gamma^\infty(\Lambda^\bullet T|_{M^\gamma} M).$$

In the following we use ideas from the paper [CO] to construct  $L_2^\gamma$ . To this end let  $\text{pr}_2 : M \times M \rightarrow M$  be the projection onto the second factor of  $M \times M$ , and  $\xi$  the vector field on  $M \times M$  which maps  $(x_1, x_2)$  to  $\exp_{x_2}^{-1}(x_1)$ . By our assumptions on the riemannian metric on  $M$  the vector field  $\xi$  is well-defined and  $\Gamma$ -invariant. According to [CO, Lemma 44], the complex  $K^\bullet = (\Gamma_{\text{cpt}}^\infty(\text{pr}_2^*(\Lambda^\bullet T^* M)), \xi_\perp)$  defines a projective resolution of  $\mathcal{C}_{\text{cpt}}^\infty(M)$ . Essentially, it is a Koszul resolution for  $\mathcal{C}_{\text{cpt}}^\infty(M)$ . Following Appendix A.4, we use the resolution  $K^\bullet$  to determine the Hochschild cohomology  $H^\bullet(\mathcal{C}_{\text{cpt}}^\infty(M), \mathcal{C}_{\text{cpt}}^\infty(M_\gamma))$  as the cohomology of the cochain complex  $\text{Hom}_{\mathcal{C}_{\text{cpt}}^\infty(M^2)}(K^\bullet, \mathcal{C}_{\text{cpt}}^\infty(M_\gamma))$ . By [CO] the following chain map is a quasi-isomorphism between the resolution  $K^\bullet$  and the Bar resolution  $\text{Bar}_\bullet(\mathcal{C}_{\text{cpt}}^\infty(M))$ :

$$\begin{aligned} \Phi : K^k &\rightarrow \text{Bar}_k(\mathcal{C}_{\text{cpt}}^\infty(M)) = \mathcal{C}_{\text{cpt}}^\infty(M^{k+2}), \\ \omega &\longmapsto \left( M^{k+2} \ni (a, b, x_1, \dots, x_k) \longmapsto \langle \xi(x_1, b) \wedge \cdots \wedge \xi(x_k, b), \omega(a, b) \rangle \right). \end{aligned}$$

Hence the dual of the chain map  $\Psi$  defines a quasi-isomorphism

$$\Phi^* : \text{Hom}_{\mathcal{C}_{\text{cpt}}^\infty(M^2)}(\mathcal{C}_{\text{cpt}}^\infty(M^{k+2}), \mathcal{C}_{\text{cpt}}^\infty) \rightarrow \text{Hom}_{\mathcal{C}_{\text{cpt}}^\infty(M^2)}(K^k, \mathcal{C}_{\text{cpt}}^\infty(M)).$$

Now consider the embedding  $\Delta_\gamma : M \rightarrow M \times M$  given by  $\Delta_\gamma(x) = (\gamma(x), x)$ . According to [NEPFOPTA, Sec. 3, Step 4], the map

$$\eta : \Gamma^\infty(\Lambda^k TM) \rightarrow \text{Hom}_{\mathcal{C}_{\text{cpt}}^\infty(M^2)}(K^k, \mathcal{C}_{\text{cpt}}^\infty(M)), \quad \tau \longmapsto \eta(\tau),$$

defined by  $\eta(\tau)(\omega) = \langle \Delta_\gamma^* \omega, \tau \rangle$  for  $\omega \in \Gamma_{\text{cpt}}^\infty \text{pr}_2^*(\Lambda^\bullet T^* M)$  is an isomorphism. So finally we can define for  $F \in C^k(\mathcal{C}_{\text{cpt}}^\infty(M), \mathcal{C}_{\text{cpt}}^\infty(M)_\gamma)$  an element  $L_2^\gamma(F) \in \Gamma^\infty(\Lambda^k TM)$  by

$$L_2^\gamma(F) = \eta^{-1} \Phi^* \Psi^{k,\varepsilon}(F),$$

where we have used the cut-off cochain map  $\Psi^{k,\varepsilon}$  defined above. Thus we obtain an isomorphism of complexes

$$L_2^\gamma : (C^k(\mathcal{C}_{\text{cpt}}^\infty(M), \mathcal{C}_{\text{cpt}}^\infty(M)), b) \rightarrow (\Lambda^\bullet TM, \kappa_\gamma \wedge),$$

where  $\kappa_\gamma$  is the restriction of the vector field  $\xi$  on  $M \times M$  to the  $\gamma$ -diagonal  $\Delta_\gamma$ .

In the case, where  $M$  is a (finite dimensional) vector space  $V$  with a linear  $\Gamma$ -action, we can write down  $L_2^\gamma$  explicitly. Choose coordinates  $x^i$ ,  $i = 1, \dots, \dim V$  on  $V$ . Then the vector field  $\xi$  on  $V \times V$  can be written as

$$\xi(x_1, x_2) = \sum_i (x_1 - x_2)^i \frac{\partial}{\partial x_2^i}. \quad (3.4)$$

Moreover,  $L_2^\gamma(F)$  is given as follows:

$$\begin{aligned} L_2^\gamma(F)(x) &= \sum_{i_1, \dots, i_k} F(\Psi_{k,\varepsilon}(-, x_1, \dots, x_k) \\ &\quad \cdot \langle \xi(x_1, x) \wedge \dots \wedge \xi(x_k, x), \text{pr}_2^*(dx^{i_1} \wedge \dots \wedge dx^{i_k}) \rangle)(x) \frac{\partial}{\partial x^{i_1}} \wedge \dots \wedge \frac{\partial}{\partial x^{i_k}} \\ &= \sum_{i_1, \dots, i_k} \left( \sum_{\sigma \in S_k} (-1)^\sigma F((x_{\sigma(1)} - x)^{i_1} \dots (x_{\sigma(k)} - x)^{i_k}) \right)(x) \frac{\partial}{\partial x^{i_1}} \wedge \dots \wedge \frac{\partial}{\partial x^{i_k}}, \end{aligned} \quad (3.5)$$

where  $x \in V$ , and where we have identified  $F$  with a bounded linear map from  $\mathcal{C}_{\text{cpt}}^\infty(V^k)$  to  $\mathcal{C}^\infty(V)$ .

To define  $L_3$  we construct for each  $\gamma \in \Gamma$  a localization map to the fixed point sub manifold  $M^\gamma$ . Recall that we have chosen a  $\Gamma$ -invariant complete riemannian metric on  $M$ , and consider the normal bundle  $N^\gamma$  to the embedding  $\iota_\gamma : M^\gamma \hookrightarrow M$ . The riemannian metric allows us to regard  $N^\gamma$  as a subbundle of the restricted tangent bundle  $T|_{M^\gamma} M$ . Now we denote by  $\text{pr}^\gamma$  the orthogonal projection from  $\Lambda^\bullet T|_{M^\gamma} M$  to  $\Lambda^{\bullet - \ell(\gamma)} TM^\gamma \otimes \Lambda^{\ell(\gamma)} N^\gamma$ . The chain map

$$L_3 : \left( \bigoplus_{\gamma \in \Gamma} \Gamma^\infty(\Lambda^\bullet TM), \kappa \wedge - \right) \longrightarrow \bigoplus_{\gamma \in \Gamma} (\Gamma^\infty(\Lambda^{\bullet - \ell(\gamma)} TM^\gamma \otimes \Lambda^{\ell(\gamma)} N^\gamma), 0).$$

is then constructed as the sum of the maps  $L_3^\gamma$  defined by

$$L_3^\gamma(X) = \text{pr}^\gamma(X|_{M^\gamma}) \quad \text{for } X \in \Gamma^\infty(\Lambda^\bullet TM).$$

In [NEPFPOTA, Sec. 3] we proved that  $L = L_3 \circ L_2 \circ L_1$  is a quasi-isomorphism of cochain complexes.

**3.2.2. The chain map  $T$ .** Under the assumption that  $M$  is a linear  $\Gamma$ -representation space  $V$  we construct in this section a quasi-inverse

$$T : \left( \bigoplus_{\gamma \in \Gamma} \Gamma^\infty(\Lambda^{\bullet - \ell(\gamma)} TM^\gamma \otimes \Lambda^{\ell(\gamma)} N^\gamma) \right)^\Gamma \longrightarrow C^\bullet(\mathcal{C}_{\text{cpt}}^\infty(M) \rtimes \Gamma, \mathcal{C}_{\text{cpt}}^\infty(M) \rtimes \Gamma).$$

to the above cochain map  $L$ . To this end we first recall the construction of the *normal twisted cocycle*

$$\Omega_\gamma \in C^{\ell(\gamma)}(\mathcal{C}_{\text{cpt}}^\infty(V), \mathcal{C}_{\text{cpt}}^\infty(V)_\gamma)$$

from [HATA]. Since  $\Gamma$  acts linearly on  $V$ ,  $V^\gamma$  is a linear subspace of  $U$  and has a normal space  $V^\perp$ . Let  $x_i$ ,  $i = 1, \dots, n - \ell(\gamma)$ , be coordinates on  $V^\gamma$ , and  $y_j$ ,  $j = 1, \dots, \ell(\gamma)$ , coordinates on  $V^\perp$ . We write  $\tilde{y} = \gamma y$ , and for every  $\sigma \in S_{\ell(\gamma)}$  we introduce the following vectors in  $V^\perp$ :

$$\begin{aligned} z^0 &= (y_1, \dots, y_{\ell(\gamma)}), & z^1 &= (y_1, \dots, \tilde{y}_{\sigma(1)}, \dots, y_{\ell(\gamma)}), \\ z^2 &= (y_1, \dots, \tilde{y}_{\sigma(1)}, \dots, \tilde{y}_{\sigma(2)}, \dots, y_{\ell(\gamma)}), & \dots & \\ z^{\ell(\gamma)-1} &= (\tilde{y}_1, \dots, y_{\sigma(\ell(\gamma))}, \dots, \tilde{y}_{\ell(\gamma)}), & z^{\ell(\gamma)} &= (\tilde{y}_1, \dots, \tilde{y}_{\ell(\gamma)}). \end{aligned}$$

Then we define a cochain  $\Omega_\gamma \in C^{\ell(\gamma)}(\mathcal{C}_{\text{cpt}}^\infty(V), \mathcal{C}_{\text{cpt}}^\infty(V)_\gamma)$  as follows:

$$\Omega_\gamma(f_1, \dots, f_{\ell(\gamma)})(x, y) := \frac{1}{\ell(\gamma)!} \cdot \sum_{\sigma \in S_{\ell(\gamma)}} \frac{(f_1(x, z^0) - f_1(x, z^1))(f_2(x, z^1) - f_2(x, z^2)) \cdots (f_{\ell(\gamma)}(x, z^{n-1}) - f_{\ell(\gamma)}(x, z^n))}{(y_1 - \tilde{y}_1) \cdots (y_{\ell(\gamma)} - \tilde{y}_{\ell(\gamma)})},$$

where  $f_1, \dots, f_{\ell(\gamma)} \in \mathcal{C}_{\text{cpt}}^\infty(V)$ ,  $x \in V^\gamma$  and  $y \in V^\perp$ . It is straightforward to check that  $\Omega_\gamma$  is a cocycle in  $C^{\ell(\gamma)}(\mathcal{C}_{\text{cpt}}^\infty(V), \mathcal{C}_{\text{cpt}}^\infty(V)_\gamma)$  indeed. Now define the cochain map

$$T_1 : \left( \bigoplus_{\gamma \in \Gamma} \Gamma^\infty(\Lambda^{\bullet-\ell(\gamma)}TV^\gamma \otimes \Lambda^{\ell(\gamma)}N^\gamma) \right)^\Gamma \longrightarrow C^\bullet(\mathcal{C}_{\text{cpt}}^\infty(V), \mathcal{C}_{\text{cpt}}^\infty(V) \rtimes \Gamma)^\Gamma$$

as the sum of maps

$$T_1^\gamma : \Gamma^\infty(\Lambda^{k-\ell(\gamma)}TV^\gamma \otimes \Lambda^{\ell(\gamma)}N^\gamma) \longrightarrow C^k(\mathcal{C}_{\text{cpt}}^\infty(V), \mathcal{C}_{\text{cpt}}^\infty(V)_\gamma),$$

defined by

$$T_1^\gamma(X \otimes Y_\gamma) = Y_\gamma(y_1, \dots, y_{\ell(\gamma)}) X \# \Omega_\gamma,$$

where  $X \in \Gamma^\infty(\Lambda^{k-\ell(\gamma)}TV^\gamma)$ ,  $Y_\gamma \in \Gamma^\infty(\Lambda^{\ell(\gamma)}N^\gamma)$ , and where  $X \# \Omega_\gamma(f_1, \dots, f_k)$  is equal to

$$X(f_1, \dots, f_{k-\ell(\gamma)}) \Omega_\gamma(f_{k-\ell(\gamma)+1}, \dots, f_{\ell(\gamma)}).$$

Observe hereby that  $X$  to act on  $f_1, \dots, f_{k-\ell(\gamma)}$ , we need to use a  $\Gamma$ -invariant connection, i.e. the Levi-Civita connection of the invariant metric, on the normal bundle of  $V^\gamma$  in  $V$  to lift  $X$  to a vector field on  $V$ .

The map

$$T : \left( \bigoplus_{\gamma \in \Gamma} \Gamma^\infty(\Lambda^{\bullet-\ell(\gamma)}TV^\gamma \otimes \Lambda^{\ell(\gamma)}N^\gamma) \right)^\Gamma \longrightarrow C^\bullet(\mathcal{C}_{\text{cpt}}^\infty(V) \rtimes \Gamma, \mathcal{C}_{\text{cpt}}^\infty(V) \rtimes \Gamma)$$

is now written as the composition of  $T_1$  and  $T_2$ , where  $T_2$  is the standard cochain map from the Eilenberg-Zilber theorem:

$$T_2 : C^\bullet(\mathcal{C}_{\text{cpt}}^\infty(V), \mathcal{C}_{\text{cpt}}^\infty(V) \rtimes \Gamma)^\Gamma \longrightarrow C^\bullet(\mathcal{C}_{\text{cpt}}^\infty(V) \rtimes \Gamma, \mathcal{C}_{\text{cpt}}^\infty(V) \rtimes \Gamma).$$

More precisely, for  $F \in C^k(\mathcal{C}_{\text{cpt}}^\infty(V), \mathcal{C}_{\text{cpt}}^\infty(V) \rtimes \Gamma)$  one has

$$T_2(F)(f_1 \delta_{\gamma_1}, \dots, f_k \delta_{\gamma_k}) = F(f_1, \gamma_1(f_2), \dots, \gamma_1 \cdots \gamma_{k-1}(f_k)) \delta_{\gamma_1 \cdots \gamma_k}.$$

The following result then holds for the composition  $T = T_2 \circ T_1$ . Its proof is performed by a straightforward check (cf. [HATA, Sec. 2] for some more details).



**Theorem 3.1.** *Let  $V$  be a finite dimensional real linear  $\Gamma$ -representation space. Then the cochain maps  $L$  and  $T$  defined above satisfy  $L \circ T = \text{id}$ . In particular,  $T$  is a quasi-inverse to  $L$ .*

By the above considerations one concludes that there is an isomorphism of vector spaces between the Hochschild cohomology of  $\mathcal{C}_{\text{cpt}}^\infty(M) \rtimes \Gamma$  and the space of smooth sections of alternating multi-vector fields on the corresponding inertia orbifold.

3.2.3. *The cup product.* In this part, we use the above constructed maps to compute the cup product on the Hochschild cohomology of the algebra  $\mathcal{C}_{\text{cpt}}^\infty(M) \rtimes \Gamma$ . By proving the following proposition, we will complete proof of Theorem IV.

**Proposition 3.2.** *For every smooth  $\Gamma$ -manifold  $M$  the cup product on the Hochschild cohomology  $H^\bullet(\mathcal{C}_{\text{cpt}}^\infty(M) \rtimes \Gamma, \mathcal{C}_{\text{cpt}}^\infty(M) \rtimes \Gamma) \cong \left( \bigoplus_{\gamma \in \Gamma} \Gamma^\infty(\Lambda^{\bullet-\ell(\gamma)} TM^\gamma \otimes \Lambda^{\ell(\gamma)} N^\gamma) \right)^\Gamma$  is given for two cochains*

$$\xi = (\xi_\alpha)_{\alpha \in \Gamma}, \eta = (\eta_\beta)_{\beta \in \Gamma} \in \left( \bigoplus_{\gamma \in \Gamma} \Gamma^\infty(\Lambda^{\bullet-\ell(\gamma)} TM^\gamma \otimes \Lambda^{\ell(\gamma)} N^\gamma) \right)^\Gamma$$

as the cochain  $\xi \cup \eta$  with components

$$(\xi \cup \eta)_\gamma = \sum_{\alpha\beta=\gamma, \ell(\alpha)+\ell(\beta)=\ell(\gamma)} \xi_\alpha \wedge \eta_\beta.$$

*Proof.* It suffices to prove the claim under the assumption that  $M$  is a linear  $\Gamma$ -representation space  $V$ . Then we have the above defined quasi-inverse  $T$  to the cochain map  $L$  at our disposal. To compute  $\xi \cup \eta$  we thus have to determine the multivector field  $L(T(\xi) \cup T(\eta))$ . Since  $L$  is the composition of  $L_1$ ,  $L_2$ , and  $L_3$ , we compute  $L_1(T(\xi) \cup T(\eta))$  first. Recall that the cochain  $L_1(T(\xi) \cup T(\eta)) \in C^{p+q}(\mathcal{C}_{\text{cpt}}^\infty(V), \mathcal{C}_{\text{cpt}}^\infty(V)_\gamma)$  is defined by

$$L_1(T(\xi) \cup T(\eta))(f_1, \dots, f_{p+q}) = \sum_{\alpha\beta=\gamma} T_1^\alpha(\xi_\alpha)(f_1, \dots, f_p) \alpha(T_1^\beta(\eta_\beta)(f_{p+1}, \dots, f_{p+q})), \quad (3.6)$$

where  $f_1, \dots, f_{p+q} \in \mathcal{C}_{\text{cpt}}^\infty(V)$ , Recall also that the cochain map

$$L_2 : C^k(\mathcal{C}_{\text{cpt}}^\infty(V), \mathcal{C}_{\text{cpt}}^\infty(V) \rtimes \Gamma) \rightarrow \bigoplus_{\gamma \in \Gamma} \Gamma^\infty(\Lambda^k T_{V^\gamma} V)$$

essentially is the anti-symmetrization of the linear terms of a cochain. Hence  $L_2(L_1(T(\xi) \cup T(\eta)))$  is equal to

$$\sum_{\alpha\beta=\gamma} L_2^\alpha(T_1^\alpha(\xi_\alpha)) \wedge \alpha(L_2^\beta(T_2^\beta(\eta_\beta))).$$

To compute  $L_2(L_1(T(\xi) \cup T(\eta)))$ , it thus suffices to determine

$$L_2^\alpha(T_1^\alpha(\xi_\alpha)) \wedge \alpha(L_2^\beta(T_2^\beta(\eta_\beta))),$$

which defines a  $(p+q)$ -multivector field  $Z$  supported in a neighborhood of  $V^\alpha \cap V^\beta$  in  $V$ . By Equation (3.5), one observes that when the restrictions of the normal bundles  $N^\alpha$  and  $N^\beta$  to  $V^\alpha \cap V^\beta$  have a nontrivial intersection, for instance along a coordinate  $x^0$ , then in Equation (3.6), the derivative  $\frac{\partial}{\partial x^0}$  shows up in both  $L_2^\alpha(T_1^\alpha(\xi_\alpha))$  and  $\alpha(L_2^\beta(T_2^\beta(\eta_\beta)))$ . Therefore their wedge product then has to vanish. This argument shows that the nontrivial contribution of the cup product  $\xi \cup \eta$  comes from those

components, where  $N^\alpha$  and  $N^\beta$  do not have a nontrivial intersection. By the following Lemma (3.3) this implies that at a point  $x \in V^\alpha \cap V^\beta$  with  $N_x^\alpha \cap N_x^\beta = \{0\}$  one has  $T_x V^\alpha + T_x V^\beta = T_x V$  and therefore  $V^{\alpha\beta} = V^\alpha \cap V^\beta$ . This last condition by Lemma 5.6 is equivalent to  $\ell(\alpha) + \ell(\beta) = \ell(\gamma)$ . When  $V^\alpha \cap V^\beta = V^{\alpha\beta}$  and  $N^\alpha \cap N^\beta = \{0\}$ , one computes  $L_3(Z)$  using the definition of  $L_3$  and obtains

$$\begin{aligned} L(T(\xi) \circ T(\eta)) &= \sum_{\substack{\alpha\beta=\gamma \\ \ell(\alpha)+\ell(\beta)=\ell(\beta)}} L_3\left(L_2^\alpha(T_1^\alpha(\xi_\alpha)) \wedge \alpha(L_2^\beta(T_2^\beta(\eta_\beta)))\right) \\ &= \sum_{\substack{\alpha\beta=\gamma \\ \ell(\alpha)+\ell(\beta)=\ell(\beta)}} \xi_\alpha \cup \alpha(\eta_\beta). \end{aligned}$$

Note that on  $V^\alpha \cap V^\beta = V^{\alpha\beta}$  one has  $\alpha(\eta_\beta) = \eta_\beta$ . This finishes the proof of the claim.  $\square$

To end this section we finally show a lemma which already has been used in the proof of the preceding result.

