The hand-in exercise is the exercise 3.

Please hand it in electronically at topologyinphysics2019@gmail.com (1 pdf!)

Deadline is Wednesday February 20, 23.59.

Please make sure your name and the week number are present in the file name.

**Exercise 1: Maxwell theory in differential form notation**

In this exercise we assume a flat Lorentzian metric, $g_{\mu\nu} = \eta_{\mu\nu}$. Recall that the current one-form can be written in components as $J = J_\mu dx^\mu = \rho dx^0 + j_i dx^i$.

a. Show that the inhomogeneous Maxwell equations of motion, $\nabla \cdot E = \rho$ and $\nabla \times B - \partial E/\partial t = j$ can be written as $d*F = *J$.

b. What does the equation $d^2 = 0$ imply for $\rho$ and $j$? Explain that the resulting equation can be interpreted as conservation of charge.

c. To gauge fix the Maxwell gauge symmetry, one often chooses the condition $\partial_\mu A^\mu = 0$. Write this condition in differential form notation.

**Exercise 2: Equations of motion for Maxwell theory**

Show that the Euler-Lagrange equations for the action $S = \int F \wedge *F + A \wedge *J$ are indeed Maxwell’s homogeneous equations of motion (2.49).

**Exercise 3: The Dirac monopole (hand-in exercise)**

In this exercise, we will work with three-dimensional polar coordinates $(r, \theta, \phi)$, defined by

\[
\begin{align*}
x &= r \sin \theta \cos \phi \\
y &= r \sin \theta \sin \phi \\
z &= r \cos \theta
\end{align*}
\]

in terms of the cartesian coordinates $(x, y, z)$. Of course, this map is not 1-to-1; in particular, $\theta$ and $\phi$ are only defined up to multiples of $2\pi$. The time coordinate $t$ will not play a role in this exercise, so you may assume we are working on $\mathbb{R}^3$.

We will consider the field strength

\[
F = \frac{g}{4\pi} \sin \theta \, d\theta \wedge d\phi
\]

with $g \neq 0$. 

1
a. Compute \( \int_{S^2} F \), where \( F \) is a two-sphere centered at the origin of \( \mathbb{R}^3 \).

b. Use the previous result and Stokes’ theorem to show that \( F \) can not be a closed form.

c. Naively computing \( dF \), one still seems to find \( dF = 0 \). How can this apparent contradiction with the result of (b) be understood?

d. One way to understand the result of (c) is to compute \( \star F \). Show that indeed \( \star F \) has a singularity at the origin of \( \mathbb{R}^3 \).

Summarizing, we have found that our field strength \( F \) is well-defined and closed only on \((\mathbb{R}^3)^* \equiv \mathbb{R}^3 \setminus \{0, 0, 0\}\). As this manifold is not topologically trivial, we cannot use Poincaré’s lemma to conclude that \( F = dA \) everywhere. However, if we define \( D^+ \) and \( D^- \) as the regions where \( \theta \neq \pi \) and \( \theta \neq 0 \) respectively (that is, \( D^+ \) is \( \mathbb{R}^3 \) excluding the negative \( z \)-axis, and \( D^- \) is \( \mathbb{R}^3 \) excluding the positive \( z \)-axis), these two regions are topologically trivial. On these regions, we now consider the 1-forms

\[
A^+ = \frac{g}{4\pi} (1 - \cos \theta) d\phi, \quad A^- = -\frac{g}{4\pi} (1 + \cos \theta) d\phi
\]

e. (Easy:) Show that \( F = dA^+ \) and \( F = dA^- \) in \( D^+ \) and \( D^- \) respectively.

f. Compute \( A^+ - A^- \). Where is this 1-form defined? In particular: is that space topologically trivial? Can \( A^+ \) be obtained from \( A^- \) using a gauge transformation?

One can slightly generalize the concept of a gauge transformation as follows. In quantum mechanics, one is interested in the wave function \( \psi(x) \) of a particle. Under a transformation of the potential

\[
A \rightarrow A + \omega
\]

with \( \omega \) a closed one-form, the wave function transforms as

\[
\psi(x) \rightarrow \psi(x) \exp \left( i \int_{\gamma} \omega \right)
\]

where \( \gamma \) is a path from an arbitrarily chosen base point to the point to \( x \). A large gauge transformation is a transformation of \( A \) by a one-form \( \omega \) such that the transformation of \( \psi(x) \) is well-defined.

g. In the previous sentence, “well-defined” means that the transformed value of \( \psi(x) \) should not depend on the choice of a path \( \gamma \). Argue that in a topologically trivial space, this condition is automatically satisfied if \( \omega \) is closed.

h. Which additional condition should \( \omega \) satisfy if the space is not topologically trivial? (In particular: if it is not simply connected?)

i. In our example, which condition should \( g \) satisfy so that \( A^+ \) and \( A^- \) are related by a large gauge transformation?
The upshot of this exercise is therefore that, assuming the condition found in (i) is satisfied, our field strength $F$ describes a “good” electromagnetic field configuration on $\mathbb{R}^3 \setminus \{0, 0, 0\}$. The interpretation of this configuration is that there is a “defect” at the singular point in the origin – an object which can be interpreted as a particle.

j. Show that this particle does not have an electric charge, but that by replacing the $E$-field with the $B$-field, it can be considered to have a “magnetic charge”. This particle (which has never been observed in nature!) is called the Dirac magnetic monopole.