**Lemma 3.3.** *Let  $\alpha, \beta$  be two linear automorphisms on the real vector space  $V$ . Let  $\langle -, - \rangle$  be a scalar product preserved by  $\alpha$  and  $\beta$  and let  $V^\alpha, V^\beta$  be the corresponding fixed point subspaces. If  $V^\alpha + V^\beta = V$ , then  $V^\alpha \cap V^\beta = V^{\alpha\beta}$ .*

*Proof.* Obviously,  $V^\alpha \cap V^\beta \subset V^{\alpha\beta}$ . It is enough to show that if  $v \in V^{\alpha\beta}$ , then  $v \in V^\alpha \cap V^\beta$ .

Since  $v \in V^{\alpha\beta}$ , one has  $\alpha\beta(v) = v$ , hence  $\beta(v) = \alpha^{-1}(v)$ . Define  $w = \beta(v) - v = \alpha^{-1}(v) - v$ . We prove that  $w$  is orthogonal to both  $V^\alpha$  and  $V^\beta$ . For every  $u \in V^\alpha$  one has

$$\begin{aligned} \langle w, u \rangle &= \langle \alpha^{-1}(v) - v, u \rangle = \langle \alpha^{-1}(v), u \rangle - \langle v, u \rangle \\ &= \langle v, \alpha(u) \rangle - \langle v, u \rangle = \langle v, u \rangle - \langle v, u \rangle = 0, \end{aligned}$$

where in the first equality of the second line we have used the fact that  $\alpha$  preserves the metric  $\langle -, - \rangle$ , and in the second equality of the second line we have used that  $u$  is  $\alpha$ -invariant. Therefore one concludes that  $w$  is orthogonal to  $V^\alpha$ . Likewise one shows that  $w$  is orthogonal to  $V^\beta$ . Therefore,  $w$  is orthogonal to  $V^\alpha + V^\beta = V$ , hence  $w$  has to be 0. This implies that  $v$  is invariant under both  $\alpha$  and  $\beta$ .  $\square$

#### 4. CUP PRODUCT ON THE HOCHSCHILD COHOMOLOGY OF THE DEFORMED CONVOLUTION ALGEBRA

In this section we compute the Hochschild cohomology together with the cup product of a formal deformation of the convolution algebra of a proper étale groupoid  $\mathbf{G}$ . For this we assume that the orbifold  $X$  is symplectic or in other words that  $\mathbf{G}_0$  carries a  $\mathbf{G}$ -invariant symplectic form  $\omega$ , i.e., satisfying  $s^*\omega = t^*\omega$ . We let  $\mathcal{A}^\hbar$  be a  $\mathbf{G}$ -invariant formal deformation quantization of  $\mathcal{A} = \mathcal{C}_{\mathbf{G}_0}^\infty$ , where the deformation parameter is denoted by  $\hbar$ . This means that  $\mathcal{A}^\hbar$  is a  $\mathbf{G}$ -sheaf over  $\mathbf{G}_0$  and the associated crossed product  $\mathcal{A}^\hbar \rtimes \mathbf{G}$  is a formal deformation of the convolution algebra, cf. [TA].

As a formal deformation the algebra  $\mathcal{A}^\hbar \rtimes \mathbf{G}$  is filtered by powers of  $\hbar$ , i.e.,  $F_k(\mathcal{A}^\hbar \rtimes \mathbf{G}) := \hbar^k(\mathcal{A}^\hbar \rtimes \mathbf{G})$  and we have

$$F_k(\mathcal{A}^\hbar \rtimes \mathbf{G}) / F_{k-1}(\mathcal{A}^\hbar \rtimes \mathbf{G}) \cong \mathcal{A} \rtimes \mathbf{G}. \quad (4.1)$$

As usual, the Hochschild cochain complex is defined by

$$C^\bullet(\mathcal{A}^\hbar \rtimes \mathbf{G}, \mathcal{A}^\hbar \rtimes \mathbf{G}) := \mathrm{Hom}_{\mathbb{C}[[\hbar]]}((\mathcal{A}^\hbar \rtimes \mathbf{G})^{\hat{\otimes}^\bullet}, \mathcal{A}^\hbar \rtimes \mathbf{G}),$$

with differential  $\beta$  defined with respect to the deformed convolution algebra. The justification for this definition comes from Proposition A.8, which also shows that the cup-product, defined by (A.6), extends this complex to a differential graded algebra (DGA). The  $\hbar$ -adic filtration of  $\mathcal{A}^\hbar \rtimes \mathbf{G}$  above induces a complete and exhaustive filtration of the Hochschild complex. Since the product in  $\mathcal{A}^\hbar \rtimes \mathbf{G}$  is a formal deformation of the convolution product, cf. equation (4.1), the associated spectral sequence has  $E_0$ -term just the undeformed Hochschild complex of the convolution algebra.

This has the following useful consequence that we will use several times in the course of the argument: Suppose that  $A_1^\hbar$  and  $A_2^\hbar$  are formal deformations of the algebras  $A_1$  and  $A_2$ , and

$$f : C^\bullet(A_1^\hbar, A_1^\hbar) \rightarrow C^\bullet(A_2^\hbar, A_2^\hbar)$$

is a morphism of filtered complexes. Then  $f$  is a quasi-isomorphism, if it induces an isomorphism at level  $E_1$ . The proof of this statement is a direct application of the Eilenberg–Moore spectral sequence comparison theorem, cf. [WE, Thm. 5.5.11].

Let us apply this to the following situation: consider the following subspace of the space of Hochschild cochains on  $\mathcal{A}^\hbar \rtimes \mathbf{G}$ :

$$C_{\mathrm{loc}}^k(\mathcal{A}^\hbar \rtimes \mathbf{G}, \mathcal{A}^\hbar \rtimes \mathbf{G}) := \left\{ \Psi \in C^k(\mathcal{A}^\hbar \rtimes \mathbf{G}, \mathcal{A}^\hbar \rtimes \mathbf{G}) \mid \pi s(\mathrm{supp} \Psi(a_1, \dots, a_k)) \subset \bigcap_{i=1}^k \pi s(\mathrm{supp} a_i) \right\}.$$

Here  $\mathrm{supp}(a)$  denotes the support of a function. These are the *local* cochains with respect to the underlying orbifold  $X$ . Notice that because of the convolution nature of the algebra  $\mathcal{A}^\hbar \rtimes \mathbf{G}$ , which involves the action of  $\mathbf{G}$ , it is unreasonable to require locality with respect to  $\mathbf{G}_0$  or  $\mathbf{G}_1$ . The important point now is:

**Proposition 4.1.** *The complex of local Hochschild cochains  $C_{\mathrm{loc}}^\bullet(\mathcal{A}^\hbar \rtimes \mathbf{G}, \mathcal{A}^\hbar \rtimes \mathbf{G})$  is a subcomplex of  $C^\bullet(\mathcal{A}^\hbar \rtimes \mathbf{G}, \mathcal{A}^\hbar \rtimes \mathbf{G})$ , and the canonical inclusion map is a quasi-isomorphism preserving cup-products.*

*Proof.* The orbifold  $X$  can be identified with the quotient space  $\mathbf{G}_0/\mathbf{G}_1$ . The deformed convolution product on  $\mathcal{A}^\hbar \rtimes \mathbf{G}$  involves the local product on the  $\mathbf{G}$ -sheaf  $\mathcal{A}^\hbar$  on  $\mathbf{G}_0$  and the groupoid action, and the product in turn defines the Hochschild complex as well as the cup-product. With this, it is easy to check that the locality condition on  $X$  is compatible with both the differential and the product.

To show that the canonical inclusion is a quasi-isomorphism, first observe that the map clearly respects the  $\hbar$ -adic filtration. It follows from Theorem IV that for the undeformed convolution algebra the local Hochschild cochain complex computes the same cohomology, since the vector fields clearly satisfy the locality condition. Therefore the inclusion map is a quasi-isomorphism at the  $E_0$ -level, and by the above, a quasi-isomorphism in general.  $\square$

**Remark 4.2.** In the following we will often consider the ring extension

$$\mathcal{A}^{((\hbar))} := \mathcal{A}^\hbar \hat{\otimes}_{\mathbb{C}[[\hbar]]} \mathbb{C}((\hbar)),$$

where  $\mathbb{C}((\hbar))$  denotes the field of formal Laurent series in  $\hbar$ , and will then regard  $\mathcal{A}^{((\hbar))}$  as an algebra over the ground field  $\mathbb{C}((\hbar))$ . By standard results from Hochschild (co)homology theory one knows that

$$H^\bullet(\mathcal{A}^{((\hbar))} \rtimes \mathfrak{G}, \mathcal{A}^{((\hbar))} \rtimes \mathfrak{G}) = H^\bullet(\mathcal{A}^\hbar \rtimes \mathfrak{G}, \mathcal{A}^\hbar \rtimes \mathfrak{G}) \hat{\otimes}_{\mathbb{C}[[\hbar]]} \mathbb{C}((\hbar)). \quad (4.2)$$

In the remainder of this article we will tacitly make use of this fact.

**4.1. Reduction to the Čech complex.** As in the undeformed case, the idea is to use a Čech complex to compute the cohomology. For  $U \subset X$ , introduce

$$\mathcal{H}_{\mathfrak{G}, \hbar}^k(U) := \text{Hom}_{\mathbb{C}[[\hbar]]}(\Gamma_{\text{cpt}}^\infty(U, \tilde{\mathcal{A}}_{\text{fc}}^\hbar)^{\hat{\otimes} k}, \tilde{\mathcal{A}}_{\text{fc}}^\hbar(U)).$$

This is clearly a deformation of the sheaf  $\mathcal{H}_{\mathfrak{G}}^\bullet$ . The sheaf  $\mathcal{H}_{\mathfrak{G}, \text{loc}, \hbar}^k$  is similarly defined. As in the undeformed case, we now have an obvious map

$$I_{\text{loc}}^\hbar : C_{\text{loc}}^\bullet(\mathcal{A}^\hbar \rtimes \mathfrak{G}, \mathcal{A}^\hbar \rtimes \mathfrak{G}) \rightarrow \mathcal{H}_{\mathfrak{G}, \text{loc}, \hbar}^\bullet(X).$$

**Proposition 4.3.** *The map  $I_{\text{loc}}^\hbar$  is a quasi-isomorphism of DGA's.*

*Proof.* By the Eilenberg-Moore spectral sequence, this follows from Theorem I.  $\square$

**4.2. Twisted cocycles on the formal Weyl algebra.** Our aim is to reduce the computation of Hochschild to sheaf cohomology. The present section can be viewed as a stalkwise computation. Let  $V = \mathbb{R}^{2n}$  equipped with the standard symplectic form  $\omega$ , and suppose that  $\Gamma \subset Sp(V, \omega)$  is a finite group acting on  $V$  by linear symplectic transformations. The action of an element  $\gamma \in \Gamma$  induces a decomposition  $V = V^\gamma \oplus V^\perp$  into symplectic subspaces. Put

$$\ell(\gamma) := \dim(V^\perp) = \dim(V) - \dim(V^\gamma).$$

Let  $\mathbb{W}_{2n}$  be the formal Weyl algebra, i.e.,  $\mathbb{W}_{2n} = \mathbb{C}[[y_1, \dots, y_n]][[\hbar]]$  equipped with the Moyal product

$$f \star g = \sum_{k=0}^{\infty} \sum_{\substack{1 \leq i_1, \dots, i_k \leq n \\ 1 \leq j_1, \dots, j_k \leq n}} \Pi^{i_1 j_1} \dots \Pi^{i_k j_k} \frac{\hbar^k}{k!} \frac{\partial^k f}{\partial y_1 \dots \partial y_k} \frac{\partial^k g}{\partial y_1 \dots \partial y_k},$$

where  $\Pi := \omega^{-1}$  is the Poisson tensor associated to  $\omega$ . With this product, the formal Weyl algebra  $\mathbb{W}_{2n}$  is a unital algebra over  $\mathbb{C}[[\hbar]]$ . It is a formal deformation of the commutative algebra  $\mathbb{C}[[y_1, \dots, y_{2n}]]$ . With an automorphism  $\gamma \in \Gamma$ , we can consider the Weyl algebra bimodule  $\mathbb{W}_{2n, \gamma}$  which equals  $\mathbb{W}_{2n}$  except for the fact that the right action of  $\mathbb{W}_{2n}$  is twisted by  $\gamma$ . With this we have:

**Proposition 4.4** (cf. [PI]). *The twisted Hochschild cohomology is given by*

$$H^k(\mathbb{W}_{2n}, \mathbb{W}_{2n, \gamma}) = \begin{cases} \mathbb{C}[[\hbar]], & \text{for } k = \ell(\gamma), \\ 0, & \text{else.} \end{cases}$$

*There exists a generator  $\Psi_\gamma$  in the reduced Hochschild complex satisfying*

$$\Psi_\gamma|_{\Delta V^*} = (\Pi_\gamma^\perp)^{\ell(\gamma)/2}.$$

*In fact,  $\Psi$  is, up to a coboundary, uniquely determined by this property.*

*Proof.* The first part of the Proposition is essentially well-known, cf. [ALFALASO, AL]. It is conveniently proved by the Koszul resolution of the Weyl algebra

$$0 \longleftarrow \mathbb{W}_{2n} \xleftarrow{\star} \mathbb{W}_{2n} \otimes \mathbb{W}_{2n}^{\text{op}} \xleftarrow{\partial} K_1 \xleftarrow{\partial} K_2 \xleftarrow{\partial} \dots$$

where

$$K_p := \mathbb{W}_{2n} \otimes \Lambda^p V^* \otimes \mathbb{W}_{2n}^{\text{op}},$$

and where the differential  $\partial : K_p \rightarrow K_{p-1}$  is defined by

$$\begin{aligned} \partial(a_1 \otimes a_2 \otimes dy_{i_1} \wedge \dots \wedge dy_{i_p}) := \\ \sum_{j=1}^p (-1)^j ((y_{i_j} \star a_1) \otimes a_2 - a_1 \otimes (a_2 \star y_{i_j})) dy_{i_1} \wedge \dots \wedge \widehat{dy}_{i_j} \wedge \dots \wedge dy_{i_p}, \end{aligned}$$

with respect to a Darboux basis of  $V$ , i.e.,  $\omega(y_i, y_{i+n}) = 1$  and zero otherwise.

To compute the Hochschild cohomology, we take  $\text{Hom}_{\mathbb{W}_{2n}}(-, \mathbb{W}_{2n, \gamma})$  to obtain the complex

$$K_\gamma^p := \Lambda^p V \otimes \mathbb{W}_{2n}, \quad (4.3)$$

with differential  $d_\gamma : K^p \rightarrow K^{p+1}$  given by

$$d_\gamma(a \otimes y_{i_1} \wedge \dots \wedge y_{i_{p+1}}) = \sum_{j=1}^{2n} (-1)^j (y_j \star a - a \star_\gamma y_j) y_j \wedge y_{i_1} \wedge \dots \wedge y_{i_p}.$$

The cohomology of this complex can easily be computed using the spectral sequence of the  $\hbar$ -adic filtration. In degree zero one finds the ordinary, i.e. commutative Koszul complex and therefore we find

$$E_1^{p,q} = \Lambda^{p+q} V^\perp.$$

The differential  $d_1 : E_1^{p,q} \rightarrow E_1^{p+1,q}$  is given by the Poisson cohomology differential, which has trivial cohomology except in maximal degree and therefore

$$E_2^{p,q} = \begin{cases} \Lambda^{\ell(\gamma)} V^\perp, & \text{for } p+q = \ell(\gamma), \\ 0, & \text{else.} \end{cases}$$

The spectral sequence degenerates at this point and the first part of the Proposition is proved. The second part is as in [PI]: the Koszul complex is naturally a subcomplex of the reduced Bar complex  $(K_\bullet, \partial) \subset (B_\bullet^{\text{red}}, b)$ , where

$$B_k^{\text{red}} = \mathbb{W}_{2n} \otimes (\mathbb{W}_{2n}/\mathbb{C}[[\hbar]])^{\otimes k} \otimes \mathbb{W}_{2n},$$

and the embedding is induced by the natural inclusion  $V^* \hookrightarrow \mathbb{W}_{2n}$  as degree one homogeneous polynomials. This leads to a natural projection

$$R_V : (C_{\text{red}}^\bullet(\mathbb{W}_{2n}, \mathbb{W}_{2n}), \beta_\gamma) \rightarrow (K^\bullet, d_\gamma)$$

given by restricting cochains to  $\Lambda V^*$ . It is easily checked that  $(\Pi_\gamma^\perp)^{\ell(\gamma)/2}$  defines a cocycle of degree  $\ell(\gamma)$  in the complex  $(K^\bullet, d_\gamma)$ , and the statement follows.  $\square$

Let  $\mu_\gamma : K_\gamma^\bullet \rightarrow \mathbb{C}[[\hbar]]$  be the morphism defined by

$$\mu_\gamma(a \otimes v_{i_1} \wedge \dots \wedge v_{i_k}) := a(0) (\omega_\gamma^\perp)^{\ell(\gamma/2)}(v_{i_1}, \dots, v_{i_k}).$$

Clearly, this map is only nontrivial in degree  $\ell(\gamma)$  and maps the differential  $d_\gamma$  on  $K_\gamma^\bullet$  to zero. Define

$$P_\gamma : C_{\text{red}}^\bullet(\mathbb{W}, \mathbb{W}_\gamma) \rightarrow \mathbb{C}[[\hbar]]$$

to be  $P_\gamma := \mu \circ R_V$ . On the other hand, choosing  $\Psi_\gamma$  as in the proposition defines a morphism  $I_{\Psi_\gamma} : \mathbb{C}[[\hbar]][\ell(\gamma)] \rightarrow C_{\text{red}}^\bullet(\mathbb{W}, \mathbb{W}_\gamma)$ . The argument in the proof of the proposition then shows:

**Corollary 4.5.** *The inclusion  $I_{\Psi_\gamma}$  and the projection  $P_\gamma$  are quasi-isomorphisms satisfying  $P_\gamma \circ I_{\Psi_\gamma} = \text{id}$ .*

Finally, we come to the full crossed product  $\mathbb{W}_{2n} \rtimes \Gamma$ . As usual, we have

$$H^\bullet(\mathbb{W}_{2n} \rtimes \Gamma, \mathbb{W}_{2n} \rtimes \Gamma) \cong \left( \bigoplus_{\gamma \in \Gamma} H^\bullet(\mathbb{W}_{2n}, \mathbb{W}_{2n, \gamma}) \right)^\Gamma,$$

where the  $\Gamma$ -action is as explained in Section 3.2.

**Corollary 4.6.** *The generators  $\Psi_\gamma$ ,  $\gamma \in \Gamma$  satisfy*

$$\gamma_1 \cdot \Psi_{\gamma_2} - \Psi_{\gamma_1 \gamma_2 \gamma_1^{-1}} = \text{exact}.$$

*Therefore they define a canonical isomorphism*

$$H^\bullet(\mathbb{W}_{2n} \rtimes \Gamma, \mathbb{W}_{2n} \rtimes \Gamma) = \bigoplus_{\langle \gamma \rangle \subset \Gamma} \mathbb{C}[[\hbar]][\ell(\langle \gamma \rangle)].$$

*Proof.* Restricting to  $\Lambda^\bullet V^*$ , we find

$$\begin{aligned} \left( \gamma_1 \cdot \Psi_{\gamma_2} - \Psi_{\gamma_1 \gamma_2 \gamma_1^{-1}} \right)_{|\Lambda^\bullet V^*} &= \gamma_1 \cdot (\Pi_{\gamma_2}^\perp)^{\ell(\gamma_2)/2} - (\Pi_{\gamma_1 \gamma_2 \gamma_1^{-1}}^\perp)^{\ell(\gamma_1 \gamma_2 \gamma_1^{-1})/2} \\ &= 0 \end{aligned}$$

in  $K_{\gamma_1 \gamma_2 \gamma_1^{-1}}^{\ell(\gamma_2)}$ . By the argument above, the cocycles  $\gamma_1 \cdot \Psi_{\gamma_2}$  and  $\Psi_{\gamma_1 \gamma_2 \gamma_1^{-1}}$  therefore differ by a coboundary, and the result follows.  $\square$

**4.3. The Fedosov–Weinstein–Xu resolution over  $B_0$ .** Let  $B_0$  be the space of loops in  $G$ :

$$B_0 := \{g \in G_1 \mid s(g) = t(g)\}.$$

We recall from [NEPFPOTA] that the canonical inclusion  $\iota : B_0 \hookrightarrow G_1$  gives  $B_0$  a symplectic form by pull-back. Denote by  $\mathcal{A}_{B_0}^\hbar := \iota^{-1} \mathcal{A}^\hbar$  the pull-back of the deformation quantization of  $G$ ; this is not quite a deformation quantization of  $(B_0, \iota^* \omega)$ , because it involves the germ of  $B_0$  inside  $G_1$ . Recall from [PFPOTA] that the sheaf  $\mathcal{A}_{B_0}^\hbar$  has a canonical local automorphism, denoted  $\theta$ , coming from the fact that  $B_0$  has a cyclic structure [CR]. This enables us to define the following complex of sheaves on  $B_0$ :

$$\mathcal{C}^0 \xrightarrow{\beta_\theta} \mathcal{C}^1 \xrightarrow{\beta_\theta} \dots$$

where

$$\mathcal{C}^k := \underline{\text{Hom}}_{\mathbb{C}[[\hbar]]} \left( (\mathcal{A}_{B_0}^\hbar)^{\hat{\otimes} k}, \mathcal{A}_{B_0, \theta}^\hbar \right)$$

is the sheaf of Hochschild  $k$ -cochains, and  $\beta_\theta : \mathcal{C}^k \rightarrow \mathcal{C}^{k+1}$  is the twisted Hochschild coboundary. We will now write down a resolution of this complex of sheaves. For this, let  $\mathcal{W}_G$  be the bundle of Weyl algebras over  $G_0$ . This is just the bundle  $F_{\text{Sp}, G_0} \times_{\text{Sp}} \mathbb{W}$  associated to the symplectic frame bundle over  $G_0$  with typical fiber  $\mathbb{W}$ . This construction shows that  $\mathcal{W}_G$  carries a canonical action of the groupoid  $G$ . In the following we will denote its sheaf of sections by the same symbol  $\mathcal{W}_G$ .

**Proposition 4.7** (cf. [FE]). *On  $\mathbf{B}_0$  there exists a resolution*

$$0 \longrightarrow \mathcal{A}_{\mathbf{B}_0}^{\hbar} \longrightarrow \Omega_{\mathbf{B}_0}^0 \otimes \mathcal{W}_{\mathbf{G}} \xrightarrow{D} \Omega_{\mathbf{B}_0}^1 \otimes \mathcal{W}_{\mathbf{G}} \xrightarrow{D} \dots, \quad (4.4)$$

where  $D$  is a Fedosov connection on  $\mathcal{W}_{\mathbf{G}}$ .

*Proof.* Let us first construct the Fedosov differential  $D$ . The sequence of maps

$$T\mathbf{B}_0 \xrightarrow{\tilde{\omega}_0} T^*\mathbf{B}_0 \rightarrow T^*\mathbf{G} \rightarrow \mathcal{W}_{\mathbf{G}}$$

determines an element  $A_0 \in \Omega^1(\mathbf{B}_0, \mathcal{W}_{\mathbf{G}})$ . One easily verifies that

$$[A_0, A_0] = \omega_0 \in \Omega_{\mathbf{B}_0}^2 \otimes \mathbb{C}[[\hbar]],$$

which is central in  $\mathcal{W}_{\mathbf{G}}$ . Therefore,  $\delta = \text{ad}(A_0)$  defines a differential on  $\Omega^\bullet(\mathbf{B}_0, \mathcal{W}_{\mathbf{G}})$ , and we have

$$H^k(\Omega_{\mathbf{B}_0}^\bullet \otimes \mathcal{W}_{\mathbf{G}}, \text{ad}(A_0)) = \begin{cases} \Omega_{\mathbf{B}_0}^0 \otimes \mathcal{W}_N, & \text{for } k = 0, \\ 0, & \text{for } k \neq 0, \end{cases}$$

because  $\text{ad}(A_0)$  is simply a Koszul differential in the tangential directions along  $\mathbf{B}_0$ .

Choose a symplectic connection  $\nabla_{\mathbf{B}_0}$  on  $\mathbf{B}_0$  and a symplectic connection  $\nabla_N$  on the normal bundle  $N \rightarrow \mathbf{B}_0$ . We will consider connections on  $\mathcal{W}_{\mathbf{G}}$  of the form

$$D = \delta + \nabla + \text{ad}(A_\bullet),$$

where  $\nabla = (\nabla_{\mathbf{B}_0} \otimes 1 + 1 \otimes \nabla_N)$  and  $A \in \Omega^1(\mathbf{B}_0, \mathcal{W}_{\mathbf{G}})$  has  $\deg(A) \geq 2$ . Notice that  $\deg(\delta) = 0$  and  $\deg(\nabla) = 1$ . Such a connection has Weyl curvature given by

$$\Omega = \omega_0 + \tilde{R} + \nabla A + \frac{1}{2}[A, A].$$

By the usual Fedosov method, we can find  $A$  such that  $\Omega$  is central, i.e.,  $\Omega \in \Omega_{\mathbf{B}_0}^2 \otimes \mathbb{C}[[\hbar]]$ . Since the differential  $D$  is a deformation of the Koszul complex above, acyclicity of the sequence (4.4) follows. This shows the existence of the resolution

$$0 \longrightarrow \Omega_{\mathbf{B}_0}^0 \otimes \mathcal{W}_N \longrightarrow \Omega_{\mathbf{B}_0}^0 \otimes \mathcal{W}_{\mathbf{G}} \xrightarrow{D} \Omega_{\mathbf{B}_0}^1 \otimes \mathcal{W}_{\mathbf{G}} \xrightarrow{D} \dots,$$

so it remains to construct an isomorphism  $\Omega_{\mathbf{B}_0}^0 \otimes \mathcal{W}_N \cong \iota^{-1}\mathcal{A}_{\mathbf{G}}^{\hbar}$ . This is done by minimal coupling in [FE].  $\square$

Consider now the following double complex  $(\mathcal{C}^{\bullet, \bullet}, \beta_\theta, D)$  of sheaves:

$$\begin{array}{ccccccc} & \vdots & & \vdots & & \vdots & \\ & \beta_\theta \uparrow & & \beta_\theta \uparrow & & \beta_\theta \uparrow & \\ \Omega_{\mathbf{B}_0}^0 \otimes \mathcal{C}^1 & \xrightarrow{D} & \Omega_{\mathbf{B}_0}^1 \otimes \mathcal{C}^1 & \xrightarrow{D} & \Omega_{\mathbf{B}_0}^2 \otimes \mathcal{C}^1 & \xrightarrow{D} & \dots \\ & \beta_\theta \uparrow & & \beta_\theta \uparrow & & \beta_\theta \uparrow & \\ \Omega_{\mathbf{B}_0}^0 \otimes \mathcal{C}^0 & \xrightarrow{D} & \Omega_{\mathbf{B}_0}^1 \otimes \mathcal{C}^0 & \xrightarrow{D} & \Omega_{\mathbf{B}_0}^2 \otimes \mathcal{C}^0 & \xrightarrow{D} & \dots \end{array}$$

where  $\mathcal{C}^\bullet$  is the sheaf of formal power series of polydifferential operators on  $\mathcal{W}_{\mathbf{G}}$ . This complex is a twisted version of the Fedosov–Weinstein–Xu resolution considered in [Do].

**Proposition 4.8.** *The total cohomology of this complex is given by*

$$H^k(\text{Tot}(\mathcal{C}^{\bullet, \bullet}), \beta_\theta + D) = \mathbb{H}^k(\mathcal{C}^\bullet, \beta_\theta).$$

*Proof.* Consider the spectral sequence by filtering the total complex by rows. This yields

$$E_1^{p,q} = H_v^p(\mathcal{C}^{\bullet,q}) = \begin{cases} \Omega_{\mathbb{B}_0}^0 \otimes \mathcal{C}^q, & \text{for } p = 0, \\ 0, & \text{for } p \neq 0, \end{cases}$$

since the resolution (4.4) is acyclic. At the second stage

$$E_2^{0,q} = \mathbb{H}^q(\mathcal{C}^\bullet, \beta_\theta) \Rightarrow H^q(\text{Tot}(\mathcal{C}^{\bullet,\bullet})).$$

Since the spectral sequence collapses at this point, this proves the statement.  $\square$

Of course the cohomology sheaf of the vertical complex is computed by Proposition 4.4. If we replace the vertical complex by its reduced counterpart, we get a natural projection

$$R : \Gamma(\mathbb{B}_0, \mathcal{C}^\ell) \rightarrow \Gamma(\mathbb{B}_0, \Omega_{\mathbb{G}}^\ell),$$

where  $\ell$  is the locally constant function on  $\mathbb{B}_0$  defined in Section 2. The transversal Poisson structure induced by  $\omega \in \Omega^2(\mathbb{G}_0)$  induces a section

$$(\Pi_\theta^\perp)^{\ell/2} \in \Gamma(\mathbb{B}_0, \Omega_{\mathbb{G}}^\ell),$$

and we choose  $\Psi \in \Gamma(\mathbb{B}_0, \mathcal{C}^\ell)$  which generates the fiberwise vertical cohomology and projects onto the section above.

**Proposition 4.9.** *The map  $I_\Psi$  extends to a morphism*

$$I_\Psi : (\Omega_{\mathbb{B}_0}^\bullet \otimes \mathbb{C}[[\hbar]][[\ell], d] \rightarrow (\text{Tot}^\bullet(\mathcal{C}^{\bullet,\bullet}), D + \beta_\theta)$$

*of cochain complexes of sheaves by the formula*

$$I_\Psi(\alpha) := \alpha \otimes \Psi.$$

*In fact, this is a quasi-isomorphism.*

*Proof.* Let us first prove that  $I_\Psi$  is a map of cochain complexes of sheaves. Consider the projection map  $P : \Omega_{\mathbb{B}_0}^\bullet \otimes \mathcal{C}_{red}^\bullet \rightarrow \Omega_{\mathbb{B}_0}^\bullet \otimes \mathbb{C}((t))$ . It satisfies  $P \circ I_\Psi = \text{id}$ . We therefore have to show that

$$P \circ D \circ I_\Psi = d.$$

Since the statement of the proposition is a local statement we can write the Fedosov connection as

$$D = d + \text{ad}(A),$$

with  $A \in \Omega_{\mathbb{B}_0}^1 \otimes \mathcal{W}_{\mathbb{G}} = \Omega_{\mathbb{B}_0}^1 \otimes \mathcal{C}^0$ . It then follows easily that  $D : \Omega_{\mathbb{B}_0}^k \otimes \mathcal{C}^\bullet \rightarrow \Omega_{\mathbb{B}_0}^{k+1} \otimes \mathcal{C}^\bullet$  is given by

$$D = d + [\beta A, \ ] ,$$

where  $[ \ , \ ]$  is the Gerstenhaber bracket on the Hochschild cochain complex. We now compute

$$\begin{aligned} PDI_\Psi(\alpha) - d\alpha &= \alpha \otimes P([\beta A, \Psi]) \\ &= \alpha \otimes P(\beta([A, \Psi]) - [A, \beta\Psi]) \\ &= 0, \end{aligned}$$

where we have used that  $\Psi$  is a cocycle, i.e.,  $\beta\Psi = 0$ , and the fact that  $P$  maps coboundaries to zero. This proves that  $I_\Psi$  is a map of cochain complexes of sheaves.  $\square$

Using Proposition 4.8 we now find:



**Corollary 4.10.** *There is a natural isomorphism*

$$\mathbb{H}^\bullet(\mathcal{C}_{\mathbb{B}_0}^{\bullet,\bullet}, \beta_\theta) \cong H^{\bullet-\ell}(\mathbb{B}_0, \mathbb{C}[[\hbar]]).$$

Finally, notice that  $\mathcal{C}_{\mathbb{B}_0}^{\bullet,\bullet}$  is in a natural way a double complex of  $\Lambda(\mathbb{G})$ -sheaves on  $\mathbb{B}_0$ , because the Fedosov resolution of Proposition 4.7 carries a natural  $\mathbb{G}$ -action. Therefore, we can define the following sheaf on  $\tilde{X}$ :

$$\mathcal{C}_{\tilde{X}}^{\bullet,\bullet}(\tilde{U}) := \mathcal{C}_{\mathbb{B}_0}^{\bullet,\bullet}(\tilde{\pi}^{-1}(\tilde{U}))^{\Lambda(\mathbb{G})|_{\tilde{U}}}.$$

Because  $\mathbb{G}$ , and therefore also  $\Lambda(\mathbb{G})$ , is proper, it follows from Corollary 4.10 that its hypercohomology is given by

$$\mathbb{H}^\bullet(\mathcal{C}_{\tilde{X}}^{\bullet,\bullet}, \beta_\theta) \cong H^{\bullet-\ell}(\tilde{X}, \mathbb{C}[[\hbar]]). \quad (4.5)$$

**4.4. Local computations.** In this section we will perform several explicit computations in some open orbifold charts. This suffices to prove the result in the case of a global quotient orbifold. The general case is treated in the next section.

Let  $U \subset \mathbb{R}^{2n}$  be an open orbifold chart with a finite group  $\Gamma_U$  acting by linear symplectic transformations, so that we have  $U/\Gamma_U \subset X$ .

**Proposition 4.11.** *There exist a natural quasi-isomorphism*

$$\mathcal{H}_{M \times \Gamma, \text{loc}, \hbar}^\bullet(M/\Gamma) \rightarrow \mathcal{C}_{\widetilde{M/\Gamma}}^{\bullet,\bullet}(\widetilde{M/\Gamma}).$$

*Proof.* We first use the natural map

$$\mathcal{H}_{\mathbb{G}, \text{loc}, \hbar}^\bullet(U/\Gamma_U) \rightarrow C_{\text{loc}}^\bullet(\mathcal{A}^\hbar(U) \rtimes \Gamma_U, \mathcal{A}^\hbar(U) \rtimes \Gamma_U)$$

of the “deformed version” of Theorem IIIb. As in the undeformed case, there is a quasi-isomorphism

$$L_1^\hbar : C_{\text{loc}}^\bullet(\mathcal{A}^\hbar(U) \rtimes \Gamma_U, \mathcal{A}^\hbar(U) \rtimes \Gamma_U) \rightarrow C_{\text{loc}}^\bullet(\mathcal{A}_{\text{cpt}}^\hbar(U), \mathcal{A}_{\text{cpt}}^\hbar(U) \rtimes \Gamma_U)^{\Gamma_U},$$

given by the same formula as for  $L_1$ . The right hand side is the space of  $\Gamma_U$ -invariants of a complex which decomposes into

$$\bigoplus_{\gamma \in \Gamma_U} C_{\text{loc}}^\bullet(\mathcal{A}^\hbar(U), \mathcal{A}^\hbar(U)_\gamma).$$

There is a natural morphism

$$C_{\text{loc}}^\bullet(\mathcal{A}^\hbar(U), \mathcal{A}^\hbar(U)_\gamma) \rightarrow C_{\text{loc}}^\bullet(\iota^{-1}\mathcal{A}^\hbar(U^\gamma), \iota^{-1}\mathcal{A}^\hbar(U^\gamma)_\gamma),$$

where  $\iota^{-1}\mathcal{A}^\hbar(U^\gamma) := \Gamma(U^\gamma, \iota^{-1}\mathcal{A}^\hbar)$  is by definition the algebra given by the jets along the embedding  $\iota : U^\gamma \hookrightarrow U$ . Indeed the locality condition for cochains  $\Psi \in C_{\text{loc}}^k$  states in this case that the value  $\Psi(f_1, \dots, f_k)(x)$  can only depend on the germs of  $f_1, \dots, f_k$  at the points of the  $\gamma$ -orbit of  $x \in U$ . Restricted to  $U^\gamma \subset U$ , such cochains therefore preserve the subalgebra  $\iota^{-1}\mathcal{A}^\hbar(U^\gamma) \subset \mathcal{A}^\hbar(U)$ , and the restriction map above is well-defined. Even stronger, it is a quasi-isomorphism because on  $E_0$ -level of the spectral sequence associated to the filtration by powers of  $\hbar$  this map is simply given by localization, which we already know to be a quasi-isomorphism, cf. Section 3.2.

As remarked above, local cochains, in the sense defined above, are truly local with respect to  $U^\gamma$ , because points in  $U^\gamma$  by definition have a trivial  $\gamma$ -orbit. From this we see that there is a canonical isomorphism

$$C_{\text{loc}}^\bullet(\iota^{-1}\mathcal{A}_{\text{cpt}}^\hbar(U^\gamma), \iota^{-1}\mathcal{A}_{\text{cpt}}^\hbar(U^\gamma)_\gamma) \cong \mathcal{C}^\bullet(U^\gamma),$$

compatible with differentials. Taking the sum over all  $\gamma \in \Gamma_U$ , and taking  $\Gamma_U$ -invariants, defines the map of the proposition. As the argument shows, it is a quasi-isomorphism.  $\square$

Using the Fedosov–Weinstein–Xu resolution, this result suffices to compute the Hochschild cohomology for a global quotient symplectic orbifold:

**Corollary 4.12.** *For a global quotient  $X = M/\Gamma$  of a finite group  $\Gamma$  acting on a symplectic manifold  $M$ , there is a natural isomorphism*

$$H^\bullet(\mathcal{A}_{\text{cpt}}^{((\hbar))}(M) \rtimes \Gamma, \mathcal{A}_{\text{cpt}}^{((\hbar))} \rtimes \Gamma) \cong \bigoplus_{(\gamma) \subset \Gamma} H_{Z(\gamma)}^{\bullet-\ell_\gamma}(M^\gamma, \mathbb{C}((\hbar))).$$

*Proof.* This follows from the isomorphism (4.5).  $\square$

Next, we consider the cup-product. An easy computation shows that the map  $L_1^{\hbar}$  induces the following product on the complex  $C_{\text{loc}}^{\bullet}(\mathcal{A}_{\text{cpt}}^{\hbar}(U), \mathcal{A}_{\text{cpt}}^{\hbar}(U) \rtimes \Gamma_U)^{\Gamma_U}$ :

$$(\psi \bullet \phi)_\gamma := \sum_{\gamma_1 \gamma_2 = \gamma} \psi_{\gamma_1} \cup_{\text{tw}} \phi_{\gamma_2}$$

where the map

$$\cup_{\text{tw}} : C^k(\mathcal{A}^{\hbar}(U), \mathcal{A}^{\hbar}(U)_{\gamma_1}) \times C^l(\mathcal{A}^{\hbar}(U), \mathcal{A}^{\hbar}(U)_{\gamma_2}) \rightarrow C^{k+l}(\mathcal{A}^{\hbar}(U), \mathcal{A}^{\hbar}(U)_{\gamma_1 \gamma_2})$$

is defined as

$$(\psi_{\gamma_1} \cup_{\text{tw}} \phi_{\gamma_2})(a_1, \dots, a_{k+l}) := \psi_{\gamma_1}(a_1, \dots, a_k) \gamma_1 \phi_{\gamma_2}(a_{k+1}, \dots, a_{k+l}).$$

Indeed one easily checks that

$$\beta_{\gamma_1 \gamma_2} (\psi_{\gamma_1} \cup_{\text{tw}} \phi_{\gamma_2}) = (\beta_{\gamma_1} \psi_{\gamma_1}) \cup_{\text{tw}} \phi_{\gamma_2} + (-1)^{\deg(\psi)} \psi_{\gamma_1} \cup_{\text{tw}} (\beta_{\gamma_2} \phi_{\gamma_2}).$$

Restricting to  $\tilde{U} = \bigsqcup_{\gamma} U^\gamma$ , this induces the following product on the Fedosov–Weinstein–Xu resolution  $\mathcal{C}_{\tilde{U}/\Gamma_U}^{\bullet, \bullet}(\tilde{U}/\Gamma_U)$ :

$$((\alpha \otimes \psi) \bullet (\beta \otimes \phi))_\gamma := \sum_{\gamma_1 \gamma_2 = \gamma} (\iota_{\gamma_1}^*(\alpha_{\gamma_1}) \wedge \iota_{\gamma_2}^*(\beta_{\gamma_2})) \otimes (\iota_{\gamma_1}^*(\psi_{\gamma_1}) \cup_{\text{tw}} (\iota_{\gamma_2}^*(\phi_{\gamma_2})),$$

where an element of  $\mathcal{C}_{\tilde{U}/\Gamma_U}^{\bullet, \bullet}(\tilde{U}/\Gamma_U)$  is written as  $\alpha \otimes \psi = \sum_{\gamma} \alpha_{\gamma} \otimes \psi_{\gamma}$ , with  $\alpha_{\gamma} \in \Omega^\bullet(U^\gamma)$  and  $\psi_{\gamma}$  is a local section of the sheaf of Hochschild cocycle on  $\mathcal{W}_{\mathbb{G}}$  over  $U^\gamma \subset U$ . We therefore have:

**Proposition 4.13.** *The map of Proposition 4.11 is compatible with products that means defines a quasi-isomorphism of sheaves of DGA's on  $U/\Gamma_U$ .*

For a global quotient orbifold, this leads immediately to:

**Corollary 4.14.** *Under the isomorphism of Corollary 4.12, the cup-product is given by*

$$\alpha \bullet \beta = \sum_{\substack{\gamma_1 \gamma_2 = \gamma \\ \ell(\gamma_1) + \ell(\gamma_2) = \ell(\gamma_1 \gamma_2)}} \iota_{\gamma_1}^* \alpha_{\gamma_1} \wedge \iota_{\gamma_2}^* \alpha_{\gamma_2}.$$

*Proof.* The isomorphism of Corollary 4.12 is induced by the quasi-isomorphism

$$I_\Psi : (\Omega_{\tilde{X}}^\bullet \otimes \mathbb{C}[[\hbar]], d) \rightarrow (\mathcal{C}_{\tilde{X}}^{\bullet, \bullet}, D + \beta_{\text{tw}})$$

of Proposition 4.9. We therefore find

$$I_\Psi(\alpha) \bullet I_\Psi(\beta) = \sum_{\gamma_1 \gamma_2 = \gamma} (\iota_\gamma^*(\alpha_{\gamma_1}) \wedge \iota_\gamma^*(\beta_{\gamma_2})) \otimes (\iota_\gamma^*(\Psi_{\gamma_1}) \cup_{\text{tw}} (\iota_\gamma^*(\Psi_{\gamma_2}))).$$

We will now consider the second component  $\iota_\gamma^*(\Psi_{\gamma_1}) \cup_{\text{tw}} (\iota_\gamma^*(\Psi_{\gamma_2}))$  for a moment. An easy calculation shows that in the Koszul complex (4.3), the product

$$\cup_{\text{tw}} : K_{\gamma_1}^p \times K_\gamma^q \rightarrow K_{\gamma_1 \gamma_2}^{p+q}$$

is given by

$$(a_1 \otimes y_{I_p}) \cup_{\text{tw}} (a_2 \otimes y_{I_q}) = (a_1 \gamma_1 a_2) \otimes y_{I_p} \wedge \gamma_1 y_{I_q},$$

where  $I_p$  and  $I_q$  are multi-indices of length  $p$  resp.  $q$ . Therefore,

$$(\Pi_{\gamma_1}^\perp)^{\ell(\gamma_1)/2} \cup_{\text{tw}} (\Pi_{\gamma_2}^\perp)^{\ell(\gamma_2)/2} = \begin{cases} (\Pi_{\gamma_1 \gamma_2}^\perp)^{\ell(\gamma_1 \gamma_2)/2}, & \text{if } \ell(\gamma_1) + \ell(\gamma_2) = \ell(\gamma_1 \gamma_2), \\ 0, & \text{else.} \end{cases}$$

In the reduced Hochschild complex, this gives

$$\Psi_{\gamma_1} \cup_{\text{tw}} \Psi_{\gamma_2} = \begin{cases} \Psi_{\gamma_1 \gamma_2} + \text{exact}, & \text{if } \ell(\gamma_1) + \ell(\gamma_2) = \ell(\gamma_1 \gamma_2), \\ \text{exact}, & \text{else.} \end{cases}$$

Taking cohomology we find the product as stated above.  $\square$

**4.5. The general case.** Recall that we have spaces and morphisms as in the following diagram:

$$\begin{array}{ccc} B_0 & \longrightarrow & G_0 \\ \downarrow \tilde{\pi} & & \downarrow \pi \\ \tilde{X} & \xrightarrow{\psi} & X \end{array}$$

As in [CHHU], define the space

$$S_G^1 := \{(g_1, g_2) \in G_1 \times G_1 \mid s(g_1) = t(g_1) = s(g_2) = t(g_2)\}.$$

It comes equipped with three maps  $\text{pr}_1, m, \text{pr}_2 : S_G^1 \rightarrow B_0$ , where  $\text{pr}_1(g_1, g_2) = g_1$ ,  $m(g_1, g_2) = g_1 g_2$  and  $\text{pr}_2(g_1, g_2) = g_2$ . For differential forms  $\alpha, \beta \in \Omega^\bullet(B_0)$ , define the following product:

$$\alpha \bullet \beta = \int_{m_\ell} \text{pr}_1^* \alpha \wedge \text{pr}_2^* \beta, \quad (4.6)$$

where  $m_\ell : S_{G,\ell}^1 \rightarrow B_0$  is the restriction of the multiplication map and

$$S_{G,\ell}^1 := \{(g_1, g_2) \in S_G^1 \mid \ell(g_1) + \ell(g_2) = \ell(g_1 g_2)\}.$$

Both  $S_{G,\ell}^1$  and  $B_0$  carry a natural action of  $G$  by conjugating loops and the quotient  $B_0/G = \tilde{X}$ . We have

$$\Omega^\bullet(\tilde{X}) = \Omega^\bullet(B_0)^G,$$

and the product (4.6) defines an associative graded product on  $\Omega^\bullet(\tilde{X})$ . Together with the de Rham differential, it turns  $\Omega^\bullet(\tilde{X})$  into a differential graded algebra. Of course,  $\Omega^\bullet(\tilde{X})$  is the global sections of a sheaf on  $\tilde{X}$ , but it is important to notice that the product (4.6) is not local. However, if we consider  $\psi_* \Omega_{\tilde{X}}^\bullet$ , the push-forward to  $X$ , we have  $\Gamma(X, \psi_* \Omega_{\tilde{X}}^\bullet) = \Omega^\bullet(\tilde{X})$  and now the product is local, i.e.,  $(\psi_* \Omega_{\tilde{X}}^\bullet, d, \bullet)$  does define a sheaf of DGA's on  $X$ .

The same can be done for the FWX-resolution on  $\tilde{X}$ . This time we introduce the product

$$(\alpha \otimes \psi) \bullet (\beta \otimes \phi) := \int_{m_e} (pr_1^* \alpha \wedge pr_2^* \beta) \otimes (pr_1^* \psi \cup_{\text{tw}} pr_2^* \phi). \quad (4.7)$$

Again, because of the integration over the fiber, this defines a local product on  $\psi^* \mathcal{C}_{\tilde{X}}^{\bullet, \bullet}$ , so that the total sheaf complex is a sheaf of DGA's.

**Lemma 4.15.** *The embedding of Proposition 4.9 defines a quasi-isomorphism*

$$I_\Psi : \psi_* \Omega_{\tilde{X}}^{\bullet - \ell} \otimes \mathbb{C}[[\hbar]] \rightarrow \psi_* \mathcal{C}_{\tilde{X}}^{\bullet, \bullet}$$

*compatible with products up to homotopy.*

*Proof.* By Corollary 4.6, if we choose  $\Psi \in \Gamma(\mathbb{B}_0, \mathcal{C}^\ell)$  to be  $\mathbb{G}$ -invariant, it descends to a morphism

$$I_\Psi : \psi_* \Omega_{\tilde{X}}^{\bullet - \ell} \otimes \mathbb{C}[[\hbar]] \rightarrow \psi_* \mathcal{C}_{\tilde{X}}^{\bullet, \bullet}$$

By assumption, the groupoid  $\mathbb{G}$  is proper, so we have

$$\mathbb{H}(\tilde{X}, (\Omega_{\tilde{X}}^\bullet, d)) \cong \mathbb{H}(\mathbb{B}_0, (\Omega_{\mathbb{B}_0}^\bullet, d))^{\mathbb{G}},$$

and similarly for  $\mathcal{C}_{\tilde{X}}^{\bullet, \bullet}$ . Therefore the morphism  $I_\Psi$  is a quasi-isomorphism because it is a quasi-isomorphism on  $\mathbb{B}_0$ . The fact that it preserves products up to a coboundary, is a simple calculation as in the proof of Corollary 4.14.  $\square$

**Proposition 4.16.** *There is a quasi-isomorphism*

$$\mathcal{H}_{\mathbb{G}, \text{loc}, \hbar}^\bullet \rightarrow \psi_* \mathcal{C}_{\tilde{X}}^{\bullet, \bullet}$$

*which maps the cup-product to the product (4.7)*

*Proof.* For any  $x \in X$ , choose a local slice to obtain a morphism

$$\mathcal{H}_{\mathbb{G}, \text{loc}, \hbar}^\bullet(U_x) \rightarrow C^\bullet(\mathcal{A}^\hbar(M_x) \rtimes \mathbb{G}_x, \mathcal{A}^\hbar(M_x) \rtimes \mathbb{G}_x),$$

as in Theorem IIIb. By this very same Theorem IIIb, one knows that on  $E_0$  of the spectral sequences associated to the  $\hbar$ -filtration, the above chain morphism is a quasi-isomorphism, and therefore it is a quasi-isomorphism on the original complexes. We now compose with the morphism of Proposition 4.11 to get a map

$$\mathcal{H}_{\mathbb{G}, \text{loc}, \hbar}^\bullet \rightarrow \psi_* \mathcal{C}_{M_x/\mathbb{G}_x}^{\bullet, \bullet}(U_x) \cong \psi_* \mathcal{C}_{\tilde{X}}^{\bullet, \bullet}(U_x).$$

Because the sheaves are fine, this in fact defines a global quasi-isomorphism over the orbifold  $X$ .  $\square$

Finally, combining Lemma 4.15 with Proposition 4.16, we have arrived at the main conclusion:

**Theorem 4.17.** *Let  $\mathbb{G}$  be a proper étale groupoid with an invariant symplectic structure, modeling a symplectic orbifold  $X$ . For any invariant deformation quantization  $\mathcal{A}^\hbar$  of  $\mathbb{G}$ , we have a natural isomorphism*

$$H^\bullet(\mathcal{A}^{((\hbar))} \rtimes \mathbb{G}, \mathcal{A}^{((\hbar))} \rtimes \mathbb{G}) \cong H^{\bullet - \ell}(\tilde{X}, \mathbb{C}((t))).$$

*With this isomorphism, the cup product is given by (4.6).*

**Remark 4.18.** We explain the product (4.6) using the orbifold language. Let  $X$  be represented by the groupoid  $\mathbf{G}$  such that  $X = \mathbf{G}_0/\mathbf{G}$ , and  $\tilde{X}$  be the corresponding inertia orbifold represented by  $\mathbf{B}_0/\mathbf{G}$ . Locally, an open chart of  $X$  is like  $U/\Gamma$  with  $\Gamma$  a finite group acting linearly on an open subset  $U$  of  $\mathbb{R}^n$ . Accordingly  $\tilde{X}$  is locally represented by  $(\coprod_{\gamma \in \Gamma} U^\gamma)/\Gamma \cong \coprod_{(\gamma) \subset \Gamma} U^\gamma/Z(\gamma)$ , where  $(\gamma)$  is the conjugacy class of  $\gamma$  in  $\Gamma$ . We usually use  $X^\gamma$  to stand for the component of  $\tilde{X}$  containing  $U^\gamma/Z(\gamma)$ . With these notations, the above  $S_{\mathbf{G}}^1$  and  $S_{\mathbf{G},\ell}^1$  are locally represented as

$$S_{\mathbf{G}}^1 = \coprod_{\gamma_1, \gamma_2} U^{\gamma_1} \cap U^{\gamma_2}, \quad S_{\mathbf{G},\ell}^1 = \coprod_{\substack{\gamma_1, \gamma_2, \\ \ell(\gamma_1) + \ell(\gamma_2) = \ell(\gamma_1 \gamma_2)}} U^{\gamma_1} \cap U^{\gamma_2}.$$

As we are considering  $\mathbf{G}$ -invariant differential forms on  $B_0$ , their pull-backs through projections  $\text{pr}_1, \text{pr}_2$  to  $S_{\mathbf{G}}^1$  and  $S_{\mathbf{G},\ell}^1$  are invariant under the following  $\mathbf{G}$ -action on  $S_{\mathbf{G}}^1$  and  $S_{\mathbf{G},\ell}^1$ , which is defined as

$$(g_1, g_2)g = (g^{-1}g_1g, g^{-1}g_2g), \quad (g_1, g_2) \in S_{\mathbf{G},\ell}^1, \quad s(g) = t(g_2) = t(g_2).$$

Locally this action can be written as a  $\Gamma$ -action on  $\coprod_{\gamma_1, \gamma_2} U^{\gamma_1} \cap U^{\gamma_2}$ ,

$$(x, \gamma_1, \gamma_2)\gamma = (\gamma^{-1}(x), \gamma^{-1}\gamma_1\gamma, \gamma^{-1}\gamma_2\gamma), \quad \gamma_1(x) = \gamma_2(x) = x, \gamma \in \Gamma.$$

The corresponding quotient space  $S_{\mathbf{G}}^1/\mathbf{G}$  is usually denoted by

$$X_3 = \{(x, (g_1, g_2, g_3)) \mid g_1, g_2 \in \text{Stab}(x), g_1g_2g_3 = \text{id}, x \in X\}. \quad (4.8)$$

One can see that locally  $X_3 = X^{g_1} \cap X^{g_2}$ . The pullbacks of  $\mathbf{G}$ -invariant differential forms on  $\mathbf{B}_0$  are differential forms on  $X_3$ . Therefore, the formula (4.6) can be interpreted as follows. For  $\alpha_1, \alpha_2 \in \Omega^\bullet(\tilde{X})$ ,

$$\alpha_1 \bullet \alpha_2|_\gamma = \sum_{\substack{\gamma_1, \gamma_2, \\ \ell(\gamma_1) + \ell(\gamma_2) = \ell(\gamma_1 \gamma_2)}} \iota_{\gamma_1}^*(\alpha_1|_{\gamma_1}) \wedge \iota_{\gamma_2}^*(\alpha_2|_{\gamma_2}),$$

where  $\iota_{\gamma_i}^*$  is the embedding of  $X_3$  in  $X^{\gamma_i}$ ,  $i = 1, 2$ .

**4.6. Frobenius algebras from Hochschild cohomology.** The product structure of Theorem 4.17 is part of a natural graded Frobenius algebra associated to  $\mathcal{A}^{(\hbar)} \rtimes \mathbf{G}$ . Recall that a Frobenius algebra is a commutative unital algebra equipped with an invariant trace. The construction of this Frobenius algebra on the Hochschild cohomology uses one additional piece of data, namely the trace on the algebra  $\mathcal{A}^\hbar \rtimes \mathbf{G}$  constructed in [PFPOTA].

Let  $A$  be a unital algebra over a field  $\mathbb{k}$  equipped with a trace  $\text{tr} : A \rightarrow \mathbb{k}$ . As we have seen, the cup-product (A.6) gives the Hochschild cohomology  $H^\bullet(A, A)$  the structure of a graded algebra. The Hochschild homology  $HH_\bullet(A)$  is a natural module over this algebra if we let a cochain  $\psi \in C^k(A, A)$  act as  $\iota_\psi : C_p(A) \rightarrow C_{p-k}(A)$  given by

$$\iota_\psi(a_0 \otimes \dots \otimes a_p) = (-1)^{\text{deg}(\psi)} a_0 \psi(a_1, \dots, a_k) \otimes a_{k+1} \otimes \dots \otimes a_p.$$

With this module structure, the trace induces a pairing

$$\langle \cdot, \cdot \rangle : H^\bullet(A, A) \times HH_\bullet(A) \rightarrow \mathbb{k}$$

which is given by

$$\langle \psi, a_0 \otimes \dots \otimes a_k \rangle = \text{tr}(\iota_\psi(a_0 \otimes \dots \otimes a_k)).$$

Let us assume, as in our case, that the Hochschild cohomology and homology are finite dimensional and that this pairing is perfect.

**Proposition 4.19.** *Under these assumptions, the ring structure on  $H^\bullet(A, A)$  is part of a natural graded Frobenius algebra structure.*

*Proof.* The pairing gives us a canonical isomorphism  $H^\bullet(A, A) \cong H_\bullet(A)^*$ . The trace defines a canonical element  $[\text{tr}] \in H_0(A)^*$  which defines the unit. Furthermore, the unit  $1 \in H_0(A) \cong H^0(A, A)^*$  defines an invariant trace.  $\square$

In the case at hand, the deformation of the convolution algebra on a symplectic orbifold, the Hochschild homology was computed in [NEPFPOTA] to be

$$H_\bullet \left( \mathcal{A}^{((\hbar))} \rtimes \mathbb{G} \right) \cong H_{\text{cpt}}^{2n-\bullet} \left( \tilde{X}, \mathbb{C}((\hbar)) \right).$$

With the trace of [PFPOTA], one checks that the pairing between Hochschild homology and cohomology above is nothing but Poincaré duality on  $\tilde{X}$ .

## 5. CHEN-RUAN ORBIFOLD COHOMOLOGY

In this section, we study  $S^1$ -equivariant Chen-Ruan orbifold cohomologies on an almost complex orbifold. In a special case, we apply the idea from [CHHU] to introduce a de Rham model (topological Hochschild cohomology) to compute this equivariant cohomology. In the last subsection, we compare this de Rham model with the previous computation of Hochschild cohomology of the quantized groupoid algebra. The Hochschild cohomology of the quantized groupoid algebra is identified as a graded algebra of the de Rham model with respect to some filtration.

**5.1.  $S^1$ -Equivariant Chen-Ruan orbifold cohomology.** In this subsection, we briefly introduce the idea of  $S^1$ -equivariant Chen-Ruan orbifold cohomology. Let  $X$  be an orbifold with an  $S^1$  action. This means that there is a morphism  $f : S^1 \times X \rightarrow X$  of orbifolds. One can think of a morphism between two orbifolds as a collection of morphisms between charts and group homomorphisms between local groups such that the morphisms between charts are equivariant with respect to local group actions, and are compatible with overlaps of charts. (See [CHRU04] and [ADLERU] for more details.) The action is assumed to be associative, which is a somewhat delicate property since the category of orbifolds is not a category but a 2-category. This means that the standard associativity diagram for a group action on an orbifold is only required to be commutative up to 2-morphisms. Generalities on group actions on categories can be found in [RO]. The  $S^1$ -equivariant cohomology is defined via the standard Borel construction:

$$H_{S^1}^\bullet(X) := H^\bullet(X \times_{S^1} ES^1).$$

We identify  $H_{S^1}^\bullet(\text{pt}) = \mathbb{C}[t]$ . With this,  $H_{S^1}^\bullet(X)$  is considered as a  $\mathbb{C}[t]$ -module. As usual, we consider the fraction field  $\mathbb{C}((t))$  and put

$$H_{S^1}^\bullet(X)((t)) := H_{S^1}^\bullet(X) \otimes_{\mathbb{C}[t]} \mathbb{C}((t)).$$

In the following, we shall use the Cartan model for equivariant cohomology to represent cohomology classes by equivariant differential forms.

As before,  $\tilde{X}$  is the inertia orbifold, and  $p : \tilde{X} \rightarrow X$  the natural projection. It is easy to check that the  $S^1$ -action lifts to  $\tilde{X}$ . Indeed, for any  $s \in S^1$  the action morphism  $f_s : s \times X \rightarrow X$  defines for each  $x \in X$  a group homomorphism

$\rho_s : \text{Stab}(x) \rightarrow \text{Stab}(f_s(x))$ . This induces the  $S^1$ -action on  $\tilde{X}$ , whose points are pairs  $(x, (\gamma))$ ,  $x \in X, \gamma \in \text{Stab}(x)$ . More precisely, the action is given by

$$S^1 \times \tilde{X} \rightarrow \tilde{X}, \quad (s, (x, (\gamma))) \mapsto (f_s(x), (\rho_s(\gamma))).$$

As a  $\mathbb{C}[t]$ -module, the  $S^1$ -Chen-Ruan orbifold cohomology can be defined exactly in the same fashion as its non-equivariant version [CHRU04], that is

$$H_{S^1}^\bullet(\tilde{X}) := H^\bullet(\tilde{X} \times_{S^1} ES^1).$$

There is a natural involution  $I : \tilde{X} \rightarrow \tilde{X}$  which maps a point  $(x, (\gamma))$  to  $(x, (\gamma^{-1}))$ . The orbifold Poincaré pairing  $\langle \cdot, \cdot \rangle$ , which is defined by

$$\langle a, b \rangle := \int_{\tilde{X}} a \wedge I^* b,$$

naturally extends to a non-degenerate pairing on  $H_{S^1}^\bullet(\tilde{X})$ .

The additional structures one defines on Chen-Ruan orbifold cohomology require a choice of an almost complex structure on the tangent bundle  $TX$ , which we now make. We also assume that the  $S^1$ -action on  $TX$  is compatible with this almost complex structure.

We will assume an  $S^1$ -action on the tangent bundle  $TX$  which commutes with the  $S^1$ -action on the base  $X$ . It should be noted that we *do not* necessarily work with the canonical action on  $TX$  induced from that on  $X$ . This will be important in what follows. Therefore the pull-back bundle  $p^*TX$  admits an  $S^1$ -action covering that on  $\tilde{X}$ . Let  $X^\gamma$  be a component of  $\tilde{X}$ . The bundle  $p^*TX|_{X^\gamma}$  splits into a direct sum of  $\gamma$ -eigenbundles. This allows one to define the age function, denoted by  $\iota(\gamma)$  (c.f. [CHRU02]). This is a locally constant function on  $\tilde{X}$ . We consider the shifted  $S^1$ -equivariant cohomology of  $\tilde{X}$ ,

$$H_{S^1}^\bullet(\tilde{X})((t))[-2\iota(\gamma)].$$

Here  $t$  is assigned degree 2.

The  $S^1$ -action on  $p^*TX|_{X^\gamma}$  restricts to an  $S^1$ -action on each eigenbundle. Now consider the tri-cyclic sector (4.8), i.e., the quotient  $\mathbb{S}_\mathbb{G}/\mathbb{G}$ . There are three evaluation maps  $e_i : X_3 \rightarrow \tilde{X}$ ,  $e_i((x, (\gamma_1, \gamma_2, \gamma_3))) = (x, (\gamma_i))$ . The  $S^1$ -action also lifts to tri-cyclic sector:

$$S^1 \times X_3 \rightarrow X_3, \quad (s, (x, (\gamma_1, \gamma_2, \gamma_3))) \mapsto (f_s(x), (\rho_s(\gamma_1), \rho_s(\gamma_2), \rho_s(\gamma_3))).$$

The evaluation maps are clearly  $S^1$ -equivariant. It follows from the above discussion that the obstruction bundle  $\Theta$  over the tri-cyclic sector  $X_3$  is an  $S^1$ -equivariant orbifold bundle on  $\tilde{X}$ . Therefore, we can define  $S^1$ -equivariant orbifold cup product  $\star_t$  by

$$\langle \alpha_1 \star_t \alpha_2, \alpha_3 \rangle = \int_{X_3} e_1^*(\alpha_1) \wedge e_2^*(\alpha_2) \wedge e_3^*(I^*(\alpha)) \wedge \text{eu}_{S^1}(\Theta),$$

where  $\text{eu}_{S^1}(\Theta)$  is the equivariant Euler class of the obstruction bundle. Many properties of the Chen-Ruan orbifold cohomology algebra holds for the algebra

$$(H_{S^1}^\bullet(\tilde{X})((t))[-2\iota(\gamma)], \star_t),$$

with the same proofs. For example, the associativity of  $\star_t$  is reduced to the rational equivalence between two points in the moduli space  $\overline{M}_{0,4}$  of genus zero stable curves with four marked points. (Note that  $\overline{M}_{0,4} \simeq \mathbb{CP}^1$ .) See [CHRU04] for more details.

**5.2. Equivariant de Rham model.** In this subsection, we define an equivariant de Rham model for a special case of the above introduced  $S^1$ -equivariant Chen-Ruan orbifold cohomology. We introduce our definition of topological Hochschild cohomology algebra with the following steps.

**Step I:** We start with an arbitrary almost complex orbifold  $X$  locally like  $M/\Gamma$ , and introduce a trivial  $S^1$  action on  $X$  and therefore also on  $\tilde{X}$  which is locally like  $(\coprod_{\gamma \in \Gamma} M^\gamma)/\Gamma$ . Accordingly, the  $S^1$ -equivariant cohomology of  $\tilde{X}$  is equal to  $H^\bullet(\tilde{X})(t)$ .

**Step II:** We introduce a “nontrivial”  $S^1$ -action on the tangent bundle  $TX$  of  $X$  which commutes with the trivial  $S^1$ -action on  $X$ . Since  $TX$  is an almost complex bundle,  $S^1$  identified with  $U(1)$  acts on  $TX$  as the center of the principal group  $\mathrm{GL}(\dim_{\mathbb{C}}(TX), \mathbb{C})$ . Geometrically, this action is simply rotation by an angle. We remark that since  $S^1$  is identified as the center of the principal group, the above  $S^1$ -action commutes with all the orbifold structure. And we have made  $TX$  into an  $S^1$ -equivariant orbifold bundle on  $X$ , and the same is for  $p^*TX$  on  $\tilde{X}$ .

**Step III:** We consider the normal bundle  $N^\gamma$  of the embedding of  $X^\gamma$  into  $X$ . Since the  $S^1$ -action on  $TX$  commutes with the  $\gamma$ -action,  $N^\gamma$  inherits an  $S^1$ -action, and becomes an  $S^1$ -equivariant vector bundle on  $X^\gamma$ . We decompose  $N^\gamma$  into a direct sum of  $S^1$ -equivariant line bundles  $\oplus_i N_i^\gamma$ , with respect to the eigenvalue of  $\gamma$ -action, i.e.  $\exp(2\pi i\theta_i)$  and  $0 \leq \theta_i < 1$ . Let  $t_i$  be the equivariant Thom form for  $N_i^\gamma$ , and the equivariant Thom class  $T_\gamma$  of  $N^\gamma$  be defined by

$$T_\gamma := \prod_i t_i.$$

For the following, it is important to remark that  $T_\gamma$  is invertible in  $\Omega_{S^1}^\bullet(N_\gamma)$ .

**Definition 5.1.** Define the topological Hochschild cohomology  $HT^\bullet(X)(t)$  of an orbifold  $X$  to be

$$\bigoplus H^\bullet(X^\gamma)(t)[- \ell(\gamma)],$$

where  $\ell(\gamma)$  is, as before, the codimension of  $X^\gamma \in X$ .

On  $HT^\bullet(X)(t)$ , we define a cup product  $\wedge_t$  as follows. First of all, the cup product is  $\mathbb{C}(t)$  linear. For  $\alpha_i \in \Omega^{\bullet - \ell(\gamma_i)}(X^{\gamma_i})(t)$ ,  $i = 1, 2$ ,  $\alpha_1 \wedge_t \alpha_2$  is defined by the following integral,

$$\langle \alpha_1 \wedge_t \alpha_2, \alpha_3 \rangle = \int_{X^{\gamma_1 \gamma_2}} \frac{\iota^*(\alpha_1 \wedge T_{\gamma_1} \wedge \alpha_2 \wedge T_{\gamma_2})}{\iota^*(T_{\gamma_1 \gamma_2})} \wedge I^*(\alpha_3),$$

for any  $\alpha_3 \in \Omega^{\bullet - \ell(\gamma_1 \gamma_2)}(X^{\gamma_1 \gamma_2})(t)$ ,

**Remark 5.2.** More explicitly, if  $\iota_*$  is the pushforward of  $\Omega^*(\tilde{X})$  into  $\Omega^*(X)$  we have that  $\alpha_1 \wedge_t \alpha_2 = \iota^*(\iota_*(\alpha_1) \wedge \iota_*(\alpha_2))$ . A more global way to write the product, in the style of Section 4.5, is as follows:

$$\alpha_1 \wedge_t \alpha_2 = \int_m \frac{\mathrm{pr}_1^*(\alpha_1 \wedge T) \wedge \mathrm{pr}_2^*(\alpha_2 \wedge T)}{m^*T},$$

where, as before  $m : \mathbb{S} \rightarrow \mathbb{B}_0$  is the multiplication and  $\int_m$  means integration over the discrete fiber.



We remark that the associativity of  $\wedge_t$  is an easy corollary of the associativity of the wedge product on differential forms of  $X$ . The following are a few simple observations of  $HT^\bullet(X)((t))$ , which we state without proofs. (They are corollaries of Theorem 5.2.)

- (1) the product  $\wedge_t$  is  $\mathbb{C}((t))$ -linear;
- (2)  $(HT^\bullet(X)((t)), \wedge_t)$  is a graded algebra.

In summary, with the  $S^1$ -action introduced in Step I and II, we can introduce two cohomology algebra structures on  $\Omega^\bullet(\tilde{X})((t))$ .

- (1)  $S^1$ -equivariant Chen-Ruan orbifold cohomology algebra as in Section 5.1;
- (2) topological Hochschild cohomology algebra as in Definition 5.1.

In the rest of this subsection, we relate the topological Hochschild cohomology  $(HT^\bullet(\tilde{X})((t))[-l], \wedge_t)$  with the  $S^1$ -equivariant Chen-Ruan orbifold cohomology  $(H_{CR}^\bullet(\tilde{X})((t))[-2l], \star_t)$ . The key ingredient connecting these two algebra structures is a certain equivariant Euler class naturally associated to the orbifold.

We consider a stringy K-group class [JAKAKI]  $\mathfrak{s}^\gamma$  associated to the normal bundle  $N^\gamma$  of an orbifold  $X$ , i.e.,

$$\mathfrak{s}^\gamma := \bigoplus_i \theta_i N_i^\gamma. \quad (5.1)$$

The equivariant Euler class  $t_\gamma$  of  $\mathfrak{s}^\gamma$  is defined to be

$$t_\gamma := \iota^* \left( \prod_i t_i^{\theta_i} \right) \in H^{2\iota(\gamma)}(X^\gamma)((t)),$$

where  $t_i$  is the  $S^1$ -equivariant Thom class of  $N_i^\gamma$ ,  $\iota^*$  is the pullback of the Thom form to  $X^\gamma$  which is embedded as the zero section. We remark that  $t_i^{\theta_i}$  and  $t_\gamma$  are well defined in the  $S^1$ -equivariant cohomology  $H^\bullet(X^\gamma)((t))$  by using the Taylor expansion of  $t_i^\theta$  for the  $\theta$ -power.

We define the following isomorphism  $J_\gamma : H^\bullet(X^\gamma)((t))[-2\iota(\gamma)] \rightarrow H^{\bullet-\iota(\gamma)}(X^\gamma)((t))$  of vector spaces,

$$J_\gamma(\alpha) = \alpha/t_{\gamma-1}, \quad \text{for all } \alpha \in H^{\bullet-2\iota(\gamma)}(X^\gamma)((t)).$$

**Remark 5.3.** We observe that  $t_\gamma$  is invertible in  $H^\bullet(X^\gamma)((t))$ , because it has a nonzero constant term. Accordingly,  $J_\gamma$  is a linear isomorphism of the vector spaces. The collection of all  $J_\gamma$  defines an isomorphism  $J = \bigoplus_\gamma J_\gamma$  on  $H^\bullet(\tilde{X})$ .

The map  $J$  preserves grading. The degree of  $t_{\gamma-1}$  is equal to  $\ell(\gamma) - 2\iota(\gamma)$ . If  $\alpha$  is an element in  $H^\bullet(X^\gamma)((t))[-2\iota(\gamma)] = H^{\bullet-2\iota(\gamma)}(X^\gamma)((t))$ ,  $J(\alpha)$  is of degree

$$\bullet - 2\iota(\gamma) - (\ell(\gamma) - 2\iota(\gamma)) = \bullet - \ell(\gamma).$$

The following theorem is a generalization of the result in [CHHU].

**Theorem VI.** *The map  $J$  is an isomorphism of  $\mathbb{C}((t))$ -algebras from the algebra  $(H_{CR}^\bullet(X)((t)), \star_t)$  to  $(HT^\bullet(X)((t)), \wedge_t)$ .*

*Proof.* As we have remarked,  $J$  is an isomorphism of vector spaces preserving the degrees. It is sufficient to show that  $J$  is compatible with the algebra structures.

For  $\alpha^i \in H^\bullet(X^{\gamma_i})((t))[-2\iota(\gamma_i)]$ ,  $i = 1, 2$ , and  $\alpha_3 \in H^\bullet(X^{\gamma_1\gamma_2})((t))[-2\iota(\gamma_1\gamma_2)]$  we have

$$\begin{aligned} & \langle J(\alpha_1) \wedge J(\alpha_2), \alpha_3 \rangle \\ &= \int_{X^{\gamma_1\gamma_2}} \frac{\iota^*\left(\frac{\alpha_1}{t^{\gamma_1-1}} \wedge T_{\gamma_1} \wedge \frac{\alpha_2}{t^{\gamma_2-1}} \wedge T_{\gamma_2}\right)}{\iota^*(T_{\gamma_1\gamma_2})} \wedge I^*(\alpha_3) \\ &= \int_{X^{\gamma_1, \gamma_2}} \frac{\iota^*\left(\frac{\alpha_1}{t^{\gamma_1-1}} \wedge T_{\gamma_1} \wedge \frac{\alpha_2}{t^{\gamma_2-1}} \wedge T_{\gamma_2}\right)}{\iota^*(T_{\gamma_1\gamma_2})} \wedge I^*(\alpha_3) \wedge \frac{\iota^*(T_{\gamma_1\gamma_2})}{\iota^*(T_{\gamma_1, \gamma_2})} \\ &= \int_{X^{\gamma_1, \gamma_2}} \iota^*(\alpha_1) \wedge \iota^*(\alpha_2) \wedge \iota^*(I^*(\alpha_3)) \wedge \frac{\iota^*(T_{\gamma_1}) \wedge \iota^*(T_{\gamma_2})}{\iota^*(t_{\gamma_1-1}) \wedge \iota^*(t_{\gamma_2-1}) \wedge \iota^*(T_{\gamma_1, \gamma_2})}, \end{aligned}$$

where  $X^{\gamma_1, \gamma_2} := X^{\gamma_1} \cap X^{\gamma_2}$ , and  $T_{\gamma_1, \gamma_2}$  is the equivariant Thom form for the normal bundle of  $X^{\gamma_1, \gamma_2}$  in  $X$ , and  $\iota^*$  is the pullback of the forms to  $X^{\gamma_1, \gamma_2}$ . And we can summarize the above computation in the following equation, for  $\gamma_3 = (\gamma_1\gamma_2)^{-1}$ ,

$$\langle J(\alpha_1) \wedge J(\alpha_2), \alpha_3 \rangle = \int_{X^{\gamma_1, \gamma_2}} \iota^*(\alpha_1) \wedge \iota^*(\alpha_2) \wedge \iota^*(I^*(\alpha_3)) \wedge \mathcal{R}_{\gamma_1, \gamma_2, \gamma_3}, \quad (5.2)$$

with

$$\mathcal{R}_{\gamma_1, \gamma_2, \gamma_3} = \frac{\iota^*(T_{\gamma_1}) \wedge \iota^*(T_{\gamma_2})}{\iota^*(t_{\gamma_1-1}) \wedge \iota^*(t_{\gamma_2-1}) \wedge \iota^*(T_{\gamma_1, \gamma_2})}$$

We now apply the result of [JAKAKI] to better understand the term  $\mathcal{R}_{\gamma_1, \gamma_2, \gamma_3}$ . By [JAKAKI][Thm. 1.2], for  $\gamma_1\gamma_2\gamma_3 = id$ , when restricted to  $X^{\gamma_1, \gamma_2} := X^{\gamma_1} \cap X^{\gamma_2}$ , the obstruction bundle  $\Theta_{\gamma_1, \gamma_2}$  as a stringy  $K$ -group class has a natural splitting

$$\Theta_{\gamma_1, \gamma_2} = T(X^{\gamma_1, \gamma_2}) \ominus TX|_{X^{\gamma_1, \gamma_2}} \oplus \mathfrak{s}^{\gamma_1}|_{X^{\gamma_1, \gamma_2}} \oplus \mathfrak{s}^{\gamma_2}|_{X^{\gamma_1, \gamma_2}} \oplus \mathfrak{s}^{\gamma_3}|_{X^{\gamma_1, \gamma_2}}, \quad (5.3)$$

where we remind that  $\mathfrak{s}^{\gamma_i}$  is an element in the stringy  $K$ -group [JAKAKI] as defined in Eq. (5.1).

We remark that the above isomorphism for  $\Theta_{\gamma_1, \gamma_2}$  again holds as  $S^1$ -equivariant bundles because the  $S^1$  actions on the respective bundles are defined by the almost complex structures and the equation (5.3) preserves almost complex structures. Now taking the equivariant Euler classes of the bundles in Eq. (5.3) on  $X^{\gamma_1, \gamma_2}$ , we have that on  $X^{\gamma_1, \gamma_2}$

$$\begin{aligned} \text{eu}_{S^1}(\Theta_{\gamma_1, \gamma_2}) &= \frac{\iota^*(t_{\gamma_1}) \wedge \iota^*(t_{\gamma_2}) \wedge \iota^*(t_{\gamma_3})}{\iota^*(T_{\gamma_1, \gamma_2})} \\ &= \frac{\iota^*(T_{\gamma_1}) \wedge \iota^*(T_{\gamma_2}) \wedge \iota^*(T_{\gamma_3})}{\iota^*(t_{\gamma_1-1}) \wedge \iota^*(t_{\gamma_2-1}) \wedge \iota^*(t_{\gamma_3-1}) \wedge \iota^*(T_{\gamma_1, \gamma_2})}, \end{aligned}$$

where in the second equality, we have used the fact that on  $X^{\gamma_1, \gamma_2}$ ,

$$\iota^*(t_{\gamma_i}) = \frac{\iota^*(T_{\gamma_i})}{\iota^*(t_{\gamma_i-1})} \quad \text{for } i = 1, 2, 3.$$

We use the above expression for  $\text{eu}_{S^1}(\Theta_{\gamma_1, \gamma_2})$  to compute  $\langle J(\alpha_1 \star_t \alpha_2), \alpha_3 \rangle$ .

$$\begin{aligned} & \langle J(\alpha_1 \star_t \alpha_2), \alpha_3 \rangle \\ &= \int_{X^{\gamma_1, \gamma_2}} \frac{\alpha_1 \star_t \alpha_2}{t_{(\gamma_1 \gamma_2)^{-1}}} \wedge I^*(\alpha_3) \\ &= \int_{X^{\gamma_1, \gamma_2}} \alpha_1 \wedge \alpha_2 \wedge \frac{\iota^*(I^*(\alpha_3))}{\iota^*(t_{(\gamma_1 \gamma_2)^{-1}})} \wedge \text{eu}_{S^1}(\Theta_{\gamma_1, \gamma_2}) \\ &= \int_{X^{\gamma_1, \gamma_2}} \alpha_1 \wedge \alpha_2 \wedge \frac{\iota^*(I^*(\alpha_3))}{\iota^*(t_{(\gamma_1 \gamma_2)^{-1}})} \wedge \frac{\iota^*(T_{\gamma_1}) \wedge \iota^*(T_{\gamma_2}) \wedge \iota^*(T_{\gamma_3})}{\iota^*(t_{\gamma_1^{-1}}) \wedge \iota^*(t_{\gamma_2^{-1}}) \wedge \iota^*(t_{\gamma_3^{-1}}) \wedge \iota^*(T_{\gamma_1, \gamma_2})}. \end{aligned}$$

Using the equality

$$\iota^*(t_{(\gamma_1 \gamma_2)^{-1}}) = \iota^*(t_{\gamma_3}) = \frac{\iota^*(T_{\gamma_3})}{\iota^*(t_{\gamma_3^{-1}})},$$

we conclude that

$$\begin{aligned} & \langle J(\alpha_1 \star_t \alpha_2), \alpha_3 \rangle \\ &= \int_{X^{\gamma_1, \gamma_2}} \iota^*(\alpha_1) \wedge \iota^*(\alpha_2) \wedge \iota^*(I^*(\alpha_3)) \wedge \frac{\iota^*(T_{\gamma_1}) \wedge \iota^*(T_{\gamma_2})}{\iota^*(t_{\gamma_1^{-1}}) \wedge \iota^*(t_{\gamma_2^{-1}}) \wedge \iota^*(T_{\gamma_1, \gamma_2})} \\ &= \langle J(\alpha_1) \wedge_t J(\alpha_2), \alpha_3 \rangle. \end{aligned}$$

The last equation, combining with Poincaré duality, implies that

$$J^{-1}(J(\alpha_1) \wedge J(\alpha_2)) = \alpha_1 \star_t \alpha_2.$$

This completes the proof.  $\square$

**Remark 5.4.** Note that when  $t$  is equal to 0, the map  $J$  is not invertible generally. However, one can solve this problem by working in the formal framework as in [CHHu]. In this case our model extends Chen-Hu's model to an arbitrary almost complex orbifold.

**5.3. Topological and algebraic Hochschild cohomology.** In the case of a symplectic orbifold  $(X, \omega)$ , we have two cohomology algebra structures from different approaches. One is the Hochschild cohomology algebra of the quantized groupoid algebra computed in Theorem 4.17, the other is the topological Hochschild cohomology  $HT^\bullet(X)((t))$  defined in Definition 5.1 using essentially a unique (up to homotopy) compatible almost complex structure to the symplectic structure on  $X$ . We observe that the algebra structure on the Hochschild cohomology of the quantized groupoid algebra is completely topological, which does not depend on the symplectic structures or the almost complex structures at all. On the other hand, the topological Hochschild cohomology  $HT^\bullet(X)((t))$  does depend on the choices of almost complex structures. Therefore, it is natural to expect that these two algebras are not isomorphic. In this subsection, we would like to study the connections between these two algebra structures. We show in the following that the graded algebra of the topological Hochschild cohomology algebra is isomorphic to the Hochschild cohomology of the corresponding quantized groupoid algebra.

We introduce a decreasing filtration on the topological Hochschild cohomology  $HT^\bullet(X)((t))$  as follows

$$\mathcal{F}^* = \{\alpha \in HT^\bullet(X^\gamma)((t)) \mid \deg(\alpha) - \ell(\gamma) \geq *\}.$$

**Lemma 5.5.**  $(HT^\bullet(X)((t)), \wedge_t, \mathcal{F}^*)$  is a filtered algebra.

*Proof.* One needs to prove that  $\mathcal{F}^k \wedge_t \mathcal{F}^l \subset \mathcal{F}^{k+l}$ . To this end let  $\alpha_1 \in \mathcal{F}^k$  and  $\alpha_2 \in \mathcal{F}^l$  and consider  $\alpha_1 \wedge_t \alpha_2$ . Without loss of generality, let us assume that  $\alpha_1 \in HT^{k_1}(X^{\gamma_1})(t)$  and  $\alpha_2 \in HT^{k_2}(X^{\gamma_2})(t)$  with  $k_1 \geq k + \ell(\gamma_1)$  and  $k_2 \geq l + \ell(\gamma_2)$ . Since  $(HT^\bullet(X)(t), \wedge_t)$  is a graded algebra respect to  $\bullet$ , we have that  $\deg(\alpha_1 \wedge \alpha_2) = k_1 + k_2 \geq k + \ell(\gamma_1) + l + \ell(\gamma_2)$ . Since  $\ell(\gamma_1) + \ell(\gamma_2) \geq \ell(\gamma_1\gamma_2)$ , we have that

$$\deg(\alpha_1 \wedge_t \alpha_2) - \ell(\gamma_1\gamma_2) \geq k + l + \ell(\gamma_1) + \ell(\gamma_2) - \ell(\gamma_1\gamma_2) \geq k + l.$$

Therefore,  $\alpha_1 \wedge_t \alpha_2$  belongs to  $\mathcal{F}^{k+l}$ .  $\square$

**Lemma 5.6.** *Let  $\Gamma$  be a finite group acting a vector space  $V$ . Then for every  $\gamma_1, \gamma_2 \in \Gamma$  one has  $\ell(\gamma_1) + \ell(\gamma_2) = \ell(\gamma_1\gamma_2)$  if and only if  $V^{\gamma_1} + V^{\gamma_2} = V$  and  $V^{\gamma_1\gamma_2} = V^{\gamma_1} \cap V^{\gamma_2}$ .*

*Proof.* By linear algebra one knows that

$$\dim(V^{\gamma_1}) + \dim(V^{\gamma_2}) = \dim(V^{\gamma_1} + V^{\gamma_2}) + \dim(V^{\gamma_1} \cap V^{\gamma_2}).$$

Moreover, one has

$$\begin{aligned} \ell(\gamma_1) + \ell(\gamma_2) &= 2 \dim(V) - (\dim(V^{\gamma_1}) + \dim(V^{\gamma_2})) \\ &= 2 \dim(V) - \dim(V^{\gamma_1} + V^{\gamma_2}) - \dim(V^{\gamma_1} \cap V^{\gamma_2}) \\ &= \dim(V) - \dim(V^{\gamma_1} + V^{\gamma_2}) + \dim(V) - \dim(V^{\gamma_1} \cap V^{\gamma_2}). \end{aligned}$$

Since  $V^{\gamma_1} + V^{\gamma_2} \subset V$  and  $V^{\gamma_1} \cap V^{\gamma_2} \subset V^{\gamma_1\gamma_2}$ , we have

$$\dim(V) - \dim(V^{\gamma_1} + V^{\gamma_2}) \geq 0, \quad \dim(V) - \dim(V^{\gamma_1} \cap V^{\gamma_2}) \geq \dim(V) - \dim(V^{\gamma_1\gamma_2}).$$

Therefore

$$\ell(\gamma_1) + \ell(\gamma_2) \geq \ell(\gamma_1\gamma_2),$$

and equality holds, if and only if  $\dim(V) = \dim(V^{\gamma_1} + V^{\gamma_2})$  and  $\dim(V^{\gamma_1} \cap V^{\gamma_2}) = \dim(V^{\gamma_1\gamma_2})$ .  $\square$

**Theorem VII.** *The graded algebra  $\text{gr}(HT^\bullet(X)(t))$  of  $(HT^\bullet(X)(t), \wedge_t)$  with respect to the filtration  $\mathcal{F}^*$  is isomorphic to the Hochschild cohomology algebra  $(H^\bullet(\mathcal{A}^{(\hbar)} \rtimes G; \mathcal{A}^{(\hbar)} \rtimes G), \cup)$  by identifying  $t$  with  $\hbar$ .*

*Proof.* Obviously, the two vector spaces over  $\mathbb{C}(t)$  are isomorphic. It is sufficient to prove that the two product structures agree.

According to the proof of Lemma 5.5, we have that for  $\alpha_1 \in \mathcal{F}^k$  and  $\alpha_2 \in \mathcal{F}^l$ , the graded product  $\text{gr}(\alpha_1 \wedge_t \alpha_2)$  is not equal to zero only when  $\ell(\gamma_1) + \ell(\gamma_2) = \ell(\gamma_1\gamma_2)$ .

In the case of  $\ell(\gamma_1) + \ell(\gamma_2) = \ell(\gamma_1\gamma_2)$ , by Lemma 5.6, we have that  $V^{\gamma_1} + V^{\gamma_2} = V$  and  $V^{\gamma_1\gamma_2} = V^{\gamma_1} \cap V^{\gamma_2}$ . This implies that  $N^{\gamma_1} \oplus N^{\gamma_2} = N^{\gamma_1\gamma_2}$  on  $X^{\gamma_1\gamma_2}$ . Therefore, the following identity of equivariant Thom classes holds true:

$$\iota^*(T_{\gamma_1} \wedge T_{\gamma_2}) = \iota^*(T_{\gamma_1\gamma_2}).$$

Hence, by Definition 5.1, one obtains

$$\langle \alpha_1 \wedge_t \alpha_2, \alpha \rangle = \int_{X^{\gamma_1, \gamma_2}} \iota_{\gamma_1}^*(\alpha_1|_{\gamma_1}) \wedge \iota_{\gamma_2}^*(\alpha_2|_{\gamma_2}) \wedge I^*(\alpha)|_{\gamma_1\gamma_2},$$

where the  $\wedge$  on the right hand side is the wedge product on differential forms. One concludes that  $\text{gr}(\alpha_1 \wedge_t \alpha_2)$  agrees with the cup product on the Hochschild cohomology algebra.  $\square$

From Theorem VII, we can view that the topological Hochschild cohomology  $(HT^\bullet(X)((t)), \wedge_t)$  as a deformation of the algebraic Hochschild cohomology

$$(H^\bullet(\mathcal{A}^{((h))} \rtimes G; \mathcal{A}^{((h))} \rtimes G), \cup).$$

It is very interesting to study this deformation using the Hochschild cohomology method again, which will illustrate the role of the almost complex structure chosen to define  $\wedge_t$ . We leave this topic for future research.

APPENDIX A. HOMOLOGICAL ALGEBRA OF BORNLOGICAL ALGEBRAS AND MODULES

**A.1. Bornologies on vector spaces.** In this appendix we recollect the basic definitions and constructions in the theory of bornological vector spaces. For further details on this see [BO] and [HO].

Let  $\mathbb{k}$  be the ground field  $\mathbb{R}$  or  $\mathbb{C}$ , and  $V$  be a vector space over  $\mathbb{k}$ . A set  $\mathfrak{B}$  of subsets of  $V$  is called a (*convex linear*) *bornology* on  $V$  and  $(V, \mathfrak{B})$  a (*convex linear*) *bornological vector space*, if the following axioms hold true:

- (BOR1) Every subset of an element of  $\mathfrak{B}$  belongs to  $\mathfrak{B}$ .
- (BOR2) Every finite union of elements of  $\mathfrak{B}$  belongs to  $\mathfrak{B}$ .
- (BOR3) The set  $\mathfrak{B}$  is *covering* for  $V$  that means every element of  $V$  is contained in some set belonging to  $\mathfrak{B}$ .
- (BOR4) For every  $B \in \mathfrak{B}$ , the *absolutely convex hull*

$$B^\diamond := \{\lambda_1 v_1 + \lambda_2 v_2 \mid v_1, v_2 \in B, \lambda_1, \lambda_2 \in \mathbb{k}, |\lambda_1| + |\lambda_2| \leq 1\}$$

is again  $\mathfrak{B}$ .

The elements of a bornology  $\mathfrak{B}$  are called its *bounded sets* or sometimes its *small sets*.

Given an absolutely convex set  $S \subset V$ , we denote its linear span by  $V_S$  and by  $\|\cdot\|_B$  the seminorm on  $V_S$  with unit ball  $\overline{S} := \bigcap_{\lambda > 1} \lambda S$ . If  $\|\cdot\|_S$  is a norm on  $V_S$ , then  $S$  is said to be *norming*, and *completant*, if  $(V_S, \|\cdot\|_S)$  is even a Banach space. A bornological vector space  $(V, \mathfrak{B})$  is called *separated* (resp. *complete*), if every bounded absolutely convex set  $B \subset V$  is norming resp. completant.

**Proposition A.1.** *Let  $V$  be a bornological vector space. Then there exists a complete bornological vector space  $\hat{V}$  together with a bounded linear map  $\iota : V \rightarrow \hat{V}$  such that the following universal property is fulfilled:*

- *For every complete bornological vector space  $W$  and every bounded linear map  $f : V \rightarrow W$  there exists a unique bounded linear map  $\hat{f} : \hat{V} \rightarrow W$  such that the diagram*

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ \downarrow \iota & \nearrow \hat{f} & \\ \hat{V} & & \end{array} \tag{A.1}$$

*commutes.*

*Proof.* For the proof of this see [ME99]. □

For  $(V, \mathfrak{B})$  and  $(W, \mathcal{D})$  two bornological vector spaces, a linear map  $f : V \rightarrow W$  is called *bounded*, if for every  $S \in \mathfrak{B}$  the image  $f(S)$  is in  $\mathcal{D}$ . The space of bounded linear maps  $V \rightarrow W$  will be denoted by  $\text{Hom}(V, W)$ . It carries itself a canonical

bornology, namely the bornology of *equibounded* sets of linear maps, i.e. of subsets  $E \subset \text{Hom}(V, W)$  such that for each  $S \in \mathfrak{B}$  the set  $E(S)$  is bounded in  $(W, \mathfrak{D})$ . Obviously, the bornological vector spaces together with the bounded linear maps then form a category. Since the direct sum  $V \oplus W$  of two bornological vector spaces obviously inherits a canonical bornological structure from its components, the category of bornological vector spaces is even an additive category. Moreover, it carries the structure of a tensor category, since the algebraic tensor product  $V \otimes W$  of two bornological vector spaces  $(V, \mathfrak{B})$  and  $(W, \mathfrak{D})$  carries a natural bornology which is generated by the sets  $S \otimes T$ , where  $S \in \mathfrak{B}$ ,  $T \in \mathfrak{D}$ .

In case  $V$  and  $W$  are both complete bornological vector spaces, the direct sum  $V \oplus W$  is obviously a complete bornological vector space as well. For the tensor product  $V \otimes W$ , though, with its canonical bornological structure, completeness need not necessarily hold. Therefore, one introduces the completed tensor product  $V \hat{\otimes} W := (V \otimes W)^\wedge$  for any pair of bornological vector spaces  $V, W$ . Note that the category of complete bornological vector spaces with  $\oplus$  and  $\hat{\otimes}$  as direct sum resp. tensor functor also satisfies the axioms of an additive tensor category. We denote the category of complete bornological vector spaces and bounded linear maps by  $\text{Bor}$ .

**Example A.2.** Let  $V$  be a locally convex topological vector space. Then

$$\begin{aligned} \mathfrak{Bnd}(V) &:= \{S \subset V \mid p(S) < \infty \text{ for every seminorm } p \text{ on } V\} \quad \text{and} \\ \mathfrak{Cpt}(V) &:= \{S \subset V \mid S \text{ is precompact in } V\} \end{aligned}$$

are two, in general different, bornologies on  $V$ , which one calls, respectively, the *von Neumann* and the *precompact* bornology.

**A.2. Bornological algebras and modules.** By a *bornological algebra* one understands a  $\mathbb{k}$ -algebra  $A$  together with a complete convex bornology  $\mathfrak{B}$  such that the product map  $m : A \otimes A \rightarrow A$  is bounded. By the universal property of the completed bornological tensor product one knows that for such an  $A$  the multiplication  $m$  lifts uniquely to a bounded map  $A \hat{\otimes} A \rightarrow A$ .

For any (real or complex) algebra  $A$  we denote by  $A^+$  the unital algebra  $A \oplus \mathbb{k}$ , and by  $A^u$  the smallest unital algebra containing  $A$ , which means that  $A^u$  coincides with  $A$ , if  $A$  is unital, and with  $A^+$  otherwise. Obviously,  $A^+$  and  $A^u$  are again bornological algebras, if that is the case already for  $A$ . For every bornological algebra  $A$  we denote by  $A^\circ$  its *enveloping algebra* which is defined as the bornological tensor product algebra  $A^u \hat{\otimes} (A^u)^{\text{op}}$ .

By a (*left*)  $A$ -*module* over a bornological algebra  $A$  one understands a complete bornological vector space  $M$  together with a bounded linear map  $A^u \hat{\otimes} M \rightarrow M$  such that the following axioms are satisfied:

(MOD1) One has  $(a_1 \cdot a_2) \cdot m = a_1 \cdot (a_2 \cdot m)$  for all  $a_1, a_2 \in A$  and  $m \in M$ .

(MOD2) The relation  $1 \cdot m = m$  holds for all  $m \in M$ .

**Example A.3.** For every complete bornological vector space  $V$  the tensor product  $A^u \hat{\otimes} V$  carries in a natural way the structure of a left  $A$ -module. Modules of this form are called *free* left  $A$ -modules; likewise one defines free right  $A$ -modules.

Given left  $A$ -modules  $M$  and  $N$  we write  $\text{Hom}_A(M, N)$  for the space of bounded  $A$ -module homomorphisms with the equibounded bornology. Obviously, the left  $A$ -modules together with these morphisms form a category, which we will denote by  $\text{Mod}(A)$ . Note that every morphism  $f : M \rightarrow N$  in  $\text{Mod}(A)$  has a kernel and a

cokernel. The kernel simply coincides with the vector space kernel equipped with the subspace bornology, where the cokernel is the quotient  $N/\widehat{f(M)}$  together with the quotient bornology. Similarly, one defines *right  $A$ -modules* over a bornological algebra  $A$  and writes  $\text{Mod}(A^{\text{op}})$  (resp.  $\text{Hom}_{A^{\text{op}}}(M, N)$ ) for the category of right  $A$ -modules (resp. the set of right  $A$ -module morphisms from  $M$  to  $N$ ). Finally, an object in the category  $\text{Mod}(A^e)$  will be called an  *$A$ -bimodule*.

For any right  $A$ -module  $M$  and any left  $A$ -module  $N$  we denote by  $M\hat{\otimes}_A N$  the  *$A$ -balanced tensor product* that means the cokernel of the bounded linear map

$$M\hat{\otimes}_A\hat{\otimes}N \rightarrow M\hat{\otimes}N, \quad m \otimes a \otimes n \mapsto m \cdot a \otimes n - m \otimes a \cdot n.$$

A bornological algebra  $A$  is said to have an *approximate identity*, if for every bounded subset  $S \subset A$  there is a bounded sequence  $(u_{S,k})_{k \in \mathbb{N}}$  and an absolutely convex bounded  $T_S \subset A$  such that the following properties hold true:

- (AID1) For every  $a \in A_S$  one has  $u_{S,k} \cdot a \in A_{T_S}$  and  $a \cdot u_{S,k} \in A_{T_S}$ .
- (AID2) For all  $a \in S$ , the sequences  $u_{S,k} \cdot a$  and  $a \cdot u_{S,k}$  converge to  $a$  in the Banach space  $A_{T_S}$ , and the convergence is uniform in  $a$ .
- (AID3) For bounded subsets  $S_1, S_2 \subset A$  such that  $S_1 \subset S_2$  one has

$$\|u_{S_2,k} \cdot a - a\|_{T_{S_2}} \leq \|u_{S_1,k} \cdot a - a\|_{T_{S_2}} \quad \text{for all } a \in A_{T_{S_1}} \text{ and } k \in \mathbb{N}.$$

In other words, an approximate identity  $(u_{S,k})_{S \in \mathcal{B}, k \in \mathbb{N}}$  is essentially a net in  $A$  such that each of the nets  $(u_{S,k} a)$  and  $(a u_{S,k})$  converges to  $a$ .

A bornological algebra  $A$  which possesses an approximate identity and which, additionally, is projective both as a left and a right  $A$ -module, is called *quasi-unital*. Note that under the assumption that  $A$  has an approximate identity, projectivity of  $A$  is equivalent to the existence of a bounded left  $A$ -module map  $l : A \rightarrow A^n \hat{\otimes} A$  and a bounded right  $A$ -module map  $r : A \rightarrow A \hat{\otimes} A^n$  which are both sections of the multiplication map (cf. [ME04]).

Given a quasi-unital bornological algebra  $A$ , a left  $A$ -module  $M$  (resp. a right  $A$ -module  $N$ ) is called *essential*, if the canonical map  $A\hat{\otimes}_A M \rightarrow M$  (resp.  $N\hat{\otimes}_A A \rightarrow N$ ) is an isomorphism. If the left  $A$ -module  $M$  (resp. the right  $A$ -module  $N$ ) has the property that the canonical map  $M \rightarrow \text{Hom}_A(A, M)$  (resp.  $N \rightarrow \text{Hom}_{A^{\text{op}}}(A, N)$ ) is an isomorphism, one calls  $M$  (resp.  $N$ ) a *rough* module. The category of essential left  $A$ -modules (resp. right  $A$ -modules) will be denoted by  $\text{Mod}_e(A)$  (resp. by  $\text{Mod}_e(A^{\text{op}})$ ). Since  $A$  is assumed to be quasi-unital, one concludes that for every  $A$ -module  $M$ , the tensor product  $A\hat{\otimes}_A M$  is an essential module.

**A.3. Resolutions and homology.** In this article we consider homology theories in the additive but in general not abelian category of modules over a bornological algebra  $A$ . This implies that we have to use methods from relative homological algebra. Essentially this means that only so-called allowable chain complexes and allowable projective resolutions are used to determine homologies and cohomologies. To define the notion of allowability precisely recall that a bounded epimorphism of left  $A$ -modules  $f : M \rightarrow N$  or in other words a short exact sequence of left  $A$ -modules and bounded maps

$$0 \rightarrow K \rightarrow M \xrightarrow{f} N \rightarrow 0$$

is called *linearly split*, if there exists a bounded linear map  $N \rightarrow M$  which is a section of  $f$ . A left  $A$ -module  $P$  is now called *projective*, if the functor  $\text{Hom}_A(P, -)$  is exact on linearly split short exact sequences in  $\text{Mod}(A)$ . Moreover, a chain complex

$(C_\bullet, \partial)$  of  $A$ -modules and bounded maps  $\partial_k : C_k \rightarrow C_{k-1}$  is called *allowable*, if for every  $k$  the image of  $\partial_k$  is in  $\text{Mod}(A)$ , i.e. is a complete bornological subspace of  $C_{k-1}$ , and if the bounded epimorphism  $\overline{\partial}_k : C_k \rightarrow \text{im } \partial_k$  induced by  $\partial_k$  is linearly split. Likewise one defines allowable cochain complexes. For homology theories in categories of modules of bornological algebras the following result now is crucial.

**Proposition A.4.** *Let  $A$  be a quasi-unital bornological algebra  $A$ . Then every free left  $A$ -module is projective. Moreover, the category  $\text{Mod}(A)$  has enough projectives, that means for every left  $A$ -module  $M$  there exists a projective left  $A$ -module  $P$  together with a split epimorphism of  $A$ -modules  $P \rightarrow M$ . Hereby,  $P$  can be chosen to be free. Finally, the functor*

$$\text{Mod}(A) \rightarrow \text{Mod}_e(A), \quad M \mapsto A \hat{\otimes} M$$

*preserves projective modules, and the category  $\text{Mod}_e(A)$  of essential left  $A$ -modules has enough projectives as well.*

*Proof.* See [ME04, Sec. 4]. □

The proposition implies that for every left  $A$ -module there exists an *allowable projective resolution* of  $M$ , i.e. an allowable acyclic complex  $(P_\bullet, \partial)$  of projective left  $A$ -modules  $P_k$ ,  $k \geq 0$  together with a quasi-isomorphism  $\varepsilon : P_\bullet \rightarrow M_\bullet$  in the category of left  $A$ -modules, where  $M_\bullet$  denotes the complex which is concentrated in degree 0 and coincides there with the  $A$ -module  $A$ . These conditions are equivalent to the requirement that  $\varepsilon$  is a split bounded  $A$ -linear surjection  $\varepsilon : P_0 \rightarrow M$  which satisfies

$$\varepsilon \circ \partial_1 = 0$$

and that there exists an  $A$ -linear splitting  $h : M \rightarrow P_0$  and a family  $(h_k)_{k \in \mathbb{N}}$  of bounded linear maps  $h_k : P_k \rightarrow P_{k+1}$  such that

$$\partial_1 h_0 = \text{id}_{P_0} - h\varepsilon \quad \text{and} \quad \partial_{k+1} h_k - h_{k-1} \partial_k = \text{id}_{P_k} \quad \text{for all } k \geq 1.$$

The proof of the following result is standard in (relative) homological algebra.

**Theorem A.5** (Comparison Theorem). *(cf. [ME99, Thm. A.9]) Assume that  $M$  and  $N$  are two left  $A$ -modules over a bornological algebra  $A$ . Let  $P_\bullet \rightarrow M_\bullet$  and  $Q_\bullet \rightarrow N_\bullet$  be allowable resolutions of  $M$  resp.  $N$ . If  $P_\bullet$  is projective, then there exists for every morphism  $f : M \rightarrow N$  of left  $A$ -modules a lifting of  $f$ , i.e. a chain map  $F : P_\bullet \rightarrow Q_\bullet$  in the category of left  $A$ -modules such that the diagram*

$$\begin{array}{ccc} P_\bullet & \longrightarrow & M_\bullet \\ F \downarrow & & \downarrow f \\ Q_\bullet & \longrightarrow & N_\bullet \end{array}$$

*commutes. Any two such liftings of  $f$  are homotopic. In particular, any two allowable projective resolutions of  $M$  are homotopy equivalent.*

The comparison theorem allows the construction of derived functors in the category of  $A$ -modules. In particular, the functors  $\text{Ext}$  and  $\text{Tor}$  can now be defined for  $A$ -modules as usual.



**A.4. Hochschild homology and Bar resolution.** Given a bornological algebra  $A$  and an  $A$ -bimodule  $M$ , the Hochschild homology  $H_\bullet(A, M)$  and cohomology  $H^\bullet(A, M)$  are defined as derived functors in the category  $\text{Mod}(A^e)$  of  $A$ -bimodules as follows:

$$H_\bullet(A, M) := \text{Tor}_\bullet^{A^e}(A, M), \quad H^\bullet(A, M) := \text{Ext}_{A^e}^\bullet(A, M). \quad (\text{A.2})$$

A particularly useful resolution of the  $A$ -bimodule  $A$  is given by the *Bar complex*  $(\text{Bar}_\bullet(A), b')$  together with the multiplication map inducing the quasi-isomorphism  $\text{Bar}_\bullet(A) \rightarrow A$ . Hereby,

$$\text{Bar}_k(A) = A \hat{\otimes} A^{\hat{\otimes} k} \hat{\otimes} A,$$

and  $b'$  is the standard boundary map on the Bar complex:

$$b'_0(a_0 \otimes a_1) = 0,$$

$$b'_k(a_0 \otimes a_1 \dots a_k \otimes a_{k+1}) = \sum_{i=0}^k (-1)^i a_0 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_{k+1}.$$

Obviously, if  $A$  is quasi-unital, then the Bar complex of  $A$  provides an allowable projective resolution of  $A$  in the category of  $A$ -bimodules. In other words, this means that a quasi-unital bornological algebra  $A$  is *H-unital* in the sense of Wodzicki (see [WO, LO]). Thus, for quasi-unital  $A$ , the Hochschild homology and cohomology groups  $H_\bullet(A, M)$ ,  $H^\bullet(A, M)$  are computed as the homology resp. cohomology of the Hochschild complexes

$$\begin{aligned} (C_\bullet(A, M), b'_*) \text{ with } C_\bullet(A, M) &:= \text{Bar}_\bullet(A) \hat{\otimes}_{A^e} M, \quad \text{and} \\ (C^\bullet(A, M), b'^*) \text{ with } C^\bullet(A, M) &:= \text{Hom}_{A^e}(\text{Bar}_\bullet(A), M). \end{aligned} \quad (\text{A.3})$$

For some applications, in particular to define the cup product on the Hochschild cochain complex of a quasi-unital bornological algebra which does not have a unit, the left and right reduced Bar complexes  $(\text{Bar}_\bullet^{\text{l-red}}(A), b')$  and  $(\text{Bar}_\bullet^{\text{r-red}}(A), b')$  are quite useful. They carry the same boundary as the Bar complex, and have components

$$\text{Bar}_k^{\text{l-red}}(A) := A^u \hat{\otimes} A^{\hat{\otimes} k} \hat{\otimes} A \quad \text{and} \quad \text{Bar}_k^{\text{r-red}}(A) := A \hat{\otimes} A^{\hat{\otimes} k} \hat{\otimes} A^u.$$

Obviously, under the assumption that  $A$  is quasi-unital, the left and right reduced Bar complexes are both allowable projective resolutions of  $A$ . Moreover, the canonical embeddings  $\text{Bar}_\bullet(A) \hookrightarrow \text{Bar}_\bullet^{\text{l-red}}(A)$  and  $\text{Bar}_\bullet(A) \hookrightarrow \text{Bar}_\bullet^{\text{r-red}}(A)$  have the following quasi-inverses:

$$\begin{aligned} r_k : \text{Bar}_k^{\text{l-red}}(A) &\rightarrow \text{Bar}_k(A), \quad a_0 \otimes \dots \otimes a_{k+1} \mapsto a_0 r(a_1) \cdot \dots \cdot r(a_{k+1}), \\ l_k : \text{Bar}_k^{\text{r-red}}(A) &\rightarrow \text{Bar}_k(A), \quad a_0 \otimes \dots \otimes a_{k+1} \mapsto l(a_0) \cdot \dots \cdot l(a_k) a_{k+1}, \end{aligned} \quad (\text{A.4})$$

where  $l : A \rightarrow A \hat{\otimes} A$  resp.  $r : A \rightarrow A \hat{\otimes} A$  is an  $A$ -left resp.  $A$ -right linear section of the multiplication map on  $A$ .

Finally in this section we consider the *reduced Hochschild chain* and *reduced Hochschild cochain complexes*. These are defined by

$$\begin{aligned} C_k^{\text{red}}(A, M) &:= \begin{cases} A \hat{\otimes}_A M \hat{\otimes}_A A, & \text{if } k = 0, \\ M \hat{\otimes} (A^u/\mathbb{k})^{\hat{\otimes} k} & \text{if } k \geq 1, \end{cases} \\ C_{\text{red}}^k(A, N) &:= \begin{cases} \text{Hom}_{A^e}(A \hat{\otimes} A, N), & \text{if } k = 0, \\ \text{Hom}_{\mathbb{k}}((A^u/\mathbb{k})^{\hat{\otimes} k}, N), & \text{if } k \geq 1, \end{cases} \end{aligned} \quad (\text{A.5})$$

and carry the same boundary resp. coboundary maps as the unreduced complexes. By construction the canonical maps

$$C_\bullet(A, M) \rightarrow C_\bullet^{\text{red}}(A, M) \quad \text{and} \quad C_{\text{red}}^\bullet(A, N) \rightarrow C^\bullet(A, N)$$

are then chain maps. If  $M$  is an essential  $A$ -bimodule (resp.  $N$  a rough  $A$ -bimodule), then the first (resp. the second) of these chain maps is a quasi-isomorphism. Since  $A$  is assumed to be quasi-unital, the first chain map is always a quasi-isomorphism for  $M = A$ . In many applications, and in particular those appearing in this article, the second chain map is also a quasi-isomorphism for  $N = A$ . In case  $A$  is unital, both chain maps are always quasi-isomorphisms.

**A.5. The cup product on Hochschild cohomology.** Under the general assumption from above that  $A$  is a (possibly nonunital) bornological algebra, we will now explain the construction of the cup product

$$\cup : H^\bullet(A, A) \times H^\bullet(A, A) \rightarrow H^\bullet(A, A).$$

One way to define  $\cup$  is via the Yoneda product on (bounded) extensions

$$0 \rightarrow A \rightarrow E_1 \rightarrow \cdots \rightarrow E_k \rightarrow A \rightarrow 0$$

and the interpretation of  $\text{Ext}_{Ae}^k(A, A)$  as the space of equivalence classes of such extensions. Alternatively, and that is the approach we will follow here, one can use the quasi-isomorphisms from Eq. (A.4) to directly define a cup product  $\cup : C^\bullet(A, A) \times C^\bullet(A, A) \rightarrow C^\bullet(A, A)$  on the Hochschild cochain complex, which on cohomology coincides with the Yoneda product. More precisely, we define for  $f \in C^k(A, A)$  and  $g \in C^l(A, A)$  the product  $f \cup g \in C^{k+l}(A, A)$  by

$$f \cup g(a_0 \otimes \cdots \otimes a_{k+l+1}) := f(l_k(a_0 \otimes \cdots \otimes a_k \otimes 1)) g(r_l(1 \otimes a_{k+1} \otimes \cdots \otimes a_{k+l+1})), \quad (\text{A.6})$$

where  $a_0, \dots, a_{k+l+1} \in A$ . It is straightforward to check that the thus defined map  $\cup$  is a chain map and associative up to homotopy. The cup-product induced on the reduced Hochschild cochain complex by the embedding  $C_{\text{red}}^\bullet(A, A) \rightarrow C^\bullet(A, A)$  is given by

$$f \cup g(a_1 \otimes \cdots \otimes a_{k+l}) := f(a_1 \otimes \cdots \otimes a_k) g(a_{k+1} \otimes \cdots \otimes a_{k+l}) \quad (\text{A.7})$$

for  $f \in C_{\text{red}}^k(A, A)$ ,  $g \in C_{\text{red}}^l(A, A)$  and  $a_1, \dots, a_{k+l} \in A$ .

#### A.6. Bornological structures on convolution algebras and their modules.

Consider a proper étale Lie groupoid  $\mathbb{G}$  and let  $\mathcal{A} \rtimes \mathbb{G}$  denote its convolution algebra (see Sec. 2). A subset  $S \subset \mathcal{A} \rtimes \mathbb{G}$  is said to be *bounded*, if there is a compact subset in  $K \subset \mathbb{G}_1$  such that  $\text{supp } a \subset K$  for every  $a \in S$ , and if for each differential operator  $D$  on  $\mathbb{G}_1$  one has

$$\sup_{a \in S} \|Da\|_K < \infty.$$

The bounded subset of  $\mathcal{A} \rtimes \mathbb{G}$  form a bornology which coincides both with the von Neumann and the precompact bornology defined in Example A.2. In this article, we always assume that  $\mathcal{A} \rtimes \mathbb{G}$  carries this bornology. By an immediate argument one checks that the convolution product is bounded and that the bornology on  $\mathcal{A} \rtimes \mathbb{G}$  is complete. Thus  $\mathcal{A} \rtimes \mathbb{G}$  becomes a bornological algebra. Let us check that it is quasi-unital. To this end choose a sequence of smooth maps  $\varphi_k : \mathbb{G}_0 \rightarrow [0, 1]$  such that the support of each  $\varphi_k$  is compact and such that  $(\varphi_k^2)_{k \in \mathbb{N}}$  is a locally finite partition of unity on  $\mathbb{G}_0$ . Obviously, one can even achieve that

$$\overline{(\text{supp } \varphi_k)}^\circ = \text{supp } \varphi_k \quad (\text{A.8})$$

holds for every  $k \in \mathbb{N}$ ; this is a property we will need later. Now extend each  $\varphi_k$  by zero to a smooth function on  $\mathbf{G}_1$  and denote the resulting element of  $\mathcal{A} \rtimes \mathbf{G}$  again by  $\varphi_k$ . Then put  $u_k := \sum_{l \leq k} \varphi_l * \varphi_k$  and check that  $(u_k)_{k \in \mathbb{N}}$  is an approximate identity. Moreover, the maps

$$\begin{aligned} l : \mathcal{A} \rtimes \mathbf{G} &\rightarrow \mathcal{A} \rtimes \mathbf{G} \hat{\otimes} \mathcal{A} \rtimes \mathbf{G}, \quad a \mapsto a * \varphi_k \otimes \varphi_k \quad \text{and} \\ r : \mathcal{A} \rtimes \mathbf{G} &\rightarrow \mathcal{A} \rtimes \mathbf{G} \hat{\otimes} \mathcal{A} \rtimes \mathbf{G}, \quad a \mapsto \sum_{k \in \mathbb{N}} \varphi_k \otimes \varphi_k * a \end{aligned}$$

are both sections of the convolution product. This proves

**Proposition A.6.** *The convolution algebra  $\mathcal{A} \rtimes \mathbf{G}$  of a proper étale Lie groupoid  $\mathbf{G}$  together with the von Neumann bornology is a quasi-unital bornological algebra.*

Next let us consider the case where  $\mathbf{G}_0$  carries a  $\mathbf{G}$ -invariant symplectic form  $\omega$  and where a  $\mathbf{G}$ -invariant local star product  $\star$  on  $\mathbf{G}_0$  has been chosen. Under these assumptions consider the crossed product algebra  $\mathcal{A}^{\hbar} \rtimes \mathbf{G}$ , where  $\mathcal{A}^{\hbar}$  denotes the sheaf  $\mathcal{C}_{\mathbf{G}_0}^{\infty}[[\hbar]]$  with product  $\star$ . A subset  $B \subset \mathcal{A}^{\hbar} \rtimes \mathbf{G}$  is said to be *bounded*, if there is a compact subset in  $K \subset \mathbf{G}_1$  such that  $\text{supp } a \subset K$  for every  $a \in B$ , and such that for each differential operator on  $\mathbf{G}_1$  and  $k \in \mathbb{N}$  one has

$$\sup_{a \in B} \|Da\|_{k,K} < \infty.$$

Hereby,  $\|\cdot\|_{k,K}$  is the seminorm on  $\mathcal{A}^{\hbar} \rtimes \mathbf{G}$  defined by

$$\|a\|_{k,K} := \sup_{g \in K} |a_k(g)|, \quad a \in \mathcal{A}^{\hbar} \rtimes \mathbf{G},$$

where the  $a_l \in \mathcal{C}_{\text{cpt}}^{\infty}(\mathbf{G}_1)$ ,  $l \in \mathbb{N}$  are the unique coefficients in the formal power series expansion  $a = \sum_{l \in \mathbb{N}} a_l \hbar^l$ . The bounded subsets of  $\mathcal{A}^{\hbar} \rtimes \mathbf{G}$  define a complete bornology which we call the *canonical bornology* on  $\mathcal{A}^{\hbar} \rtimes \mathbf{G}$ . One immediately checks that the convolution product  $\star_c$  defined by Eq. (2.2) is bounded, hence  $\mathcal{A}^{\hbar} \rtimes \mathbf{G}$  is a bornological algebra. Obviously, the family  $(u_k)_{k \in \mathbb{N}}$  from above forms an approximate unit also for  $\mathcal{A}^{\hbar} \rtimes \mathbf{G}$ . By the assumption (A.8) it is clear that each of the functions  $\varphi_k$  has only zeros of infinite order. Now check the following lemma by using standard arguments from the theory of deformation quantization.

**Lemma A.7.** *Let  $\varphi : \mathbf{G}_0 \rightarrow [0, 1]$  be a smooth function which has only zeros of infinite order, and put  $u = \varphi^2$ . Then  $u$  has a star product root, that means there exists an element  $\Phi = \sum_{l \in \mathbb{N}} \Phi_l \hbar^l \in \mathcal{C}^{\infty}[[\hbar]]$  such that*

$$\Phi \star \Phi = u, \quad \Phi_0 = \varphi, \quad \text{and} \quad \text{supp } \Phi \subset \text{supp } \varphi.$$

Using this result choose  $\Phi_k \in \mathcal{A}^{\hbar} \rtimes \mathbf{G}$  with support in  $\mathbf{G}_0$  such that  $\Phi_k \star_c \Phi_k = \varphi_k^2$  and  $\Phi_k - \varphi_k \in \hbar \mathcal{A}^{\hbar} \rtimes \mathbf{G}$ . The maps

$$\begin{aligned} l : \mathcal{A}^{\hbar} \rtimes \mathbf{G} &\rightarrow \mathcal{A}^{\hbar} \rtimes \mathbf{G} \hat{\otimes} \mathcal{A}^{\hbar} \rtimes \mathbf{G}, \quad a \mapsto \sum_{k \in \mathbb{N}} a \star_c \Phi_k \otimes \Phi_k \quad \text{and} \\ r : \mathcal{A}^{\hbar} \rtimes \mathbf{G} &\rightarrow \mathcal{A}^{\hbar} \rtimes \mathbf{G} \hat{\otimes} \mathcal{A}^{\hbar} \rtimes \mathbf{G}, \quad a \mapsto \sum_{k \in \mathbb{N}} \Phi_k \otimes \Phi_k \star_c a \end{aligned}$$

then are both sections of the convolution product. Hence we obtain

**Proposition A.8.** *The crossed product algebra  $\mathcal{A}^{\hbar} \rtimes \mathbf{G}$  associated to an invariant local deformation quantization on the space of objects of a proper étale Lie groupoid  $\mathbf{G}$  with an invariant symplectic form is a quasi-unital bornological algebra.*

**A.7. Morita equivalence for bornological algebras.** Assume that  $A$  and  $B$  are two bornological algebras. Recall that by an  $A$ - $B$ -bimodule one understands an element of the category  $\text{Mod}(A^u \hat{\otimes} (B^u)^{\text{op}})$ . Under the condition that the bornological algebras  $A$  and  $B$  are both quasi-unital, one calls  $A$  and  $B$  *Morita equivalent*, if there exist bimodules  $P \in \text{Mod}(A^u \hat{\otimes} (B^u)^{\text{op}})$  and  $Q \in \text{Mod}(B^u \hat{\otimes} (A^u)^{\text{op}})$  such that the following axioms hold true:

- (MOR1)  $P$  is essential both as an  $A$ -left module and as a  $B$ -right module.
- (MOR2)  $Q$  is essential both as a  $B$ -left module and as an  $A$ -right module.
- (MOR3) There exist bounded bimodule isomorphisms

$$u : P \hat{\otimes}_B Q \rightarrow A \quad \text{and} \quad v : Q \hat{\otimes}_A P \rightarrow B.$$

- (MOR4)  $P$  is projective as a  $B$ -right module, and  $Q$  is projective as an  $A$ -right module.

We sometimes say in this situation that  $(A, B, P, Q, u, v)$  is a *Morita context*. The following result follows easily from the definition of a Morita context.

**Proposition A.9.** *Let  $(A, B, P, Q, u, v)$  be a Morita context. Then the functors*

$$\begin{aligned} \text{Mod}(A) &\rightarrow \text{Mod}(B), & M &\mapsto Q \hat{\otimes}_A M \\ \text{Mod}(B) &\rightarrow \text{Mod}(A), & N &\mapsto P \hat{\otimes}_B N \end{aligned}$$

*are both exact and quasi-inverse to each other. In particular this means that  $\text{Mod}(A)$  and  $\text{Mod}(B)$  are equivalent categories.*

**Example A.10.** Let  $\varphi : \mathbf{H} \rightarrow \mathbf{G}$  be a weak equivalence of proper étale Lie groupoids. By [MR, Cor. 3.2] it follows that the convolution algebras  $A := \mathcal{A} \rtimes \mathbf{G}$  and  $B := \mathcal{A} \rtimes \mathbf{H}$  are Morita equivalent. A Morita context is given by the bimodules  $P = \mathcal{C}_{\text{cpt}}^\infty(\langle \varphi \rangle)$  and  $Q = \mathcal{C}_{\text{cpt}}^\infty(\langle \varphi \rangle^-)$ , where  $\langle \varphi \rangle := \mathbf{G}_1 \times_{(s, \varphi)} \mathbf{H}_0$  and  $\langle \varphi \rangle^- := \mathbf{H}_0 \times_{(\varphi, t)} \mathbf{G}_1$ . Let us provide the details for the case, where  $\varphi$  is even an open embedding. Then,  $\langle \varphi \rangle$  is the open subset  $s^{-1}(\varphi(\mathbf{H}_0)) \subset \mathbf{G}_1$ , and  $\langle \varphi \rangle^- = t^{-1}(\varphi(\mathbf{H}_0)) \subset \mathbf{G}_1$ . Moreover, the  $A$ - $B$ -bimodule structure on  $P$  is given by

$$a * p * b(g) = \sum_{\substack{g_1, g_2 \in \langle \varphi \rangle \\ g_1 \varphi(g_2) = g}} a(g_1) p(g_2) b(g),$$

where  $a \in \mathcal{C}_{\text{cpt}}^\infty(\mathbf{G}_1)$ ,  $b \in \mathcal{C}_{\text{cpt}}^\infty(\mathbf{H}_1)$  and  $p \in \mathcal{C}_{\text{cpt}}^\infty(s^{-1}(\varphi(\mathbf{H}_1)))$ . The bimodule structure for  $Q$  is defined analogously.

**Theorem A.11.** *Under the assumption that the weak equivalence  $\varphi : \mathbf{H} \hookrightarrow \mathbf{G}$  is an open embedding, the following holds true:*

- (1) *The bimodules  $P = \mathcal{C}_{\text{cpt}}^\infty(s^{-1}(\varphi(\mathbf{H}_0)))$  and  $Q = \mathcal{C}_{\text{cpt}}^\infty(t^{-1}(\varphi(\mathbf{H}_0)))$  satisfy axioms (MOR1) and (MOR2).*
- (2)  *$P$  resp.  $Q$  is projective both as an  $A$ - as a  $B$ -module.*
- (3) *The maps*

$$\begin{aligned} u : \mathcal{C}_{\text{cpt}}^\infty(s^{-1}(\varphi(\mathbf{H}_0))) \hat{\otimes}_{\mathcal{C}_{\text{cpt}}^\infty(\mathbf{H}_1)} \mathcal{C}_{\text{cpt}}^\infty(t^{-1}(\varphi(\mathbf{H}_0))) &\rightarrow \mathcal{C}_{\text{cpt}}^\infty(\mathbf{G}_1), \quad a \otimes \tilde{a} \mapsto a * \tilde{a}, \\ v : \mathcal{C}_{\text{cpt}}^\infty(t^{-1}(\varphi(\mathbf{H}_0))) \hat{\otimes}_{\mathcal{C}_{\text{cpt}}^\infty(\mathbf{G}_1)} \mathcal{C}_{\text{cpt}}^\infty(s^{-1}(\varphi(\mathbf{H}_0))) &\rightarrow \mathcal{C}_{\text{cpt}}^\infty(\mathbf{H}_1), \quad \tilde{a} \otimes a \mapsto \varphi^*(\tilde{a} * a). \end{aligned}$$

*are bounded bimodule isomorphisms.*

*This means in particular that the tuple  $(\mathcal{A} \rtimes \mathbf{G}, \mathcal{A} \rtimes \mathbf{H}, P, Q, u, v)$  is a Morita context between bornological algebras.*

*Proof.* Consider the family  $(\varphi_k)_{k \in \mathbb{N}}$  of elements of  $A = \mathcal{A} \rtimes \mathbf{G}$  from above. Then the map

$$l_P : P \rightarrow A \hat{\otimes} P, p \mapsto \sum_k p * \varphi_k \otimes \varphi_k \quad \text{resp.}$$

$$r_Q : Q \rightarrow Q \hat{\otimes} A, q \mapsto \sum_k \varphi_k \otimes \varphi_k * q$$

is a section of the left (resp. right)  $A$ -action on  $P$  (resp.  $Q$ ). Since  $A$  is quasi-unital, this implies that  $P$  (resp.  $Q$ ) is essential and projective as a left (resp. right)  $A$ -module. Likewise, one proves the existence of a section  $l_Q$  (resp.  $r_P$ ) of the left (resp. right)  $B$ -action on  $Q$  (resp.  $P$ ). Hence,  $Q$  (resp.  $P$ ) is essential and projective as a left (resp. right)  $B$ -module. Thus, we have proved (1) and (2).

Next we show that there exists a bounded section of the map

$$\hat{v} : \mathcal{C}_{\text{cpt}}^\infty(t^{-1}(\varphi(\mathbf{H}_0))) \hat{\otimes} \mathcal{C}_{\text{cpt}}^\infty(s^{-1}(\varphi(\mathbf{H}_0))) \rightarrow \mathcal{C}_{\text{cpt}}^\infty(\mathbf{H}_1), \tilde{a} \otimes a \mapsto \varphi^*(\tilde{a} * a).$$

To this end choose a family  $(\psi_k)_{k \in \mathbb{N}}$  in  $B = \mathcal{A} \rtimes \mathbf{H}$  which has support in  $\mathbf{H}_0$ , and such that  $(\psi_k^2)_{k \in \mathbb{N}}$  is a locally finite partition of unity on  $\mathbf{H}_0$ . For each element  $b \in B$  define elements  $\varphi_* b$  in  $P$  and  $Q$  by extension by zero. Then the map

$$\hat{v} : \mathcal{C}_{\text{cpt}}^\infty(\mathbf{H}_1) \rightarrow \mathcal{C}_{\text{cpt}}^\infty(t^{-1}(\varphi(\mathbf{H}_0))) \hat{\otimes} \mathcal{C}_{\text{cpt}}^\infty(s^{-1}(\varphi(\mathbf{H}_0))), b \mapsto \sum_{k \in \mathbb{N}} \varphi_* b * \psi_k \otimes \varphi_* \psi_k$$

is a bounded section of  $\hat{v}$  and a morphism of left  $B$ -modules. Hence  $\nu := \pi \hat{v}$ , where  $\pi : Q \hat{\otimes} P \rightarrow Q \hat{\otimes}_A P$  denotes the canonical projection, is a bounded section of  $v$ . Note that by construction the image of  $\nu$  lies in the algebraic tensor product  $Q \otimes_A P$ , and that the image of  $\nu$  is a complete bornological subspace of  $Q \otimes_A P$  which has to be isomorphic to  $B$ . Since by [MR, Thm. 4] the restriction to the algebraic tensor product  $v|_{Q \otimes_A P} : Q \otimes_A P \rightarrow B$  is an (algebraic) isomorphism of  $B$ -bimodules, one then concludes that  $Q \otimes_A P = Q \hat{\otimes}_A P$  and that  $v$  is a (bornological) isomorphism of  $B$ -bimodules.

The proof that  $u$  is a (bornological) isomorphism of  $A$ -bimodules is more complicated. We show this claim under the additional assumption that  $\mathbf{H}_0$  is connected. The general case is slightly more involved, but can be proved along the same lines. Denote by  $\mathbf{G}_\alpha$  the connected components of  $\mathbf{G}_0$ , and by  $\mathbf{G}_{\alpha_0}$  the image  $\varphi(\mathbf{H}_0)$ . Let  $\mathbf{G}_{\alpha,\beta} = s^{-1}(\mathbf{G}_\alpha) \cap t^{-1}(\mathbf{G}_\beta)$ . Then  $\mathbf{G}_1$  is the disjoint union of the  $\mathbf{G}_{\alpha,\beta}$ . Since  $\varphi$  is a weak equivalence, one derives the following:

- (i) The image  $\varphi(\mathbf{H}_1)$  coincides with  $\mathbf{G}_{\alpha_0,\alpha_0}$ .
- (ii) The bibundle  $\langle \varphi \rangle = s^{-1}(\varphi(\mathbf{H}_0))$  coincides with the disjoint union of the components  $\mathbf{G}_{\alpha_0,\alpha}$ , and  $\langle \varphi \rangle^- = t^{-1}(\varphi(\mathbf{H}_0))$  with the disjoint union of the components  $\mathbf{G}_{\alpha,\alpha_0}$ .
- (iii) There exist unique open embeddings  $\sigma_{\alpha,\beta} : \mathbf{G}_{\alpha,\beta} \hookrightarrow \mathbf{G}_{\alpha,\alpha_0}$  and  $\tau_{\alpha,\beta} : \mathbf{G}_{\alpha,\beta} \hookrightarrow \mathbf{G}_{\alpha_0,\beta}$  such that  $s \circ \sigma_{\alpha,\beta} = s|_{\mathbf{G}_{\alpha,\beta}}$  resp.  $t \circ \tau_{\alpha,\beta} = t|_{\mathbf{G}_{\alpha,\beta}}$ .

Next choose for every  $\beta$  functions  $\psi_{\beta,k} \in \mathcal{C}_{\text{cpt}}^\infty(\mathbf{G}_{\beta,\beta} \cap \mathbf{G}_0)$  such that each family  $(\psi_{\beta,k}^2)_{k \in \mathbb{N}}$  is a locally finite partition of unity. Extend  $\psi_{\beta,k}$  by 0 to a smooth function with compact support in  $\mathbf{G}_{\beta,\beta}$ . Define  $\psi_{\beta,k}^{(1)} \in \mathcal{C}_{\text{cpt}}^\infty(\mathbf{H}_1)$  and  $\psi_{\beta,k}^{(2)} \in \mathcal{C}_{\text{cpt}}^\infty(t^{-1}(\varphi(\mathbf{H}_0)))$  by

$$\psi_{\beta,k}^{(1)} = \begin{cases} \varphi^*((\tau_{\alpha_0,\beta} \sigma_{\beta,\beta}) * (\psi_{\beta,k})) & \text{over } \mathbf{H}_0 \\ 0 & \text{else.} \end{cases}$$

and

$$\psi_{\beta,k}^{(2)} = \begin{cases} (\tau_{\beta,\beta})_*(\psi_{\beta,k}) & \text{over } \mathbf{G}_{\alpha_0,\beta} \\ 0 & \text{else.} \end{cases}$$

Now we have the means to construct a bounded section of the map

$$\hat{u} : \mathcal{C}_{\text{cpt}}^\infty(s^{-1}(\varphi(\mathbf{H}_0))) \hat{\otimes} \mathcal{C}_{\text{cpt}}^\infty(t^{-1}(\varphi(\mathbf{H}_0))) \rightarrow \mathcal{C}_{\text{cpt}}^\infty(\mathbf{G}_1), \quad a \otimes \tilde{a} \mapsto a * \tilde{a}.$$

Put

$$\begin{aligned} \hat{\mu} : \mathcal{C}_{\text{cpt}}^\infty(\mathbf{G}_1) &\rightarrow \mathcal{C}_{\text{cpt}}^\infty(s^{-1}(\varphi(\mathbf{H}_0))) \hat{\otimes} \mathcal{C}_{\text{cpt}}^\infty(t^{-1}(\varphi(\mathbf{H}_0))), \\ \mathcal{C}_{\text{cpt}}^\infty(\mathbf{G}_{\alpha,\beta}) \ni a &\mapsto \sum_{k \in \mathbb{N}} (\sigma_{\alpha,\beta})_*(a) * \psi_{\beta,k}^{(1)} \otimes \psi_{\beta,k}^{(2)}. \end{aligned}$$

and check that  $\hat{\mu}$  is a bounded section of  $\hat{u}$  indeed. One proceeds now exactly as for  $v$  to show that  $P \otimes_B Q = P \hat{\otimes}_B Q$  and that  $u$  is a bornological isomorphism of  $A$ -bimodules.  $\square$

Hochschild (co)homology of bornological algebras and their bimodules is invariant under a Morita context as the following result shows.

**Theorem A.12** (cf. [LO, Thm. 1.2.7]). *Assume that  $A$  and  $B$  are quasi-unital bornological algebras and assume that there is a Morita context  $(A, B, P, Q, u, v)$  with the additional property that*

$$qu(p \otimes q') = v(q \otimes p)q' \text{ and } pv(q \otimes p') = u(p \otimes q)p' \text{ for all } p, p' \in P \text{ and } q, q' \in Q. \quad (\text{A.9})$$

Let  $M$  be an essential  $A$ -bimodule. Then there are natural isomorphisms

$$H_\bullet(A, M) \cong H_\bullet(B, Q \hat{\otimes}_A M \hat{\otimes}_A P) \quad \text{and} \quad H^\bullet(A, M) \cong H^\bullet(B, Q \hat{\otimes}_A M \hat{\otimes}_A P) \quad (\text{A.10})$$

*Proof.* We only prove the homology case. The cohomology case is proven similarly. To prove the claim first choose approximate identities  $(u_{S,k})_{S \in \mathcal{B}, k \in \mathbb{N}}$  for  $A$  and  $(v_{T,l})_{T \in \mathcal{D}, l \in \mathbb{N}}$  for  $B$ , where  $\mathcal{B}$  and  $\mathcal{D}$  are the bornologies of  $A$  and  $B$  respectively. Since  $P$  and  $Q$  are essential modules over  $A$  and  $B$ , there exists a bounded section  $\hat{\mu} : A \rightarrow P \hat{\otimes} Q$  (resp.  $\hat{\nu} : B \rightarrow Q \hat{\otimes} P$ ) of the composition of  $u : P \hat{\otimes}_B Q \rightarrow A$  with the canonical projection  $P \hat{\otimes} Q \rightarrow P \hat{\otimes}_B Q$  (resp. of  $V : Q \hat{\otimes}_A P \rightarrow B$  with  $Q \hat{\otimes} P \rightarrow Q \hat{\otimes}_A P$ ). The section  $\hat{\mu}$  (resp.  $\hat{\nu}$ ) can even be chosen to be a morphism of left  $A$ -modules (resp. of left  $B$ -modules). For every bounded  $S \subset A$  and  $k \in \mathbb{N}$  (resp. bounded  $T \subset B$  and  $l \in \mathbb{N}$ ) let

$$\sum_{i \in \mathbb{N}} p_{S,k,i} \otimes q_{S,k,i} := \hat{\mu}(u_{S,k}) \text{ and } \sum_{j \in \mathbb{N}} q'_{T,l,j} \otimes p'_{T,l,j} := \hat{\nu}(v_{T,l}).$$

Then define for every  $n \in \mathbb{N}$  a bounded map  $\varrho_n : M \hat{\otimes} A^{\hat{\otimes} n} \rightarrow (Q \hat{\otimes}_A M \hat{\otimes}_A P) \hat{\otimes} B^{\hat{\otimes} k}$  by

$$\begin{aligned} \varrho_n(m \otimes a_1 \otimes \dots \otimes a_n) = \\ \lim_{(S,k) \in \mathcal{B} \times \mathbb{N}} \sum_{i_0, i_1, \dots, i_n} (q_{S,k,i_0} \otimes m \otimes p_{S,k,i_1}) \otimes v(q_{S,k,i_1} \otimes a_1 p_{S,k,i_2}) \otimes \dots \\ \dots \otimes v(q_{S,k,i_n} \otimes a_n p_{S,k,i_0}). \end{aligned}$$

Note that this map is well-defined since  $(u_{S,k})$  is an approximate identity,  $\hat{\mu}$  is a bounded morphism of left  $A$ -modules, and since  $M$  is essential. Analogously we define maps  $\theta_n : (Q \hat{\otimes}_A M \hat{\otimes}_A P) \hat{\otimes} B^{\hat{\otimes} k} \rightarrow M \hat{\otimes} A^{\hat{\otimes} n}$  by

$$\begin{aligned} \theta_n (q \otimes m \otimes p \otimes b_1 \otimes \dots \otimes b_n) = \\ \lim_{(T,l) \in \mathcal{D} \times \mathbb{N}} \sum_{j_0, j_1, \dots, j_n} u(p'_{T,l,j_0} \otimes q) m u(p \otimes q'_{T,l,j_1}) \otimes u(p'_{T,l,j_1} \otimes b_1 q'_{T,l,j_2}) \otimes \dots \\ \dots \otimes u(p'_{T,l,j_n} \otimes b_n q'_{T,l,j_0}). \end{aligned}$$

Again, convergence is guaranteed by the fact that  $A$  and  $B$  are quasi-unital and that  $M$  is essential. By assumption (A.9) on  $u$  and  $v$  the maps  $\varrho$  and  $\theta$  are both chain maps. The composites  $\theta \circ \psi$  and  $\varrho \circ \theta$  are homotopic to the identity. A simplicial homotopy between  $\theta \circ \varrho$  and the identity is given by

$$\begin{aligned} h_i (m \otimes a_1 \otimes \dots \otimes a_n) = \\ \lim_{\substack{(S,k) \in \mathcal{B} \times \mathbb{N} \\ (T,l) \in \mathcal{D} \times \mathbb{N}}} \sum_{\substack{i_0, i_1, \dots, i_i \\ j_0, j_1, \dots, j_i}} m u(p_{S,k,i_0} \otimes q'_{T,l,j_0}) \otimes u(p'_{T,l,j_0} \otimes q_{S,k,i_0}) a_1 u(p_{S,k,i_1} \otimes q'_{T,l,j_1}) \otimes \\ \dots \otimes u(p'_{T,l,j_{i-2}} \otimes q_{S,k,i_{i-2}}) a_{i-1} u(p_{S,k,i_{i-1}} \otimes q'_{T,l,j_{i-1}}) \otimes \\ \otimes u(p'_{T,l,j_{i-1}} \otimes q_{S,k,i_{i-1}}) \otimes a_i \otimes a_{i+1} \otimes \dots \otimes a_n. \end{aligned}$$

By a straightforward though somewhat lengthy argument (cf. [HA, Chap. 5]) one checks that the thus defined  $h_i$  are well-defined and form a bounded simplicial homotopy indeed. Similarly, one constructs a bounded homotopy between  $\psi \circ \theta$  and the identity.

Since  $[\varrho] = [\theta]^{-1}$ , where  $[\varrho]$  and  $[\theta]$  denote the induced maps on the Hochschild homology of  $\mathcal{A} \rtimes \mathbb{G}$  resp.  $\mathcal{A} \rtimes \mathbb{H}$ , and since  $\theta$  neither depends on the particular choice of an approximate identity on  $\mathcal{A} \rtimes \mathbb{G}$  nor on the choice of the elements  $p_{S,k,i}$  and  $q_{S,k,i}$ , one concludes that  $[\varrho]$  is independent of the particular choice of these data. Likewise one shows that  $[\theta]$  does not depend on the choice of an approximate identity on  $\mathcal{A} \rtimes \mathbb{H}$  and of the elements  $p'_{T,l,j}$  and  $q'_{T,l,j}$ . One concludes from this that  $[\theta]$  and  $[\varrho]$  are natural isomorphisms. This finishes the proof of the claim.  $\square$

**Remark A.13.** If one represents the Hochschild cohomology groups  $H^n(A, A)$  as equivalence classes of bounded extensions

$$0 \rightarrow A \rightarrow E_1 \rightarrow \dots \rightarrow E_k \rightarrow A \rightarrow 0, \quad (\text{A.11})$$

the isomorphism  $H^\bullet(A, A) \rightarrow H^\bullet(B, B)$  between the Hochschild cohomologies of two Morita equivalent bounded algebras is given by tensoring (A.11) with  $P$  and  $Q$ , i.e. by mapping it to the bounded extension

$$0 \rightarrow B \rightarrow Q \hat{\otimes}_A E_1 \hat{\otimes}_A P \rightarrow \dots \rightarrow Q \hat{\otimes}_A E_k \hat{\otimes}_A P \rightarrow B \rightarrow 0,$$

Let us now apply the preceding theorem to the situation of Example A.10. Then we can prove the following result.

**Theorem A.14.** *Assume that  $\varphi : \mathbb{H} \hookrightarrow \mathbb{G}$  is a weak equivalence of proper étale Lie groupoids which also is an open embedding, and let  $(\mathcal{A} \rtimes \mathbb{G}, \mathcal{A} \rtimes \mathbb{H}, P, Q, u, v)$  denote the Morita context from Theorem A.11. The resulting natural isomorphisms in Hochschild homology and cohomology are implemented by the following natural*

chain maps

$$\begin{aligned} \varphi_* : C_\bullet(\mathcal{A} \rtimes \mathbf{H}, \mathcal{A} \rtimes \mathbf{H}) &\rightarrow C_\bullet(\mathcal{A} \rtimes \mathbf{G}, \mathcal{A} \rtimes \mathbf{G}), \\ a_0 \otimes a_1 \otimes \dots \otimes a_n &\mapsto \varphi_*(a_0) \otimes \varphi_*(a_1) \otimes \dots \otimes \varphi_*(a_n) \end{aligned}$$

and

$$\begin{aligned} \varphi^* : C^\bullet(\mathcal{A} \rtimes \mathbf{G}, \mathcal{A} \rtimes \mathbf{G}) &\rightarrow C^\bullet(\mathcal{A} \rtimes \mathbf{H}, \mathcal{A} \rtimes \mathbf{H}), \\ F &\mapsto \left( (\mathcal{A} \rtimes \mathbf{H})^{\hat{\otimes} n} \ni a_1 \otimes \dots \otimes a_n \mapsto F(\varphi_*(a_1) \otimes \dots \otimes \varphi_*(a_n)) \circ \varphi \right). \end{aligned}$$

*Proof.* We only show the claim in the homology case. The claim for Hochschild cohomology can be proved by a dual argument.

As above let  $A = \mathcal{A} \rtimes \mathbf{G}$  and  $B = \mathcal{A} \rtimes \mathbf{H}$ . Choose a locally finite family  $(\psi_k)_{k \in \mathbb{N}} \in \mathcal{C}_{\text{cpt}}^\infty(\mathbf{H}_1)$  of smooth functions  $\psi_k : \mathbf{H}_1 \rightarrow [0, 1]$  with compact and connected support on  $\mathbf{H}_0$  such that  $(\psi_k^2)$  is a partition of unity on  $\mathbf{H}_0$ . Put for  $m \in \mathbb{N}$ :

$$\psi_{[m],k} := \begin{cases} \sqrt{\sum_{l=0}^m \psi_l^2} & \text{for } k = 0, \\ \psi_{k+m} & \text{for } k \geq 1. \end{cases} \quad (\text{A.12})$$

and

$$u_{[m],k} = \sum_{l=0}^k (\psi_{[m],l})^2. \quad (\text{A.13})$$

Then  $(u_{[m],k})_{k \in \mathbb{N}}$  is an approximate unit in  $\mathcal{A} \rtimes \mathbf{H}$  for every  $m \in \mathbb{N}$ . Denote by  $K_{[m]}$  the compact set  $\overline{(u_{[m],0}^{-1}(1))^\circ}$ , i.e. the closure of the set of all points  $x \in \mathbf{H}_0$  on a neighborhood of which  $u_{[m],0}$  (and thus  $\psi_{[m],0}$  also) has value 1. Then for every compact  $K \subset \mathbf{H}_0$  there exists an  $m_K \in \mathbb{N}$  such that  $K \subset K_{[m]}^\circ$  for all  $m \geq m_K$ . Hence one has

$$\psi_{[m],0}|_K = u_{[m],0}|_K = 1 \quad \text{for all } m \geq m_K. \quad (\text{A.14})$$

Let  $K_{[m]} = \bigcup_{i=0}^{j_m} K_{[m],i}$  be the decomposition of  $K_{[m]}$  in connected components, i.e. each  $K_{[m],i}$  is compact and connected and  $K_{[m],i} \neq K_{[m],i'}$  for  $i \neq i'$ . By the proof of Theorem A.11 one can construct for every fixed  $m \in \mathbb{N}$  a section

$$\hat{\nu}_{[m]} : \mathcal{C}_{\text{cpt}}^\infty(\mathbf{H}_1) \rightarrow \mathcal{C}_{\text{cpt}}^\infty(t^{-1}(\varphi(\mathbf{H}_0))) \hat{\otimes} \mathcal{C}_{\text{cpt}}^\infty(s^{-1}(\varphi(\mathbf{H}_0)))$$

and elements  $p'_{[m],0,j} \in \mathcal{C}_{\text{cpt}}^\infty(s^{-1}(\varphi(\mathbf{H}_0)))$ ,  $q'_{[m],0,j} \in \mathcal{C}_{\text{cpt}}^\infty(t^{-1}(\varphi(\mathbf{H}_0)))$ ,  $j = 0, \dots, j_m$  such that every  $p'_{[m],0,j}$  and every  $q'_{[m],0,j}$  has connected support in  $\varphi(\mathbf{H}_0)$ , such that

$$\hat{\nu}_{[m]}(u_{[m],0}) = \sum_{j=0}^{j_m} q'_{[m],0,j} \otimes p'_{[m],0,j} \quad (\text{A.15})$$

and finally such that

$$\begin{aligned} (p'_{[m],0,j})|_{\varphi(K_{[m],j'})} &= \begin{cases} 1 & \text{if } j = j', \\ 0 & \text{else,} \end{cases} \\ (q'_{[m],0,i})|_{\varphi(K_{[m],j'})} &= \begin{cases} 1 & \text{if } j = j', \\ 0 & \text{else.} \end{cases} \end{aligned} \quad (\text{A.16})$$

Put  $p'_{[m],0,j} = 0$  and  $q'_{[m],0,j} = 0$  for  $j > j_m$  and let

$$\sum_{j \in \mathbb{N}} q'_{[m],k,j} \otimes p'_{[m],k,j} := \hat{\nu}_{[m]}(u_{[m],k}) \quad \text{for } k \geq 1. \quad (\text{A.17})$$



For every fixed  $m \in \mathbb{N}$  we now have the data to construct the quasi-isomorphism  $\theta_{[m]} : C_{\bullet}(\mathcal{A} \rtimes \mathbf{H}, \mathcal{A} \rtimes \mathbf{H}) \rightarrow C_{\bullet}(\mathcal{A} \rtimes \mathbf{G}, \mathcal{A} \rtimes \mathbf{G})$  as in the proof of the preceding theorem. Note that the induced maps  $[\theta_{[m]}]$  between the Hochschild homology groups all coincide.

Now let  $K \subset \mathbf{H}_0$  be compact and consider

$$a_0, a_1, \dots, a_n \in \mathcal{C}_{\text{cpt}}^{\infty}(s^{-1}(K) \cap t^{-1}(K)) \subset \mathcal{A} \rtimes \mathbf{H}.$$

Then one for  $m \geq m_k$  by construction:

$$\begin{aligned} \theta_{[m]}(a_0 \otimes a_1 \otimes \dots \otimes a_n) &= \\ &= \sum_{j_0, \dots, j_n} u(p'_{[m],0,j_0} \otimes a_0 q'_{[m],0,j_1}) \otimes u(p'_{[m],0,j_1} \otimes a_1 q'_{[m],0,j_2}) \otimes \dots \\ &\quad \dots \otimes u(p'_{[m],0,j_n} \otimes a_n q'_{[m],0,j_0}) \\ &= \varphi_*(a_0) \otimes \varphi_*(a_1) \otimes \dots \otimes \varphi_*(a_n). \end{aligned}$$

Since  $\mathcal{A} \rtimes \mathbf{H}$  is the inductive limit of the subspaces  $\mathcal{C}_{\text{cpt}}^{\infty}(s^{-1}(K) \cap t^{-1}(K))$  the claim follows.  $\square$

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